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Renormalized solutions of nonlinear parabolic problems in generalized Musielak-Orlicz spaces

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Abstract

We will present the proof of existence of renormalized solutions to a nonlinear parabolic problem $\partial_t u - \mathrm{div} a(\cdot, Du) = f$ with right hand side f and initial data u_0 in L^1. The growth and coercivity conditions on the monotone vector field a are prescribed by a generalized $\mathcal{N}$-function M which is anisotropic and inhomogenous with respect to the space variable. In particular, M does not have to satisfy an upper growth bound described by a Δ_2-condition. Therefore we work with generalized Musielak-Orlicz spaces which are not necessarily reflexive. Moreover we provide a weak sequential stability result for a more general problem: $\partial_t \beta(\cdot, u) - \mathrm{div}(a(\cdot, Du) + F(u)) = f$, where β is a monotone function with respect to the second variable and F is locally Lipschitz continuous. Within the proof we use truncation methods, Young measure techniques, the integration by parts formula and monotonicity arguments which have been adapted to nonreflexive Musielak-Orlicz spaces.

Key words: parabolic equations, renormalized solutions, integration by parts, generalized Musielak-Orlicz spaces, monotonicity arguments, biting lemma, Young measures.

1 Introduction

1.1 Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with Lipschitz boundary $\partial\Omega$ if $d \geq 2$ and let $[0, T]$ be finite time interval, and $Q_T = (0, T) \times \Omega$. We are interested in existence of renormalized solutions to the following nonlinear parabolic problem

$$\begin{aligned} \partial_t \beta(x, u(t, x)) - \mathrm{div}(a(x, Du(t, x)) + F(u(t, x)) = f \quad & \text{in } Q_T, \\ u = 0 \quad & \text{on } (0, T) \times \partial\Omega, \\ \beta(x, u(0, x)) = b_0 \quad & \text{in } \Omega, \end{aligned} \qquad (P, f, b_0)$$

where $f \in L^1(Q_T)$, $F : \mathbb{R} \to \mathbb{R}^d$ is locally Lipschitz continuous and

B1: $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone (with respect to the second argument), single-valued Carathéodory function.

B2: $\beta(x, 0) = 0$ for a.a. $x \in \Omega$.

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B3: for all $l \in \mathbb{R}$ $\beta(\cdot, l) \in L^1(\Omega)$.

Moreover we assume that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following conditions:

A1: $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function.

A2: there exist generalized $\mathcal{N}$-functions $M, M^* : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, where M^* is a conjugate function to M (for definitions see Section 2), a constant $c_a \in (0, 1]$ and a nonnegative integrable function a_0 such that

$$a(x, \xi) \cdot \xi \geq c_a \{M^*(x, a(x, \xi)) + M(x, \xi)\} - a_0(x) \tag{1}$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

A3: $a(\cdot, \cdot)$ is monotone, i.e.

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq 0 \tag{2}$$

for a.a. $x \in \Omega$ and all ξ, $\eta \in \mathbb{R}^d$.

Additionally we assume that

M1: there exists $c_M > 0$, $\nu > 0$ and $\xi_0 \in \mathbb{R}^d$ such that

$$M(x, \xi) \geq c_M |\xi|^{1+\nu} \qquad \text{for a.a. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^d,\ |\xi| \geq |\xi_0|. \tag{3}$$

M2: the conjugate function

$$M^* \text{ satisfies the } \Delta_2\text{–condition,} \tag{4}$$

i.e. there exists some nonnegative, integrable on Ω function g_{M^*} and a constant $C_{M^*} > 0$ such that

$$M^*(x, 2\xi) \leq C_{M^*} M^*(x, \xi) + g_{M^*}(x) \quad \text{for all } \xi \in \mathbb{R}^d \text{ and a.a. } x \in \Omega. \tag{5}$$

M3: the conjugate function M^* satisfies

$$\lim_{|\xi| \to \infty} \operatorname*{ess\,inf}_{x \in \Omega} \frac{M^*(x, \xi)}{|\xi|} = \infty. \tag{6}$$

Example 1.1 *The following examples of $\mathcal{N}$-functions fit into our setting:*

- $M(x, \xi) = |\xi|^{p(x)}$, *with* $p : \Omega \to (p^-, \infty)$ *measurable and* $p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) > 1$.
- $M(x, \xi) = \sum_{i=1}^d |\xi_i|^{p_i(x)}$, $p_i : \Omega \to (p_i^-, \infty)$ *measurable,* $p_i^- := \operatorname*{ess\,inf}_{x \in \Omega} p_i(x) > 1$, $i = 1, \ldots, d$ *and* $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.
- $M(x, \xi) = -1 + \frac{1}{d} \sum_{i=1}^d \exp(a_i(x)\xi_i^2))$, *where* $i = 1, \ldots, d$, $a_i : \Omega \to \mathbb{R}$ *are measurable, a.e. positive functions and* $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

Let us denote by $L_M(Q_T; \mathbb{R}^d)$ a generalized Musielak-Orlicz space (which we define in Section 2). Let $\mathcal{D}([0, T) \times \Omega)$ be a space of smooth functions with compact support in $[0, T) \times \Omega$ and let $C_c^1(\mathbb{R})$ be a space of continuously differentiable functions with compact support in $\mathbb{R}$. Moreover, for any given $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(r) := \begin{cases} -k & \text{if } r \leq -k \\ r & \text{if } |r| < k \\ k & \text{if } r \geq k. \end{cases}$$

Definition 1.1 *For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ a renormalized solution to (P, f, b_0) we call a pair of functions (u, b) satisfying the following conditions:*

R1: *$u : Q_T \to \mathbb{R}$ is a measurable, $b \in L^1(Q_T)$, where $b(t,x) = \beta(x, u(t,x))$, $(t,x) \in Q_T$.*

R2: *for each $k > 0$, $T_k(u) \in L^{1+\nu}(0,T; W_0^{1+\nu}(\Omega))$, $DT_k(u) \in L_M(Q_T;\mathbb{R}^d)$ and $a(\cdot, DT_k(u)) \in L_{M^*}(Q_T;\mathbb{R}^d)$.*

R3:

$$-\int_{Q_T} \partial_t\varphi \int_{b_0}^{b(t,x)} h \circ (\beta^{-1})^0(r)\,\mathrm{d}r\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} (a(x,Du) + F(u))) \cdot D(h(u)\varphi)\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} fh(u)\varphi\,\mathrm{d}x\mathrm{d}t$$

holds for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in \mathcal{D}([0,T)\times\Omega)$.

R4: *$\int_{Q_T\cap\{k<|u|<k+1\}} a(x,Du)\cdot Du\,\mathrm{d}x\mathrm{d}t \to 0$ as $k \to \infty$.*

An immediate observation from the above definition (**R3**) is that essential difficulties in the studies on renormalized solutions concentrate around showing weak convergence in L^1 of the product $a(x, DT_k(u_n)) \cdot DT_k(u_n)$, where u_n is an approximate sequence, such that Du_n is bounded in $L_M(Q_T;\mathbb{R}^d)$. The way we are dealing with this step is a novel approach in comparison to earlier studies and allows for relaxing the assumption of strict monotonicity to merely monotone vector fields a. Note that in case of working in reflexive spaces (such as e.g. $L^{p(x)}$ spaces), the monotonicity is a sufficient argument to conclude from $(a(x, DT_k(u_n)) - a(x, DT_k(u)) \cdot (DT_k(u_n) - x, DT_k(u))\cdot) \to 0$ in L^1 that $a(x, DT_k(u_n)) \cdot DT_k(u_n) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u)$. However, once the space is not reflexive, as the case of Orlicz space $L_M(Q_T;\mathbb{R}^d)$ may be (we discuss this issue in a sequel), then the convergence $a(x, DT_k(u_n)) \cdot DT_k(u) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u)$ may fail. As far as we know up to date studies on renormalized solutions concentrated on the case of strictly monotone vector fields, [18, 40]. The methods based on showing modular convergence of the sequences $a(x, DT_k(u_n))$ and $DT_k(u_n)$ what together allowed to conclude strong convergence in L^1 of the product. In the current paper we are using the biting lemma [1, 6, 38] and methods of Young measures to show that $a(x, DT_k(u_n)) \cdot DT_k(u_n)$ converges weakly and hence the renormalization identity (**R3**) is satisfied by the limit of approximation.

The existence proof of the full problem, as stated above, requires a very technical construction of multi-stage approximation of the solution. In particular it is based on nonlinear semigroup theory of m-accretive operators, it consists of construction of monotone approximation families of solutions, doubling of variables w.r.t. time and many others, see e.g. [9]. For sake of readability of the current paper we provide the complete proof of existence of renormalized solutions for a simplified problem, i.e. $\beta(\cdot, u) = u$ and $F(u) = 0$ what allows to avoid many technical difficulties and shorten the proof significantly. For the problem in its full generality we provide the weak sequential stability result. Namely we show that if there exists a sequence of approximate solutions converging in proper topologies, then its limit is a renormalized solution to (P, f, b_0). The remaining construction of approximate solutions to the full problem, i.e. that there exists a proper sequence which approximates a solution as well as the proof of uniqueness is postponed to a forthcoming paper. We believe that such strategy will allow the reader to understand better the main difficulties one has to face considering even the simplified problem.

The main result of this work is the following theorem:

Theorem 1.1 (The existence of renormalized solutions) *Let assumptions (**A1** - **A3**) and (**M1**-**M3**) be satisfied. Let $b_0 = u_0$ in Ω, $\beta(\cdot, u) = u$ and $F(u) = 0$ in (P, f, b_0). Then for each $f \in L^1(Q_T)$ and $b_0 \in L^1(\Omega)$ there exists a renormalized solution to (P, f, b_0).*

1.2 State of the art

For most equations that are significant from the point of view of mathematical physics, it is impossible to show the existence of classical solutions, i.e., solutions which are continuously differentiable as many times as the order of derivatives in equations under consideration. On the other hand, the concept of distributional solutions (or weak solutions) does not allow us to pose certain problems correctly. As an example, we can point out the lack of uniqueness in the class of distributional solutions to the simplest Burgers equation or other pathologies which appear in the case of distributional solutions to many partial differential equations. Hence, as a certain more general idea, we can use the notion of entropy and/or renormalization, that is to postulate that instead of (or in addition to) the weak formulation of the problem the equation satisfies certain additional entropy inequalities or renormalized equations for a sufficiently rich family of entropies or renormalizations.

The notion of renormalized solutions was introduced by DiPerna and Lions in [16] for the study of the Boltzmann equation. The concept was also applied to fluid mechanics models by Lions, cf. [33]. The notion was then adapted to elliptic equations by Boccardo et al. in [12] and by Murat in [36]. The existence of renormalized solutions to parabolic problems was considered by Blanchard et al. in [10, 11]. At the same time, for nonlinear elliptic problems with right-hand side in L^1, the notion of entropy solution has been developed by Benilan et al. in [8] and the notion of *solution obtained as limit of approximations* has been developed by Dall'Aglio in [15]. These three notions of solution, which turn out to be equivalent, allow to show well-posedness, namely existence, uniqueness, and continuity of the solution with respect to the data.

Due to prescribed growth conditions (1), the current studies are related with two directions in the research on nonlinear parabolic problems: classical Orlicz spaces and generalized Lebesgue spaces. The first one concerns equations having solutions in classical Orlicz spaces, namely generated by an isotropic $\mathcal{N}$-function which is independent of $x \in \Omega$, i.e. $M(\cdot) = m(|\cdot|)$, with $m : \mathbb{R}_+ \to \mathbb{R}_+$. The first existence result was provided by Donaldson in [17]. The existence of solutions for a parabolic equation in Orlicz space with a given right-hand side in the space of bounded Radon measures was shown by Meskine in [34]. The existence of solutions to some unilateral parabolic problems in the framework of Orlicz spaces with given data in L^1 is shown by Meskine et al. in [30]. Elmahi and Meskine in [18] study the Cauchy-Dirichlet problem and prove the existence of at least one entropy solution to strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces (isotropic, homogenous) provided that the data is in L^1. The extension in the direction of anisotropic $\mathcal{N}$-function is considered in [23]. The authors showed existence of weak solutions for homogeneous equation without assuming Δ_2 condition for M^*. For the convergence of a numerical schema see [20]. The main effort was concentrated at the issue of density of compactly supported smooth functions in appropriate space with respect to modular convergence of the gradients. This example allows us to sketch a general scheme of proceeding with the limit passage from the approximate problem. In case when Δ_2 condition for M^* is assumed, the limit passage follows by means of weak-$*$ convergence. However, once this assumption is omitted, the arguments of approximation theorems in modular convergence are exploit. The classical results related to the second approach are due to Gossez, [22, 21].

Following the studies for standard Leray-Lions operators (namely under assumption of polynomial growth conditions for the nonlinear term) on renormalized solutions [36, 10, 12], there appeared also results on renormalized solutions for parabolic problems in Orlicz spaces. The existence result of a renormalized solution for a class of nonlinear parabolic equations in classical Orlicz spaces with no growth assumption made on the nonlinearities and an $\mathcal{N}$-function which does not have to satisfy the Δ_2-condition was achieved in [5]. The arguments used to prove the convergence of approximate solutions to a renormalized solution are based on an approximation property in Orlicz-Sobolev spaces proved by Gossez in [22, Theorem 4].

The author shows that it is possible to approximate the gradient of an $W^1_0 L_M(\Omega)$-function in modular convergence by a sequence of gradients of smooth functions, compactly supported in Ω.

As we allow the $\mathcal{N}-$function to depend on x, the solutions will belong to Musielak-Orlicz space. A basic example of a Musielak-Orlicz space is a variable exponent Lebesgue space generated by the $\mathcal{N}-$function

$$M(x,\xi) = |\xi|^{p(x)}. \tag{7}$$

Therefore we wish to place our studies in the context of wide literature available for $L^{p(x)}$ framework. We omit a survey of results on existence of weak solutions and already concentrate on renormalized solutions. In [7] the authors studied well-posedness of a parabolic problem with the following variable exponent growth and coercivity assumptions:

(i) There exist a continuous function $p : \overline{\Omega} \to (1,\infty)$, $1 < \min_{x\in\overline{\Omega}} p(x) \le d$ and $c > 0$ such that

$$a(x,\xi)\cdot\xi \ge c|\xi|^{p(x)}$$

holds for all $\xi \in \mathbb{R}^d$ and a.a. $x \in \Omega$.

(ii) For a.a. $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$

$$|a(x,\xi)| \le h(x) + |\xi|^{p(x)-1}$$

where h is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ and $p'(x) := p(x)/(p(x)-1)$ for a.e. $x \in \Omega$.

The corresponding elliptic $p(x)$-Laplace problem with a variable exponent was considered in [46]. The proper function spaces to capture these problems are the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$. Thanks to the assumption (i), the spaces are reflexive and separable what significantly facilitates the considerations (see [7, 46]) since classical monotonicity methods are applicable. Moreover, in this case the weak-$*$ closure of compactly supported functions (which is equivalent to a weak one) coincides with strong (norm) closure. This is a direct consequence of the Mazur lemma. It is worth to underline here that this result does not hold true in the case of nonreflexive generalized Musielak-Orlicz spaces and it is one of the essential difficulties we have to face in our considerations.

Finally, we want to generalize an $\mathcal{N}$-function to include anisotropic and inhomogeneous dependence. Again, there are two ways that can be followed in the limiting procedure. Due to the x-dependence of the $\mathcal{N}$-function, a result similar to Gossez [22] related with approximation theorems can in general not be achieved. It is already well-known in the theory of variable exponent Lebesgue spaces that convolution with a smooth, compactly supported kernel may fail to be a bounded operator unless the variable exponent satisfies the log-Hölder continuity condition. A generalization of log-Hölder condition of exponents for $M(x,\xi)$ was assumed in [44] and [45], whereas the first paper concerns isotropic $(t,x)-$dependent $\mathcal{N}$-function and the second one $x-$dependent anisotropic $\mathcal{N}$-function. See also [25] for an analogue condition in elliptic case, however for anisotropic $\mathcal{N}$-function.

Most of essential and necessary tools of functional analysis for classical Orlicz spaces are already deeply understood, for example: the density of smooth functions in modular topology ([22]) and the integration by parts formula for classical Orlicz spaces ([19]). But many structures for anisotropic Musielak-Orlicz spaces have not been developed or are not understood purely yet.

The setting considered in this paper includes and generalizes known results in the variable exponent, anisotropic and classical Orlicz setting. The function M which describes the growth condition on the vector field a is a so-called *generalized* $\mathcal{N}$-function (see Definition 2.1 below). The corresponding generalized Musielak-Orlicz spaces $L_M(\Omega;\mathbb{R}^d)$, have been introduced in [42], [43]. In general, if M and M^* do not satisfy a Δ_2-condition these spaces fail to be separable or reflexive.

Our techniques to overcome these difficulties are inspired by former works motivated by fluid dynamics [13, 26, 27, 47, 48, 49]. The authors considered equations involving vector fields satisfying general non-standard growth conditions of type (**A2**) with a generalized $\mathcal{N}$-function $M(x,\xi)$.

Gwiazda et al. in [24] studied a steady and in [26] a dynamic model for non-Newtonian fluids under an additional strict monotonicity assumption on the vector field. The author used Young measures techniques in place of a monotonicity method. The additional assumption of strict monotonicity allowed to conclude that the measure-valued solution is a family of Dirac measures and hence a weak solution. A similar method is used for the variable exponent setting in [4].

Later the existence of weak solutions in the framework of non-Newtonian fluids without the assumption of strict monotonicity of the vector field a was obtained in [47, 27]. The authors used a Browder-Minty trick adapted to nonreflexive generalized Musielak-Orlicz spaces which we also have to employ in our current studies. In [28] we have studied the existence and uniqueness of renormalized solutions of the corresponding elliptic problem to (P, f, b_0).

2 Orlicz spaces - notation and properties

Definition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^d$. A function $M : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is said to be* a generalized $\mathcal{N}$-function *if it satisfies the following conditions:*

1. *M is a Carathéodory function such that $M(x,\xi) = M(x,-\xi)$ a.e. in Ω and $M(x,\xi) = 0$ if and only if $\xi = 0$.*
2. *$M(x,\xi)$ is a convex function w.r.t. ξ.*
3. $$\lim_{|\xi|\to 0} \frac{M(x,\xi)}{|\xi|} = 0 \quad \text{for a.a. } x \in \Omega. \tag{8}$$
4. $$\lim_{|\xi|\to \infty} \frac{M(x,\xi)}{|\xi|} = \infty \quad \text{for a.a. } x \in \Omega. \tag{9}$$

The *complementary function* M^* to a generalized $\mathcal{N}$-function M is defined by

$$M^*(x,\xi) = \sup_{\eta\in\mathbb{R}^d} (\xi \cdot \eta - M(x,\eta)) \qquad \text{for } \xi \in \mathbb{R}^d,\ x \in \Omega. \tag{10}$$

Let $Q_T = (0,T) \times \Omega$. The *generalized Orlicz class* $\mathcal{L}_M(Q_T;\mathbb{R}^d)$ is the set of all measurable functions $\xi : Q_T \to \mathbb{R}^d$ such that

$$\int_{Q_T} M(x,\xi(t,x))\,\mathrm{d}x\mathrm{d}t < \infty.$$

Let us notice that $\mathcal{L}_M(Q_T;\mathbb{R}^d)$ is a convex set and it is not necessary a linear space.

The *generalized Orlicz space* (or Musielak-Orlicz space) $L_M(Q_T;\mathbb{R}^d)$ is defined as the set of all measurable functions $\xi : Q_T \to \mathbb{R}^d$ which satisfy

$$\int_{Q_T} M(x,\lambda\xi(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \lambda \to 0.$$

Obviously there holds that $L_M(Q_T;\mathbb{R}^d) \subseteq L^1(Q_T;\mathbb{R}^d)$.

If $\xi \in L_M(Q_T;\mathbb{R}^d)$, then we may consider the Luxemburg norm defined by

$$\|\xi\|_M = \inf\left\{\lambda > 0\ :\ \int_{Q_T} M\left(x, \frac{\xi(t,x)}{\lambda}\right) \mathrm{d}x\mathrm{d}t \le 1\right\}. \tag{11}$$

In general, $L_M(Q_T;\mathbb{R}^d)$ is neither separable nor reflexive.

By $E_M(Q_T;\mathbb{R}^d)$ we denote the closure of all bounded measurable functions defined on Q_T with respect to the Luxemburg norm $\|\cdot\|_M$ (see Remark 2.2). The space $E_M(Q_T;\mathbb{R}^d)$ is the largest linear space such that $E_M(Q_T;\mathbb{R}^d) \subseteq \mathcal{L}_M(Q_T;\mathbb{R}^d) \subseteq L_M(Q_T;\mathbb{R}^d)$, where the inclusion is in general strict. The space $E_M(Q_T;\mathbb{R}^d)$ is separable and $C_0^\infty(Q_T;\mathbb{R}^d)$ is dense in $E_M(Q_T;\mathbb{R}^d)$. Moreover the space $L_{M^*}(Q_T;\mathbb{R}^d)$ is the dual space of $E_M(Q_T;\mathbb{R}^d)$, i.e. $(E_M(Q_T;\mathbb{R}^d))^* = L_{M^*}(Q_T;\mathbb{R}^d)$.

If M is an $\mathcal{N}$-function and M^* is the complementary to M, then the following *Fenchel-Young* inequality is satisfied

$$|\xi\cdot\eta| \le M(x,\xi) + M^*(x,\eta) \qquad \text{for all } \xi,\eta\in\mathbb{R}^d \text{ and a.a. } x\in\Omega. \tag{12}$$

Moreover if M is an $\mathcal{N}$-function and M^* its complementary, then the *generalized Hölder inequality* holds, e.g.

$$\left|\int_{Q_T} \xi\cdot\eta \,\mathrm{d}x\mathrm{d}t\right| \le 2\|\xi\|_M\|\eta\|_{M^*} \qquad \text{for } \xi\in L_M(Q_T;\mathbb{R}^d) \text{ and } \eta\in L_{M^*}(Q_T;\mathbb{R}^d). \tag{13}$$

The functional

$$\varrho(\xi) = \int_{Q_T} M(x,\xi(t,x))\,\mathrm{d}x\mathrm{d}t$$

is a modular in the space of measurable functions $\xi : Q_T \to \mathbb{R}^d$ in the sense of [31, p. 208]. We say that a sequence $\{\xi^j\}_{j=1}^\infty$ converges *modularly* to ξ in $L_M(Q_T;\mathbb{R}^d)$ if there exists $\lambda > 0$ such that

$$\int_{Q_T} M\left(x, \frac{\xi^j - \xi}{\lambda}\right)\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } j\to\infty.$$

We will write $\xi^j \xrightarrow{M} \xi$ for the modular convergence in $L_M(Q_T;\mathbb{R}^d)$.

Remark 2.1 *The assumption* (6) *provides that*

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} M(x,\xi) < \infty \qquad \textit{for all } 0 < R < +\infty, \tag{14}$$

and (3) *provides analogous condition to* (6) *for M, consequently M^* satisfies also condition analogous to* (14).

Indeed: For any $R > 0$ we have

$$\begin{aligned}
\sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} M(x,\xi) &= \sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\eta\in\mathbb{R}^d\setminus\{0\}} \{\xi\cdot\eta - M^*(x,\eta)\} \\
&\le \sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\eta\in\mathbb{R}^d\setminus\{0\}} |\eta| \left\{ \frac{\xi\cdot\eta}{|\eta|} - \operatorname*{ess\,inf}_{x\in\Omega} \frac{M^*(x,\eta)}{|\eta|} \right\}. \qquad (15)
\end{aligned}$$

From (6) it follows that there exists $R_0 > 0$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} \frac{M^*(x,\eta)}{|\eta|} > R + 1$$

for all $\eta\in\mathbb{R}^d$ such that $|\eta| \ge R_0$. Therefore

$$\sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\eta\in\mathbb{R}^d, |\eta|>R_0} |\eta| \left\{ \frac{\xi\cdot\eta}{|\eta|} - \frac{M^*(x,\eta)}{|\eta|} \right\} \le 0 \tag{16}$$

and from (16) it follows that

$$\begin{aligned}
\sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} M(x,\xi) &\le \sup_{|\xi|\le R} \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\eta\in\mathbb{R}^d, |\eta|\le R_0} |\eta| \left\{ \frac{\xi\cdot\eta}{|\eta|} - \frac{M^*(x,\eta)}{|\eta|} \right\}. \\
&\le RR_0. \qquad (17)
\end{aligned}$$

Remark 2.2 *Note that from* (14) *it follows that* $L^\infty(Q_T) \subset \mathcal{L}_M(Q_T)$*, since* $|Q_T| < \infty$*.*

Proofs of the following properties the reader can find in [26].

Lemma 2.1 *Let* $\xi^j : Q_T \to \mathbb{R}^d$ *be a sequence of measurable functions. Then* $\xi^j \xrightarrow{M} \xi$ *in* $L_M(Q_T;\mathbb{R}^d)$ *modularly as* $j \to \infty$ *if and only if* $\xi^j \to \xi$ *in measure and there exists some* $\lambda > 0$ *such that the sequence* $\{M(\cdot, \lambda\xi^j)\}_{j=1}^\infty$ *is uniformly integrable, i.e.,*

$$\lim_{R\to\infty}\left(\sup_{j\in\mathbb{N}}\int_{\{(t,x):|M(x,\lambda\xi^j)|\geq R\}} M(x,\lambda\xi^j)\,\mathrm{d}x\mathrm{d}t\right) = 0.$$

Lemma 2.2 *Let* M *be an* $\mathcal{N}-$*function such that*

$$\lim_{|\xi|\to\infty} \operatorname{ess\,inf}_{x\in\Omega} \frac{M(x,\xi)}{|\xi|} = \infty$$

and for all $j \in \mathbb{N}$ *let* $\int_{Q_T} M(x, z^j)\,\mathrm{d}x\mathrm{d}t \leq c$*. Then the sequence* $\{z^j\}_{j=1}^\infty$ *is uniformly integrable.*

Proposition 2.3 *Let* $\{\varrho^j\}_{j=1}^\infty$ *a sequence of probability measures, i.e.,* ϱ *is a measurable nonnegative function and* $\int_{\mathbb{R}} \varrho(\tau)\,d\tau = 1$*. We define* $\varrho^j(t) = j\varrho(jt)$*. Moreover let* $*$ *denote a convolution in the variable* t*. Then for any function* $\psi : Q_T \to \mathbb{R}^d$ *such that* $\psi \in L^1(Q_T;\mathbb{R}^d)$ *it holds*

$$(\varrho^j * \psi)(t,x) \to \psi(t,x) \quad \text{in measure as } j \to \infty.$$

Proposition 2.4 *Let* ϱ^j *be defined as in Proposition 2.3, let* M *be an* $\mathcal{N}$*–function and* $\psi : Q_T \to \mathbb{R}^d$ *be such that* $\psi \in \mathcal{L}_M(Q_T;\mathbb{R}^d)$*. Then the sequence* $\{M(x, \varrho^j * \psi)\}_{j=1}^\infty$ *is uniformly integrable.*

Remark 2.3 *The Proposition 2.3 and 2.4 holds for standard regularising kernels, a Landes regularisation and Steklov averages as well.*

Proposition 2.5 *Let* M *be an* $\mathcal{N}-$*function and* M^* *its complementary function. Suppose that the sequences* $\psi^j : Q \to \mathbb{R}^d$ *and* $\phi^j : Q \to \mathbb{R}^d$ *are uniformly bounded in* $L_M(Q;\mathbb{R}^n)$ *and* $L_{M^*}(Q;\mathbb{R}^d)$ *respectively. Moreover* $\psi^j \xrightarrow{M} \psi$ *modularly in* $L_M(Q;\mathbb{R}^d)$ *and* $\phi^j \xrightarrow{M^*} \phi$ *modularly in* $L_{M^*}(Q;\mathbb{R}^d)$ *as* $j \to \infty$*. Then* $\psi^j \cdot \phi^j \to \psi \cdot \phi$ *strongly in* $L^1(Q)$ *as* $j \to \infty$*.*

3 Notation

Let us define the following linear space

$$\begin{aligned}\mathcal{V} := \{\varphi \in L^1_{\text{loc}}(Q_T);\ \exists\{\varphi_j\}_{j=1}^\infty \subset \mathcal{D}([0,T)\times\Omega) \text{ such that } \ D\varphi_j \overset{*}{\rightharpoonup} D\varphi \text{ in } L_M(Q_T;\mathbb{R}^d),\\ \text{and } \varphi_j \to \varphi \text{ in } L^1(Q_T) \text{ and } \varphi_j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(Q_T) \text{ as } j \to \infty\}.\end{aligned} \tag{18}$$

We will use also the following notation: for any $u : Q_T \to \mathbb{R}$ and $k \geq 0$, we denote $\{|u| \leq (<,>,\geq,=)k\}$ for the set $\{(t,x) \in Q_T \ : \ |u(t,x)| \leq (<,>,\geq,=)k\}$. For $r \in \mathbb{R}$ by $\mathrm{sign}_0(r)$ we mean the usual (single-valued) sign function, $\mathrm{sign}_0^+(r) = 1$ if $r > 0$ and $\mathrm{sign}_0^+(r) = 0$ if $r \leq 0$. Let $h_l(r) : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) = \min((l+1-|r|)^+, 1) \tag{19}$$

for each $r \in \mathbb{R}$.

4 Existence of renormalized solutions

4.1 Integration-by-parts-formula

In the next Lemma, for the sake of completeness, we prove the integration-by-parts-formula. The idea of the proof is the same as in [2] and the generalizations considered in [14] and [37]. The essential point is that the Stekhlov average of a function $v \in \mathcal{V} \cap L^\infty(Q_T)$ defined by $v_\eta(\cdot) = \frac{1}{\eta}\int_\cdot^{\cdot+\eta} v(\sigma)d\sigma$, $\eta > 0$, (appropriately prolongated outside $(0,T)$) still belongs to $\mathcal{V} \cap L^\infty(Q_T)$ and converges strongly in $L^1(Q_T)$, weakly-$*$ in $L^\infty(Q_T)$ to v and their gradients in modular in $L_M(Q_T;\mathbb{R}^d)$ to the gradient of v as $\eta \downarrow 0$.

Lemma 4.1 *Let $\beta(\cdot,\cdot)$ be a Carathéodory function and let $\beta(x,\cdot)$ be monotone for a.a. $x \in \Omega$. Moreover, let $u : Q_T \to \mathbb{R}$ be a measurable function such that, for every $k \geq 0$, $T_k(u) \in \mathcal{V}$, $b \in L^\infty([0,T]; L^1(\Omega))$ such that $b(t,x) = \beta(x, u(t,x))$ almost everywhere in Q_T, $b_0 \in L^1(\Omega)$ and $u_0 : \Omega \to \overline{\mathbb{R}}$ be a measurable function such that $b_0 = \beta(x,u_0)$ almost everywhere in Ω. Furthermore, we assume that there exist $G_1 \in E_{M^*}(Q_T;\mathbb{R}^d)$, $G_2 \in L^\infty(Q_T;\mathbb{R}^d)$ and $G_3 \in L^1(Q_T)$ satisfying*

$$\int_{Q_T} (b - b_0)\partial_t \xi \,\mathrm{d}x\mathrm{d}t = \int_{Q_T} (G_1 + G_2)\cdot D\xi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3 \xi \,\mathrm{d}x\mathrm{d}t, \tag{20}$$

for all $\xi \in \mathcal{D}([0,T) \times \Omega)$. Then,

$$\int_{Q_T} \partial_t \xi \int_{b_0}^{b(t,x)} h((\beta^{-1})^0(\sigma))d\sigma \,\mathrm{d}x\mathrm{d}t = \int_{Q_T} (G_1 + G_2)\cdot D(h(u)\xi)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3 h(u)\xi \,\mathrm{d}x\mathrm{d}t \tag{21}$$

i) *for all $h \in W^{1,\infty}(\mathbb{R})$, where* supp$\,(h')$ *is compact and for all $\xi \in \mathcal{V}$ such that $\partial_t \xi \in L^\infty(Q_T)$,.*

or

ii) *for all $h \in W^{1,\infty}(\mathbb{R})$, where* supp$\,(h')$ *is compact and $h(0) = 0$ and for all $\xi \in \mathcal{D}([0,T) \times \overline{\Omega})$.*

Proof of Lemma 4.1: First note that there exist Lipschitz continuous functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ of the following form: $h_1(t) = \int_{-\infty}^t (h')^+(s)\,\mathrm{d}s$, $h_2(t) = \int_{-\infty}^t (h')^-(s)$ (where $h^+ := \max(0,h)$ and $h^- := \min(0,h)$). Additionally h_1 is non-decreasing, h_2 is non-increasing, $h = h_1 + h_2$ and for the case *(ii)* $h_1(0) = 0$ and $h_2(0) = 0$. Furthermore for both cases, there exists $k > 0$ such that $\operatorname{supp} h' \subset [-k,k]$, hence $h(u) = h(T_k(u)) = h_1(T_k(u)) + h_2(T_k(u))$, moreover $h_1(T_k(u)),\ h_2(T_k(u)) \in L^\infty(Q_T)$ and $D(h_1(T_k(u))),\ D(h_2(T_k(u))) \in L_M(Q_T;\mathbb{R}^d)$. Let us assume that ξ is nonnegative, and satisfies conditions form the case *(i)) or (*ii)) respectively. Moreover let $\eta > 0$ and let us set

$$\zeta := h_1(T_k(u))\xi \tag{22}$$

and

$$\zeta_\eta(t,x) := \frac{1}{\eta}\int_t^{t+\eta} \zeta(\sigma,x)\,\mathrm{d}\sigma. \tag{23}$$

Note that $\zeta \in \mathcal{V}$. Indeed, for the case (i) the crucial is to show that $\varphi_1\varphi_2 \in \mathcal{V}$ for each $\varphi_1 \in \mathcal{V}$ and each $\varphi_2 \in \mathcal{V}$. This is provided by the fact that due to definition of $\mathcal{V}$ there exist $\{\varphi_1^j\}_j$, $\{\varphi_2^j\}_j \subset C_c^\infty(Q_T)$ s.t. $\varphi_i^j \to \varphi_i$ a.e. in Q_T and weakly-(*) in $L^\infty(Q_T)$, $D\varphi_i^j \overset{*}{\rightharpoonup} D\varphi_i$ in $L_M(Q_T;\mathbb{R}^d)$ for $i = 1, 2$ and thus also $D\varphi_i^j \rightharpoonup D\varphi_i$ in L^1. Then since $\{\varphi_1^j D\varphi_2^j\}_j$, $\{\varphi_2^j D\varphi_1^j\}_j$ are uniformly integrable in $L^1(Q_T;\mathbb{R}^d)$, we get that $\varphi_1^j D\varphi_2^j \rightharpoonup \varphi_1 D\varphi_2$ and $\varphi_2^j D\varphi_1^j \rightharpoonup \varphi_2 D\varphi_1$ in $L^1(Q_T;\mathbb{R}^d)$. Thus as $\{\varphi_1^j D\varphi_2^j\}_j$, $\{\varphi_2^j D\varphi_1^j\}_j$ are bounded in $L_M(Q_T;\mathbb{R}^d)$, we get that

$\varphi_1^j D\varphi_2^j \overset{*}{\rightharpoonup} \varphi_1 D\varphi_2$ and $\varphi_2^j D\varphi_1^j \overset{*}{\rightharpoonup} \varphi_2 D\varphi_1$ in $L_M(Q_T;\mathbb{R}^d)$, what gives the assertion. For the case (ii)) it is enough to notice that for each $\varphi_1 \in W^{1,\infty}([0,T)\times\overline{\Omega})$ and each $\varphi_2 \in \mathcal{V}$, it holds that $\varphi_1\varphi_2 \in \mathcal{V}$. In both cases the function $\zeta_\eta : Q_T \to \mathbb{R}$ is such that $\zeta_\eta \in \mathcal{V}$, $(\zeta_\eta)_t \in L^\infty(Q_T)$, with $\xi_\eta(T,x) = 0$ for all $x \in \Omega$ and $\eta > 0$. Noticing that ξ_η is an admissible test function in (20) we have

$$\begin{aligned}\int_{Q_T} (G_1+G_2)\cdot D\zeta_\eta \,\mathrm{d}x\mathrm{d}t + &\int_{Q_T} G_3\zeta_\eta \,\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_T} \frac{1}{\eta}(\zeta(t+\eta,x)-\zeta(t,x))(b(t,x)-b(0,x))\,\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{\eta}(I_1+I_2+I_3),\end{aligned} \tag{24}$$

where $\zeta(t,x)=0$ for $t>T$, $b(t,x)=b_0(x)$ for $t<0$, and

$$\begin{aligned} I_1 &= \int_0^T \int_\Omega \zeta(t+\eta,x)b(t,x) = \int_\eta^T \int_\Omega \zeta(t,x)b(t-\eta,x)\,\mathrm{d}x\mathrm{d}t \\ I_2 &= -\int_0^T \int_\Omega \zeta(t,x)b(t,x)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{25}$$

$$\begin{aligned} I_3 &= -\int_0^T \int_\Omega \zeta(t+\eta,x)-\zeta(t,x))b(0,x)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \int_t^{t+\eta} \zeta(\sigma,x)\,\mathrm{d}\sigma\cdot 0 - \int_\Omega \int_t^{t+\eta} \zeta(\sigma,x)d\sigma\cdot b(0,x)\Big|_0^T \,\mathrm{d}x\mathrm{d}t \\ &= \int_0^\eta \int_\Omega \zeta(t,x)b(t-\eta,x)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{26}$$

From (25) and (26) we get

$$\int_{Q_T} (G_1+G_2)\cdot D\zeta_\eta \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3\zeta_\eta \,\mathrm{d}x\mathrm{d}t = \frac{1}{\eta}\int_{Q_T} \zeta(t,x)(b(t-\eta,x)-b(t,x))\,\mathrm{d}x\mathrm{d}t. \tag{27}$$

Now, let us notice that $b(t,x) \in \beta(x,u(t,x))$ almost everywhere in Q_T, $h_1\circ T_k$ nondecreasing and $\xi \geq 0$. Moreover, if we define $\phi_{h_1} : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ by

$$\phi_{h_1}(r) := \begin{cases} \int_0^r h_1(T_k((\beta^{-1})^0(\sigma)))\,\mathrm{d}\sigma, & r \in \overline{R(\beta)}, \\ +\infty, & \text{otherwise}, \end{cases}$$

then let us note that $\phi_{h_1}(\cdot)$ is convex, and

$$\phi_{h_1}(r_2) - \phi_{h_1}(r_1) \geq \chi(r_2-r_1)$$

for any $\chi \in \partial_r\phi_{h_1}(r_1)$. Then it follows that

$$\frac{1}{\eta}\int_{Q_T} \zeta(t,x)(b(t-\eta,x)-b(t,x))\,\mathrm{d}x\mathrm{d}t \leq \frac{1}{\eta}\int_{Q_T} \xi(t,x)\int_{b(t,x)}^{b(t-\eta,x)} h_1(T_k((\beta^{-1})^0(\sigma)))\,\mathrm{d}\sigma\,\mathrm{d}x\mathrm{d}t. \tag{28}$$

From (27) and (28) it follows with the same arguments as in (26) that

$$\begin{aligned}\int_{Q_T}(G_1+G_2)\cdot D\zeta_\eta \,\mathrm{d}x\mathrm{d}t+\int_{Q_T} G_3\zeta_\eta\,\mathrm{d}x\mathrm{d}t\\
&\leq \frac{1}{\eta}\int_{Q_T}\xi(t,x)(\phi_{h_1}(b(t-\eta,x))-\phi_{h_1}(b(t,x)))\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\eta}\int_{Q_T}(\xi(t+\eta,x)-\xi(t,x))(\phi_{h_1}(b(t,x))-\phi_{h_1}(b(0,x))\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\eta}\int_{Q_T}(\xi(t+\eta,x)-\xi(t,x))\int_{b_0}^{b(t,x)}h_1(T_k((\beta^{-1})^0(\sigma)))\,\mathrm{d}\sigma\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{29}$$

Passing to a subsequence if necessary, we have $\zeta_\eta \stackrel{*}{\rightharpoonup} \xi h_1(T_k(u))$ weakly-$*$ in $L^\infty(Q_T)$ as $\eta\downarrow 0$. Note that $D\zeta_\eta=[D\xi h_1(T_k(u))]_\eta+[\xi D(h_1(T_k(u)))]_\eta$, and, obviously, $[D\xi h_1(T_k(u))]_\eta \stackrel{*}{\rightharpoonup} D\xi h_1(T_k(u))$ weakly-$*$ in $L^\infty(Q_T;\mathbb{R}^d)$. Moreover, according to Proposition 2.3, Proposition 2.4 and Lemma 2.1 we obtain that

$$[\xi D(h_1(T_k(u)))]_\eta \xrightarrow{M} \xi D(h_1(T_k(u)))$$

in modular in $L_M(Q_T;\mathbb{R}^d)$ as $\eta\downarrow 0$. Hence passing to the limit in (29) yields

$$\begin{aligned}\int_{Q_T}(G_1+G_2)\cdot D(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t+&\int_{Q_T}G_3(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t\\
&\leq \int_{Q_T}\xi_t\int_{b_0}^{b(t,x)}h_1\circ T_k\circ(\beta^{-1})^0(\sigma)d\sigma\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{30}$$

Note that since $T_k(u_0)\in L^\infty(\Omega)$, there exists a sequence $\{u_{0,n}\}_n\subset\mathcal{D}(\Omega)$ such that $T_k(u_{0,n})\to T_k(u_0)$ in $L^1(\Omega)$ and almost everywhere in Ω as $n\to\infty$. For $t<0$ and all $x\in\Omega$ we write $u(t,x)=u_{0,n}$ for some $n\in\mathbb{N}$ and $b(t,x)=b_0$. For $\xi\in\mathcal{D}^+([0,T)\times\overline{\Omega})$ we define $\xi(t,x):=\xi(-t,x)$ for $t<0$ and all $x\in\Omega$. If $\zeta:=h_1(T_k(u))\xi$, for $(t,x)\in Q_T$ and $\eta>0$ we define

$$\tilde{\zeta}_\eta(t,x):=\frac{1}{\eta}\int_{t-\eta}^{t}\zeta(\sigma,x)d\sigma,\tag{31}$$

then $\tilde{\zeta}_\eta:Q_T\to\mathbb{R}$ is such that $\tilde{\zeta}_\eta$, $(\tilde{\zeta}_\eta)_t\in L^\infty(Q_T)$, $\tilde{\zeta}_\eta\in\mathcal{V}$ and $\tilde{\zeta}_\eta(T,x)=0$ for all $x\in\Omega$ and $\eta>0$ sufficiently small. Since $\tilde{\zeta}_\eta$ is an admissible test function in (20),

$$\begin{aligned}\int_{Q_T}(G_1+G_2)\cdot D\tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t+&\int_{Q_T}G_3\tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t\\
&=\int_{Q_T}(b-b_0)\partial_t(\tilde{\xi}_\eta)\,\mathrm{d}x\mathrm{d}t\\
&=\int_{Q_T}\frac{1}{\eta}(\zeta(t,x)-\zeta(t-\eta,x))(b(t,x)-b(0,x))\,\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{\eta}(J_1+J_2+J_3),\end{aligned}\tag{32}$$

where

$$\begin{aligned}J_1&=\int_0^T\int_\Omega\zeta(t,x)b(t,x)\,\mathrm{d}x\mathrm{d}t=\int_\eta^{T+\eta}\int_\Omega\zeta(t-\eta,x)b(t-h,x)\,\mathrm{d}x\mathrm{d}t\\
J_2&=-\int_0^T\int_\Omega\zeta(t-\eta,x)b(t,x)\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{33}$$

and, for $\eta > 0$ sufficiently small,

$$\begin{aligned} J_3 &= -\int_0^T \int_\Omega (\zeta(t,x) - \zeta(t-\eta,x)) b(0,x)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \int_{t-\eta}^t \zeta(\sigma,x) d\sigma \cdot 0\,\mathrm{d}x\mathrm{d}t - \int_\Omega \int_{t-\eta}^t \zeta(\sigma,x) d\sigma \cdot b(0,x)\big|_0^T\,\mathrm{d}x\mathrm{d}t \\ &= \int_{-\eta}^0 \int_\Omega \zeta(t,x) b(t,x)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^\eta \int_\Omega \zeta(t-\eta,x) b(t-\eta,x)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{34}$$

From (33) and (34) we get

$$\int_{Q_T} (G_1 + G_2) \cdot D\tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3 \tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t = \frac{1}{\eta}(I_1 + I_2), \tag{35}$$

where

$$\begin{aligned} I_1 &= \int_\eta^T \int_\Omega \zeta(t-\eta,x)(b(t-\eta,x) - b(t,x)), \\ I_2 &= \int_0^\eta \int_\Omega h_1(T_k(u_0))\xi(b(t-\eta,x) - b(t,x)) \\ &+ \int_0^\eta \int_\Omega (h_1(T_k(u_{0,n})) - h_1(T_k(u_0)))\xi(b(t-\eta,x) - b(t,x)). \end{aligned}$$

Since $-(h_1 \circ T_k \circ (\beta^{-1})^0)$ is nonincreasing, for any $\eta > 0$ we have

$$\int_{b(t-\eta,x)}^{b(t,x)} -(h_1 \circ T_k \circ (\beta^{-1})^0)(\sigma) d\sigma \le -(b(t,x) - b(t-\eta,x)) h_1(T_k(u(t-\eta,x))) \tag{36}$$

almost everywhere in $(\eta, T) \times \Omega$ and

$$\int_{b(t-\eta,x)}^{b(t,x)} -(h_1 \circ T_k \circ (\beta^{-1})^0)(\sigma) d\sigma \le -(b(t,x) - b_0) h_1(T_k(u_0)) \tag{37}$$

almost everywhere in $(0,\eta) \times \Omega$. Now, putting together (35), (36) and (37) yields

$$\begin{aligned} \int_{Q_T} (G_1 + G_2) \cdot D\tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t + &\int_{Q_T} G_3 \tilde{\zeta}_\eta\,\mathrm{d}x\mathrm{d}t \\ &\ge \frac{1}{\eta} \int_{Q_T} \xi(t-\eta,x)(\phi_{h_1}(b(t,x)) - \phi_{h_1}(b(t-\eta,x)))\,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^\eta \int_\Omega (h_1(T_k(u_{0,n})) - h_1(T_k(u_0)))\xi(b_0 - b(t,x))\,\mathrm{d}x\mathrm{d}t \\ &\ge \frac{1}{\eta} \int_{Q_T} (\xi(t-\eta,x) - \xi(t,x))(\phi_{h_1}(b(t,x)) - \phi_{h_1}(b(0,x)))\,\mathrm{d}x\mathrm{d}t \\ &- \int_{Q_T} |h_1(T_k(u_{0,n})) - h_1(T_k(u_0))||\xi|(|b_0| + |b(t,x)|)\,\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{\eta} \int_{Q_T} (\xi(t-\eta,x) - \xi(t,x)) \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma) d\sigma\,\mathrm{d}x\mathrm{d}t \\ &- \int_{Q_T} |h_1(T_k(u_{0,n})) - h_1(T_k(u_0))||\xi|(|b_0| + |b(t,x)|)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{38}$$

The left hand side is treated in the same way as in (30). Passing to the limit with $\eta \downarrow 0$ and then with $n \to \infty$ in (38) by the Lebesgue Dominated Convergence Theorem we get

$$\begin{aligned}\int_{Q_T} (G_1 + G_2) \cdot D(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t \\ \geq \int_{Q_T} \partial_t \xi \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma) d\sigma\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{39}$$

Combining (30) and (39) finally we get

$$\begin{aligned}\int_{Q_T} (G_1 + G_2) \cdot D(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} G_3(h_1(T_k(u))\xi)\,\mathrm{d}x\mathrm{d}t \\ = \int_{Q_T} \partial_t \xi \int_{b_0}^{b(t,x)} h_1 \circ T_k \circ (\beta^{-1})^0(\sigma) d\sigma\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{40}$$

for all $h_1 : \mathbb{R} \to \mathbb{R}$ non-decreasing and Lipschitz continuous and for all nonnegative ξ satisfying *(i)*) or *(ii)*) respectively.

Next replacing $h_1(T_k(u))$ by $-h_2(T_k(u))$ in (40) it follows that (40) also holds for $-h_2(T_k(u))$ and $h_2(T_k(u))$, hence we can also replace $h_1(T_k(u))$ by $h(T_k(u)) = h(u)$ in (40).

For ξ satisfying *(i)*) or *(ii)*) we have $\xi = \xi^+ + \xi^-$, where $\xi^+ := \max(0, \xi)$, $\xi^- := \min(0, \xi)$ are in $\mathcal{V}$ or in $W^{1,\infty}(Q_T)$ respectively (for the cases *(i)*) or *(ii)*)).

4.2 Existence of weak solutions to a regularised problem

The idea of the approximation is the same as in [25] and is based on adding a regularizing term which provides higher integrability of gradients. Such a regularized problem approximates (P, f, b_0) with bounded given data.

4.2.1 An auxiliary $\mathcal{N}$−function m

First we construct a new isotropic and homogenous $\mathcal{N}$–function $m : \mathbb{R}_+ \to \mathbb{R}_+$ which grows essentially more rapidly than M and whose conjugate satisfies the Δ_2-condition. To this end let us define a new $\mathcal{N}$–function $m : \mathbb{R}_+ \to \mathbb{R}_+$, which is radially symmetric and

$$m(r) = \left((\tilde{m}_*(r))^*\right)^2,$$

where $\tilde{m}_*$ is a solution to the following differential equation

$$\tilde{m}_*'(s) = \begin{cases} m_*' & \text{for } s \text{ such that } m_*'(s) \leq \alpha \frac{\tilde{m}_*(s)}{s} \\ \alpha \frac{\tilde{m}_*(s)}{s} & \text{for } s \text{ such that } m_*'(s) > \alpha \frac{\tilde{m}_*(s)}{s}, \end{cases} \tag{41}$$

with the initial condition $\tilde{m}_*(0) = 0$ and where $\alpha > 1$,

$$m_*(r) = \left(\operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=r} M^*(x, \xi) \right)^{**} \qquad \text{for } r \in \mathbb{R}_+.$$

In the above $(\cdot)^{**}$ denotes is biconjugate function. Let us notice that by the Fenchel-Moreau Theorem $m_*(r)$ is a convex function (let us notice that $\operatorname{ess\,inf}_{x\in\Omega} \min_{|\xi|=r} M^*(x, \xi)$ is not necessary convex). Since $\tilde{m}_*'$ is a monotone function as consisting of pieces which are non-decreasing, the function $\tilde{m}_*$ is convex. Moreover, by [39, Chapter II.2.3] (Theorem 3, point 1. (ii)) $\tilde{m}_*$ satisfies the Δ_2−condition.

Let us notice that m_* may be zero on a finite interval $[0, r_0]$ ($r_0 < \infty$ as a consequence of (**M3**)), therefore $(\tilde{m}_*)^*(r)$ may behave as a linear function of r around zero, but then $\left((\tilde{m}_*(r))^*\right)^2$ possesses superlinear growth around zero.

The complementary function to m, i.e. m^*, satisfies the Δ_2-condition, as a consequence of [39, Chapter II.2.3] (Theorem 3, point 1. (iii)). Namely, as $(m^{\frac{1}{2}})^* = \tilde{m}_*$ satisfies the Δ_2-condition, by [39] (Theorem 3, point 1. (iii)), there exists $\gamma > 1$ and $s_0 \geq 0$ such that

$$\frac{(m^{\frac{1}{2}})'(s)s}{(m^{\frac{1}{2}})(s)} > \gamma$$

for all $s \geq s_0$, which is equivalent to the inequality

$$\frac{m'(s)s}{m(s)} > 2\gamma$$

for all $s \geq s_0$. Thus, using again [39] (Theorem 3, point 1. (iii)), we can conclude that m^* satisfies the Δ_2-condition.

Remark 4.1 *Since m grows essentially more rapidly than M (namely setting $\overline{M}(r) = \operatorname{ess\,sup}_{x\in\Omega} \max_{|\xi|=r} M(x,\xi)$, m satisfies the following: $\frac{\overline{M}(r)}{m(r)} \to 0$ as $r \to \infty$) and therefore $L_m(\Omega;\mathbb{R}^d) \subset E_M(\Omega;\mathbb{R}^d)$ and $L_m(Q_T;\mathbb{R}^d) \subset E_M(Q_T;\mathbb{R}^d)$.*

4.2.2 A regularized problem

Let us introduce the following notation

$$\bar{\nabla} m(\xi) := \nabla_\xi m(|\xi|).$$

Moreover, let us observe that m as an $\mathcal{N}$–function satisfies the Fenchel-Young inequality as an equality

$$\bar{\nabla} m(\xi)\cdot\xi = m(|\xi|) + m^*(|\bar{\nabla} m(\xi)|) \tag{42}$$

and is strictly monotone as a gradient of a strictly convex function, i.e.

$$(\bar{\nabla} m(\xi) - \bar{\nabla} m(\eta))\cdot(\xi-\eta) > 0 \quad \forall \xi,\eta\in\mathbb{R}^d.$$

Let us define the following operators

$$A_\theta(x,\xi) := a(x,\xi) + \theta\bar{\nabla} m(\xi) \quad \text{ for all } \xi\in\mathbb{R}^d \text{ and a.a.} x\in\Omega. \tag{43}$$

Lemma 4.2 *Let A_θ be as above, m and M be $\mathcal{N}$–functions s.t. m grows essentially more rapidly then M and m^* satisfy the Δ_2-condition. Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Then for every fixed $\theta\in(0,1)$ and $n\in\mathbb{N}$ there exists a weak solution to the problem*

$$\begin{aligned} \partial_t u_n^\theta - \operatorname{div} A_\theta(x, Du_n^\theta) &= T_n(f) \quad \text{in } Q_T \\ u_n^\theta(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ u_n^\theta(0,\cdot) = u_{n,0}(\cdot) &= T_n(u_0) \quad \text{in } \Omega \end{aligned} \tag{P, A_θ}$$

i.e., there exists $u_n^\theta \in C([0,T];L^2(\Omega)) \cap L^{1+\nu}(0,T;W_0^{1+\nu}(\Omega))$ with $Du_n^\theta \in L_m(Q_T;\mathbb{R}^d)$ such that

$$-\int_{Q_T} u_n^\theta \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_n^\theta(T)\varphi(T)\,\mathrm{d}x - \int_\Omega u_n^\theta(0)\varphi(0)\,\mathrm{d}x \tag{44}$$

$$+ \quad \int_{Q_T} A_\theta(x, Du_n^\theta)\cdot D\varphi \,\mathrm{d}x\mathrm{d}t = \int_{Q_T} T_n(f)\varphi\,\mathrm{d}x\mathrm{d}t. \tag{45}$$

holds for $\varphi \in C^\infty([0,T];\mathcal{D}(\Omega))$.

Moreover, the energy equality is satisfied, i.e.

$$\frac{1}{2}\int_\Omega (u_n^\theta(\tau))^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega (u_{n,0})^2\,\mathrm{d}x + \int_{Q_\tau} A_\theta(x, Du_n^\theta)\cdot Du_n^\theta\,\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} T_n(f)u_n^\theta\,\mathrm{d}x\mathrm{d}t, \tag{46}$$

where $\tau\in[0,T]$ and $Q_\tau = (0,\tau)\times\Omega$.

Proof: The above result follows from direct application of [19, Theorem 2]. For the convenience of the reader we check if all assumptions of this theorem are satisfied.

First, let us show that A_θ meets assumptions (7), (9) and (10) from [19, p. 418] (obviously the operator A_θ is a Carathéodory function).

The assumption (10) from [19] in our case follows from the condition: $A_\theta(x,\xi)\cdot\xi \geq \theta m(|\xi|) - a_0(x)$, which is satisfied for nonnegative $a_0 \in L^1(\Omega)$ and all $\xi \in \mathbb{R}^d$.

The operator A_θ is strictly monotone as a consequence of strict convexity of $m(\xi)$ and monotonicity of $a(x,\xi)$, therefore (9) from [19] is satisfied.

It remains to show that A_θ satisfies assumption (7) from [19], i.e.

$$|A_\theta(x,Du_n^\theta)| \leq k_1(x) + c_1(m^*)^{-1}(m(|c_2 Du_n^\theta|)) + c_1(p^*)^{-1}(m(c_2|u_n^\theta|)) \tag{47}$$

with some $k_1 \in E_{m^*}(\Omega)$ and an $\mathcal{N}$–function p s.t. $p << m$ (m growths essentially more rapidly then p). In order to show (47) it is sufficient to prove the following condition

$$m^*\left(\left|c_1 A_\theta(x,Du_n^\theta)\right|\right) \leq m\left(\left|c_2 Du_n^\theta\right|\right) + a_0(x), \tag{48}$$

where $a_0 \in L^1(\Omega)$. To provide (48) let us notice that according to the Fenchel-Young inequality and the convexity of m^* we obtain

$$\begin{aligned} A_\theta(x,Du_n^\theta)\cdot Du_n^\theta &\leq m\left(\left|\frac{2}{c_a}Du_n^\theta\right|\right) + m^*\left(\left|\frac{c_a}{2}A_\theta(x,Du_n^\theta)\right|\right) \\ &\leq m\left(\left|\frac{2}{c_a}Du_n^\theta\right|\right) + c_a m^*\left(\left|\frac{1}{2}A_\theta(x,Du_n^\theta)\right|\right) \end{aligned} \tag{49}$$

for $0 < \theta < 2$. On the other hand using (43), (**A2**), (42), the convexity of m^*, the nonnegativity of $\mathcal{N}$–functions and knowing that $m \geq M$ implies $M^* \geq m^*$ we have

$$\begin{aligned} A_\theta(x,Du_n^\theta)\cdot Du_n^\theta &\geq c_a M(x,Du_n^\theta) + c_a M^*(x,a(x,Du_n^\theta)) + \theta m(|Du_n^\theta|) \\ &\quad + \theta m^*(|\bar{\nabla} m(Du_n^\theta)|) - a_0(x) \\ &\geq 2c_a\left(\frac{1}{2}m^*(|a(x,Du_n^\theta)|) + \frac{1}{2}m^*(|\theta\bar{\nabla} m(Du_n^\theta)|)\right) - a_0(x) \\ &\geq 2c_a m^*\left(\left|\frac{1}{2}A_\theta(x,Du_n^\theta)\right|\right) - a_0(x). \end{aligned} \tag{50}$$

According to (49) and (50) we get

$$c_a m^*\left(\left|\frac{1}{2}A_\theta(x,Du_n^\theta)\right|\right) \leq m\left(\left|\frac{2}{c_a}Du_n^\theta\right|\right) + a_0(x).$$

From the convexity of m^* for $0 < \theta \leq 1$ we infer

$$\frac{c_a}{2}|A_\theta(x,Du_n^\theta)| \leq (m^*)^{-1}\left(m\left(\left|\frac{2}{c_a}Du_n^\theta\right|\right)\right) + (m^*)^{-1}\left(a_0(x)\right). \tag{51}$$

Since $a_0 \in L^1$ and m^* satisfies the Δ_2–condition, the last term on the right hand side of (51) belongs to E_{m^*}. This provides that A_θ fulfils assumption (7) of [19, Theorem 2].

Finally we conclude that [19, Theorem 2] assures the existence of weak solutions to the problem (P, A_θ) and the proof of Lemma 4.2 is complete.

4.3 Existence of weak solutions for bounded data

Our goal now is to show the existence of weak solutions to the simplified problem ($F = 0$, $\beta(\cdot,u) = u$) and for bounded data, i.e. with right hand side $T_n(f)$ and bounded initial data $T_n(u_0)$. To this end we pass to the limit with $\theta \to 0$ in problem (P, A_θ).

Lemma 4.3 *Let assumptions (**A1** - **A3**) and (**M1** - **M3**) be satisfied. Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Then for every fixed $n \in \mathbb{N}$ there exists a weak solution to the problem*

$$\begin{aligned} \partial_t u_n - \mathrm{div} a(x, Du_n) &= T_n(f) \quad \text{in } Q_T \\ u_n(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ u_n(0,x) = u_{n,0} &= T_n(u_0) \quad \text{in } \Omega. \end{aligned} \tag{52}$$

More precisely, there exists $u_n \in L^\infty(0,T;L^2(\Omega)) \cap L^{1+\nu}(0,T;W_0^{1+\nu}(\Omega))$ with $Du_n \in L_M(Q_T;\mathbb{R}^d)$ and $a(x,Du_n) \in L_{M^*}$ such that

$$-\int_{Q_T} u_n \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_{n,0}\varphi(0)\,\mathrm{d}x + \int_{Q_T} a(x,Du_n)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} T_n(f)\varphi\,\mathrm{d}x\mathrm{d}t. \tag{53}$$

holds for $\varphi \in \mathcal{D}([0,T)\times\Omega)$.

4.3.1 Uniform estimates. Part I.

The first step in the proof of the Lemma is to obtain uniform bounds with respect to θ.

By energy equality (46) and (**A2**), (42) we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u_n^\theta(\tau))^2\,\mathrm{d}x + \int_{Q_\tau} c_a M(x,Du_n^\theta) + c_a M^*(x,a(x,Du_n^\theta))\,\mathrm{d}x\mathrm{d}t - T\int_\Omega a_0\,\mathrm{d}x \\ &+ \int_{Q_\tau} \theta m(|Du_n^\theta|) + \theta m^*(|\bar\nabla m(Du_n^\theta)|)\,\mathrm{d}x\mathrm{d}t \le \int_{Q_\tau} T_n(f)u_n^\theta\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega (u_{n,0})^2\,\mathrm{d}x \end{aligned} \tag{54}$$

Then by the Young inequality, the Poincaré inequality and by the assumption (**M1**) we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u_n^\theta(\tau))^2\,\mathrm{d}x + \int_{Q_\tau} \frac{c_a}{2} M(x,Du_n^\theta) + c_a M^*(x,a(x,Du_n^\theta))\,\mathrm{d}x\mathrm{d}t - T\int_\Omega a_0\,\mathrm{d}x \\ &+ \int_{Q_\tau} \theta m(|Du_n^\theta|) + \theta m^*(|\bar\nabla m(Du_n^\theta)|)\,\mathrm{d}x\mathrm{d}t \le \int_{Q_\tau} C|T_n(f)|^{(1+\nu)'}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega (T_n(u_0))^2\,\mathrm{d}x \end{aligned} \tag{55}$$

where $C = ((c_a(1+\nu)')/(2c_M c_p^{1/(1+\nu)}))^{-(1+\nu)/(1+\nu)'}(1+\nu)^{-1}$, here c_p denotes constant from Poincaré inequality with $p = 1+\nu$.

Let us notice that for each fixed $n \in \mathbb{N}$ the right hand side of (55) is bounded and consequently, we obtain

$$\sup_{\tau\in[0,T]} \|u_n^\theta(\tau)\|^2_{L^2(\Omega)} \le Cn^{(1+\nu)'}T|\Omega| + \frac{1}{2}n^2|\Omega| + T\|a_0\|_{L^1(\Omega)}. \tag{56}$$

Consequently for a subsequence if necessary we have

$$u_n^\theta \overset{*}{\rightharpoonup} u_n \text{ weakly-* in } L^\infty(0,T;L^2(\Omega)). \tag{57}$$

Since τ is arbitrary we have also

$$\int_{Q_T} M(x,Du_n^\theta)\,\mathrm{d}x\mathrm{d}t \le Cn^{(1+\nu)'}T|\Omega| + \frac{1}{2}n^2|\Omega| + T\|a_0\|_{L^1(\Omega)}, \tag{58}$$

$$\int_{Q_T} M^*(x,a(x,Du_n^\theta))\,\mathrm{d}x\mathrm{d}t \le Cn^{(1+\nu)'}T|\Omega| + \frac{1}{2}n^2|\Omega| + T\|a_0\|_{L^1(\Omega)}, \tag{59}$$

$$\int_{Q_T} \theta m\left(|Du_n^\theta|\right)\,\mathrm{d}x\mathrm{d}t \le Cn^{(1+\nu)'}T|\Omega| + \frac{1}{2}n^2|\Omega| + T\|a_0\|_{L^1(\Omega)}, \tag{60}$$

$$\int_{Q_T} \theta m^*\left(|\bar\nabla m(Du_n^\theta)|\right)\,\mathrm{d}x\mathrm{d}t \le Cn^{(1+\nu)'}T|\Omega| + \frac{1}{2}n^2|\Omega| + T\|a_0\|_{L^1(\Omega)}. \tag{61}$$

Thus, there exists a subsequence $\theta \to 0$, such that

$$Du_n^\theta \overset{*}{\rightharpoonup} Du_n \text{ weakly-* in } L_M(Q_T;\mathbb{R}^d) \tag{62}$$

and there exists $\alpha^n \in L_{M^*}(Q_T;\mathbb{R}^d)$ such that

$$a(\cdot, Du_n^\theta) \overset{*}{\rightharpoonup} \alpha^n \text{ weakly-* in } L_{M^*}(Q_T;\mathbb{R}^d). \tag{63}$$

4.3.2 Uniform estimates. Part II

In the following section we provide estimates needed in the proof of Theorem 4.4.

By Lemma 4.1, ii.) applied to (44) with $h(\cdot) = T_k(\cdot)$ and $\xi = \kappa^{\tau,\delta} = \omega^\delta * \mathbb{1}_{[0,\tau)}$, where ω_δ is a standard regularising kernel ($\omega \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\omega_\delta \subset (-\delta,\delta)$) we get

$$\begin{aligned}
&\int_{Q_T} \partial_t \kappa^{\tau,\delta} \int_{u_{n,0}(x)}^{u_n^\theta(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}x\mathrm{d}t = \\
&\int_{Q_T} A_\theta(x, Du_n^\theta)\cdot DT_k(u_n^\theta)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} T_n(f)T_k(u_n^\theta)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then, as $\theta \to 0$, we get

$$\begin{aligned}
&\int_{Q_T} \partial_t \kappa^{\tau,\delta} \left\{ \int_0^{u_n(t,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_{n,0}(x)} T_k(\sigma)\,\mathrm{d}\sigma \right\} \mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} \alpha^n \cdot DT_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} T_n(f)T_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

On the other hand, passing first with $\delta \to 0$ for a.a. $\tau \in [0,T]$ we get

$$\begin{aligned}
&\int_\Omega \left\{ \int_0^{u_n^\theta(\tau,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_{n,0}(x)} T_k(\sigma)\,\mathrm{d}\sigma \right\} \mathrm{d}x \\
&+ \int_{Q_\tau} A_\theta(x, Du_n^\theta)\cdot DT_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} T_n(f)T_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\|T_k(u_n^\theta(\tau))\|_{L^2(\Omega)}^2 - \frac{1}{2}\|T_k(u_{n,0})\|_{L^2(\Omega)}^2 + \int_{Q_\tau} a(x, DT_k(u_n^\theta))\cdot DT_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} \theta\bar{\nabla} m(DT_k(u_n^\theta))\cdot DT_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t = \int_{Q_\tau} T_n(f)T_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By coercivity condition (1) and (42) it follows that

$$\begin{aligned}
&\frac{1}{2}\|T_k(u_n^\theta(\tau))\|_{L^2(\Omega)}^2 - \frac{1}{2}\|T_k(u_{n,0})\|_{L^2(\Omega)}^2 + \int_{Q_\tau} c_a\{M^*(x, a(x, DT_k(u_n^\theta)) + M(x, DT_k(u_n^\theta))\}\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} \theta m^*(\bar{\nabla} m(DT_k(u_n^\theta))) + \theta m(DT_k(u_n^\theta))\,\mathrm{d}x\mathrm{d}t \le \int_{Q_\tau} T_n(f)T_k(u_n^\theta)\,\mathrm{d}x\mathrm{d}t + T\|a_0\|_{L^1(\Omega)}.
\end{aligned}$$

As the above holds for arbitrary $\tau \in [0,T]$ we get

$$\int_{Q_T} \theta m(DT_k(u_n^\theta))\,\mathrm{d}x\mathrm{d}t \le k\|f\|_{L^1(Q_T)} + T\|a_0\|_{L^1(\Omega)} + \frac{1}{2}k^2|\Omega|, \tag{64}$$

$$c_a \int_{Q_T} M(DT_k(u_n^\theta))\,\mathrm{d}x\mathrm{d}t \le k\|f\|_{L^1(Q_T)} + T\|a_0\|_{L^1(\Omega)} + \frac{1}{2}k^2|\Omega|, \tag{65}$$

$$c_a \int_{Q_T} M^*(a(x, DT_k(u_n^\theta)) \,\mathrm{d}x\mathrm{d}t \le k\|f\|_{L^1(Q_T)} + T\|a_0\|_{L^1(\Omega)} + \frac{1}{2}k^2|\Omega| \tag{66}$$

and by assumption (3) we have also

$$c_M \int_{Q_T} |DT_k(u_n^\theta))|^{1+\nu} \,\mathrm{d}x\mathrm{d}t \le k\|f\|_{L^1(Q_T)} + T\|a_0\|_{L^1(\Omega)} + \frac{1}{2}k^2|\Omega|. \tag{67}$$

Since m growths essentially more rapidly then M, by (64) we have that for each fixed $\theta \in (0,1)$

$$DT_k(u_n^\theta) \in E_M(Q_T; \mathbb{R}^d). \tag{68}$$

As the above estimates are independent of $n \in \mathbb{N}$ and $\theta \in (0,1)$, one can choose by a diagonal method the subsequence $\{u^{n(\theta_0),\theta_0}\}_{\theta_0}$ denoted by $\{u^{\theta_0}\}_{\theta_0}$ and satisfying

$$T_k(u^{\theta_0}) \overset{*}{\rightharpoonup} T_k(u) \text{ weakly-* in } L^\infty(Q_T; \mathbb{R}^d), \tag{69}$$

$$DT_k(u^{\theta_0}) \overset{*}{\rightharpoonup} DT_k(u) \text{ weakly-* in } L_M(Q_T; \mathbb{R}^d) \tag{70}$$

$$DT_k(u^{\theta_0}) \rightharpoonup DT_k(u) \text{ weakly in } L^{1+\nu}(Q_T; \mathbb{R}^d) \tag{71}$$

and for each fixed $\theta_0 \in (0,1)$ $DT_k(u^{\theta_0}) \in E_M(Q_T; \mathbb{R}^d)$.

Next we show that $T_k(u) \in \mathcal{V}$. Let us denote by

$$m_\gamma := ((\tilde{m}_*)^*)^\gamma \quad \text{for some } 1 < \gamma < 2 \quad (\text{for } \tilde{m}_* \text{ see section 4.2.1})$$

Then m_γ is an homogenous isotropic $\mathcal{N}$-function s.t. $m >> m_\gamma$ and $DT_k(u^{\theta_0}) \in E_{m_\gamma}(Q_T; \mathbb{R}^d)$ and obviously for all $\lambda \in \mathbb{R}^+$ and $\theta_0 \in (0,1)$ we have that $\lambda DT_k(u^{\theta_0}) \in \mathcal{L}_{m_\gamma}(Q_T; \mathbb{R}^d)$.

Let us note that due to [22, Theorem 4] there exists sequence $\{(T_k(u^{\theta_0}))_i\}_{i\in\mathbb{N}} \subset \mathcal{D}([0,T) \times \Omega)$ s.t.

$$\int_{Q_T} m_\gamma \left(\frac{D(T_k(u^{\theta_0}))_i - DT_k(u^{\theta_0})}{\lambda} \right) \to 0 \quad \text{as } i \to \infty$$

for every $\lambda \in \mathbb{R}^+$, in particular for $\lambda = 1$

Therefore for all $\theta_0 \in (0,1)$ we can choose i_{θ_0} s.t. $\{(T_k(u^{\theta_0}))_{i_{\theta_0}}\}_{i_{\theta_0}\in\mathbb{N}} \subset \mathcal{D}([0,T) \times \Omega)$

$$\int_{Q_T} m_\gamma \left(D\left(T_k(u^{\theta_0})\right)_{i_{\theta_0}} - DT_k(u^{\theta_0}) \right) \le \theta_0.$$

Therefore

$$\left(D(T_k(u^{\theta_0}))_{i_{\theta_0}} - DT_k(u^{\theta_0}) \right) \xrightarrow{m_\gamma} 0 \quad \text{as } \theta_0 \to 0$$

and hence also weakly-* in $L_{m_\gamma}(Q_T; \mathbb{R}^d)$ and $L_M(Q_T; \mathbb{R}^d)$ and finally

$$D(T_k(u^{\theta_0}))_{i_{\theta_0}} \overset{*}{\rightharpoonup} DT_k(u) \quad \text{weakly-* in } L_M(Q_T; \mathbb{R}^d)$$

Together with (69) - (71) it yields that

$$T_k(u) \in \mathcal{V}, \tag{72}$$

where $\mathcal{V}$ is defined by (18).

4.3.3 Passing with $\theta \to 0$ for fixed n

Our aim now is to pass with θ to zero and show that the additional term $\theta\bar{\nabla}m$ vanishes and that $\alpha^n = a(x, Du^n)$ a.e. in Q_T. First we need to to show the following inequality

$$\limsup_{\theta\to 0} \int_{Q_T} a(x, Du_n^\theta) \cdot Du_n^\theta \,\mathrm{d}x\mathrm{d}t \le \int_{Q_T} \alpha^n \cdot Du_n \,\mathrm{d}x\mathrm{d}t. \tag{73}$$

To this end we start with the weak formulation (44) with $\varphi \in \mathcal{D}([0,T) \times \Omega)$, i.e.

$$-\int_{Q_T} u_n^\theta \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_{n,0}\varphi(0)\,\mathrm{d}x + \int_{Q_T} A_\theta(x, Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} T_n(f)\varphi\,\mathrm{d}x\mathrm{d}t \tag{74}$$

and with the energy equality (46)

$$\frac{1}{2}\int_\Omega (u_n^\theta(\tau))^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega (u_{n,0})^2\,\mathrm{d}x + \int_{Q_T} A_\theta(x, Du_n^\theta)\cdot Du_n^\theta\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} T_n(f)u_n^\theta\,\mathrm{d}x\mathrm{d}t. \tag{75}$$

Next, we pass to the limit with $\theta \downarrow 0^+$ in (74), (75). In order to pass to the limit in (75) we use the weak lower semi-continuity of the L^2-norm and (57). Then neglecting the nonnegative term $\int_{Q_T} \theta\bar{\nabla}m(Du_n^\theta)\cdot Du_n^\theta\,\mathrm{d}x\mathrm{d}t$ we infer

$$\frac{1}{2}\|u_n(\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_{n,0}\|^2_{L^2(\Omega)} + \limsup_{\theta\downarrow 0}\int_{Q_T} a(x, Du_n^\theta)\cdot Du_n^\theta\,\mathrm{d}x\mathrm{d}t \le \int_{Q_T} T_n(f)u_n\,\mathrm{d}x\mathrm{d}t. \tag{76}$$

Let us concentrate now on (74). The first term on the left hand side is treated by (57) and we get

$$\lim_{\theta\downarrow 0}\int_{Q_T} u_n^\theta\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} u_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{77}$$

A subsequent step is to show that $\int_{Q_T}\theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi$ vanishes as $\theta\downarrow 0^+$. Indeed, note that by (61) $\theta m^*(\bar{\nabla}m(Du_n^\theta))$ is uniformly bounded in $L^1(Q_T)$ with respect to θ. We divide the domain into two parts

$$\begin{aligned}\int_{Q_T}\theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t = &\int_{Q_{T,R}^\theta}\theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t\\ &+\int_{(Q_{T,R}^\theta)^C}\theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{78}$$

where

$$Q_{T,R}^\theta = \{(t,x)\in Q_T : |Du_n^\theta| < R\}.$$

Let us notice that by (58) we obtain uniform boundedness in $L^1(Q_T)$ of $\{Du_n^\theta\}_\theta$ and therefore

$$\sup_{0\le\theta\le 1}|(Q_{T,R}^\theta)^c| \le \frac{C}{R}. \tag{79}$$

For the first term on the r.h.s. of (78), by the continuity of $\bar{\nabla}m(\cdot)$, we have

$$\begin{aligned}\int_{Q_{T,R}^\theta}\theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t &\le \int_{Q_{T,R}^\theta}\theta|\bar{\nabla}m(Du_n^\theta)||D\varphi|\,\mathrm{d}x\mathrm{d}t \le \theta|Q_T|\|D\varphi\|_\infty\sup_{|\xi|=R}|\bar{\nabla}m(\xi)|\\ &\le \theta c_{m,R}|Q_T|\|D\varphi\|_\infty \xrightarrow{\theta\downarrow 0} 0.\end{aligned}$$

The definition of an $\mathcal{N}$–function together with (61) provide the uniform integrability of $\{\theta\bar{\nabla}m(Du_n^\theta)\}_{\theta\in(0,1)}$. Then by (79) we treat the second term in (78) as follows

$$\int_{(Q^\theta_{T,R})^C} \theta\bar{\nabla}m(Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t \le \|D\varphi\|_\infty \sup_{\theta\in(0,1)} \int_{(Q^\theta_{T,R})^C} \theta|\bar{\nabla}m(Du_n^\theta)|\,\mathrm{d}x\mathrm{d}t \xrightarrow{R\to\infty} 0.$$

As $D\varphi \in E_M(Q_T;\mathbb{R}^d)$, we obtain by (63) that

$$\lim_{\theta\downarrow 0}\int_{Q_T} a(x,Du_n^\theta)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} \alpha^n\cdot D\varphi\,\mathrm{d}x\mathrm{d}t.$$

Summarising above arguments we pass to the limit with $\theta\to 0$ in (74) and get

$$-\int_{Q_T} u_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_{n,0}\varphi(0)\,\mathrm{d}x + \int_{Q_T}\alpha^n\cdot D\varphi\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} T_n(f)\varphi\,\mathrm{d}x\mathrm{d}t \tag{80}$$

for $\varphi\in\mathcal{D}([0,T)\times\Omega)$. Let us notice that in order to accomplish the proof of Lemma 4.3 it remains to characterise the nonlinear term α^n. By Lemma 4.1, ii.) with $h(\cdot) = T_k(\cdot)$ and $\xi = \kappa^{\tau,\delta} = \omega^\delta * 1\!\!1_{[0,\tau)}$ where ω_δ is a standard regularising kernel we get

$$\int_{Q_T}\partial_t\kappa^{\tau,\delta}\int_{u_{n,0}(x)}^{u_n(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}x\mathrm{d}t = \int_{Q_T}\alpha^n\cdot DT_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} T_n(f)T_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t.$$

Then

$$\begin{aligned}&\int_{Q_T}\partial_t\kappa^{\tau,\delta}\left\{\int_0^{u_n(t,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_{n,0}(x)} T_k(\sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}x\mathrm{d}t\\ &= \int_{Q_T}\alpha^n\cdot DT_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} T_n(f)T_k(u_n)\kappa^{\tau,\delta}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Passing with $\delta\to 0$ for almost all $\tau\in[0,T]$ we get

$$\begin{aligned}&\int_\Omega\left\{\int_0^{u_n(\tau,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_{n,0}(x)} T_k(\sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}x\\ &= -\int_{Q_\tau}\alpha^n\cdot DT_k(u_n)\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} T_n(f)T_k(u_n)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Letting $k\to\infty$ by the Lebesgue dominated convergence theorem we have

$$\frac{1}{2}\|u_n(\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_{n,0}\|^2_{L^2(\Omega)} = -\int_{Q_\tau}\alpha^n\cdot Du_n\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} T_n(f)u_n\,\mathrm{d}x\mathrm{d}t.$$

Then together with (76) it gives (73).

Our aim now is to identify the limit $\alpha^n = a(x,Du_n)$ by the monotonicity argument adapted to non-reflexive generalized Orlicz spaces.

Since $a(x,\cdot)$ is monotone (see (**A3**)),

$$(a(x,\zeta) - a(x,Du_n^\theta))\cdot(\zeta - Du_n^\theta) \ge 0 \tag{81}$$

holds a.e. in Q_T and for all $\zeta\in L^\infty(Q_T;\mathbb{R}^d)$. Integrating (81) over Q_T, using $a(x,\zeta)\in\mathcal{L}_{M^*}(Q_T;\mathbb{R}^d) = E_{M^*}(Q_T;\mathbb{R}^d)$ and (73) to pass to the limit with $\theta\downarrow 0$ we obtain

$$\int_{Q_T}(a(x,\zeta)-\alpha^n)\cdot(\zeta - Du_n)\,\mathrm{d}x\mathrm{d}t \ge 0. \tag{82}$$

For $l > 0$ let us introduce

$$Q_T^l := \{(t,x) \in Q_T : |Du^\varepsilon(t,x)| \le l \quad \text{a.e. in } Q_T\}.$$

Now let $0 < j < i$ be arbitrary, $z \in L^\infty(Q_T;\mathbb{R}^d)$ and $\lambda > 0$. Plugging

$$\zeta = (Du_n)\mathbb{1}_{Q_T^i} + \lambda z \mathbb{1}_{Q_T^j}$$

into (82) we get

$$-\int_{Q_T\setminus Q_T^i} (a(x,0) - \alpha^n)\cdot Du_n^\theta \,\mathrm{d}x\mathrm{d}t + \lambda \int_{Q_T^j} (a(x,Du_n+\lambda z) - \alpha^n)\cdot z \,\mathrm{d}x\mathrm{d}t \ge 0. \tag{83}$$

Note that by coercivity assumption (1), $M^*(x,a(x,0)) \le a_0(x)$ a.e. in Ω and from (12) it follows that

$$\int_{Q_T} |a(x,0)\cdot Du_n| \,\mathrm{d}x\mathrm{d}t \le \int_{Q_T} (a_0 + M(x,Du_n)) \,\mathrm{d}x\mathrm{d}t. \tag{84}$$

Since $Du_n \in \mathcal{L}_M(Q_T;\mathbb{R}^d)$ the right-hand side of (84) is finite and consequently

$$a(x,0)\cdot Du_n \in L^1(Q_T).$$

As $\alpha^n \in \mathcal{L}_{M^*}(Q_T;\mathbb{R}^d)$ and $Du_n \in \mathcal{L}_M(Q_T;\mathbb{R}^d)$ it follows immediately by the Fenchel-Young inequality (12) that $\alpha^n\cdot Du_n$ is in $L^1(Q_T)$. Therefore, by the Lebesgue Dominated Convergence Theorem, the first term on the left-hand of (83) vanishes for $i\to\infty$. Passing to the limit with $i \to \infty$ in (83) and dividing by λ we get

$$\int_{Q_T^j} (a(x,Du_n+\lambda z) - \alpha^n)\cdot z \,\mathrm{d}x\mathrm{d}t \ge 0.$$

Note that $a(x,Du_n+\lambda z) \to a(x,Du_n)$ a.e. in Q_T^j when $\lambda \downarrow 0$. Moreover, for $0<\lambda<1$

$$\int_{Q_T^j} M^*(x,a(x,Du_n+\lambda z)) \,\mathrm{d}x\mathrm{d}t \le \frac{2}{c_a} \sup_{0<\lambda<1} \int_{Q_T^j} M(x,\frac{2}{c_a}(Du_n+\lambda z)) + a_0(x) \,\mathrm{d}x\mathrm{d}t \tag{85}$$

and the right-hand side of (85) is bounded since $Du_n+\lambda z$ is uniformly (in λ) bounded in $L^\infty(Q_T^j;\mathbb{R}^d)$, $L^\infty(Q_T) \subseteq \mathcal{L}_M(Q_T)$ and since $M(x,\frac{2}{c_a}(Du_n+\lambda z))$ is bounded. Hence it follows from Lemma 2.2 that $\{a(x,Du_n+\lambda z)\}_\lambda$ is uniformly integrable. Note that $|\Omega_j|<\infty$, hence by the Vitali Lemma it follows that

$$a(x,Du_n+\lambda z) \to a(x,Du_n) \quad \text{in } L^1(Q_T^j;\mathbb{R}^d) \quad \text{for } \lambda\downarrow 0$$

and therefore

$$\int_{Q_T^j} (a(x,Du_n+\lambda z) - \alpha^n)\cdot z \,\mathrm{d}x\mathrm{d}t \to \int_{Q_j} (a(x,Du_n) - \alpha^n)\cdot z \,\mathrm{d}x\mathrm{d}t \qquad \text{for } \lambda\downarrow 0.$$

Consequently,

$$\int_{Q_T^j} (a(x,Du_n) - \alpha^n)\cdot z \,\mathrm{d}x\mathrm{d}t \ge 0 \qquad \text{for all } z \in L^\infty(Q_T;\mathbb{R}^d).$$

Substituting

$$z = \begin{cases} -\frac{a(x,Du_n)-\alpha^n}{|a(x,Du_n)-\alpha^n|} & \text{if } a(x,Du_n)-\alpha^n \neq 0 \\ 0 & \text{if } a(x,Du_n)-\alpha^n = 0 \end{cases}$$

into the above, we obtain

$$\int_{Q_T^j} |a(x, Du_n) - \alpha^n| \,\mathrm{d}x\mathrm{d}t \le 0.$$

Hence

$$a(x, Du_n) = \alpha^n \quad \text{a.e. in } Q_T^j. \tag{86}$$

Since j is arbitrary (86) holds a.e. in Q_T.

The above argument together with (80) finish the proof of Lemma 4.3. □

4.4 Weak sequential stability result

In the following section we prove the so-called weak sequential stability result for the general problem (P, f, b_0). Assuming existence of a sequence of approximate solutions which converge in proper topologies we show existence of renormalized solution to the problem (P, f, b_0).

Before we formulate the result let us denote by $(P, f_n, b_{0,n})$ the problem of a (P, f, b_0)-type with given right hand side f_n and initial data $b_{0,n}$.

Theorem 4.4 (The weak sequential stability) *Let assumptions (***A1 - A3***) and (***M1 - M3***) be satisfied.*

Let $f \in L^1(Q_T)$ *and* $b_0 \in \left\{ b \in L^1(\Omega) : b(x) \in \overline{R(\beta(x,\cdot))} \text{ a.e. in } \Omega \right\}$. *Let* $\{(u_n, b_n)\}_{n\in\mathbb{N}}$ *be a sequence of renormalized solutions to* $(P, f_n, b_{0,n})$ *such that*

i.) $f_n \to f$ *in* $L^1(Q_T)$,

ii.) $b_{0,n} \to b_0$ *in* $L^1(\Omega)$, $b_n \to b$ *a.e. in* Q_T *and* $b_n = \beta(x, u_n)$ *almost everywhere in* Q_T

iii.) $u_n \to u$ *almost everywhere in* Q_T,

iv.)

$$\lim_{l\to\infty} \sup_n \int_{\{l<|u_n|<l+1\}} a(x, Du_n) \cdot Du_n = 0 \tag{87}$$

and there exists a constant $C > 0$, *not depending on* $n, l \in \mathbb{R}$ *such that*

$$|\{|u_n| \ge l\}| \le Cl^{-\nu}. \tag{88}$$

Moreover, for any $k > 0$, *let*

v.) $T_k(u_n) \rightharpoonup T_k(u)$ *in* $L^{1+\nu}(0, T, W_0^{1,1+\nu}(\Omega))$,

vi.) $DT_k(u_n) \overset{*}{\rightharpoonup} DT_k(u)$ *weakly-* in* $L_M(Q_T; \mathbb{R}^d)$,

vii.) $a(x, DT_k(u_n)) \overset{*}{\rightharpoonup} \Phi_k$ *weakly-* in* $L_{M^*}(Q_T; \mathbb{R}^d)$.

Moreover, assume that there exists a sequence $\{u^{\theta_0}\}_{\theta_0>0}$ *such that*

$$DT_k(u^{\theta_0}) \in E_M(Q_T; \mathbb{R}^d) \textit{ for each } \theta_0 \in (0, 1) \tag{89}$$

and

$$u^{\theta_0} \to u \textit{ a.e. in } Q_T, \tag{90}$$

$$T_k(u^{\theta_0}) \overset{*}{\rightharpoonup} T_k(u) \textit{ weakly-* in } L^\infty(Q_T; \mathbb{R}^d), \tag{91}$$

$$DT_k(u^{\theta_0}) \overset{*}{\rightharpoonup} DT_k(u) \textit{ weakly-(*) in } L_M(Q_T; \mathbb{R}^d) \tag{92}$$

$$DT_k(u^{\theta_0}) \rightharpoonup DT_k(u) \textit{ weakly in } L^{1+\nu}(Q_T; \mathbb{R}^d) \tag{93}$$

Then (u, b) *is a renormalized solution to* (P, f, b_0).

Proof: The proof of the Theorem 4.4 is divided into several steps.
Step 1.
One of the main steps is to show that

$$\Phi_k = a(x, DT_k(u)) \text{ a.e. in } Q_T. \tag{94}$$

To this end we proceed combining methods from [28, 29, 3] and defining a special time regularisation of $T_k(u)$ by the regularisation method of Landes ([32]). For $\mu > 0$, we denote this time regularized function by $(T_k(u))_\mu : Q_T \to \mathbb{R}$ defined by

$$(T_k(u))_\mu(t,x) := \mu \int_{-\infty}^{t} e^{\mu(s-t)} T_k(u(s,x)) ds \tag{95}$$

where, for $s \leq 0$ and $\mu > 0$ we extend $u(s,x)$ by a function $\omega_0^\mu \in W_0^{1,1+\nu}(\Omega) \cap L^\infty(\Omega)$ with $D\omega_0^\mu \in L_M(\Omega)$ such that $\{\omega_0^\mu\}_\mu$ is a sequence of functions with $\|\omega_0^\mu\|_{L^\infty(\Omega)} \leq k$ for all $\mu > 0$, $\frac{1}{\mu}\|\omega_0^\mu\|_{W_0^{1,1+\nu}(\Omega)} \to 0$ as $\mu \to \infty$, $\omega_0^\mu \to T_k(u_0)$ almost everywhere in Ω as $\mu \to \infty$ where $u_0 : \Omega \to \overline{\mathbb{R}}$ is a measurable function such that $b_0 = \beta(x, u_0)$ almost everywhere in Ω. We can check that $(T_k(u))_\mu \in \mathcal{V} \cap L^\infty(Q_T)$ is the unique solution of the equation

$$\partial_t (T_k(u))_\mu + \mu((T_k(u))_\mu - T_k(u)) = 0 \quad \text{in } \mathcal{D}'(Q_T)$$

satisfying the initial condition $(T_k(u))_\mu(0,x) = \omega_0^\mu$ almost everywhere in Ω. In particular, $(T_k(u))_\mu$ is differentiable for almost every $t \in (0,T)$ with $\partial_t(T_k(u))_\mu = \mu(T_k(u) - (T_k(u))_\mu) \in \mathcal{V} \cap L^\infty(Q_T)$. Moreover, we have $D(T_k(u))_\mu = (DT_k(u))_\mu$ in $\mathcal{D}'([0,T) \times \Omega)$, $\|(T_k(u))_\mu\|_{L^\infty(Q_T)} \leq k$ for all $\mu > 0$, $(T_k(u))_\mu \to T_k(u)$ almost everywhere in Q_T, weakly-(*) in $L^\infty(Q_T)$. Moreover $(DT_k(u))_\mu$ converges in modular in L_M to $DT_k(u)$ (as consequence of Proposition 2.3, Proposition 2.4 and Lemma 2.1). Now, for $\kappa \in \mathcal{D}^+([0,T))$ we show that

$$\limsup_{\mu\to\infty} \limsup_{n\to\infty} \int_{Q_T} \kappa a(x, DT_k(u_n)) \cdot D(T_k(u_n) - (T_k(u))_\mu) \leq 0. \tag{96}$$

To this end we will need an auxiliary sequence $\{u^{\theta_0}\}_{\theta_0 > 0}$ satisfying (91) - (92) such that $DT_k(u^{\theta_0}) \in E_M(Q_T; \mathbb{R}^d)$ for each $\theta_0 > 0$. Its existence is assured by the assumptions of Theorem 4.4.

Let us notice that if we apply regularisation (95) to $T_k(u^{\theta_0})$ then the above convergence properties hold also for $\{(T_k(u^{\theta_0}))_\mu\}_\theta$. In particular $D(T_k(u^{\theta_0}))_\mu \in E_M$. Note that the operator L s.t. $\xi \overset{L}{\mapsto} (\xi)_\mu$ is linear and bounded from $L^\infty(Q_T; \mathbb{R}^d)$ into $L^\infty(Q_T; \mathbb{R}^d)$ and from $L_M(Q_T; \mathbb{R}^d)$ into $L_M(Q_T; \mathbb{R}^d)$ (the last holds by the Jensen inequality, Proposition 2.3 and independence of M of the time variable). Hence by definition of $E_M(Q_T; \mathbb{R}^d)$ we have $L(E_M(Q_T; \mathbb{R}^d) \subseteq E_M(Q_T; \mathbb{R}^d)$. Then we apply Lemma 4.1 to $(P, f_n, b_{0,n})$ twice: first with $h(\cdot) = h_l(\cdot)T_k(\cdot)$, where h_l is defined by (19), and with $\xi = \kappa$ and again with $h(\cdot) = -h_l(\cdot)$ and $\xi = \kappa(T_k(u^{\theta_0}))_\mu$. Summarising those two results we are allowed to write the following:

$$I_1^{n,\theta_0,\mu,l} + I_2^{n,\theta_0,\mu,l} + I_3^{n,\theta_0,\mu,l} + I_4^{n,\theta_0,\mu,l} = I_5^{n,\theta_0,\mu,l}, \tag{97}$$

where

$$\begin{aligned} I_1^{n,\theta_0,\mu,l} = &- \int_{Q_T} \partial_t \kappa \int_{b_0^n}^{b_n} h_l((\beta^{-1})^0(\sigma)) T_k((\beta^{-1})^0) \mathrm{d}\sigma \, \mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} \partial_t \left(\kappa T_k(u^{\theta_0})_\mu\right) \int_{b_0^n}^{b_n} h_l((\beta^{-1})^0(\sigma)) \mathrm{d}\sigma \, \mathrm{d}x\mathrm{d}t \end{aligned} \tag{98}$$

$$
\begin{aligned}
I_2^{n,\theta_0,\mu,l} &= \int_{Q_T} \kappa h_l(u_n) a(x, Du_n) \cdot D(T_k(u_n) - (T_k(u^{\theta_0}))_\mu) \,\mathrm{d}x\mathrm{d}t, \\
I_3^{n,\theta_0,\mu,l} &= \int_{Q_T} \kappa h_l'(u_n)(T_k(u_n) - (T_k(u^{\theta_0}))_\mu) a(x, Du_n) \cdot Du_n \,\mathrm{d}x\mathrm{d}t, \\
I_4^{n,\theta_0,\mu,l} &= \int_{Q_T} \kappa F(u_n) \cdot D(h_l(u_n)(T_k(u_n) - (T_k(u^{\theta_0}))_\mu)) \,\mathrm{d}x\mathrm{d}t, \\
I_5^{n,\theta_0,\mu,l} &= \int_{Q_T} \kappa f_n h_l(u_n)(T_k(u_n) - (T_k(u^{\theta_0}))_\mu) \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{99}
$$

Now, we want to pass to the limit with $n \to \infty$ and then with $\theta_0 \to 0$ and $\mu \to \infty$ and finally with $l \to \infty$. Using the convergence assumptions of Theorem 4.4, (91), (92), properties of Landes regularisation we obtain:

$$
\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_5^{n,\theta_0,\mu,l} = 0. \tag{100}
$$

Now we show the following

$$
\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_4^{n,\theta_0,\mu,l} = 0. \tag{101}
$$

Let us write

$$
I_4^{n,\theta_0,\mu,l} = I_{4,1}^{n,\theta_0,\mu,l} + I_{4,2}^{n,\theta_0,\mu,l}
$$

with

$$
I_{4,1}^{n,\theta_0,\mu,l} = \int_{Q_T} \kappa F(T_{l+1}(u_n)) \cdot D(T_k(u_n) - T_k(u^{\theta_0})_\mu) h_l(u_n) \,\mathrm{d}x\mathrm{d}t.
$$

Therefore by assumption (iii.), (v.) of Theorem 4.4, uniform integrability in L^1 of $\{DT_k(u_n)\}_n$, $\{(DT_k(u^{\theta_0}))_\mu\}_{\theta_0}$ and $\{(DT_k(u))_\mu\}_\mu$, and uniform boundedness in L^∞ of $\{F(T_{l+1}(u_n))h_l(u_n)\}_n$ follows that

$$
\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_{4,1}^{n,\theta_0,\mu,l} = 0.
$$

Now let us write

$$
I_{4,2}^{n,\theta_0,\mu,l} = \int_{Q_T} \operatorname{div}\left(\int_0^{T_{l+1}(u_n)} F(r) h_l'(r) \,\mathrm{d}r \right) (T_k(u_n) - (T_k(u^{\theta_0}))_\mu) \,\mathrm{d}x\mathrm{d}t,
$$

hence from Gauss-Green Theorem for Sobolev functions it follows that

$$
I_{4,2}^{n,\theta_0,\mu,l} = -\int_{Q_T} \int_0^{T_{l+1}(u_n)} F(r) h_l'(r) \,\mathrm{d}r \cdot \nabla(T_k(u_n) - (T_k(u^{\theta_0}))_\mu) \,\mathrm{d}x\mathrm{d}t.
$$

Therefore by similar arguments as in the above we get

$$
\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_{4,2}^{n,\theta_0,\mu,l} = 0.
$$

Applying the uniform renormalized condition (87), it follows that

$$
\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_3^{n,\theta_0,\mu,l} = \gamma_1(l) \quad \text{ and } \gamma_1(l) \to 0 \text{ as } l \to \infty. \tag{102}
$$

To handle $I_2^{n,\theta_0,\mu,l}$, we choose $l > k$ and we notice that

$$
\begin{aligned}
I_2^{n,\theta_0,\mu,l} &= \int_{Q_T} \kappa h_l(u_n) a(x, Du_n) \cdot D(T_k(u_n) - (T_k(u^{\theta_0}))_\mu) \,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} \kappa h_l(u_n) a(x, DT_k(u_n)) \cdot DT_k(u_n) \,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_T} \kappa h_l(u_n) a(x, DT_{l+1}(u_n)) \cdot D(T_k(u^{\theta_0}))_\mu \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{103}
$$

Then passing with $n\to\infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} I_2^{n,\theta_0,\mu,l} = \lim_{n\to\infty}\int_{Q_T} \kappa h_l(u_n) a(x,DT_k(u_n))\cdot DT_k(u_n)\,\mathrm{d}x\mathrm{d}t\\ -\int_{Q_T}\kappa h_l(u)\Phi_{l+1}\cdot D(T_k(u^{\theta_0}))_\mu\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{104}$$

Indeed, since $h_l(u_n)a(\cdot,DT_{l+1}(u_n)) \stackrel{*}{\rightharpoonup} \Psi$ in $L_{M^*}(Q_T;\mathbb{R}^d)$ and $a(x,DT_{l+1}(u_n)) \stackrel{*}{\rightharpoonup} \Phi_{l+1}$ in $L_{M^*}(Q_T;\mathbb{R}^d)$, $h_l(u_n)\to h_l(u)$ a.e. in Q_T and $\|h_l\|_{L^\infty(Q_T)}\le 1$, we infer that $\Psi = \Phi_{l+1}h_l(u)$ a.e. in Q_T and thus $h_l(u_n)a(\cdot,DT_{l+1}(u_n)) \rightharpoonup \Phi_{l+1}h_l(u)$ in $L^1(Q_T;\mathbb{R}^d)$ and $h_l(u_n)a(\cdot,DT_{l+1}(u_n)) \stackrel{*}{\rightharpoonup} \Phi_{l+1}h_l(u)$ in $L_{M^*}(Q_T;\mathbb{R}^d)$. It gives (104) as $D(T_k(u^{\theta_0}))_\mu\in E_M(Q_T;\mathbb{R}^d)$.

Next since $\kappa h_l(u)\Phi_{l+1}\in E_{M^*}(Q_T;\mathbb{R}^d)$ and $D(T_k(u^{\theta_0}))_\mu \stackrel{*}{\rightharpoonup} D(T_k(u))_\mu$ in $L_M(Q_T;\mathbb{R}^d)$ as $\theta_0\downarrow 0$ and $D(T_k(u))_\mu \stackrel{*}{\rightharpoonup} D(T_k(u))_\mu$ in $L_M(Q_T;\mathbb{R}^d)$ as $\mu\to\infty$, we get

$$\begin{aligned}\lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty} I_2^{n,\theta,\mu,l} &= \lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\int_{Q_T}\kappa h_l(u_n)a(x,DT_k(u_n))\cdot DT_k(u_n)\,\mathrm{d}x\mathrm{d}t\\ &-\int_{Q_T}\kappa h_l(u)\Phi_{l+1}\cdot DT_k(u)\,\mathrm{d}x\mathrm{d}t\\ &= \lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\int_{Q_T}\kappa h_l(u_n)a(x,DT_k(u_n))\cdot DT_k(u_n)\,\mathrm{d}x\mathrm{d}t\\ &-\int_{Q_T}\kappa h_l(u)\Phi_{l+1}\cdot DT_k(u)\mathbb{1}_{\{|u|<k\}}\,\mathrm{d}x\mathrm{d}t\end{aligned}\tag{105}$$

Now let observe the following

$$a(\cdot,DT_k(u_n)) - a(\cdot,DT_{l+1}(u_n))\to 0 \quad \text{a.e. in } \{(t,x)\in Q_T\,|\,|u|<k\} \text{ as } n\to\infty.$$

Since $\{a(\cdot,DT_k(u_n)) - a(\cdot,DT_{l+1}(u_n))\}_n$ is uniformly integrable in $L^1(Q_T;\mathbb{R}^d)$ we have

$$a(\cdot,DT_k(u_n)) - a(\cdot,DT_{l+1}(u_n))\to 0 \quad \text{in } L^1 \text{ on the set } \{t,x\in Q_T\,|\,|u|<k\}.$$

On the other hand $a(\cdot,DT_k(u_n))\rightharpoonup\Phi_k$ and $a(\cdot,DT_{l+1}(u_n))\rightharpoonup\Phi_{l+1}$ in $L^1(Q_T;\mathbb{R}^d)$. Summarising we get

$$\Phi_k = \Phi_{l+1} \quad \text{a.e. in } \{(t,x)\in Q_T\,|\,|u|<k\}.$$

Above considerations provide

$$\begin{aligned}\lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty} I_2^{n,\theta,\mu,l} &= \lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\int_{Q_T}\kappa h_l(u_n)a(x,DT_k(u_n))\cdot DT_k(u_n)\,\mathrm{d}x\mathrm{d}t\\ &-\int_{Q_T}\kappa h_l(u)\Phi_k\cdot DT_k(u)\,\mathrm{d}x\mathrm{d}t\\ &= \lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\int_{Q_T}\kappa h_l(u_n)a(x,DT_k(u_n))\cdot D\left[T_k(u_n)-T_k(u^{\theta_0})_\mu\right]\,\mathrm{d}x\mathrm{d}t\\ &= \lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\tilde I_2^{n,\theta_0,\mu,l},\end{aligned}\tag{106}$$

as

$$\lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\lim_{n\to\infty}\int_{Q_T}\kappa h_l(u_n)a(x,DT_k(u_n))\cdot D(T_k(u^\theta))_\mu\,\mathrm{d}x\mathrm{d}t = \int_{Q_T}\kappa h_l(u)\Phi_k\cdot DT_k(u)\,\mathrm{d}x\mathrm{d}t.$$

Therefore by (106) instead to consider $I_2^{n,\theta,\mu,l}$ in the form (103) we concentrate now on $\tilde I_2^{n,\theta_0,\mu,l}$. Let us observe that for $l>k$

$$\tilde I_2^{n,\theta_0,\mu,l} = \tilde I_{2,1}^{n,\theta_0,\mu,l} + \tilde I_{2,2}^{n,\theta_0,\mu,l},\tag{107}$$

where

$$\tilde{I}_{2,1}^{n,\theta_0,\mu,l} = \int_{Q_T} \kappa a(x, DT_k(u_n)) \cdot D(T_k(u_n) - (T_k(u^{\theta_0}))_\mu)\, \mathrm{d}x\mathrm{d}t, \tag{108}$$

$$\tilde{I}_{2,2}^{n,\theta_0,\mu,l} = \int_{\{|u_n|>l\}} \kappa(1 - h_l(u_n)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu\, \mathrm{d}x\mathrm{d}t.$$

Let us focus now on $\tilde{I}_{2,2}^{n,\theta,\mu,l}$ and notice that

$$\mathbb{1}_{\{|u_n|>k\}} \stackrel{*}{\rightharpoonup} \chi \quad \text{weakly-(*) in } L^\infty(Q_T),$$

where $\chi \in L^\infty(Q_T)$ and $\chi \in \operatorname{sign}^+(|u| - k)$ a.e. in Q_T. Since $u_n \to u$ a.e. in Q_T and h_l is bounded, $a(x,0) \in E_{M^*}(Q_T;\mathbb{R}^d)$ and for fixed θ_0 and μ, $DT_k(u^{\theta_0})_\mu \in E_M(Q_T;\mathbb{R}^d)$, we obtain that

$$(1 - h_l(u_n)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu \to (1 - h_l(u)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu \quad \text{in } L^1(Q_T).$$

Therefore we have

$$\begin{aligned} &\lim_{n\to\infty} \int_{Q_T} \mathbb{1}_{\{|u_n|>l\}} \kappa(1 - h_l(u_n)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu\, \mathrm{d}x\mathrm{d}t \\ = &\int_{Q_T} \chi(1 - h_l(u)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu\, \mathrm{d}x\mathrm{d}t. \end{aligned}$$

As $(1 - h_l(u_n)) a(x,0) \in E_{M^*}(Q_T;\mathbb{R}^d)$, we get

$$\lim_{\mu\to\infty} \lim_{\theta_0\downarrow 0} \int_{Q_T} \chi(1 - h_l(u)) a(x,0) \cdot D(T_k(u^{\theta_0}))_\mu\, \mathrm{d}x\mathrm{d}t = \int_{Q_T} \chi(1 - h_l(u)) a(x,0) \cdot DT_k(u)\, \mathrm{d}x\mathrm{d}t. \tag{109}$$

Since $h_l(u) = 1$ on the set $\{|u| < l\}$, the right-hand side in (109) vanishes as we pass with $l \to \infty$, namely

$$\lim_{\mu\to\infty} \lim_{\theta_0\downarrow 0} \lim_{n\to\infty} \tilde{I}_{2,2}^{n,\theta_0,\mu,l} = \gamma_2(l), \qquad \text{where } \gamma_2(l) \to 0 \text{ as } l \to \infty. \tag{110}$$

To handle the parabolic term I_1, we will need the following

Lemma 4.5

$$\begin{aligned} &\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} I_1^{n,\theta_0,\mu,l} = \\ &\liminf_{\mu\to\infty} \liminf_{\theta_0\downarrow 0} \liminf_{n\to\infty} \Big\{ -\int_{Q_T} \partial_t \kappa \int_{b_0^n}^{b_n} h_l((\beta^{-1})^0(\sigma)) T_k((\beta^{-1})^0(\sigma)) \mathrm{d}\sigma\, \mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} \partial_t \left(\kappa T_k(u^{\theta_0})_\mu\right) \int_{b_0^n}^{b_n} h_l((\beta^{-1})^0(\sigma)) \mathrm{d}\sigma\, \mathrm{d}x\mathrm{d}t \Big\} \geq 0 \end{aligned} \tag{111}$$

where $b_n = \beta(x, u_n)$, $b_0^n = \beta(x, u_0^n)$.

Proof of Lemma 4.5: First we note that

$$\begin{aligned} \int_{b_0^n}^{b_n} h_l((\beta^{-1})^0(x,\sigma)) T_k((\beta^{-1})^0(x,\sigma)) \mathrm{d}\sigma &= \int_{u_0^n}^{u_n} h_l(\sigma) T_k(\sigma) \mathrm{d}\beta(x,\sigma) \\ &= \int_{u_0^n}^{u_n} T_k(\sigma) \mathrm{d}B_{h_l}(x,\sigma) \end{aligned}$$

where

$$B_{h_l}(x,\sigma) = \int_0^\sigma h_l(r)\ \mathrm{d}\beta(x,r) = \int_0^{\beta(x,\sigma)} h_l((\beta^{-1})^0(x,r))\ \mathrm{d}r \tag{112}$$

for all $\sigma \in \mathbb{R}$ and almost all $x \in \Omega$. Passing to the limit with $n \to \infty$, from (111) we get

$$\lim_{n\to\infty} I_1^{n,\theta_0,\mu,l} = I_{1,1}^{\theta_0,\mu,l} + I_{1,2}^{\theta_0,\mu,l} + I_{1,3}^{\theta_0,\mu,l}, \tag{113}$$

where

$$\begin{aligned} I_{1,1}^{\theta_0,\mu,l} = I_{1,1}^{l} &= -\int_{Q_T} \partial_t\kappa \int_{u_0}^{u} T_k(\sigma)\mathrm{d}B_{h_l}(\sigma,x) \\ &= -\int_{Q_T} \partial_t\kappa \left(T_k(u)B_{h_l}(x,u) - \int_0^{T_k(u)} B_{h_l}(\sigma)\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}t \\ &+ \int_\Omega \kappa(0) \int_0^{T_k(u_0)} B_{h_l}(x,\sigma)\ \mathrm{d}\sigma\,\mathrm{d}x - \int_\Omega \kappa(0)T_k(u_0)B_{h_l}(x,u_0)\,\mathrm{d}x, \end{aligned} \tag{114}$$

$$I_{1,2}^{\theta_0,\mu,l} = \int_{Q_T} (B_{h_l}(x,u) - B_{h_l}(x,u_0))\kappa_t(T_k(u^{\theta_0}))_\mu\,\mathrm{d}x\mathrm{d}t,$$

$$I_{1,3}^{\theta_0,\mu,l} = \int_{Q_T} (B_{h_l}(x,u) - B_{h_l}(x,u_0))\kappa\mu(T_k(u^{\theta_0}) - (T_k(u^{\theta_0}))_\mu)\,\mathrm{d}x\mathrm{d}t.$$

Since $(T_k(u^{\theta_0}))_\mu \overset{*}{\rightharpoonup} (T_k(u))_\mu$ in $L^\infty(Q_T)$ and (91), we get that

$$\lim_{\theta_0\to 0} I_{1,2}^{\theta_0,\mu,l} = I_{1,2}^{\mu,l} = \int_{Q_T} B_{h_l}(x,u)\partial_t\kappa(T_k(u))_\mu\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \partial_t\kappa B_{h_l}(x,u_0)(T_k(u))_\mu\,\mathrm{d}x\mathrm{d}t. \tag{115}$$

Next we pass to the limit with $\mu \to \infty$ in $I_{1,2}^{\mu,l}$. More precisely we have

$$\lim_{\mu\to\infty} I_{1,2}^{\mu,l} = I_{1,2}^{l} = \int_{Q_T} B_{h_l}(x,u)\partial_t\kappa T_k(u)\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \partial_t\kappa B_{h_l}(x,u_0)T_k(u)\,\mathrm{d}x\mathrm{d}t. \tag{116}$$

Similarly

$$\lim_{\theta_0\to 0} I_{1,3}^{\theta_0,\mu,l} = I_{1,3}^{\mu,l} = \int_{Q_T} (B_{h_l}(x,u) - B_{h_l}(x,u_0))\kappa\mu(T_k(u) - (T_k(u))_\mu)\,\mathrm{d}x\mathrm{d}t. \tag{117}$$

Now we set $u_\mu := (T_k(u))_\mu$ and write

$$I_{1,3}^{\mu,l} = I_{1,3,1}^{\mu,l} + I_{1,3,2}^{\mu,l} + I_{1,3,3}^{\mu,l} + I_{1,3,4}^{\mu,l}, \tag{118}$$

where

$$\begin{aligned} \lim_{\mu\to\infty} I_{1,3,1}^{\mu,l} &= \lim_{\mu\to\infty} \left(-\int_{Q_T} B_{h_l}(x,u_0)\kappa\mu(T_k(u) - (T_k(u))_\mu)\,\mathrm{d}x\mathrm{d}t \right) \\ &= I_{1,3,1}^{l} = \int_{Q_T} \partial_t\kappa B_{h_l}(x,u_0)T_k(u)\,\mathrm{d}x\mathrm{d}t + \int_\Omega B_{h_l}(x,u_0)\kappa(0)T_k(u_0)\,\mathrm{d}x \end{aligned} \tag{119}$$

Thanks to $|u_\mu| \le k$ a.e. in Q_T and the monotonicity of $B_{h_l}(x,\cdot)$ a.e. in Ω,

$$\begin{aligned} I_{1,3,2}^{\mu,l} &= \int_{Q_T} (B_{h_l}(x,u) - B_{h_l}(x,T_k(u)))\kappa\mu(T_k(u) - u_\mu)\,\mathrm{d}x\mathrm{d}t \\ &= \int_{\{|u|>k\}} (B_{h_l}(x,u) - B_{h_l}(x,k\,\mathrm{sign}(u)))\kappa\mu(k\,\mathrm{sign}(u) - u_\mu)\,\mathrm{d}x\mathrm{d}t \ge 0 \end{aligned}$$

and

$$I^{\mu,l}_{1,3,3} \;=\; \int_{Q_T} (B_{h_l}(x,T_k(u)) - B_{h_l}(x,u_\mu))\kappa\mu(T_k(u)-u_\mu)\,\mathrm{d}x\mathrm{d}t \geq 0.$$

Finally,

$$\begin{aligned}
I^{\mu,l}_{1,3,4} &= \int_{Q_T} \kappa B_{h_l}(x,u_\mu)\frac{d}{dt}u_\mu \mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} \kappa \frac{d}{dt}\int_0^{u_\mu} B_{h_l}(x,\sigma)\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t \\
&= -\int_{Q_T} \partial_t\kappa \int_0^{u_\mu} B_{h_l}(x,\sigma)\mathrm{d}\sigma - \int_\Omega \kappa(0)\int_0^{(T_k(u_0))_\mu} B_{h_l}(x,\sigma)\mathrm{d}\sigma
\end{aligned} \tag{120}$$

and therefore

$$\lim_{\mu\to\infty} I^{\mu,l}_{1,3,4} = I^l_{1,3,4} = -\int_{Q_T} \partial_t\kappa \int_0^{T_k(u)} B_{h_l}(x,\sigma)\;\mathrm{d}\sigma\,\mathrm{d}x\mathrm{d}t - \int_\Omega \kappa(0)\int_0^{T_k(u_0)} B_{h_l}(x,\sigma)\;\mathrm{d}\sigma\,\mathrm{d}x. \tag{121}$$

From (114), (116), (119) and (121) it follows that

$$\liminf_{\mu\to\infty}\liminf_{\theta_0\downarrow 0}\liminf_{n\to\infty} I_1^{n,\theta_0,\mu,l} \geq I^l_{1,1} + I^l_{1,2} + I_{1,3,4}, \tag{122}$$

more precisely we have

$$\begin{aligned}
&\liminf_{\mu\to\infty}\liminf_{\theta_0\downarrow 0}\liminf_{n\to\infty} I_1^{n,\theta_0,\mu,l} \qquad (123)\\
\geq\;& -\int_{Q_T} \partial_t\kappa\left(T_k(u)B_{h_l}(x,u) - \int_0^{T_k(u)} B_{h_l}(x,\sigma)\mathrm{d}\sigma\right)\mathrm{d}x\mathrm{d}t \\
+\;& \int_\Omega \kappa(0)\int_0^{T_k(u_0)} B_{h_l}(x,\sigma)\mathrm{d}\sigma\mathrm{d}x - \int_\Omega \kappa(0)T_k(u_0)B_{h_l}(x,u_0)\mathrm{d}x \\
+\;& \int_{Q_T} \partial_t\kappa T_k(u)B_{h_l}(x,u)\mathrm{d}x\mathrm{d}t - \int_{Q_T} \partial_t\kappa B_{h_l}(x,u_0)T_k(u)\mathrm{d}x\mathrm{d}t \\
+\;& \int_{Q_T} \partial_t\kappa B_{h_l}(x,u_0)T_k(u)\mathrm{d}x\mathrm{d}t + \int \kappa(0)B_{h_l}(x,u_0)T_k(u_0)\mathrm{d}x \\
-\;& \int_{Q_T} \partial_t\kappa \int_0^{T_k(u)} B_{h_l}(x,\sigma)\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t - \int_\Omega \kappa(0)\int_0^{T_k(u_0)} B_{h_l}(x,\sigma)\mathrm{d}\sigma\mathrm{d}x \\
=\;& 0 \qquad (124)
\end{aligned}$$

and Lemma 4.5 is proved. □

Now we summarise above considerations. From (97) by employing (100), (101), (102), (105), (108), (106), (107), (110) and (111) we get

$$\lim_{\mu\to\infty}\lim_{\theta_0\downarrow 0}\limsup_{n\to\infty}\int_{Q_T} \kappa a(x,DT_l(u_n))\cdot D(T_k(u_n) - T_k(u^{\theta_0})_\mu)\,\mathrm{d}x\mathrm{d}t + \gamma_1(l) + \gamma_2(l) \leq 0. \tag{125}$$

Rewriting the above, passing to the limit with $n\to\infty$, $\mu\to 0$, $\theta_0\to 0$ and $l\to\infty$ we have

$$\limsup_{n\to\infty}\int_{Q_T} \kappa a(x,DT_k(u_n))\cdot D(T_k(u_n))\,\mathrm{d}x\mathrm{d}t \leq \int_{Q_T} \kappa\Phi_k\cdot D(T_k(u))\,\mathrm{d}x\mathrm{d}t. \tag{126}$$

Then (94) follows from (126) by the Minty-Browder argument adapted to non-reflexive generalized Orlicz spaces as in the proof of Lemma (4.3) which we have applied to show (86), namely we obtain that

$$\Phi_k = a(x, DT_k(u)) \quad \text{a.e. in } Q_T$$

Step 2.

Next we have to provide that

$$a(x, DT_k(u_n)) \cdot DT_k(u_n) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u) \text{ weakly in } L^1(Q_T). \tag{127}$$

Since we work with non-reflexive Musielak-Orlicz spaces this is a new difficulty due to the lack of right duality pairing between spaces where sequences $\{a(x, DT_k(u_n))\}_n$ and $\{DT_k(u_n)\}_n$ converge. To show (127) we need the following auxiliary lemma which can be found in [38, Lemma 6.9], but we recall it for the completeness of the proof.

Lemma 4.6 *Let $a_n \in L^1(Q_T)$ and let $0 \le a_0 \in L^1(Q_T)$ such that $a_n(t,x) \ge -a_0(t,x)$ a.e. on Q_T,*

$$a_n \xrightarrow{b} a \text{ and } \limsup_{n\to\infty} \int_{Q_T} a_n \,\mathrm{d}x\mathrm{d}t \le \int_{Q_T} a \,\mathrm{d}x\mathrm{d}t,$$

then

$$a_n \rightharpoonup a \quad \text{weakly in } L^1(Q_T).$$

Here $\xrightarrow{b}$ stands for a biting limit used in the Chacon's lemma (see [1, 6]).
Proof of Lemma 4.6:

$$a_{n_{\sup}} \overset{*}{\rightharpoonup} a + \bar{a} \quad \text{in } \mathcal{M}(Q_T) \tag{128}$$

where by [1, Theorem 2.5] $\bar{a} \ge 0$, where $\bar{a}$ stands for a concentration measure and $n_{\sup}$ denotes a weakly-(*) converging subsequence which gives $\limsup \int_{Q_T} a_n \,\mathrm{d}x\mathrm{d}t$. Then we obtain that

$$\int_{Q_T} a_{n_{\sup}} \,\mathrm{d}x\mathrm{d}t \to \int_{Q_T} a \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \bar{a} \,\mathrm{d}x\mathrm{d}t \quad \text{as } n \to \infty$$

and then by assumptions of the lemma $\int_{Q_T} \bar{a} \,\mathrm{d}x\mathrm{d}t = 0$. Therefore $\bar{a} = 0$ as a measure. Then by [1, Theorem 2.9 (ii)] the sequence in (128) converges weakly in $L^1(Q_T)$ and weak limit coincides with biting limit. □

In order to show (127) at first let us notice that monotonicity of $a(x, \cdot)$ and uniform boundedness of the sequence $\{[a(\cdot, DT_k(u_n)) - a(\cdot, DT_k(u))] \cdot [DT_k(u_n) - DT_k(u)]\}_n$ in $L^1(Q_T)$ gives (see [6]) the following (for the subsequence if necessary)

$$\begin{aligned} 0 \le \kappa\, &[a(x, DT_k(u_n)) - a(x, DT_k(u))] \cdot [DT_k(u_n) - DT_k(u)] \\ &\xrightarrow{b} \int_{\mathbb{R}^d} \kappa\, [a(x,\lambda) - a(x, DT_k(u))] \cdot [\lambda - DT_k(u)] \,\mathrm{d}\nu_{t,x}(\lambda), \end{aligned} \tag{129}$$

where $\nu_{t,x}$ denotes a Young measure generated by the sequence $\{DT_k(u_n)\}_n$.

Since $\int_{\mathbb{R}^d} \lambda \,\mathrm{d}\nu = DT_k(u)$ for a.a. $t \in (0,T)$ and a.a. $x \in \Omega$ (as a consequence of $DT_k(u_n) \rightharpoonup DT_k(u)$ in $L^1(Q_T)$, it holds that

$$\int_{\mathbb{R}^d} a(x, DT_k(u)) \cdot [\lambda - DT_k(u)] \,\mathrm{d}\nu_{t,x}(\lambda) = 0$$

and consequently on the right-hand side of (129) we obtain

$$\begin{aligned} &\kappa \int_{\mathbb{R}^d} [a(x,\lambda) - a(x, DT_k(u))] \cdot [\lambda - DT_k(u)] \,\mathrm{d}\nu_{t,x}(\lambda) \\ &= \kappa \int_{\mathbb{R}^d} a(x,\lambda) \cdot \lambda \,\mathrm{d}\nu_{t,x}(\lambda) - \kappa \int_{R^d} a(x,\lambda) DT_k(u) \,\mathrm{d}\nu_{t,x}(\lambda) \quad \text{a.e. in } Q_T. \end{aligned} \tag{130}$$

Uniform boundedness of $\{a(x, DT_k(u_n) \cdot DT_k(u_n)\}_n$ in L^1 again provides that

$$\kappa a(x, DT_k(u_n)) \cdot DT_k(u_n) \xrightarrow{b} \kappa \int_{\mathbb{R}^d} a(x,\lambda)\cdot\lambda \,\mathrm{d}\nu_{t,x}(\lambda),$$

but since $a(x, DT_k(u_n)) \cdot DT_k(u_n) \geq -a_0$, where $a_0 \in L^1(\Omega)$, by [35, Corollary 3.3] we obtain

$$\limsup_{n\to\infty} \int_{Q_T} \kappa a(x, DT_k(u_n)) \cdot DT_k(u_n) \,\mathrm{d}x\mathrm{d}t \geq \int_{Q_T} \kappa \int_{\mathbb{R}^d} a(x,\lambda)\cdot\lambda \,\mathrm{d}\nu_{t,x}(\lambda)\,\mathrm{d}x\mathrm{d}t.$$

Then since $\Phi_k = a(x, DT_k(u)) = \int_{\mathbb{R}^d} a(x,\lambda)\,\mathrm{d}\nu_{t,x}(\lambda)$, by (126) we get

$$\int_{Q_T} \kappa \int_{\mathbb{R}^d} a(x,\lambda)\cdot\lambda \,\mathrm{d}\nu_{t,x}(\lambda)\,\mathrm{d}x\mathrm{d}t \leq \int_{Q_T} \kappa DT_k(u) \int_{\mathbb{R}^d} a(x,\lambda)\,\mathrm{d}\nu_{t,x}(\lambda)\,\mathrm{d}x\mathrm{d}t. \tag{131}$$

Summarising, by (130) and (131), we obtain that the limit in (129) is not positive, i.e.

$$[\kappa\,(a(x, DT_k(u_n)) - a(x, DT_k(u)))] \cdot [DT_k(u_n) - DT_k(u)] \xrightarrow{b} 0. \tag{132}$$

Now note that

$$a(x, DT_k(u)) \cdot (DT_k(u_n) - DT_k(u)) \rightharpoonup 0 \quad \text{weakly in } L^1(Q_T). \tag{133}$$

On the other hand

$$a(x, DT_k(u_n)) \cdot DT_k(u) \xrightarrow{b} a(x, DT_k(u)) \cdot DT_k(u). \tag{134}$$

Hence by (132), (133), (134) we have that

$$a(x, DT_k(u_n)) \cdot DT_k(u_n) \xrightarrow{b} a(x, DT_k(u)) \cdot DT_k(u). \tag{135}$$

As $\{a(x, DT_k(u_n)) \cdot DT_k(u_n)\}_n \geq -a_0$ a.e. together with (135) and (126) provide by Lemma 4.6 that

$$a(x, DT_k(u_n)) \cdot DT_k(u_n) \rightharpoonup a(x, DT_k(u)) \cdot DT_k(u) \quad \text{weakly in } L^1(Q_T). \tag{136}$$

Step 3.

Now, we are able to conclude the proof of Theorem 4.4. By assumptions of Theorem 4.4 it follows immediately that (**R1**) and (**R2**) of the Definition 1.1 hold for all $k > 0$. For $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\xi \in \mathcal{D}([0,T)\times\Omega)$ we can plug $h(u_n)\xi$ into $(P, f_n, b_{0,n})$ by the integration-by-parts formula of Lemma 4.1 and obtain

$$I_1 + I_2 + I_3 = I_4, \tag{137}$$

where

$$\begin{aligned}
I_1 &= -\int_{Q_T} \partial_t \xi \int_{b_0^n}^{b_n} h \circ (\beta^{-1})^0(s)\,\mathrm{d}s\,\mathrm{d}x\mathrm{d}t,\\
I_2 &= \int_{Q_T} a(x, Du_n) \cdot D(h(u_n)\xi)\,\mathrm{d}x\mathrm{d}t,\\
I_3 &= \int_{Q_T} F(u_n) \cdot D(h(u_n)\xi)\,\mathrm{d}x\mathrm{d}t,\\
I_4 &= \int_{Q_T} f_n h(u_n)\xi\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thanks to the convergence assumption, we can pass to the limit with $n \to \infty$ in $I_1, \ldots, I_4$: It follows immediately that

$$\lim_{n\to\infty} I_1 = -\int_{Q_T} \partial_t \xi \int_{b_0}^{b} h \circ (\beta^{-1})^0(s)\, \mathrm{d}s\, \mathrm{d}x\mathrm{d}t. \tag{138}$$

Now we choose $R > 0$ such that $\operatorname{supp} h \subset [-R, R]$. Next, we write

$$I_2 = I_{2,1} + I_{2,2}, \tag{139}$$

where

$$I_{2,1} \quad = \quad \int_{(0,\tau)\times\Omega} h'(T_R(u_n))\xi a(x, DT_R(u_n)) \cdot DT_R(u_n)\, \mathrm{d}x\mathrm{d}t,$$

for $0 < \tau < T$ is such that $\operatorname{supp} \xi \subset [0, \tau) \times \Omega$. By (136), $a(x, DT_R(u_n)) \cdot DT_R(u_n) \rightharpoonup a(x, DT_R(u)) \cdot DT_R(u)$ weakly in $L^1([0, \tau) \times \Omega)$ and since $h'(u_n)\xi \to h(u)\xi$ almost everywhere in Q_T and $\|h(u_n)\xi\|_{L^\infty((0,\tau)\times\Omega} \leq \|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)}$, we may pass to the limit in $I_{2,1}$ and obtain

$$\lim_{n\to\infty} I_{2,1} = \int_{Q_T} h'(u)\xi a(x, Du) \cdot Du\, \mathrm{d}x\mathrm{d}t, \tag{140}$$

where Du denotes the generalized gradient in the sense of [8]. By assumption vii) we have weak-(*) convergence of the $a(x, DT_k(u_n))$ in L_{M^*} and therefore also weak in L^1. Moreover, $h'(T_R(u_n))$ converges a.e. to $h'(T_R(u))$ and is uniformly bounded in L^∞, therefore we may pass to the limit in

$$I_{2,2} = \int_{Q_T} h(T_R(u_n))a(x, DT_R(u_n)) \cdot D\xi\, \mathrm{d}x\mathrm{d}t, \tag{141}$$

and find

$$\lim_{n\to\infty} I_{2,2} = \int_{Q_T} h(u)a(x, Du) \cdot D\xi\, \mathrm{d}x\mathrm{d}t. \tag{142}$$

Next we write $I_3 = I_{3,1} + I_{3,2}$ where

$$I_{3,1} = \int_{Q_T} h'(T_R(u_n))\xi F(T_R(u_n)) \cdot DT_R(u_n)\, \mathrm{d}x\mathrm{d}t,$$
$$I_{3,2} = \int_{Q_T} h(T_R(u_n))F(T_R(u_n)) \cdot D\xi\, \mathrm{d}x\mathrm{d}t.$$

Since $h'(T_M(u_n))F(T_M(u_n))$ converges to $h'(T_M(u))F(T_M(u))$ in $(L^{1+\nu}(Q_T; \mathbb{R}^d))$ as $n \to \infty$ using *vi.*) we have (by weak L^1 convergence combined with a.e. convergence of an L^∞ bounded sequence)

$$\lim_{n\to\infty} I_{3,1} = \int_{Q_T} h'(u)\xi F(u) \cdot Du\, \mathrm{d}x\mathrm{d}t \tag{143}$$

and moreover

$$\lim_{n\to\infty} I_{3,2} = \int_{Q_T} h(u)F(u) \cdot D\xi\, \mathrm{d}x\mathrm{d}t. \tag{144}$$

Finally, we have

$$\lim_{n\to\infty} I_4 = \int_{Q_T} fh(u)\xi\, \mathrm{d}x\mathrm{d}t \tag{145}$$

and from (137)-(145) it follows that (u, b) satisfies the renormalized formulation (**R3**).

Next we show our limit meets (**R4**). To this end let us notice first that $Du_n = 0$ a.e. on $A = \{(x,t) \in Q_T : u_n = \{l, l+1\}\}$. Hence by (87) we have

$$\lim_{l\to\infty} \sup_{n\in\mathbb{N}} \int_{\{l-1<u_n<l+2\}} a(x, Du_n) \cdot Du_n \,\mathrm{d}x\mathrm{d}t = 0 \tag{146}$$

Let $g_l(r) : \mathbb{R} \to \mathbb{R}$ be an auxiliary function such that

$$g_l(r) := \begin{cases} 1 & \text{if } l \le r \le l+1 \\ 0 & \text{if } r < l-1 \text{ or } r > l+2 \\ \text{is an affine function} & \text{otherwise.} \end{cases}$$

Since $a_0 \in L^1(\Omega)$ is nonnegative we notice that

$$\begin{aligned} \int_{\{l<u<l+1\}} a(x, Du) \cdot Du \,\mathrm{d}x\mathrm{d}t &\le \int_{\{l-1<u<l+2\}} g_l(u) \left(a(x, DT_{l+2}(u)) \cdot DT_{l+2}(u) + a_0\right) \mathrm{d}x\mathrm{d}t \\ &= \int_{Q_T} g_l(u) \left(a(x, DT_{l+2}(u)) \cdot DT_{l+2}(u) + a_0\right) \mathrm{d}x\mathrm{d}t = I. \end{aligned} \tag{147}$$

By assumption iii.) of the Theorem 4.4, (136), the continuity and boundedness of g_l, and in the last step by (146), (88) and since $(a(x, DT_{l+2}(u_n)) \cdot DT_{l+2}(u_n) + a_0)$ is nonnegative we infer that

$$\begin{aligned} \lim_{l\to\infty} I &= \lim_{l\to\infty} \lim_{n\to\infty} \int_{Q_T} g_l(u_n) \left(a(x, DT_{l+2}(u_n)) \cdot DT_{l+2}(u_n) + a_0\right) \mathrm{d}x\mathrm{d}t \\ &\le \lim_{l\to\infty} \sup_{n\in\mathbb{N}} \int_{\{l-1<u<l+2\}} \left(a(x, DT_{l+2}(u_n)) \cdot DT_{l+2}(u_n) + a_0\right) \mathrm{d}x\mathrm{d}t \\ &= 0. \end{aligned} \tag{148}$$

Then (147) together with (148) give (**R3**).

5 Proof of the Theorem 1.1

The existence of renormalized solutions to the simplified problem, namely assuming that $\beta(\cdot, u) = u$ and $F(u) = 0$ in (P, f, b_0), is now a direct consequence of the Theorem 4.4. It remains to prove only that the sequence of solutions $\{u_n\}_n$ to the approximate problem (52) satisfies the assumptions of the above theorem.

Step 1. The assumption $i)$ of Theorem 4.4, i.e.

$$f_n \to f \text{ in } L^1(Q_T) \tag{149}$$

is a consequence of the Lebesgue dominated convergence theorem and that $f_n = T_n(f)$. Since $u_{0,n} = T_n(u_0)$, in the same way we obtain the first part of $ii)$ of the Theorem 4.4, namely that

$$u_{n,0} \to u_0 \text{ in } L^1(\Omega). \tag{150}$$

Step 2. The estimates (65), (67) and weak lower semicontinouity of convex functional provide the existence of a subsequence, which satisfies assumptions $v.)$ and $vi.)$ of Theorem 4.4. Moreover, (66) gives existence of $\Phi_k \in L_{M^*}(Q_T; \mathbb{R}^d)$ such that $vii.)$ is satisfied.

The existence of the sequence satisfying (89), (91), (92), (93) is ensured again by arguments explained in Section 4.3.2, namely by (68), (69), (70), (71).

Step 3. Now let us check if the assumption $iv.)$ of the Theorem 4.4 is fulfilled.

Let us start with the following remark

Remark 5.1 *Using estimates* (65) *and* (66), *we show with analogous arguments as in the elliptic case (see [28, Corollary 5.2, Proposition 5.3]) that* u^n *and* u *are finite almost everywhere in* Q_T, *more precisely there exists a constant* $C > 0$, *not depending on* $n, l \in \mathbb{R}$ *such that*

$$|\{|u_n| \geq l\}| \leq Cl^{-\nu}, \tag{151}$$

and from (151) *it follows that*

$$\lim_{l\to\infty} |\{|u| > l\}| = 0. \tag{152}$$

To prove the uniform renormalized condition, we take $T_k(u_n)\phi_\delta$, $\phi_\delta(t) := \min(\frac{1}{\delta}\max(T-\delta-t, 0), 1)$ as a test function and apply Lemma 4.1 in the weak formulation for (P, f_n, u_0^n).

Then we set $k = l+1$ and after $k = l$. Subtracting the corresponding equalities and neglecting positive terms we obtain

$$\int_{\{l<|u_n|<l+1\}} \phi_\delta a(x, Du_n)\cdot Du_n \,\mathrm{d}x\mathrm{d}t \leq \int_{\{|u_n|>l\}} |f|\,\mathrm{d}x\mathrm{d}t + \int_\Omega \int_0^{|b_0|} G_l((\beta^{-1})^0)(s)\,\mathrm{d}s\,\mathrm{d}x \tag{153}$$

where $G_l = T_{l+1} - T_l$ and the uniform renormalized condition follows applying (151) in (153). Note that by (151) the first term on the right hand side of (153) vanishes as $l \to \infty$.

Step 4. The second part of *ii*) of Theorem 4.4 is equivalent to *iii*.) by assumption $\beta(\cdot, u) = u$. In the last step we have to show only *iii*.) i.e.

$$u_n \to u \text{ a.e. in } Q_T. \tag{154}$$

To provide (154) we use the Lions-Aubin arguments and we need to obtain the estimates for $\{T_k^\delta(u_n)_t\}_n$, where T_k^δ is a smooth approximation of the truncation function T_k (namely, $T_k^\delta = \omega_\delta * T_k$, where ω_δ is a standard regularising kernel).

Note that a weak solution to the problem (52) is also a renormalized solution. Then by integration by parts formula with $h = (T_k^\delta)'$ we can obtain the following for all $\varphi \in C_c^1([0,T); C_c^1(\Omega))$

$$\int_{Q_T} \partial_t\varphi(T_k^\delta(u_n) - T_k^\delta(u_{n,0}))\,\mathrm{d}x\mathrm{d}t\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(x, Du_n)\cdot D[(T_k^\delta)'(u_n)\varphi]\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} f(T_k^\delta)'(u^n)\varphi\,\mathrm{d}x\mathrm{d}t \tag{155}$$

where $\operatorname{supp}(T_k^\delta)' \subseteq [-(k+\delta), (k+\delta)]$ and $\|(T_k^\delta)''\|_{L^\infty(\mathbb{R})} < \infty$ for any fixed $\delta > 0$ and $k \in \mathbb{N}$. Then

$$\begin{aligned}
&\int_{Q_T} a(x, Du_n)\cdot D[(T_k^\delta)'(u_n)\varphi]\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_T} a(x, DT_{k+\delta}(u_n))\cdot (T_k^\delta)''(u_n)Du^n\varphi\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(x, DT_{k+\delta}(u_n))\cdot (T_k^\delta)'(u_n)D\varphi\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

where we notice that

$$\begin{aligned}
&\int_{Q_T} |a(x, DT_{k+\delta}(u_n))\cdot (T_k^\delta)''(u_n)Du_n\varphi|\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_{Q_T} |a(x, DT_{k+\delta}(u_n))\cdot DT_{k+\delta}(u_n)|\,\mathrm{d}x\mathrm{d}t \|(T_k^\delta)''\|_{L^\infty}\|\varphi\|_\infty.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_T} |a(x, DT_{k+\delta}(u_n))\cdot (T_k^\delta)'(u_n)D\varphi|\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_{Q_T} |a(x, DT_{k+\delta}(u_n))|\,\mathrm{d}x\mathrm{d}t \|(T_k^\delta)'(u_n)\|_{L^\infty}\|D\varphi\|_\infty.
\end{aligned}$$

As $\{|a(x, DT_{k+\delta}(u_n))\cdot DT_{k+\delta}(u_n)|\}_n$ and $\{|a(x, DT_{k+\delta}(u_n))|\}_n$ are bounded in $L^1(0,T;L^1(\Omega))$ and in $\mathcal{M}([0,T);\mathcal{M}(\Omega))$ and as $(C_0(\Omega))^* = \mathcal{M}(\Omega)$ we obtain that $\left\{\frac{\mathrm{d}T_k^\delta(u_n)}{\mathrm{d}t}\right\}$ is bounded in $\mathcal{M}([0,T);(C_0^1(\Omega)^*)$. Here $C_0^1(\Omega)$ denotes a space of continuously differentiable function vanishing at the boundary, $C_0(\Omega)$ stands for the space of continuous function vanishing at the boundary, $\mathcal{M}(\Omega)$ is a space of bounded Radon measures and the following dense embedding holds $C_c^1(\Omega) \hookrightarrow C_0^1(\Omega) \hookrightarrow C_0(\Omega)$.

Moreover let us notice that for all $s \in \mathbb{R}$ $|(T_k^\delta)'(s)| \le |T'_{k+\delta}(s)|$. Therefore $|DT_{k+\delta}(u)|^{1+\nu} \ge |DT_k^\delta(u)|^{1+\nu}$.

Let us recall now the following result:

Lemma 5.1 ([41, Corollary 7.9]) *Assuming $V_1 \subset\subset V_2 \subset V_3$, V_1 reflexive, the Banach space V_3 having a predual space V_3', i.e. $V_3 = (V_3')^*$, and $1 < p < \infty$ it holds that*

$$\left\{\phi \in L^p([0,T];V_1);\ \frac{\mathrm{d}\phi}{\mathrm{d}t} \in \mathcal{M}([0,T];V_3)\right\} \subset\subset L^p([0,T];V_2).$$

Applying the above lemma with $V_3' = C_0^1(\Omega)$, $V_2 = L^{1+\nu}(\Omega)$ and $V_1 = W^{1+\nu}(\Omega)$, $p = 1+\nu$ we obtain that $T_k^\delta(u_n) \to T_k^\delta(u)$ strongly in $L^{1+\nu}(Q_T)$. .

Let us now notice the following:

Remark 5.2 *Let ω_δ be a standard regularising kernel and by T_k^δ we mean $\omega_\delta * T_k(\cdot)$. If*

$$T_k^\delta(u_n) \to T_k^\delta(u) \text{ as } n \to \infty \text{ for any } k \in \mathbb{N} \text{ and } \delta \in (0,1) \tag{156}$$

and $|\{(t,x) \in Q_T : |u^n(t,x)| > l\}| \le cl^{-\nu}$, then $u^n \to u$ a.e. in Q_T as $n \to \infty$.

The above remark allow as to conclude that (154) is satisfied.

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