THERMO-VISCO-ELASTICITY FOR NORTON-HOFF-TYPE MODELS WITH COSSERAT EFFECTS

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Abstract. We consider the quasi-static evolution of thermo-visco-elastic material. The main goal of this paper is to present how taking into account the additional effects may improve the result of solutions' existence. We added a micropolarity effect to thermo-visco-elastic model regarding Norton-Hoff-type constitutive function. This additional phenomenon improves the regularity of solution.

1. Introduction

This paper is devoted to extending thermo-visco-elastic model so that it includes micropolar effects. In the series of papers [7, 24–26] it has been shown that in the classical metal perfect plasticity models at infinitesimal strains a coupling with Cosserat elasticity may also regularize the ill-posedness of Prandtl-Reuss plasticity model. This is possible because Cosserat coupling leads to coercivity. Therefore, the question arises naturally, whether adding Cosserat microrotations to the thermo-visco-elastic model is still enough to regularize the problem in the way to satisfy the equations in a standard pointwise sense. From a modelling perspective, adding microrotations means to consider material made up of individual particles which can rotate and interact with each other [18, 19, 23, 27]. Adding Cosserat effects is arguably a physically motivated regularization for phenomenological polycrystalline plasticity: the individual crystal grains are rotating and interacting with each other.

The extension of thermo-visco-elastic model so that it includes Cosserat effects follows the lines proposed in [24], where authors added Cosserat effect to the classical elastoplasticity model with a monotone flow rule. In both the mentioned and our present approach only the elasticity relation is augmented with Cosserat effects, the plastic constitutive equation is left unchanged. Regarding the effect of Cosserat-modification for classical plasticity models, it has been proved that the new model is thermodynamically admissible and that there exists a unique, global in time solution to Cosserat elasto-plasticity. In [8] an H^1_{loc}- regularity result for the stresses and strains was proved, cf. [28].

It is well-known that the cyclic hardening plasticity model formulated by Armstrong and Frederick [3] is non-coercive, is of non-monotone type and not of gradient type (non-associated flow rule). Hence, the mathematical analysis is quite challenging, there are no encompassing existence results for this model in the literature. In the article [9] authors proposed an extension of Armstrong-Frederick model which includes micropolar effects. Quite a satisfactory result of the existence of solutions for this model was received. It was shown that the limit in Yosida approximation process satisfies this energy inequality. The limit functions have better

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regularity than previously known in the literature, where the original Armstrong-Frederick model has been studied. Moreover, in the article [10] the first regularity result for a new model was obtained. The similar results are expected for thermo-visco-elastic model with Cosserat effects included.

The thermo-visco-elastic model, which was considered in [15] is non-coercive and monotone, however in this model a different type of problems occur. Problem regarding low regularity of right-hand side of heat equation appears in thermo-visco-elastic models with Norton-Hoff-type constitutive function or with Mróz constitutive function, see [16]. In such case the standard methods fail. To deal with it we use Boccardo's and Gallouët's approach, for parabolic problem with Dirichlet boundary condition see [6], and for parabolic problem with Neumann boundary condition see [15]. Another approach which may be used here, is a renormalisation approach. These methods were derived for parabolic equation in [4, 5] and this problem was considered also in [20, 21].

This article presents the first existence result for a new model. It was shown that the weak solution has better regularity than in the article [15] and a number of very technical proofs regarding some theorems can be proved in the easier way than in [15]. In our opinion the new model and methods shown in this article are a starting point in the mathematical analysis of several problems from the theory of inelastic deformation with Cosserat effects in which the temperature affects the inelastic response of the considered material.

The derivation of thermo-visco-elastic model (without the Cosserat effect) was presented in [15]. This system of equations contains balance of momentum, balance of energy and two constitutive relations: definition of Cauchy stress tensor (Hooke's law) and evolutionary equation for visco-elastic deformation (flow rule). In [15] authors discussed all types of simplifications which are made to obtain this model, i.e. quasi-static evolution with small displacements, lack of material's thermal expansion and isochoric visco-elastic deformation. Here we add Cosserat effects to their model, which appear in our model in two places. Firstly, the appearance of microrotation implies that balance of angular momentum should be also taken into account. This equation describes changes of microrotation, see $(1.1)_{(3)}$. Secondly, the microrotation has also influence on the Cauchy stress tensor and it occurs by adding a term associated with this phenomenon.

Using the mechanical results for Cosserat plasticity (see for example [24], [25]) we conclude that we deal with the following initial-boundary value problem: let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class C^2. $\mathcal{S}^3$ and $\mathcal{S}^3_{\mathrm{dev}}$ denote the set of symmetric 3×3-matrices and the set of symmetric 3×3-matrices with vanishing trace, respectively. Moreover, $\mathfrak{so}(3)$ denotes the set of skew-symmetric 3×3 matrices. For fixed $T > 0$ we have to find the displacement field $u : \Omega \times [0, T] \to \mathbb{R}^3$, the temperature of the material $\theta : \Omega \times [0, T] \to \mathbb{R}_+$, the microrotation matrix $\boldsymbol{A} : \Omega \times [0, T] \to \mathfrak{so}(3)$ and the visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}}$:

$\Omega \times [0,T] \to \mathcal{S}^3_{\mathrm{dev}}$ satisfying the following system of equations

$$
\begin{aligned}
\mathrm{div}\boldsymbol{T} &= -F,\\
\boldsymbol{T} &= 2\mu(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + 2\mu_c(\mathrm{skew}(\nabla u) - \boldsymbol{A}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}})\mathbb{1},\\
-l_c\Delta \mathrm{axl}(\boldsymbol{A}) &= \mu_c \mathrm{axl}(\mathrm{skew}(\nabla u) - \boldsymbol{A}),\\
\boldsymbol{\varepsilon}^{\mathbf{p}}_t &= \boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E)),\\
\boldsymbol{T}^E &= 2\mu(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}})\mathbb{1},\\
\theta_t - \Delta\theta &= \mathrm{dev}(\boldsymbol{T}^E) : \boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E)),
\end{aligned}
\tag{1.1}
$$

where $\boldsymbol{\varepsilon}(u) = \mathrm{sym}(\nabla u)$ denotes the symmetric part of the gradient of the displacement and $\mathrm{dev}(\boldsymbol{T}^E)$ stands for deviatoric part of Cauchy stress tensor elastic part $\boldsymbol{T}^E$, i.e. $\mathrm{dev}(\boldsymbol{T}^E) = \boldsymbol{T}^E - \frac{1}{3}\mathrm{tr}(\boldsymbol{T}^E)\mathbb{1}$. The above equations are studied for $x \in \Omega$ and $t \in [0,T]$.

Notice that micropolar effects in the system (1.1) remain purely elastic. The function $F : \Omega \times [0,T] \to \mathbb{R}^3$ describes density of the applied body forces, parameters μ, λ are positive Lamé constants, $\mu_c > 0$ is the Cosserat couple modulus and $l_c > 0$ is a material parameter with dimensions $[m^2]$, describing a length scale of the model due to the Cosserat effects. The operator $\mathrm{skew}(\boldsymbol{T}) = \frac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^T)$ denotes the skew-symmetric part of a 3×3-tensor. The operator $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ establishes identification of a skew-symmetric matrix with vectors in $\mathbb{R}^3$. This means that if we take $\boldsymbol{A} \in \mathfrak{so}(3)$, which is in the form $\boldsymbol{A} = ((0,\alpha,\beta),(-\alpha,0,\gamma),(-\beta,-\gamma,0))$, then $\mathrm{axl}(\boldsymbol{A}) = (\alpha,\beta,\gamma)^T$.

The given function $\boldsymbol{G} : \mathbb{R}_+ \times \mathcal{S}^3_{\mathrm{dev}} \to \mathcal{S}^3_{\mathrm{dev}}$ is a generalisation of the flow rule regarding isochoric visco-elastic deformation with Norton-Hoff constitutive equation considered by Alber and Chełmiński in [1]. Assumptions on the function $\boldsymbol{G}$ are in the following form:

A1 The function $\boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E))$ is continuous with respect to θ and $\mathrm{dev}(\boldsymbol{T}^E)$.

A2 For all $\mathrm{dev}(\boldsymbol{T}^E_1)$, $\mathrm{dev}(\boldsymbol{T}^E_2) \in \mathcal{S}^3_{\mathrm{dev}}$ and $\theta \in \mathbb{R}_+$ the following inequality holds:

$$\Big(\boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E_1)) - \boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E_2))\Big) : \Big(\mathrm{dev}(\boldsymbol{T}^E_1) - \mathrm{dev}(\boldsymbol{T}^E_2)\Big) \geq 0.$$

A3 $|\boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E))| \leq C(1 + \mathrm{dev}(\boldsymbol{T}^E))^{p-1}$ for all $\mathrm{dev}(\boldsymbol{T}^E) \in \mathcal{S}^3_{\mathrm{dev}}$ and $\theta \in \mathbb{R}_+$, where the constant $C > 0$ does not depend on θ ($p \geq 2$).

A4 $\boldsymbol{G}(\theta, \mathrm{dev}(\boldsymbol{T}^E)) : \mathrm{dev}(\boldsymbol{T}^E) \geq \alpha\,|\,\mathrm{dev}(\boldsymbol{T}^E)|^p$ for all $\mathrm{dev}(\boldsymbol{T}^E) \in \mathcal{S}^3_{\mathrm{dev}}$ and $\theta \in \mathbb{R}_+$, where the constant $\alpha > 0$ does not depend on θ ($p \geq 2$).

The system (1.1) is considered with Dirichlet boundary condition for displacement and microrotation

$$
\begin{aligned}
u(x,t) &= g_D(x,t) \qquad \text{for } x \in \partial\Omega \text{ and } t \geq 0,\\
\boldsymbol{A}(x,t) &= \boldsymbol{A}_D(x,t) \qquad \text{for } x \in \partial\Omega \text{ and } t \geq 0,
\end{aligned}
\tag{1.2}
$$

and with Neumann boundary condition for the temperature

$$\frac{\partial\theta}{\partial n}(x,t) = g_\theta(x,t) \qquad \text{for } x \in \partial\Omega \text{ and } t \geq 0, \tag{1.3}$$

where n denotes the outer normal vector to the boundary $\partial\Omega$. Finally, we consider system (1.1) with the following initial conditions

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,0) = \boldsymbol{\varepsilon}^{\mathbf{p},0}(x), \qquad \theta(x,0) = \theta^0(x). \tag{1.4}$$

2. Main result

Here we deal with developing the definitions of the solution for the system (1.1). Next we formulate the main result of this paper. Let us start with the notation which is used in definition of weak solution of the system (1.1). We define

$$
\begin{aligned}
W^{1,2}_{g_D}(\Omega;\mathbb{R}^3) &:= \left\{u \in W^{1,2}(\Omega,\mathbb{R}^3) : u = g_D \text{ on } \partial\Omega\right\}, \\
W^{1,2}_{\boldsymbol{A}_D}(\Omega;\mathfrak{so}(3)) &:= \left\{\boldsymbol{A} \in W^{1,2}(\Omega;\mathfrak{so}(3)) : \boldsymbol{A} = \boldsymbol{A}_D \text{ on } \partial\Omega\right\}.
\end{aligned} \tag{2.5}
$$

Definition 2.1 (Solution concept)**.** *Fix $T > 0$. Let us assume that $p \geq 2$ and $q \leq \frac{5}{4}$. We say that a vector*

$$(u, \boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\varepsilon}^{\mathbf{p}}) \in L^\infty(0,T; W^{1,2}_{g_D}(\Omega;\mathbb{R}^3) \times W^{1,2}_{\boldsymbol{A}_D}(\Omega;\mathfrak{so}(3)) \times L^2(\Omega;\mathbb{R}^9) \times L^2(\Omega;\mathcal{S}^3_{\text{dev}}))$$

and function

$$\theta \in L^q(0,T; W^{1,q}(\Omega)) \cap C([0,T], W^{-2,2}(\Omega))$$

are a weak solution of the system (1.1) if

(1) *functions u, $\boldsymbol{A}$ and $\boldsymbol{T}$ satisfy the system of equations:*

$$
\begin{aligned}
&\int_0^T\int_\Omega \boldsymbol{T} : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega F\cdot\phi\,\mathrm{d}x\,\mathrm{d}t,\\
&\boldsymbol{T} = 2\mu(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + 2\mu_c(\operatorname{skew}(\nabla u) - \boldsymbol{A}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}})\mathbb{1},\\
&l_c\int_0^T\int_\Omega \nabla\operatorname{axl}(\boldsymbol{A}) : \nabla\operatorname{axl}(\tilde{\boldsymbol{\phi}})\,\mathrm{d}x\,\mathrm{d}t = \mu_c\int_0^T\int_\Omega \operatorname{axl}\Big(\operatorname{skew}(\nabla u) - \boldsymbol{A}\Big)\cdot\operatorname{axl}(\tilde{\boldsymbol{\phi}})\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where the first equation holds for all $\phi \in C^\infty([0,T]; C^\infty_0(\Omega;\mathbb{R}^3))$ and the third one is satisfied for all $\tilde{\boldsymbol{\phi}} \in C^\infty([0,T]; C^\infty_0(\Omega;\mathfrak{so}(3)))$.

(2) *the function θ satisfies the following equation*

$$
\begin{aligned}
-\int_0^T\int_\Omega \theta\varphi_t\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_0(x)\varphi(0,x)\,\mathrm{d}x + \int_0^T\int_\Omega \nabla\theta : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t&\\
-\int_0^T\int_{\partial\Omega} g_\theta\varphi\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \operatorname{dev}(\boldsymbol{T}^E) : \boldsymbol{G}(\theta,\operatorname{dev}(\boldsymbol{T}^E))\varphi\,\mathrm{d}x\,\mathrm{d}t&,
\end{aligned} \tag{2.1}
$$

where $\varphi \in C^\infty_0([-\infty,T); C^\infty(\Omega))$.

(3) *the function $\boldsymbol{\varepsilon}^p$ satisfies the following equation*

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t) = \boldsymbol{\varepsilon}^{\mathbf{p},0}(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau), \operatorname{dev}(\boldsymbol{T}^E)(x,\tau))\,\mathrm{d}\tau, \tag{2.2}$$

for almost all $x \in \Omega$ and $t \in [0,T)$. Furthermore, $\boldsymbol{\varepsilon}^{\mathbf{p}}_t \in L^{p'}(0,T; L^{p'}(\Omega;\mathcal{S}^3_{\text{dev}}))$.

Theorem 2.2 (Main existence result)**.** *Let us assume that $F \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))$, $g_D \in L^\infty(0,T; H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3))$, $\boldsymbol{A}_D \in L^\infty(0,T; H^{\frac{3}{2}}(\partial\Omega;\mathfrak{so}(3)))$, $g_\theta \in L^2(0,T; L^2(\partial\Omega))$, $\theta_0 \in L^2(\Omega)$ and function $\boldsymbol{G}$ satisfies the assumptions* **A1**–**A4***. Then there exists a weak solution (in the sense of Definition 2.1) of the system (1.1)–(1.4).*

In the proof we use two-level Galerkin approximation to construct the approximated solution with independent parameters associated with number of basis elements used to construct the approximate solution to displacement and temperature. This is due to the fact that right-hand side of heat equation is only an integrable function and to prove the existence of its solutions we use Boccardo's & Gallouët's approach, see [6, 15]. In the limit passages we also have to characterize the weak limit on nonlinear function which is done by usage of Minty-Browder trick.

3. Transformation to a homogeneous boundary-value problem

Let us consider the following linear elliptic system

$$
\begin{aligned}
\mathrm{div}\tilde{\boldsymbol{T}} &= -F,\\
\tilde{\boldsymbol{T}} &= 2\mu\,\boldsymbol{\varepsilon}(\tilde{u}) + 2\mu_c(\mathrm{skew}(\nabla\tilde{u}) - \tilde{\boldsymbol{A}}) + \lambda\mathrm{tr}(\boldsymbol{\varepsilon}(\tilde{u}))\mathbb{1},\\
-l_c\,\Delta\mathrm{axl}\,(\tilde{\boldsymbol{A}}) &= \mu_c\,\mathrm{axl}\,(\mathrm{skew}(\nabla\tilde{u}) - \tilde{\boldsymbol{A}})
\end{aligned}
\tag{3.1}
$$

with boundary conditions

$$
\begin{aligned}
\tilde{u}(x,t) &= g_D(x,t) \qquad \text{for } x\in\partial\Omega \text{ and } t\geq 0,\\
\tilde{\boldsymbol{A}}(x,t) &= \boldsymbol{A}_D(x,t) \qquad \text{for } x\in\partial\Omega \text{ and } t\geq 0.
\end{aligned}
\tag{3.2}
$$

System (3.1) is a linear system of equations, hence it is sufficient to consider it with homogeneous boundary conditions. Let us define the bilinear form

$$
B:\Big(H^1_0(\Omega;\mathbb{R}^3)\times H^1_0(\Omega;\mathfrak{so}(3))\Big)\times\Big(H^1_0(\Omega;\mathbb{R}^3)\times H^1_0(\Omega;\mathfrak{so}(3))\Big)\to\mathbb{R}
$$

as follows

$$
\begin{aligned}
B[(u,\boldsymbol{A}),(v,\boldsymbol{w})](t) = {}& 2\mu\int_\Omega \boldsymbol{\varepsilon}(u(t))\boldsymbol{\varepsilon}(v(t))\,\mathrm{d}x + \lambda\int_\Omega \mathrm{tr}(\boldsymbol{\varepsilon}(u(t)))\mathrm{tr}(\boldsymbol{\varepsilon}(v(t)))\,\mathrm{d}x\\
&+ 2\mu_c\int_\Omega (\mathrm{skew}(\nabla u(t)) - \boldsymbol{A}(t))(\mathrm{skew}(\nabla v(t)) - \boldsymbol{w}(t))\,\mathrm{d}x\\
&+ 4l_c\int_\Omega \nabla\mathrm{axl}(\boldsymbol{A}(t))\nabla\mathrm{axl}(\boldsymbol{w}(t))\,\mathrm{d}x\,,
\end{aligned}
$$

where $t\in(0,T)$ and $u,v\in H^1_0(\Omega;\mathbb{R}^3)$ and $\boldsymbol{A},\boldsymbol{w}\in H^1_0(\Omega;\mathfrak{so}(3))$. From Lax-Milgram theorem we conclude that for a.a. $t\in(0,T)$ there exists a unique solution $u(t)\in H^1_0(\Omega;\mathbb{R}^3)$ and $\boldsymbol{A}(t)\in H^1_0(\Omega;\mathfrak{so}(3))$. Since $F\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, then we obtain that $u\in L^\infty(0,T;H^1_0(\Omega;\mathbb{R}^3))$ and $\boldsymbol{A}\in L^\infty(0,T;H^1_0(\Omega;\mathfrak{so}(3)))$. Standard methods from regularity theory for elliptic systems (difference quotients) imply that $\boldsymbol{A}\in L^\infty(0,T;H^2(\Omega;\mathfrak{so}(3)))$.

The above information yields that for $g_D\in L^\infty(0,T;H^{\frac12}(\partial\Omega;\mathbb{R}^3))$ and $\boldsymbol{A}_D\in L^\infty(0,T;H^{\frac32}(\partial\Omega;\mathfrak{so}(3)))$ there exists a unique solution

$$
(\tilde{u},\tilde{\boldsymbol{A}})\in L^\infty(0,T;W^{1,2}_{g_D}(\Omega;\mathbb{R}^3)\times H^2(\Omega;\mathfrak{so}(3))\cap W^{1,2}_{\boldsymbol{A}_D}(\Omega;\mathfrak{so}(3)))
\tag{3.3}
$$

of the linear elliptic system (3.1) with boundary conditions (3.2) and the following inequality is satisfied

$$\|\tilde{u}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))} + \|\tilde{\boldsymbol{A}}\|_{L^\infty(0,T;H^2(\Omega;\mathfrak{so}(3)))} \leq C\Big(\|F\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \|g_D\|_{L^\infty(0,T;H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3))} + \|\boldsymbol{A}_D\|_{L^\infty(0,T;H^{\frac{3}{2}}(\partial\Omega;\mathfrak{so}(3)))}\Big).$$

More information can be found in [12, 13, 17]. Next, let us consider the following linear parabolic system

$$\tilde{\theta}_t(x,t) - \Delta\tilde{\theta}(x,t) = 0 \tag{3.4}$$

with boundary-initial conditions

$$\begin{aligned} \frac{\partial\tilde{\theta}}{\partial n}(x,t) &= g_\theta(x,t) \quad \text{for} \quad x\in\partial\Omega \quad \text{and} \quad t\geq 0\,, \\ \tilde{\theta}(x,0) &= \tilde{\theta}^0(x) \quad \text{for} \quad x\in\Omega. \end{aligned} \tag{3.5}$$

Assuming that $g_\theta \in L^2(0,T;L^2(\partial\Omega))$ and $\theta_0 \in L^2(\Omega)$ the system (3.5) has a solution $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, cf. [11]. Additionally the following estimate holds

$$\|\tilde{\theta}\|_{L^\infty(0,T;L^2(\Omega))} + \|\tilde{\theta}\|_{L^2(0,T;H^1(\Omega))} \leq D(\|g_\theta\|_{L^2(0,T;L^2(\partial\Omega))} + \|\tilde{\theta}^0\|_{L^2(\Omega)}). \tag{3.6}$$

Finally, we define $u = \hat{u} - \tilde{u}$ and $\theta = \hat{\theta} - \tilde{\theta}$. Note that to find the solution $(\hat{u}, \hat{\theta})$ of the problem (1.1)-(1.4) we need to find solution (u,θ) of the following problem

$$\begin{aligned} \mathrm{div}\boldsymbol{T} &= 0, \\ \boldsymbol{T} &= 2\mu(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + 2\mu_c(\mathrm{skew}(\nabla u) - \boldsymbol{A}) + \lambda\mathrm{tr}(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}})\mathbb{1}, \\ -l_c\,\Delta\mathrm{axl}\,(\boldsymbol{A}) &= \mu_c\,\mathrm{axl}\,(\mathrm{skew}(\nabla u) - \boldsymbol{A}), \\ \boldsymbol{\varepsilon}^{\mathbf{p}}_t &= \boldsymbol{G}(\tilde{\theta} + \theta, \mathrm{dev}(\tilde{\boldsymbol{T}}_E) + \mathrm{dev}(\boldsymbol{T}_E)), \\ \boldsymbol{T}^E &= 2\mu(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + \lambda\mathrm{tr}(\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}^{\mathbf{p}})\mathbb{1} \\ \theta_t - \Delta\theta &= (\mathrm{dev}(\tilde{\boldsymbol{T}}_E) + \mathrm{dev}(\boldsymbol{T}_E)) : \boldsymbol{G}(\tilde{\theta} + \theta, \mathrm{dev}(\tilde{\boldsymbol{T}}_E) + \mathrm{dev}(\boldsymbol{T}_E)) \end{aligned} \tag{3.7}$$

with the initial and boundary conditions

$$\begin{aligned} \theta(0) &= \hat{\theta}^0 - \tilde{\theta}^0 = \theta_0, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(0) &= \boldsymbol{\varepsilon}^{\mathbf{p},0}, \\ u_{|\partial\Omega} &= 0, \\ \boldsymbol{A}_{|\partial\Omega} &= 0, \\ \frac{\partial\theta}{\partial n}_{|\partial\Omega} &= 0. \end{aligned} \tag{3.8}$$

4. Approximation procedure

We are going to approximate a solution of (3.7) with the use of Galerkin method. The procedure is similar as the one presented in the article [15]. Let us define a scalar product in the space $L^2(\Omega;\mathcal{S}^3)$ by formula

$$(\boldsymbol{\Psi},\boldsymbol{\Phi})_{\mathbb{C}} := \int_\Omega \mathbb{C}^{\frac12}\boldsymbol{\Psi} : \mathbb{C}^{\frac12}\boldsymbol{\Phi}\,\mathrm{d}x, \tag{4.1}$$

where $\mathbb{C}\boldsymbol{S} = 2\mu\boldsymbol{S} + \lambda\mathrm{tr}(\boldsymbol{S})\mathbb{1}$ and (4.1) holds for $\boldsymbol{\Psi},\boldsymbol{\Phi}\in L^2(\Omega;\mathcal{S}^3)$. We choose a basis in the space $H^1_0(\Omega;\mathbb{R}^3)$ associated with the scalar product

$$(\boldsymbol{\varepsilon}(u),\boldsymbol{\varepsilon}(v))_{\mathbb{C}} := \int_\Omega \mathbb{C}^{\frac12}\boldsymbol{\varepsilon}(u) : \mathbb{C}^{\frac12}\boldsymbol{\varepsilon}(v)\,\mathrm{d}x, \tag{4.2}$$

for $u,\, v\in H^1_0(\Omega;\mathbb{R}^3)$. Let $\{w_i\}_{i=1}^{\infty}$ be a set of eigenvectors of the operator $-\mathrm{div}(2\mu\,\boldsymbol{\varepsilon}(\cdot) + \lambda\mathrm{tr}\boldsymbol{\varepsilon}(\cdot))$ with the domain $H^1_0(\Omega;\mathbb{R}^3)$ associated with eigenvalues $0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots$, orthonormal in $L^2(\Omega;\mathbb{R}^3)$ and orthogonal in $H^1_0(\Omega;\mathbb{R}^3)$ (with respect to scalar product (4.2)).

Let $\{\tilde{w}_i\}_{i=1}^{\infty}$ be a set of eigenvectors of the operator $-\Delta$ with the domain $H^1_0(\Omega;\mathfrak{so}(3))\cap H^2(\Omega;\mathfrak{so}(3))$ associated with eigenvalues $0<\tilde\lambda_1\le\tilde\lambda_2\le\tilde\lambda_3\le\ldots$ orthonormal in $L^2(\Omega;\mathfrak{so}(3))$ and orthogonal in $H^1_0(\Omega;\mathfrak{so}(3))$.

Moreover, let $\{v_i\}_{i=1}^{\infty}$ be a set which consists of orthonormal in $L^2(\Omega)$ and orthogonal in the space $H^1(\Omega)$ functions v_i solving

$$\int_\Omega (\nabla v_i\cdot\nabla\phi - \eta_i v_i\phi)\,\mathrm{d}x = 0, \tag{4.3}$$

for every function $\phi\in C^\infty(\overline{\Omega})$, see [2, 29]. Above $0=\eta_1<\eta_2\le\eta_3\le\ldots$ are eigenvalues associated with functions $v_1, v_2, v_3, ...$, respectively.

Now, we follow the idea of [15] and we are going to construct a basis for approximating the visco-elastic strain tensor. It is known that $\boldsymbol{\varepsilon}(w_i)\in H^s(\Omega;\mathcal{S}^3)$, where $H^s(\Omega;\mathcal{S}^3)$ is the fractional Sobolev space with a scalar product denoted by $(\!(\cdot,\cdot)\!)_s$ and $2\ge s>\frac32$.

Using the argumentation presented in [15, Theorem B.1] we obtain that there exists orthonormal basis $\{\boldsymbol{\zeta}^k_n\}_{n=1}^{\infty}$ of $V_k := \big(\mathrm{span}\{\boldsymbol{\varepsilon}(w_1),...,\boldsymbol{\varepsilon}(w_k)\}\big)^{\perp}$, where $\cdot^{\perp}$ is an orthogonal complementation in $L^2(\Omega,\mathcal{S}^3)$ taken with respect to the scalar product $(\cdot,\cdot)_{\mathbb{C}}$. The set $\{\boldsymbol{\zeta}^k_n\}_{n=1}^{\infty}$ is also an orthogonal basis of $V^s_k := V_k\cap H^s(\Omega,\mathcal{S}^3)$.

For a fixed positive number $k, l, \in\mathbb{N}$ we will find the following functions

$$\begin{aligned}
u_{k,l} &= \sum_{n=1}^{k}\alpha^n_{k,l}(t)w_n,\\
\boldsymbol{A}_{k,l} &= \sum_{n=1}^{k}\tilde\alpha^n_{k,l}(t)\tilde{\boldsymbol{w}}_n,\\
\theta_{k,l} &= \sum_{m=1}^{l}\beta^m_{k,l}(t)v_m,\\
\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l} &= \sum_{n=1}^{k}\gamma^n_{k,l}(t)\boldsymbol{\varepsilon}(w_n) + \sum_{m=1}^{l}\delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m,
\end{aligned} \tag{4.4}$$

such that they satisfy

$$
\begin{aligned}
&\int_\Omega \boldsymbol{T}_{k,l} : \nabla w_n \,\mathrm{d}x = 0,\\
&\boldsymbol{T}_{k,l} = \mathbb{C}(\boldsymbol{\varepsilon}(u_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) + 2\mu_c(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l}),\\
&l_c \int_\Omega \nabla \mathrm{axl}(\boldsymbol{A}_{k,l}) : \nabla \mathrm{axl}(\tilde{\boldsymbol{w}}_n)\,\mathrm{d}x = \mu_c \int_\Omega \mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l}) \cdot \mathrm{axl}(\tilde{\boldsymbol{w}}_n)\,\mathrm{d}x,\\
&\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \mathbb{C}\boldsymbol{\varepsilon}(w_n)\,\mathrm{d}x = \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \mathbb{C}\boldsymbol{\varepsilon}(w_n)\,\mathrm{d}x,\\
&\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \mathbb{C}\boldsymbol{\zeta}^k_m\,\mathrm{d}x = \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \mathbb{C}\boldsymbol{\zeta}^k_m\,\mathrm{d}x,\\
&\int_\Omega (\theta_{k,l})_t v_m\,\mathrm{d}x + \int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\,\mathrm{d}x =\\
&\quad\int_\Omega \mathcal{T}_k\Big(\big(\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\big) : \boldsymbol{G}\big(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\big)\Big) v_m\,\mathrm{d}x,
\end{aligned}
\tag{4.5}
$$

for all $n = 1, \dots, k$ and $m = 1, \dots, l$, where $\tilde{\boldsymbol{T}}^E = \mathbb{C}\boldsymbol{\varepsilon}(\tilde{u})$ and $\tilde{u}$ is a solution to (3.1). The operator $\mathcal{T}_k$ is the truncation at height $k > 0$ i.e. $\mathcal{T}_k(r) = \min\{k, \max(r, -k)\}$. The system (4.5) is considered with the following initial data

$$
\begin{aligned}
\big(\theta_{k,l}(x,0), v_m\big) &= \big(\mathcal{T}_k(\theta_0), v_m\big)\\
\big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\varepsilon}(w_n)\big)_{\mathbb{C}} &= \big(\boldsymbol{\varepsilon}^{\mathbf{p},0}, \boldsymbol{\varepsilon}(w_n)\big)_{\mathbb{C}}\\
\big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\zeta}^k_m\big)_{\mathbb{C}} &= \big(\boldsymbol{\varepsilon}^{\mathbf{p},0}, \boldsymbol{\zeta}^k_m\big)_{\mathbb{C}},
\end{aligned}
\tag{4.6}
$$

for all $n = 1, \dots, k$ and $m = 1, \dots, l$, where $\big(\cdot,\cdot\big)$ denotes the standard scalar product in $L^2(\Omega)$ and $\big(\cdot,\cdot\big)_{\mathbb{C}}$ denotes the scalar product in $H^1_0(\Omega; \mathcal{S}^3)$.

Let us denote by

$$
\begin{aligned}
\xi_1(t) &= (\alpha^1_{k,l}, \dots, \alpha^k_{k,l}, \tilde{\alpha}^1_{k,l}, \dots, \tilde{\alpha}^k_{k,l})^T,\\
\xi_2(t) &= (\beta^1_{k,l}, \dots, \beta^l_{k,l}, \gamma^1_{k,l}, \dots, \gamma^l_{k,l}, \delta^1_{k,l}, \dots, \delta^l_{k,l})^T,
\end{aligned}
\tag{4.7}
$$

and let us fix the vector $\xi_2(t)$ for $t \in [0, T)$. The choice of Galerkin bases and definition of approximate solution (4.4) yields that equations $(4.5)_1$ and $(4.5)_3$ can be rewritten in the following form

$$
\begin{aligned}
&\int_\Omega \mathbb{C}\boldsymbol{\varepsilon}(u_{k,l}(t)) : \boldsymbol{\varepsilon}(w_n)\,\mathrm{d}x + 2\mu_c \int_\Omega (\mathrm{skew}(\nabla u_{k,l}(t)) - \boldsymbol{A}_{k,l}(t)) : \mathrm{skew}(\nabla w_n)\,\mathrm{d}x\\
&\qquad = \int_\Omega \mathbb{C}\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(t) : \boldsymbol{\varepsilon}(w_n)\,\mathrm{d}x,\\
&l_c \int_\Omega \nabla \mathrm{axl}\boldsymbol{A}_{k,l}(t) \cdot \nabla \mathrm{axl}(\tilde{\boldsymbol{w}}_n)\,\mathrm{d}x = \mu_c \int_\Omega \mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}(t)) - \boldsymbol{A}_{k,l}(t)) \cdot \mathrm{axl}(\tilde{\boldsymbol{w}}_n)\,\mathrm{d}x,
\end{aligned}
\tag{4.8}
$$

hence,

$$
\begin{aligned}
&\lambda_n\alpha^n_{k,l}(t)+2\mu_c\int_\Omega(\mathrm{skew}(\nabla u_{k,l}(t))-\boldsymbol{A}_{k,l}(t)):\mathrm{skew}(\nabla w_n)\,\mathrm{d}x=\gamma^n_{k,l}(t)\lambda_n,\\
&\frac{1}{2}l_c\tilde{\lambda}_n\tilde{\alpha}^n_{k,l}(t)=\mu_c\int_\Omega\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}(t))-\boldsymbol{A}_{k,l}(t))\cdot\mathrm{axl}(\tilde{\boldsymbol{w}}_n)\,\mathrm{d}x.
\end{aligned}
\tag{4.9}
$$

Multiplying $(4.9)_1$ by $\alpha^n_{k,l}(t)$, $(4.9)_2$ by $4\tilde{\alpha}^n_{k,l}(t)$ and summing this two equations over $n=1,\ldots,k$ we conclude that (notice that $\boldsymbol{A}:\boldsymbol{B}=2\mathrm{axl}(\boldsymbol{A})\cdot\mathrm{axl}(\boldsymbol{B})$ if $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{so}(3)$)

$$
\begin{aligned}
\sum_{n=1}^k\lambda_n(\alpha^n_{k,l}(t))^2+4\mu_c\int_\Omega|\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}(t))-\boldsymbol{A}_{k,l}(t))|^2\,\mathrm{d}x+\sum_{n=1}^k2l_c\tilde{\lambda}_n(\tilde{\alpha}^n_{k,l}(t))^2&\\
=\sum_{n=1}^k\gamma^n_{k,l}(t)\lambda_n\alpha^n_{k,l}(t)\,.&
\end{aligned}
\tag{4.10}
$$

The inequality (4.10) implies the following inequality

$$
\sum_{n=1}^k(\alpha^n_{k,l}(t))^2+\sum_{n=1}^k(\tilde{\alpha}^n_{k,l}(t))^2\le C\sum_{n=1}^k(\gamma^n_{k,l}(t))^2
\tag{4.11}
$$

for $t\in[0,T)$, where constant $C>0$ does not depend on $\alpha^n_{k,l}(t)$. The inequality (4.11) yields that for every $n\le k$ there exists a unique solution $(\alpha^n_{k,l}(t),\tilde{\alpha}^n_{k,l}(t))$ of the algebraic system (4.11), provided that $\xi_2(t)$ is fixed for $t\in[0,T)$.

Lemma 4.1. *Let us assume that $\boldsymbol{\varepsilon}^p_0\in L^2(\Omega;\mathcal{S}^3_{\mathrm{dev}})$ and $\theta_0\in L^1(\Omega)$. Then there exists an absolutely continuous in time solution of system* (4.5).

Proof. Notice that last three equations in (4.5) can be written in the following form

$$
\begin{aligned}
&(\gamma^n_{k,l}(t))_t=\frac{1}{\lambda_n}\int_\Omega\tilde{\boldsymbol{G}}(x,t,\xi_1(t),\xi_2(t)):\mathbb{C}\boldsymbol{\varepsilon}(w_n)\,\mathrm{d}x,\\
&(\gamma^m_{k,l}(t))_t=\int_\Omega\tilde{\boldsymbol{G}}(x,t,\xi_1(t),\xi_2(t)):\mathbb{C}\boldsymbol{\zeta}^k_m\,\mathrm{d}x,\\
&(\beta^m_{k,l}(t))_t=\int_\Omega\mathcal{T}_k\Big(\big(\mathrm{dev}(\tilde{\boldsymbol{T}}^E)+\mathrm{dev}(\boldsymbol{T}^E_{k,l})\big):\tilde{\boldsymbol{G}}(x,t,\xi_1(t),\xi_2(t))\Big)v_m\,\mathrm{d}x-\eta_m\beta^m_{k,l}(t),
\end{aligned}
\tag{4.12}
$$

where

$$
\tilde{\boldsymbol{G}}(x,t,\xi_1(t),\xi_2(t)):=\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\mathrm{dev}(\tilde{\boldsymbol{T}}^E)+\mathrm{dev}(\boldsymbol{T}^E_{k,l})).
\tag{4.13}
$$

The system (4.12) with initial conditions (4.6) is equivalent to the following initial value problem

$$
\begin{aligned}
\frac{\mathrm{d}\,\xi_2(t)}{\mathrm{d}t}&=F(t,\xi_1(t),\xi_2(t))\,,\qquad t\in[0,T)\,,\\
\xi_2(0)&=\xi_{2,0}\,,
\end{aligned}
\tag{4.14}
$$

where $F(t,\xi_1(t),\xi_2(t))$ is obtained from right-hand side of (4.12) and $\xi_2(0)$ is obtained from (4.6). The inequality (4.11) implies that the function $F(t,\xi_1(t),\xi_2(t))$ can be treated as a function, which depends only on $\xi_2(t)$. Carathéodory theorem finishes the proof, see [22, Theorem 3.4] or [30, Appendix (61)]. $\square$

5. Energy estimets

Theorem 5.1. *There exists a constant $C > 0$, which does not depend on $k > 0$ and $l > 0$ such that the following inequality*

$$\begin{aligned}&\|\boldsymbol{\varepsilon}(u_{k,l}(t)) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(t)\|^2_{L^2(\Omega;\mathcal{S}^3)} + \|\mathrm{skew}(\nabla u_{k,l}(t)) - \boldsymbol{A}_{k,l}(t)\|^2_{L^2(\Omega;\mathfrak{so}(3))}\\ &+ \|\mathrm{tr}(\boldsymbol{\varepsilon}(u_{k,l}(t)))\|^2_{L^2(\Omega)} + \|\boldsymbol{A}_{k,l}(t)\|^2_{H^1_0(\Omega;\mathfrak{so}(3))} + \|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))} \le C,\end{aligned}$$

holds for almost all $t \in [0, T)$.

Proof. Multiplying $(4.5)_1$ by $\big(\alpha^n_{k,l}(t)\big)_t$, $(4.5)_3$ by $\big(\tilde{\alpha}^n_{k,l}(t)\big)_t$ and summing these two equations over $n = 1, \ldots, k$ we conclude that

$$\tag{5.1}\begin{aligned}\int_\Omega \boldsymbol{T}^E_{k,l} : \big(\boldsymbol{\varepsilon}(u_{k,l})\big)_t \,\mathrm{d}x + 2\mu_c \int_\Omega (\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l}) : \big(\mathrm{skew}(\nabla u_{k,l})\big)_t \,\mathrm{d}x = 0,\\ l_c \int_\Omega \nabla\cdot \mathrm{axl}(\boldsymbol{A}_{k,l})\,\big(\nabla\cdot \mathrm{axl}(\boldsymbol{A}_{k,l})\big)_t \,\mathrm{d}x - \mu_c \int_\Omega \mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l}) \cdot \big(\mathrm{axl}((\boldsymbol{A}_{k,l})\big)_t \,\mathrm{d}x = 0.\end{aligned}$$

Using again the equality $\boldsymbol{A} : \boldsymbol{B} = 2\mathrm{axl}(\boldsymbol{A}) \cdot \mathrm{axl}(\boldsymbol{B})$ for the $(5.1)_1$ and adding the abovementioned equations we conclude that

$$\tag{5.2}\begin{aligned}&\int_\Omega \boldsymbol{T}^E_{k,l} : \big(\boldsymbol{\varepsilon}(u_{k,l})\big)_t \,\mathrm{d}x + 4\mu_c \int_\Omega \mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l}) \cdot \Big(\big(\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}))\big)_t - \big(\mathrm{axl}(\boldsymbol{A}_{k,l})\big)_t\Big) \,\mathrm{d}x\\ &\qquad + 4l_c \int_\Omega \nabla\mathrm{axl}(\boldsymbol{A}_{k,l}) : \big(\nabla\mathrm{axl}(\boldsymbol{A}_{k,l})\big)_t \,\mathrm{d}x = 0.\end{aligned}$$

Next, multiplying $(4.5)_4$ by $\alpha^n_{k,l}(t) - \gamma^n_{k,l}(t)$, $(4.5)_5$ by $\delta^m_{k,l}(t)$ and summing over $n = 1, \ldots, k$ and $m = 1, \ldots, l$ we obtain

$$\tag{5.3}\begin{aligned}&\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \sum_{n=1}^k \big(\alpha^n_{k,l} - \gamma^n_{k,l}\big)\mathbb{C}\boldsymbol{\varepsilon}(w_n) \,\mathrm{d}x\\ &\qquad = \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \sum_{n=1}^k \big(\alpha^n_{k,l} - \gamma^n_{k,l}\big)\mathbb{C}\boldsymbol{\varepsilon}(w_n) \,\mathrm{d}x,\\ &\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \sum_{n=1}^l \delta^m_{k,l}\mathbb{C}\boldsymbol{\zeta}^k_m \,\mathrm{d}x = \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \sum_{n=1}^l \delta^m_{k,l}\mathbb{C}\boldsymbol{\zeta}^k_m \,\mathrm{d}x.\end{aligned}$$

Subtracting these two equations we have

$$\tag{5.4}\begin{aligned}\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{T}^E_{k,l} \,\mathrm{d}x &= \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \boldsymbol{T}^E_{k,l} \,\mathrm{d}x\\ &= \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \mathrm{dev}(\boldsymbol{T}^E_{k,l}) \,\mathrm{d}x.\end{aligned}$$

Putting the equation (5.4) into (5.2) and integrating with respect to time we conclude that

(5.5)
$$\begin{aligned}&\frac{1}{2}\int_\Omega \boldsymbol{T}^E_{k,l}(t) : \big(\boldsymbol{\varepsilon}(u_{k,l}(t)) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(t)\big)\,\mathrm{d}x + 2\mu_c\int_\Omega |\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}(t)) - \boldsymbol{A}_{k,l}(t))|^2\,\mathrm{d}x\\
&\qquad + 2l_c\int_\Omega |\nabla \mathrm{axl}(\boldsymbol{A}_{k,l}(t))|^2\,\mathrm{d}x\\
&\qquad + \int_0^t\int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : (\mathrm{dev}(\boldsymbol{T}^E_{k,l}) + \mathrm{dev}(\tilde{\boldsymbol{T}}^E))\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \int_0^t\int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \mathrm{dev}(\tilde{\boldsymbol{T}}^E)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \frac{1}{2}\int_\Omega \boldsymbol{T}^E_{k,l}(0) : \big(\boldsymbol{\varepsilon}(u_{k,l}(0)) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(0)\big)\,\mathrm{d}x + 2\mu_c\int_\Omega |\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}(0)) - \boldsymbol{A}_{k,l}(0))|^2\,\mathrm{d}x\\
&\qquad + 2l_c\int_\Omega |\nabla \mathrm{axl}(\boldsymbol{A}_{k,l}(0))|^2\,\mathrm{d}x.\end{aligned}$$

The assumptions on the initial data and the proof of Lemma 4.1 yield that the initial integrals occurring in (5.5) are bounded independently on k and l. Moreover, the assumptions **A3** and **A4** imply the following inequality

$$\begin{aligned}&\frac{1}{2}\int_\Omega \boldsymbol{T}^E_{k,l} : \big(\boldsymbol{\varepsilon}(u_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}\big)\,\mathrm{d}x + 2\mu_c\int_\Omega |\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l})|^2\,\mathrm{d}x\\
&\qquad + 2l_c\int_\Omega |\nabla \mathrm{axl}(\boldsymbol{A}_{k,l})|^2\,\mathrm{d}x + \alpha\|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))}\\
&\le C + \|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E)\|_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))}\|\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l}))\|_{L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3_{\mathrm{dev}}))}\\
&\le C + C(\epsilon)\|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E)\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))} + \epsilon\|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))},\end{aligned} \tag{5.6}$$

where constant C does not depend on k, l and ϵ is any positive real number. Choosing $\epsilon > 0$ sufficiently small (e.g. $\epsilon = \frac{\alpha}{2}$) we complete the proof:

$$\begin{aligned}&\frac{1}{2}\int_\Omega \boldsymbol{T}^E_{k,l} : \big(\boldsymbol{\varepsilon}(u_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}\big)\,\mathrm{d}x + 2\mu_c\int_\Omega |\mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l})|^2\,\mathrm{d}x\\
&\qquad + 2l_c\int_\Omega |\nabla \mathrm{axl}(\boldsymbol{A}_{k,l})|^2\,\mathrm{d}x + \frac{\alpha}{2}\|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))}\\
&\le C + C(\epsilon)\|\,\mathrm{dev}(\tilde{\boldsymbol{T}}^E)\|^p_{L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}}))}.\end{aligned} \tag{5.7}$$

□

Remark. *From Theorem 5.1 we immediately observe that the sequence* $\{\mathrm{dev}(\boldsymbol{T}^E_{k,l})\}$ *is uniformly bounded in the space* $L^p(0,T;L^p(\Omega,\mathcal{S}^3_{\mathrm{dev}}))$ *with respect to* k *and* l*. Additionally, combining this results with Assumption* **A3** *we conclude the uniform boundedness of the sequence* $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l}))\}$ *in the space* $L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3_{\mathrm{dev}}))$ *and hence the uniform boundedness of the sequence* $\{(\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l})) : \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_{k,l}))\}$ *in* $L^1(0,T;L^1(\Omega))$.

Theorem 5.2. *The sequences $\{u_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;H^1_0(\Omega;\mathbb{R}^3))$ with respect to k and l.*

Proof. Using a well-know estimate (see [14, p.36])

$$\|\nabla u\|^2_{L^2(\Omega)} \le C^{\mathrm{curl}}_{\mathrm{div}}(\|\mathrm{div} u\|^2_{L^2(\Omega)} + \|\mathrm{curl} u\|^2_{L^2(\Omega)}), \tag{5.8}$$

where the constant $C^{\mathrm{curl}}_{\mathrm{div}}$ does not depend on u and 'curl' is the rotation operator. It is sufficiently to prove the uniform estimates for terms from the right-hand side. Note that $\mathrm{div} u_{k,l} = \mathrm{tr}(\boldsymbol{\varepsilon}(u_{k,l}))$ hence, the sequence $\{\mathrm{div} u_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Now, let us define $v = (v_1, v_2, v_3) := \mathrm{curl} u$ and $\tilde{v} = (-v_3, v_2, -v_1)$. Then we may observe that

$$\mathrm{axl}(\mathrm{skew}(\nabla u)) = \tilde{v}. \tag{5.9}$$

Moreover, $\|\tilde{v}\|_{L^2(\Omega)} = \|\mathrm{curl} u\|_{L^2(\Omega)}$ and $\|\mathrm{axl}(\mathrm{skew}(\nabla u))\|^2_{L^2(\Omega)} = \frac{1}{2}\|\mathrm{skew}(\nabla u)\|^2_{L^2(\Omega)}$. By Theorem 5.1 the sequence $\{\mathrm{skew}(\nabla u_{k,l})\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, which complete the proof.

□

Theorem 5.3. *The sequences $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega;\mathcal{S}^3_{\mathrm{dev}}))$ with respect to k and l.*

Proof. Let us notice that for almost all $t \in [0,T]$ it holds

$$\|\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(t)\|^2_{L^2(\Omega;\mathcal{S}^3)} \le \|\boldsymbol{\varepsilon}(u_{k,l}(t))\|^2_{L^2(\Omega;\mathcal{S}^3)} + \|\boldsymbol{\varepsilon}(u_{k,l}(t)) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(t)\|^2_{L^2(\Omega;\mathcal{S}^3)} \tag{5.10}$$

Combining results of Theorem 5.1 and Theorem 5.2 we complete the proof. □

Let us recall that $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$ an orthonormal of $V_k := \big(\mathrm{span}\{\boldsymbol{\varepsilon}(w_1), ..., \boldsymbol{\varepsilon}(w_k)\}\big)^\perp$ and orthogonal basis of $V^s_k := V_k \cap H^s(\Omega;\mathcal{S}^3)$. Its construction was presented in [15, Appendix B] and the set $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$ contains the eigenvalue of problem

$$(\!(\boldsymbol{\zeta}_i, \boldsymbol{\Phi})\!)_s = \lambda_i(\boldsymbol{\zeta}_i, \boldsymbol{\Phi})_{\mathbb{C}} \qquad \forall\, \boldsymbol{\Phi} \in V^s_k. \tag{5.11}$$

where by $(\!(\cdot,\cdot)\!)_s$ we denote the scalar product in $H^s(\Omega;\mathcal{S}^3)$ and $(\cdot,\cdot)_{\mathbb{C}}$ is a scalar product in $L^2(\Omega;\mathcal{S}^3)$ defined by (4.1). We define the following projections:

$$\begin{aligned}
&P^l_{H^s} : H^s(\Omega;\mathcal{S}^3) \to \mathrm{lin}\{\boldsymbol{\zeta}^k_1, ..., \boldsymbol{\zeta}^k_l\}, && P^l_{H^s}\boldsymbol{v} := \sum_{i=1}^{l} (\!(\boldsymbol{v}, \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}})\!)_s \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}},\\
&P^l_{L^2} : L^2(\Omega;\mathcal{S}^3) \to \mathrm{lin}\{\boldsymbol{\zeta}^k_1, ..., \boldsymbol{\zeta}^k_l\}, && P^l_{L^2}\boldsymbol{v} := \sum_{i=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}^k_i)_{\mathbb{C}} \boldsymbol{\zeta}^k_i,\\
&P^k : L^2(\Omega;\mathcal{S}^3) \to \mathrm{lin}\{\boldsymbol{\varepsilon}(w_1), ..., \boldsymbol{\varepsilon}(w_k)\}, && P^k\boldsymbol{v} := \sum_{i=1}^{k} (\boldsymbol{v}, \boldsymbol{\varepsilon}(w_i))_{\mathbb{C}} \boldsymbol{\varepsilon}(w_i).
\end{aligned} \tag{5.12}$$

Thus, we may observe that for $\boldsymbol{v} \in H^s(\Omega;\mathcal{S}^3)$ it holds that

$$P^l_{H^s}(Id - P^k)\boldsymbol{v} = \sum_{i=1}^{l} (\!((Id - P^k)\boldsymbol{v}, \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}})\!)_s \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}} = \sum_{i=1}^{l} ((Id - P^k)\boldsymbol{v}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}} \boldsymbol{\zeta}^k_i = \sum_{i=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}} \boldsymbol{\zeta}^k_i = P^l_{L^2}\boldsymbol{v}. \tag{5.13}$$

Let us observe that since P^k is the projection which does not dependent on l, then there exists $c(k)$ (depending only on k) such that for every $\boldsymbol{\varphi} \in H^s(\Omega;\mathcal{S}^3)$ it holds

$$\max(\|P^k\boldsymbol{\varphi}\|_{H^s}, \|(Id - P^k)\boldsymbol{\varphi}\|_{H^s}) \leq c(k)\|\boldsymbol{\varphi}\|_{H^s}. \tag{5.14}$$

Lemma 5.4. *For every fixed k the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^{p'}(0,T;(H^s(\Omega;\mathcal{S}^3))')$ with respect to l.*

Proof. For a fixed $k \in \mathbb{N}$ notice that, by $(4.4)_{(4)}$ we have $(P^k + P^l_{L^2})(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t = (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t$. Let $\boldsymbol{\varphi} \in L^p(0,T;H^s(\Omega,\mathcal{S}^3))$, then we may estimate as follows

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t &= \int_0^T |((P^k + P^l_{L^2})(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\mathbb{C}}|\,\mathrm{d}t \\ &= \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, (P^k + P^l_{L^2})\boldsymbol{\varphi})_{\mathbb{C}}|\,\mathrm{d}t \\ &\leq \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^k\boldsymbol{\varphi})_{\mathbb{C}}|\,\mathrm{d}t + \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^l_{L^2}\boldsymbol{\varphi})_{\mathbb{C}}|\,\mathrm{d}t,\end{aligned} \tag{5.15}$$

where the second equality holds because the projections are self-adjoint operators and the inequality is a consequence of the orthogonality of subspaces $\mathrm{lin}\{\boldsymbol{\varepsilon}(w_1),\ldots,\boldsymbol{\varepsilon}(w_k)\}$ and $\mathrm{lin}\{\boldsymbol{\zeta}^k_1,\ldots,\boldsymbol{\zeta}^k_l\}$ in the sense of $(\cdot,\cdot)_{\mathbb{C}}$. Thus,

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \varphi)_{\mathbb{C}}|\,\mathrm{d}t \leq& \int_0^T |\int_\Omega \mathbb{C}\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : P^k\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &+ \int_0^T |\int_\Omega \mathbb{C}\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : P^l_{L^2}\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\leq d\int_0^T |\int_\Omega \boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : P^k\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &+ d\int_0^T |\int_\Omega \boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : (P^l_{H^s}\circ(Id - P^k))\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t.\end{aligned} \tag{5.16}$$

The estimates of this first term on right-hand side of abovementioned inequality are obvious

$$\begin{aligned}\int_0^T |\int_\Omega \boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, &\boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : P^k\boldsymbol{\varphi}|\,\mathrm{d}t \\ &\leq \int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|P^k\boldsymbol{\varphi}\|_{L^p(\Omega;\mathcal{S}^3)}\,\mathrm{d}t \\ &\leq \tilde{c}\int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|P^k\boldsymbol{\varphi}\|_{H^s(\Omega;\mathcal{S}^3)}\,\mathrm{d}t \\ &\leq c(k)\tilde{c}\int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|\boldsymbol{\varphi}\|_{H^s(\Omega;\mathcal{S}^3)}\,\mathrm{d}t \\ &\leq c(k)\tilde{c}\|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)\|_{L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3))}\|\boldsymbol{\varphi}\|_{L^p(0,T;H^s(\Omega;\mathcal{S}^3))},\end{aligned} \tag{5.17}$$

where $\tilde{c}$ is an optimal embedding constant of $H^s(\Omega;\mathcal{S}^3)\subset L^p(\Omega;\mathcal{S}^3)$. Now, let us focus on the second term from (5.16). We obtain

(5.18)
$$
\begin{aligned}
\int_0^T |\int_\Omega &\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d):(P^l_{H^s}\circ(Id-P^k))\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t\\
&\le \int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|(P^l_{H^s}\circ(Id-P^k))\boldsymbol{\varphi}\|_{L^p(\Omega;\mathcal{S}^3)}\,\mathrm{d}t\\
&\le \tilde{c}\int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|(P^l_{H^s}\circ(Id-P^k))\boldsymbol{\varphi}\|_{H^s(\Omega;\mathcal{S}^3)}\,\mathrm{d}t\\
&\le \tilde{c}\int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|(Id-P^k)\boldsymbol{\varphi}\|_{H^s(\Omega;\mathcal{S}^3)}\,\mathrm{d}t\\
&\le \tilde{c}c(k)\int_0^T \|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|_{L^{p'}(\Omega;\mathcal{S}^3)}\|\boldsymbol{\varphi}\|_{H^s(\Omega;\mathcal{S}^3)}\,\mathrm{d}t\\
&\le \tilde{c}c(k)\|\boldsymbol{G}(\theta_{k,l}+\tilde{\theta},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|_{L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3))}\|\boldsymbol{\varphi}\|_{L^p(0,T;H^s(\Omega;\mathcal{S}^3))}.
\end{aligned}
$$

Consequently, there exists $C(k)>0$ such that

$$
\sup_{\substack{\boldsymbol{\varphi}\in L^p(0,T;H^s(\Omega;\mathcal{S}^3))\\ \|\boldsymbol{\varphi}\|_{L^p(0,T;H^s(\Omega;\mathcal{S}^3))}\le 1}} \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t \le C(k), \tag{5.19}
$$

and hence sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^{p'}(0,T;(H^s(\Omega;\mathcal{S}^3))')$ with respect to l. □

The assumptions on heat equations are the same as the assumptions in [15]. This implies that the estimates on approximate sequence of temperature remain unchanged. Hence, we only mention about this results and for a proof we refer the reader to [15].

Lemma 5.5. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ with respect to k and l.*

The sequence of right-hand sides of heat equation is uniformly bounded only in $L^1(Q)$. Due to this fact in the following lemma the bound is uniform only with respect to l. This implies that using the standard method we may pass the limit only with parameter associated with basis for temperature. To make the second limit passage we use different tools.

Lemma 5.6. *There exists a constant C, depending on the domain Ω and the time interval $(0,T)$, such that for every $k\in\mathbb{N}$*

(5.20)
$$
\begin{aligned}
\sup_{0\le t\le T}&\|\theta_{k,l}(t)\|^2_{L^2(\Omega)}+\|\theta_{k,l}\|^2_{L^2(0,T;W^{1,2}(\Omega))}+\|(\theta_{k,l})_t\|^2_{L^2(0,T;W^{-1,2}(\Omega))}\\
&\le C\Big(\|\mathcal{T}_k\Big((\mathrm{dev}(\tilde{\boldsymbol{T}}^E)+\mathrm{dev}(\boldsymbol{T}^E_{k,l})):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\mathrm{dev}(\tilde{\boldsymbol{T}}^E)+\mathrm{dev}(\boldsymbol{T}^E_{k,l}))\Big)\|^2_{L^2(0,T;L^2(\Omega))}\\
&\qquad+\|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\Big).
\end{aligned}
$$

The uniform boundedness of sequences regarding approximate solutions we obtain the following convergences

$$
\begin{aligned}
\operatorname{dev}(\boldsymbol{T}^E_{k,l}) &\rightharpoonup \operatorname{dev}(\boldsymbol{T}^E_k) && \text{weakly in } L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}})),\\
\boldsymbol{G}(\tilde\theta+\theta_{k,l},\operatorname{dev}(\tilde{\boldsymbol{T}})+\operatorname{dev}(\boldsymbol{T}_{k,l})) &\rightharpoonup \boldsymbol{\chi}_k && \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3_{\mathrm{dev}})),\\
\theta_{k,l} &\rightharpoonup \theta_k && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\\
\theta_{k,l} &\to \theta_k && \text{a.e. in } \Omega\times(0,T),\\
\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l} &\rightharpoonup \boldsymbol{\varepsilon}^{\mathbf{p}}_k && \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathcal{S}^3)),\\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t &\rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t && \text{weakly in } L^{p'}(0,T;(H^s(\Omega;\mathcal{S}^3))'),\\
u_{k,l} &\rightharpoonup u_k && \text{weakly in } L^\infty(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3)),\\
\boldsymbol{A}_{k,l} &\rightharpoonup \boldsymbol{A}_k && \text{weakly in } L^\infty(0,T;H^1(\Omega;\mathfrak{so}(3))),\\
\boldsymbol{T}_{k,l} &\rightharpoonup \boldsymbol{T}_k && \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^9)),
\end{aligned}
\tag{5.21}
$$

with $l\to\infty$. Passing to the limit in (4.5) we have similar difficulties as in problem without microrotations. Similar to [15,20] we use three-step method. Due to this method we identify weak limit $\boldsymbol{\chi}_k$ and make a limit passage in right-hand side of heat equation.

Since we assume that function $\boldsymbol{G}$ satisfies assumptions **A1** – **A4** we may repeat exact calculations of three-step method from [15]. Then we obtain the following results

1) for every $k\in\mathbb{N}$ the following inequality holds

$$
\begin{aligned}
\limsup_{l\to\infty}\int_0^t\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\operatorname{dev}(\tilde{\boldsymbol{T}})+\operatorname{dev}(\boldsymbol{T}_{k,l})):\operatorname{dev}(\boldsymbol{T}_{k,l})\,\mathrm{d}x\,\mathrm{d}t&\\
\le \int_0^t\int_\Omega \boldsymbol{\chi}_k:\operatorname{dev}(\boldsymbol{T}_k)\,\mathrm{d}x\,\mathrm{d}t.&
\end{aligned}
\tag{5.22}
$$

For a proof see [15, Lemma 3.8].

2) using Minty-Browder trick we obtain that

$$
\boldsymbol{\chi}_k=\boldsymbol{G}(\tilde\theta+\theta_k,\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_k))\quad \text{a.e. in } (0,T)\times\Omega,
\tag{5.23}
$$

and consequently for every $k\in\mathbb{N}$

$$
\boldsymbol{G}(\tilde\theta+\theta_{k,l},\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_{k,l}))\rightharpoonup \boldsymbol{G}(\tilde\theta+\theta_k,\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_k)),
\tag{5.24}
$$

in $L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3_{\mathrm{dev}}))$ with $l\to\infty$;

3) for each $k\in\mathbb{N}$ it holds

$$
\begin{aligned}
\lim_{l\to\infty}\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_{k,l})):(\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_{k,l}))\,\mathrm{d}x\,\mathrm{d}t&\\
=\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k,\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_k)):(\operatorname{dev}(\tilde{\boldsymbol{T}}^E)+\operatorname{dev}(\boldsymbol{T}^E_k))\,\mathrm{d}x\,\mathrm{d}t.&
\end{aligned}
\tag{5.25}
$$

Finally, we obtain

$$
\begin{aligned}
&\int_\Omega \boldsymbol{T}_k : \nabla\phi_1 \,\mathrm{d}x = 0,\\
&l_c\int_\Omega \nabla\mathrm{axl}(\boldsymbol{A}_k) : \nabla\mathrm{axl}(\boldsymbol{\Phi}_1)\,\mathrm{d}x = 2\mu_c\int_\Omega \mathrm{axl}(\mathrm{skew}(\nabla u_{k,l}) - \boldsymbol{A}_{k,l})\cdot\mathrm{axl}(\boldsymbol{\Phi}_1)\,\mathrm{d}x,\\
&\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\Phi}_2\,\mathrm{d}x = \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_k, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_k)) : \boldsymbol{\Phi}_2\,\mathrm{d}x,\\
&\int_\Omega (\theta_k)_t v\,\mathrm{d}x + \int_\Omega \nabla\theta_k\cdot\nabla v\,\mathrm{d}x =\\
&\qquad\int_\Omega \mathcal{T}_k\Big(\big(\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_k)\big) : \boldsymbol{G}\big(\tilde\theta + \theta_k, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_k)\big)\Big)v\,\mathrm{d}x,
\end{aligned}
\tag{5.26}
$$

where $\phi \in C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^3))$, $\boldsymbol{\Phi}_1 \in C^\infty([0,T]; C_0^\infty(\Omega;\mathfrak{so}(3)))$, $\boldsymbol{\Phi}_2 \in C^\infty([0,T]; C_0^\infty(\Omega;\mathcal{S}^3_{\mathrm{dev}}))$ and $v \in C_0^\infty([-\infty,T); C^\infty(\Omega))$ are test functions. Moreover, $\tilde{\boldsymbol{T}}^E = \mathbb{C}\boldsymbol{\varepsilon}(\tilde u)$ where $\tilde u$ is defined in (3.1).

To make the second limit passage let us observe that results from Theorem 5.1, Theorem 5.2 and Theorem 5.3 are uniform with respect to k and l. Hence, we obtain

$$
\begin{aligned}
\mathrm{dev}(\boldsymbol{T}^E_k) &\rightharpoonup \mathrm{dev}(\boldsymbol{T}^E) && \text{weakly in } L^p(0,T;L^p(\Omega;\mathcal{S}^3_{\mathrm{dev}})),\\
\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \mathrm{dev}(\tilde{\boldsymbol{T}}) + \mathrm{dev}(\boldsymbol{T}_k)) &\rightharpoonup \boldsymbol{\chi} && \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega;\mathcal{S}^3_{\mathrm{dev}})),\\
\boldsymbol{\varepsilon}^{\mathbf{p}}_k &\rightharpoonup \boldsymbol{\varepsilon}^{\mathbf{p}} && \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathcal{S}^3)),\\
u_k &\rightharpoonup u && \text{weakly in } L^\infty(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3)),\\
\boldsymbol{A}_k &\rightharpoonup \boldsymbol{A} && \text{weakly in } L^\infty(0,T;H^1(\Omega;\mathfrak{so}(3))),\\
\boldsymbol{T}_k &\rightharpoonup \boldsymbol{T} && \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^9)),
\end{aligned}
\tag{5.27}
$$

with $k \to \infty$. Moreover, the uniform boundedness of right-hand side of heat equations implies that we may use Boccardo and Gallouët approach for Neumann boundary condition (presented in [15, Appendix A]) to obtain the existence of temperature's sequence limit, i.e.

$$
\begin{aligned}
\theta_{k,l} &\rightharpoonup \theta_k && \text{weakly in } L^q(0,T;W^{1,q}(\Omega)),\\
\theta_{k,l} &\to \theta_k && \text{a.e. in } \Omega\times(0,T).
\end{aligned}
\tag{5.28}
$$

Moreover, $\theta \in C([0,T], W^{-2,2}(\Omega))$. Repeating the three steps method we may identify the weak limit of nonlinear functions, i.e.

$$
\boldsymbol{\chi} = \boldsymbol{G}(\tilde\theta + \theta, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E))
\tag{5.29}
$$

a.e. in Q. Furthermore, the following limit exists

$$
\begin{aligned}
\lim_{k\to\infty}\int_0^T\int_\Omega &\boldsymbol{G}(\tilde\theta + \theta_k, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_k)) : (\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E_k))\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta + \theta, \mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E)) : (\mathrm{dev}(\tilde{\boldsymbol{T}}^E) + \mathrm{dev}(\boldsymbol{T}^E))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{5.30}
$$

This allow us to make the limit passage (5.26) with $k \to \infty$, which complete the proof.

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