THERMO-VISCO-ELASTICITY FOR MODELS WITH GROWTH CONDITIONS IN ORLICZ SPACES

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Preprint no. 2014 - 046

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY
FNP
Foundation for Polish Science
EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND

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FILIP Z. KLAWE

Abstract. We study a quasi-static evolution of thermo-visco-elastic model. We act with external forces on non-homogeneous material body, which is a subject of our research. Such action may cause deformation of this body and may change its temperature. Mechanical part of the model contains two kinds of deformation: elastic and visco-elastic. Mechanical deformation is coupled with the temperature and they may influence each other. Since constitutive function on evolution of visco-elastic deformation depends on temperature, the visco-elastic properties of material also depend on temperature. We consider the thermodynamically complete model related to hardening rule with growth condition in generalized Orlicz spaces. We provide the proof of existence of solutions for such class of models.

1. Introduction

The objective of this paper is to show the existence of solution to special class of thermo-visco-elastic models. We consider reaction of material body treated by external forces and heat flux through the boundary. In the case of ideal elastic deformations, the body should return to its initial state after termination of external forces activity. However, if deformations are not elastic, i.e. there is a loss of potential energy, we deal with special kind of inelastic deformations. Potential energy lost during the process may be transformed into thermal energy. We focus on the visco-elastic type of deformations, which for instance may be observed in polymers. Both deformations are coupled in physical phenomena and they may be observed at the same time. Consequently, these two types of deformations appear in the models considered in this paper. Elastic deformation is reversible, whereas visco-elastic one irreversible.

The thermo-visco-elastic system of equations, as a consequence of balance of momentum and balance of energy, cf. [16, 29], see also [19], captures displacement, temperature and visco-elastic strain. Since these two principles do not take into account the material properties of considered body, we may complement it by adding constitutive relations which complete missing information. The standard technique in the solid body deformation is to work with two constitutive relations. First one describes the dependency between stress and strains, i.e. this is an equation for the Cauchy stress tensor. Second one is a constitutive equation which is characterized by the evolution of visco-elastic strain tensor.

We assume that the body $\Omega \subset \mathbb{R}^3$ is an open bounded set with a C^2 boundary. Then quasi-static evolution problem is formulated by the following system of equations

$$\left\{ \begin{array}{rcll} -\operatorname{div} \boldsymbol{T} &=& \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \boldsymbol{T} &=& \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) & \text{in } \Omega \times (0,T), \\ \boldsymbol{\varepsilon}^{\mathbf{p}}_t &=& \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T), \\ \theta_t - \Delta\theta &=& \boldsymbol{T}^d : \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T). \end{array} \right. \tag{1.1}$$

By the solution of this system we understand finding the displacement of material $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$, the temperature of material $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and the visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}} : \Omega \times \mathbb{R}_+ \to \mathcal{S}^3_d$. We denote by $\mathcal{S}^3$ the set of symmetric 3×3-matrices with real entries and by $\mathcal{S}^3_d$ a subset of $\mathcal{S}^3$ which contains traceless matrices. The function $\boldsymbol{T} : \Omega \times \mathbb{R}_+ \to \mathcal{S}^3$ stays for the Cauchy stress tensor. By $\boldsymbol{I}$ we mean the identity matrix from $\mathcal{S}^3$, thus $\boldsymbol{T}^d$ is a deviatoric (traceless) part of the tensor $\boldsymbol{T}$, i.e. $\boldsymbol{T}^d = \boldsymbol{T} - \frac{1}{3}tr(\boldsymbol{T})\boldsymbol{I}$. Additionally, we denote by $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the deformation tensor associated to $\boldsymbol{u}$, i.e. $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$.

2000 *Mathematics Subject Classification.* 74C10, 35Q74, 74F05.

Key words and phrases. visco-elasticity, thermal effects, Galerkin approximation, monotonicity method, renormalizations, generalized Orlicz space.

The motivation for current paper is to extend results presented in [18], where we proved the existence of solution to Norton-Hoff model, i.e. the model with growth condition on visco-elastic strain tensor in Lebesgue spaces. Model with growth condition in the generalized Orlicz spaces is a natural extension of Norton-Hoff model as a next step to make an approximation of Prandtl-Reuss model. Use of generalized Orlicz spaces takes into consideration more rapid growth than in the case of growth condition in Lebesgue spaces. Furthermore, choice of generalized Orlicz space allow us to consider non-homogeneous materials. Since the N-function depends on the spatial variable x, different regions of Ω may have different growth condition. Consideration of non-homogeneous materials implies that the operator $\boldsymbol{D}$ may also depend on the spatial variable x. In previous papers, see [18, 19] we considered only homogeneous materials.

Studying mechanical problems in Orlicz spaces is not an isolated issue. In the case of visco-elastic deformation, the problem involving Orlicz spaces was considered in [12], but only in the case of N-function independent on spatial variable x. In the case of N-function which depends on the spatial variable x some accurate assumptions must be done. There are two possible ways to make it. Firstly, we may assume the regularity with respect to x, e.g. log-Hölder continuity in [35, 36], secondly upper and lower growth condition of an N-function with respect to the last variable can be considered, e.g. see [17, 22, 23, 38]. There are no results for thermo-visco-elastic problems without any upper and lower growth condition on N-function with respect to the last variable.

System of equations (1.1) is a mathematical simplification of more general model. We consider the quasi-static evolution with small displacement. It means that we omit acceleration term in momentum equation as a consequence of long-term character of external forces. Small displacement allows us to use the Hooke's law in the definition of Cauchy stress tensor $(1.1)_{(2)}$. Moreover, the material does not change its volume with the temperature, i.e. there is no thermal expansion of body, thus the Cauchy stress tensor does not depend on temperature.

System (1.1) may be completed by formulating the initial

$$\left\{\begin{array}{rcl}\theta(x,0) & = & \theta_0(x),\\ \boldsymbol{\varepsilon}^{\mathbf{p}}(x,0) & = & \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x),\end{array}\right. \tag{1.2}$$

in Ω and boundary conditions

$$\left\{\begin{array}{rcl}\boldsymbol{u} & = & \boldsymbol{g},\\ \frac{\partial\theta}{\partial\boldsymbol{n}} & = & g_\theta,\end{array}\right. \tag{1.3}$$

on $\partial\Omega\times(0,T)$. We control the shape of Ω and the heat flux through the boundary.

To discuss two other equations and to formulate the statement of this paper, we need to use some definitions which are mentioned below for better readability of the paper. Let us begin with presenting the notion of generalized Orlicz spaces. For more general concept of Orlicz space we refer the reader to [1, 28, 32, 33]. We start with defining N-function.

Definition 1. *Let Ω be a bounded open domain in $\mathbb{R}^3$. A function $M:\Omega\times\mathcal{S}^3\to\mathbb{R}_+$ is said to be N-function if it satisfies the following conditions:*

1) *M is a Carathéodory function (measurable with respect to x and continuous with respect to $\boldsymbol{\xi}$) such that $M(x,\boldsymbol{\xi})=0$ if and only if $\boldsymbol{\xi}=\mathbf{0}$;*
2) *$M(x,\boldsymbol{\xi})=M(x,-\boldsymbol{\xi})$ a.e. in Ω;*
3) *$M(x,\boldsymbol{\xi})$ is a convex function with respect to $\boldsymbol{\xi}$;*
4) *$\lim_{|\boldsymbol{\xi}|\to 0} M(x,\boldsymbol{\xi})/|\xi|=0$ for all $x\in\Omega$;*
5) *$\lim_{|\boldsymbol{\xi}|\to \infty} M(x,\boldsymbol{\xi})/|\xi|=\infty$ for all $x\in\Omega$;*

Definition 2. *Function M^* which is complementary to function M is defined by*

$$M^*(x,\boldsymbol{\eta})=\sup_{\boldsymbol{\eta}\in\mathcal{S}^3}(\boldsymbol{\xi}:\boldsymbol{\eta}-M(x,\boldsymbol{\xi})), \tag{1.4}$$

for $\boldsymbol{\eta}\in\mathcal{S}^3, x\in\Omega$.

Remark. *A complementary function M^* to N-function M is also an N-function.*

Let us denote by $Q = \Omega \times (0,T)$. The generalized Orlicz class $\mathcal{L}_M(Q)$ is the set of all measurable function $\boldsymbol{\xi} : Q \to \mathcal{S}^3$ such that

$$\int_Q M(x, \boldsymbol{\xi}(x,t))\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{1.5}$$

The generalized Orlicz space $L_M(Q)$ can be defined as the smallest linear space containing $\mathcal{L}_M(Q)$. By $E_M(Q)$ we denote the closure of the set of bounded functions in L_M-norm. The generalized Orlicz space $L_M(Q)$ is a Banach space with respect to the Orlicz norm

$$\|\boldsymbol{\xi}\|_{O,M} = \sup\left\{\int_Q \boldsymbol{\xi} : \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t : \ \boldsymbol{\eta} \in L_{M^*}(Q),\ \int_Q M^*(x,\boldsymbol{\eta})\,\mathrm{d}x\,\mathrm{d}t \le 1\right\}, \tag{1.6}$$

or equivalently with respect to Luxemburg norm

$$\|\boldsymbol{\xi}\|_{L,M} = \inf\left\{\lambda > 0 : \ \int_Q M\left(x, \frac{\boldsymbol{\xi}}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t \le 1\right\}. \tag{1.7}$$

Definition 3. *We say that an N-function M satisfies Δ_2-condition if for almost all $x \in \Omega$ and for all $\boldsymbol{\xi} \in \mathcal{S}^3$, there exists a constant c and nonnegative integrable function $h : \Omega \to \mathbb{R}$ such that*

$$M(x, 2\boldsymbol{\xi}) \le cM(x,\boldsymbol{\xi}) + h(x). \tag{1.8}$$

Remark. *For every M the following concluding holds*

$$E_M(Q) \subseteq \mathcal{L}_M(Q) \subseteq L_M(Q). \tag{1.9}$$

In particular, if M satisfies the Δ_2-condition, $E_M(Q) = L_M(Q)$. If Δ_2-condition fails, we lose numerous properties of the space $L_M(Q)$ like separability, reflexivity and many others, cf. [1, 32] and particular [20] for generalized Orlicz spaces.

The space $L_{M^*}(Q)$ is the dual space of $E_M(Q)$. The functional

$$\rho(\boldsymbol{\xi}) = \int_Q M(x,\boldsymbol{\xi})\,\mathrm{d}x\,\mathrm{d}t \tag{1.10}$$

is a modular.

Definition 4. *We say that a sequence $\{\boldsymbol{\xi}_i\}_{i=1}^{\infty}$ converges modularly to $\boldsymbol{\xi}$ in $L_M(Q)$ if there exists $\lambda > 0$ such that*

$$\int_Q M\left(x, \frac{\boldsymbol{\xi}_i - \boldsymbol{\xi}}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t \to 0, \tag{1.11}$$

when i is going to ∞. We use the notation $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ for modular convergence in $L_M(Q)$.

In Appendix B we present several lemmas related to Orlicz spaces. We use these lemmas to prove the existence of thermo-visco-elasticity model solution.

After these few definitions we may discuss the constitutive relations used to complement the system (1.1). Relation between the Cauchy stress tensor and the strain tensor is defined by operator $\boldsymbol{D} : \mathcal{S}^3 \to \mathcal{S}^3$, which is linear, positively definite and bounded. Moreover, $\boldsymbol{D}$ is a four-index matrix, i.e. $\boldsymbol{D} = \boldsymbol{D}(x) = \{d_{i,j,k,l}(x)\}_{i,j,k,l=1}^3$ and the following equalities hold

$$d_{i,j,k,l}(x) = d_{j,i,k,l}(x), \quad d_{i,j,k,l}(x) = d_{i,j,l,k}(x) \quad \text{and} \quad d_{i,j,k,l}(x) = d_{k,l,i,j}(x), \tag{1.12}$$

for all $i,j,k,l = 1,2,3$ and for a.a. $x \in \Omega$. Additionally, function $d_{i,j,k,l}$ belongs to $W^{1,p}(\Omega)$ for each $i,j,k,l = 1,2,3$ and for some $p > 3$.

Second constitutive relation is an evolutionary equation for visco-elastic strain tensor. Function $\boldsymbol{G} : \mathbb{R}_+ \times \mathcal{S}^3_d \to \mathcal{S}^3_d$ is a function of temperature and deviatoric part of Cauchy stress tensor. We discuss more precisely the concept of such a choice in [18]. The properties of considered material imply the choice of specific function. Various different models were considered, e.g. Bodner-Partom model [5, 10, 11], Mróz model [9, 19, 27], Norton-Hoff model [2, 18], Prandtl-Reuss model with linear kinematic hardening [13].

Assumption 1. *The function $\boldsymbol{G}(\theta, \boldsymbol{T}^d)$ is continuous with respect to θ and $\boldsymbol{T}^d$ and satisfies the following conditions:*

a) $(\boldsymbol{G}(\theta,\boldsymbol{T}_1^d)-\boldsymbol{G}(\theta,\boldsymbol{T}_2^d)):(\boldsymbol{T}_1^d-\boldsymbol{T}_2^d)\geq 0$, *for all* $\boldsymbol{T}_1^d,\boldsymbol{T}_2^d\in\mathcal{S}^3_d$ *and* $\theta\in\mathbb{R}_+$*;*

b) $\boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d\geq c\Big(M(x,\boldsymbol{T}^d)+M^*(x,\boldsymbol{G}(\theta,\boldsymbol{T}^d))\Big)$, *where* $\boldsymbol{T}^d\in\mathcal{S}^3_d$, $\theta\in\mathbb{R}_+$ *and* c *is a positive constant independent of temperature* θ.

Moreover, we assume that the generalized Orlicz spaces fulfill:

1) *the following inequality holds*

$$\int_Q M^*(x,\boldsymbol{A}(x,t))\,\mathrm{d}x\,\mathrm{d}t\leq\int_Q|\boldsymbol{A}|^2\,\mathrm{d}x\,\mathrm{d}t\qquad\forall\boldsymbol{A}\in L_{M^*}(Q);\tag{1.13}$$

2) M^* *satisfies the* Δ_2*-condition.*

Dealing with such assumption on function $\boldsymbol{G}(\cdot,\cdot)$ implies the displacement space.

Definition 5. *Bounded deformation space, see [24]*
Let us define the space of bounded deformation $BD_M(Q,\mathbb{R}^3)$

$$BD_M(Q,\mathbb{R}^3)=\left\{\boldsymbol{u}\in L^1(\Omega,\mathbb{R}^3):\quad\boldsymbol{\varepsilon}(\boldsymbol{u})\in L_M(\Omega,\mathcal{S}^3)\right\}.\tag{1.14}$$

The space $BD_M(Q,\mathbb{R}^3)$ *is a Banach space with a norm*

$$\|\boldsymbol{u}\|_{BD_M(Q)}=\|\boldsymbol{u}\|_{L^1(Q)}+\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{O,M}.\tag{1.15}$$

According to [26,Theorem 1.1] there exists a unique continuous operator γ_0 *from* $BD_M(Q)$ *onto* $L^1((0,T)\times\partial\Omega)$ *such that the generalized Green formula*

$$2\int_Q\phi\boldsymbol{\varepsilon}_{i,j}(\boldsymbol{u})\,\mathrm{d}x\,\mathrm{d}t=-\int_Q(u_i\frac{\partial\phi}{\partial x_i}+u_j\frac{\partial\phi}{\partial x_j})+\int_0^T\int_{\partial\Omega}\phi(\gamma_0(u_i)n_j+\gamma_0(u_j)n_i)d\mathcal{H}^{n-1}\,\mathrm{d}t\tag{1.16}$$

holds for every $\phi\in C^1(\overline{Q})$ *and where* $\boldsymbol{n}=(n_1,n_2,n_3)^T$ *is an unit outward normal vector on* $\partial\Omega$ *and* $\mathcal{H}^{n-1}$ *is the* $(n-1)$*-Hausdorff measure.*

Furthermore, we understand $\boldsymbol{v}\in BD_{M^*}(Q,\mathbb{R}^3)+L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$ in the following way: There exists a decomposition $\boldsymbol{v}=\boldsymbol{v}_1+\boldsymbol{v}_2$, where $\boldsymbol{v}_1\in BD_{M^*}(Q,\mathbb{R}^3)$ and $\boldsymbol{v}_2\in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$.

In the contrast to [18] or [27] we use another approach to heat equation. By Assumption 1 we know that the right hand side function $\boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d$ is only integrable. Following Boccardo and Gallouët, cf. [8], we proved in [18] that the solution to the heat equation belongs to $L^q(0,T,W^{1,q}(\Omega))$ for all $q\in(1,\frac{5}{4})$. A weak point of this approach is lack of uniqueness. Hence, we change the approach and following Blanchard and Murat we prove the existence of a renormalised solution. The concept of renormalised solutions to parabolic equation with Dirichlet boundary condition was presented in [6,7]. In the Appendix A we prove existence of a renormalised solution in case of Neumann boundary condition.

During the modelling of physical phenomena we should not forget about their physical properties. Losses of energy or admission of negative temperature causes that mathematical result has no physical interpretation. In the case of solid mechanics, the model should fulfill the principle of thermodynamics. Considered model is thermodynamically complete, i.e. that the principle of thermodynamics is fulfilled. In [18,19] we discussed conservation of energy, positivity of temperature and existence of entropy, which has a positive rate of production. Considering the quasi-static evolution causes that the energy of system consists of internal (thermal) and potential energy. Lack of acceleration term in balance of momentum implies that the kinetic energy of Ω fails.

Definition 6. *Weak-renormalised solution of the system* (1.1)
The triple of functions $\boldsymbol{u}\in BD_{M^*}(Q,\mathbb{R}^3)+L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$, $\boldsymbol{T}\in L^2(0,T,L^2(\Omega,\mathcal{S}^3))$ *and* $\theta\in C([0,T],L^1(\Omega))$ *such that for every* $K\in\mathbb{N}$, $\mathcal{T}_K(\theta)\in L^2(0,T,W^{1,2}(\Omega))$ *is a weak-renormalised solution of the system* (1.1) *when*

$$\int_0^T\int_\Omega\boldsymbol{T}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t,\tag{1.17}$$

where

$$\boldsymbol{T}=\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}),\tag{1.18}$$

holds for every test function $\boldsymbol{\varphi} \in C^\infty([0,T], C_c^\infty(\Omega,\mathbb{R}^3))$ *and*

$$
\begin{aligned}
-\int_Q S(\theta-\tilde\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0-\tilde\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta-\tilde\theta)\nabla(\theta-\tilde\theta)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\
+\int_Q S''(\theta-\tilde\theta)|\nabla(\theta-\tilde\theta)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d S'(\theta-\tilde\theta)\phi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{1.19}
$$

holds for every test function $\phi \in C_c^\infty([-\infty,T), C^\infty(\Omega))$*, for every function* $S\in C^\infty(\mathbb{R})$ *such that* $S'\in C_0^\infty(\mathbb{R})$ *and for* $\tilde\theta$ *which is a solution of problem*

$$
\left\{
\begin{array}{rcll}
\tilde\theta_t - \Delta\tilde\theta &=& 0 & \text{in } \Omega\times(0,T),\\
\frac{\partial\tilde\theta}{\partial\boldsymbol{n}} &=& g_\theta & \text{on } \partial\Omega\times(0,T),\\
\tilde\theta(x,0) &=& \tilde\theta_0 & \text{in } \Omega.
\end{array}
\right.
\tag{1.20}
$$

Furthermore, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.

$$
\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t) = \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau),\boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau,
\tag{1.21}
$$

for a.e. $x\in\Omega$ *and* $t\in[0,T)$*. Moreover* $\boldsymbol{\varepsilon}^{\mathbf{p}}, \boldsymbol{\varepsilon}_t^{\mathbf{p}} \in L_{M^*}(Q)$*.*

Theorem 1. *Let initial conditions satisfy* $\theta_0\in L^1(\Omega)$*,* $\boldsymbol{\varepsilon}_0^{\mathbf{p}}\in L_{M^*}(\Omega,\mathcal{S}_d^3)$*, boundary conditions satisfy* $g_\theta \in L^2(0,T,L^2(\partial\Omega))$*, for* $p>3$ *the function* $\boldsymbol{g}\in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$ *and volume force* $\boldsymbol{f}\in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$*, also function* $\boldsymbol{G}(\cdot,\cdot)$ *satisfies the same conditions as in Assumption 1. Then there exists a weak solution to system* (1.1)*.*

The idea of proof is similar as in [18]. We use Galerkin approximations. Usage of growth condition in Orlicz spaces instead of growth condition in Lebesgue spaces implies utilization of different analytic tools, e.g. Minty-Browder trick for Orlicz spaces, which appear here to be non-reflexive, to identify the weak limit of nonlinear term and biting limit to show the convergences in $L^1(Q)$ of right hand side in heat equation. Moreover, Young measures tools are used and some important lemmas for the Young measure are presented in Appendix C.

2. Proof of Theorem 1

The proof of Theorem 1 consist of few steps. Each steps is presented in separate section.

2.1. **Transformation to a homogeneous boundary-value-problem.** First step of the proof is to transform the system into homogeneous boundary-value problem. Construction of solution is more clear in this case. Moreover, we also cut off the right hand side function in the elastic problems. Thereby, instead of volume force and boundary values we receive the same influence of exterior by using the shifts of solutions. It allows us to focus on the important issues instead of the calculation difficulties.

Let us consider two independent systems of equations with given initial conditions and boundary data. The boundary conditions are the same as in (1.3).

$$
\left\{
\begin{array}{rcll}
-\mathrm{div}\,\tilde{\boldsymbol{T}} &=& \boldsymbol{f} & \text{in } \Omega\times(0,T),\\
\tilde{\boldsymbol{T}} &=& \boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) & \text{in } \Omega\times(0,T),\\
\tilde{\boldsymbol{u}} &=& \boldsymbol{g} & \text{on } \partial\Omega\times(0,T),
\end{array}
\right.
\tag{2.1}
$$

and

$$
\left\{
\begin{array}{rcll}
\tilde\theta_t - \Delta\tilde\theta &=& 0 & \text{in } \Omega\times(0,T),\\
\frac{\partial\tilde\theta}{\partial\boldsymbol{n}} &=& g_\theta & \text{on } \partial\Omega\times(0,T),\\
\tilde\theta(x,0) &=& \tilde\theta_0 & \text{in } \Omega.
\end{array}
\right.
\tag{2.2}
$$

Lemma 1. *For* $p>3$*, let* $\tilde\theta_0\in L^2(\Omega)$*,* $\boldsymbol{g}\in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$*,* $g_\theta\in L^2(0,T,L^2(\partial\Omega))$ *and* $\boldsymbol{f}\in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$*. Then there exists a solution to systems* (2.1) *and* (2.2)*. Additionally, the following estimates hold:*

$$
\|\tilde{\boldsymbol{u}}\|_{L^\infty(0,T,W^{2,p}(\Omega))} \le C_1\left(\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^p(\Omega))}\right),
$$

$$
\|\tilde\theta\|_{L^\infty(0,T,L^1(\Omega))} + \|\tilde\theta\|_{L^2(0,T,W^{1,2}(\Omega))} \le C_2\left(\|g_\theta\|_{L^2(0,T,L^2(\partial\Omega))} + \|\tilde\theta_0\|_{L^2(\Omega)}\right).
$$

Moreover, the following estimate holds for the Cauchy stress tensor

$$\|\tilde{\boldsymbol{T}}\|_{L^\infty(Q)} \le C_3 \left(\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^{\cdot p}(\Omega))} \right). \tag{2.3}$$

Results for temperature are straightforward, hence let us discuss only existence of solution to the elastic system of equations.

Proof. Rewriting the solution in the form $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_1 + \boldsymbol{g}$, instead of looking for $\tilde{\boldsymbol{u}}$ we may search for $\tilde{\boldsymbol{u}}_1$, where $\tilde{\boldsymbol{u}}_1$ is a solution of the system

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_1) & = & \boldsymbol{f} + \mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g}) & \text{in } \Omega \times (0,T), \\ \tilde{\boldsymbol{u}}_1 & = & 0 & \text{on } \partial\Omega \times (0,T), \end{array} \right. \tag{2.4}$$

where function $\boldsymbol{f} + \mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g})$ belongs to $L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$. By [37, Theorem 7.1] we know that there exists an unique solution to elasticity problem. By condition $p > 3$ and by using the general Sobolev inequalities [15, Theorem 6, p. 270] we obtain the inequality (2.3). This estimate is crucial in the next steps of the proof. □

Instead of finding $(\widehat{\boldsymbol{u}}, \widehat{\theta})-$ the solution to problem (1.1)-(1.2)-(1.3) we shall search for $(\boldsymbol{u}, \theta)$, where $\boldsymbol{u} = \widehat{\boldsymbol{u}} - \tilde{\boldsymbol{u}}$ and $\theta = \widehat{\theta} - \tilde{\theta}$ where $(\tilde{\boldsymbol{u}}, \tilde{\theta})$ solves (2.1) and (2.2). Furthermore, we consider the following system of equations

$$\left\{ \begin{array}{l} -\mathrm{div}\,\boldsymbol{T} = 0, \\ \qquad \boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \\ \qquad \boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(\theta + \tilde{\theta}, \boldsymbol{T}^d + \tilde{\boldsymbol{T}}^d), \\ \theta_t - \Delta\theta = \left(\boldsymbol{T}^d + \tilde{\boldsymbol{T}}^d\right) : \boldsymbol{G}(\theta + \tilde{\theta}, \boldsymbol{T}^d + \tilde{\boldsymbol{T}}^d), \end{array} \right. \tag{2.5}$$

with boundary and initial conditions:

$$\left\{ \begin{array}{rcll} \boldsymbol{u} & = & 0 & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial\theta}{\partial\boldsymbol{n}} & = & 0 & \text{on } \partial\Omega \times (0,T), \\ \theta(\cdot,0) & = & \widehat{\theta}_0 - \tilde{\theta}_0 \equiv \theta_0 & \text{in } \Omega, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(\cdot,0) & = & \boldsymbol{\varepsilon}^{\mathbf{p}}_0 & \text{in } \Omega, \end{array} \right. \tag{2.6}$$

where $\widehat{\theta}_0$ is an initial condition for whole temperature and $\tilde{\theta}_0$ is the initial condition for the system (2.2).

2.2. **Approximate solution.** Construction of approximate solutions does not differ from the one presented in [18]. Let us define the standard truncation operator $\mathcal{T}_k(\cdot)$ by

$$\mathcal{T}_k(x) = \left\{ \begin{array}{cl} k & x > k \\ x & |x| \le k \\ -k & x < -k, \end{array} \right. \tag{2.7}$$

for $k \in \mathbb{N}$. Use of truncation is implied only by integrability of the right hand side of the heat equation and initial condition for temperature. In the proof of solutions' existence we use the truncations of solution as a test function. This truncation does not need to be a linear combination of basis functions. Thus, we use two level approximation, i.e. independent approximation parameters in the displacement and temperature. Due to this construction the limit passage in each approximation level may be done independently. As the first step we pass to the limit with temperature approximations parameter, i.e. with $l \to \infty$, and latterly we pass to the limit with displacement approximation parameter. Moreover, we construct the approximate solution for visco-elastic strain tensor. After the first limit passage the visco-elastic strain tensor is an infinite dimensional approximation. The low regularity of data implies that the second limit passage requires closer attention.

Construction of approximate solution requires usage of three different bases, i.e. bases for temperature, displacement and visco-elastic strain.

Let $\{v_i\}_{i=1}^\infty$ be the set of eigenfunctions of Laplace operator with the domain $W_n^{1,2}(\Omega) = \{v \in W^{1,2}(\Omega) : \frac{\partial v}{\partial \boldsymbol{n}} = 0\}$. Let $\{\mu_i\}$ be the set of corresponding eigenvalues, let $\{v_i\}$ be orthogonal in $W_n^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$.

To construct the basis functions for approximation let us start from considering the space $L^2(\Omega, \mathcal{S}^3)$ with the scalar product defined by

$$(\boldsymbol{\xi}, \boldsymbol{\eta})_{\boldsymbol{D}} := \int_\Omega \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\xi} \cdot \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\eta} \,\mathrm{d}x \quad \text{for } \boldsymbol{\xi}, \boldsymbol{\eta} \in L^2(\Omega, \mathcal{S}^3) \tag{2.8}$$

where $\boldsymbol{D}^{\frac{1}{2}} \circ \boldsymbol{D}^{\frac{1}{2}} = \boldsymbol{D}$. Moreover, let $\{\boldsymbol{w}_i\}_{i=1}^\infty$ be the set of eigenfunctions of elasto-static operator $-\mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\cdot)$ with the domain $W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\{\lambda_i\}$ be the corresponding eigenvalues such that $\{\boldsymbol{w}_i\}$ is orthogonal in $W_0^{1,2}(\Omega, \mathbb{R}^3)$ with the inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{W_0^{1,2}(\Omega)} = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\boldsymbol{D}} \tag{2.9}$$

and orthonormal in $L^2(\Omega, \mathbb{R}^3)$. Moreover, $\|\cdot\|_{\boldsymbol{D}}$ is a norm of $L^2(\Omega, \mathcal{S}^3)$, i.e.

$$\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{\boldsymbol{D}}^2 = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\boldsymbol{D}}. \tag{2.10}$$

Furthermore, by using the symmetry of operator $\boldsymbol{D}$ the following equality holds for basis functions $\boldsymbol{w}_i, \boldsymbol{w}_j$

$$\int_\Omega \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_i) : \boldsymbol{\varepsilon}(\boldsymbol{w}_j) \,\mathrm{d}x = \lambda_i \int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \,\mathrm{d}x = 0, \tag{2.11}$$

when $i \neq j$.

The idea of constructing visco-elastic strain approximations was presented in [18] and we hereby refer the reader to this paper for more details. We observe that $\boldsymbol{\varepsilon}(\boldsymbol{w}_i)$ are elements of $H^s(\Omega, \mathcal{S}^3)$ by regularity of eigenfunctions, where $H^s(\Omega, \mathcal{S}^3)$ is a fractional Sobolev space with scalar product denoted by $(\!(\cdot,\cdot)\!)_s$ and $s > \frac{3}{2}$. Let us define the orthogonal complement in $L^2(\Omega, \mathcal{S}^3)$

$$V_k := (\mathrm{span}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), ..., \boldsymbol{\varepsilon}(\boldsymbol{w}_k)\})^\perp, \tag{2.12}$$

taken with respect to the scalar product $(\cdot,\cdot)_{\boldsymbol{D}}$. Moreover, let us define

$$V_k^s := V_k \cap H^s(\Omega, \mathcal{S}^3). \tag{2.13}$$

Using [30, Theorem 4.11], which was also used in [18], there exists an orthonormal basis $\{\boldsymbol{\zeta}_n^k\}_{n=1}^\infty$ of V_k which is also an orthogonal basis of V_k^s. The basis for visco-elastic strain consists of two subsets. First subset is a set of first k symmetric gradients from the basis $\{\boldsymbol{w}_i\}_{i=1}^\infty$. The second subset consists of first l functions from $\{\boldsymbol{\zeta}_n^k\}_{n=1}^\infty$. Thus, for each step of approximation we use $k+l$ functions to construct visco-elastic strain.

For $k, l \in \mathbb{N}$ we define

$$\begin{aligned}
\boldsymbol{u}_{k,l} &= \sum_{n=1}^k \alpha_{k,l}^n(t)\boldsymbol{w}_n,\\
\theta_{k,l} &= \sum_{m=1}^l \beta_{k,l}^m(t) v_m,\\
\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l} &= \sum_{n=1}^k \gamma_{k,l}^n(t)\boldsymbol{\varepsilon}(\boldsymbol{w}_n) + \sum_{m=1}^l \delta_{k,l}^m(t)\boldsymbol{\zeta}_m^k,
\end{aligned} \tag{2.14}$$

such that $\boldsymbol{u}_{k,l}$, $\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}$ and $\theta_{k,l}$ solve the system of equations

$$
\begin{aligned}
\int_\Omega \boldsymbol{T}_{k,l} : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x &= 0 && n=1,...,k,\\
\boldsymbol{T}_{k,l} &= \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}),\\
\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x &= \int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x && n=1,...,k,\\
\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x &= \int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x && m=1,...,l,\\
\int_\Omega (\theta_{k,l})_t v_m\,\mathrm{d}x + \int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\,\mathrm{d}x & \\
= \int_\Omega \mathcal{T}_k((\boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d) : &\ \boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))v_m\,\mathrm{d}x && m=1,...,l.
\end{aligned}
\tag{2.15}
$$

for a.a. $t\in(0,T)$. Moreover, the solutions fulfill initial conditions in the following form

$$
\begin{cases}
(\theta_{k,l}(x,0), v_m) = (\mathcal{T}_k(\theta_0), v_m) & m=1,..,l,\\
\Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\Big)_{\boldsymbol{D}} = (\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\varepsilon}(\boldsymbol{w}_n))_{\boldsymbol{D}} & n=1,..,k,\\
\Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\zeta}^k_m)\Big)_{\boldsymbol{D}} = \Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\zeta}^k_m\Big)_{\boldsymbol{D}} & m=1,..,l,
\end{cases}
\tag{2.16}
$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $(\cdot,\cdot)_{\boldsymbol{D}}$ denotes the inner product in $L^2(\Omega,\mathcal{S}^3)$.

Let us notice that $\alpha^n_{k,l}(t) = \gamma^n_{k,l}(t)$ by the selection of Galerkin bases and representation of approximate solution (2.14). To present it more clearly let

$$
\boldsymbol{\xi}(t) = (\beta^1_{k,l}(t), ..., \beta^l_{k,l}(t), \gamma^1_{k,l}(t), ..., \gamma^k_{k,l}(t), \delta^1_{k,l}(t), ..., \delta^l_{k,l}(t))^T. \tag{2.17}
$$

Then, the system of equations (2.15) may be rewritten in the form of ODE's system

$$
\begin{cases}
(\gamma^n_{k,l}(t))_t = \dfrac{1}{\lambda_n}\displaystyle\int_\Omega \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x,\\
(\delta^m_{k,l}(t))_t = \displaystyle\int_\Omega \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x,\\
(\beta^m_{k,l}(t))_t = \displaystyle\int_\Omega \mathcal{T}_k\Big((-(\boldsymbol{D}\sum_{m=1}^{l}\delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m)^d + \tilde{\boldsymbol{T}}^d) : \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t))\Big)v_m\,\mathrm{d}x + \mu_m\beta^m_{k,l}(t),
\end{cases}
\tag{2.18}
$$

for $n=1,...,k$ and $m=1,...,l$, where

$$
\begin{aligned}
\tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) &:= \boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\\
&= \boldsymbol{G}\Big(\sum_{m=1}^{l}\beta^m_{k,l}(t)v_j(x) + \tilde\theta, -\boldsymbol{D}\Big(\sum_{m=1}^{l}\delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m\Big)^d + \tilde{\boldsymbol{T}}^d\Big).
\end{aligned}
$$

Hence, the existence of solution to approximate system is equivalent to existence of solution to the following ODE's system

$$
\begin{aligned}
&\frac{d\boldsymbol{\xi}}{dt} = \boldsymbol{F}(\boldsymbol{\xi}(t),t), \qquad t\in[0,T),\\
&\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0,
\end{aligned}
\tag{2.19}
$$

where $\boldsymbol{\xi}_0$ is a vector of initial conditions obtained from (2.16).

Lemma 2. *(Existence of approximate solution)*
For initial condition satisfying $\boldsymbol{\varepsilon}^{\mathbf{p}}_0 \in L_{M^*}(\Omega,\mathcal{S}^3_d)$ *and* $\theta_0\in L^1(\Omega)$ *there exists a local solution to* (2.19) *which is absolutely continuous in time.*

The proof of Lemma 2 is a consequence of application of Carathéodory Theorem, see [30, Theorem 3.4] or [39, Appendix (61)]. We obtain the existence of unique absolutely continuous solution for some time interval $[0,t^*]$.

2.3. **Boundedness of energy.** Since we consider the physical model, the total energy of system should be finite. Omission of the kinetic effect implies that the total energy consists of thermal energy and potential energy. We start with consideration devoted to potential energy. The part devoted to thermal energy estimates is similar to one presented in [18], hence we recall the lemmas without the proofs.

Definition 7. *We say that $\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}^{\mathbf{p}})$ is the potential energy if*

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}^{\mathbf{p}}) := \frac{1}{2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})\,\mathrm{d}x.$$

Lemma 3. *There exists a constant C (uniform with respect to k and l) such that*

$$\begin{aligned}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) &+ \frac{2c-d}{2}\int_Q M^*(x,\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\,\mathrm{d}t\\ &+ c\int_Q M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \le C,\end{aligned}\tag{2.20}$$

where c is a constant from Assumption 1 and $d=\min(1,c)$. Moreover, the constant C depends on solution of additional problem (2.1) *and potential energy at the initial time*

$$C=\int_Q M(x,\frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t+\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0).\tag{2.21}$$

Proof. Let us start with calculating the time derivative of the potential energy $\mathcal{E}(t)$. For a.a. $t\in[0,T]$ we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\,\mathrm{d}x\\ &-\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\,\mathrm{d}x.\end{aligned}$$

The terms on the right hand side of abovementioned equation may be rewritten with usage of approximate system of equations (2.15). Firstly, for each $n\le k$ let us multiply $(2.15)_1$ by $(\alpha^n_{k,l})_t$. After summing over $n=1,...,k$ we get

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\,\mathrm{d}x=0.\tag{2.22}$$

Then for each $n\le k$ let us multiply $(2.15)_3$ by $\gamma^n_{k,l}$ and for each $m\le l$ let us multiply $(2.15)_4$ by $\delta^n_{k,l}$. Summing over $n=1,..,k$ and $m=1,...,l$ we obtain

$$\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\mathrm{d}x=\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{T}_{k,l}\,\mathrm{d}x.\tag{2.23}$$

Hence

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = -\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{T}_{k,l}\,\mathrm{d}x,\tag{2.24}$$

and then using the property of traceless matrices we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &-\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\\ &+\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \tilde{\boldsymbol{T}}^d\,\mathrm{d}x.\end{aligned}$$

Thus, using Assumption 1 and Fenchel-Young inequality we estimate the changes of potential energy by

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) \le &-c\left(\int_\Omega M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x+\int_\Omega M^*(x,\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\right)\\ &+\int_\Omega M(x,\frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x+\int_\Omega M^*(x,\frac{d}{2}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x,\end{aligned}$$

where $d = \min(1, c)$. Then by convexity of N-function we obtain

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) \leq -c\left(\int_\Omega M(x, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x + \int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\right) + \int_\Omega M(x, \frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x + \frac{d}{2}\int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x.$$

Finally, after integration over time interval $(0, t)$, with $0 \leq t \leq T$ we obtain

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + c\int_Q M(x, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t + \frac{2c - d}{2}\int_Q M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q M(x, \frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0),$$

which completes the proof. □

Remark. *From Lemma 3 we know that the sequence* $\{\boldsymbol{T}^d_{k,l}\}$ *is uniformly bounded in* $L_M(Q, \mathcal{S}^3)$ *with respect to* k *and* l*, also the sequence* $\{\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ *in the space* $L_{M^*}(Q, \mathcal{S}^3)$ *with respect to* k *and* l*. Hence using the Fenchel-Young inequality the sequence* $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ *is uniformly bounded in* $L^1(Q)$*.*

Remark. *From Lemma 3 we know that the sequence* $\{\boldsymbol{T}_{k,l}\}$ *is uniformly bounded in* $L^\infty(0, T, L^2(\Omega, \mathcal{S}^3))$ *and in particular in* $L^2(0, T, L^2(\Omega, \mathcal{S}^3))$*.*

Let P^l be a projection from $H^s(\Omega)$ on $\mathrm{lin}\{\boldsymbol{\zeta}^k_1, \ldots, \boldsymbol{\zeta}^k_l\}$, defined by $P^l(\boldsymbol{v}) := \sum_{i=1}^l (\boldsymbol{v}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}}\boldsymbol{\zeta}_i$, then $\|P^l\varphi\|_{H^s} \leq \|\varphi\|_{H^s}$. Furthermore, let P^k be a projection from $H^s(\Omega)$ on $\mathrm{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), \ldots, \boldsymbol{\varepsilon}(\boldsymbol{w}_1)\}$, defined by $P^k(\boldsymbol{v}) := \sum_{i=1}^k (\boldsymbol{v}, \boldsymbol{\varepsilon}(\boldsymbol{w}_i))_{\boldsymbol{D}}\boldsymbol{\varepsilon}(\boldsymbol{w}_i)$. Since P^k is the projection of a finite dimensional space, and the dimension of this space is independent of l, there exists a constant, also independent of l such that $\|P^k\varphi\|_{H^s} \leq c\|\varphi\|_{H^s}$.

Lemma 4. *The sequence* $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ *is uniformly bounded in* $L^1(0, T, (H^s(\Omega, \mathcal{S}^3))')$ *with respect to* k *and* l*.*

Proof. Let $\varphi \in L^\infty(0, T, H^s(\Omega, \mathcal{S}^3))$ and we may estimate as follows

$$\begin{aligned}\int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \varphi\rangle|\,\mathrm{d}t &= \int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, (P^k + P^l)\varphi\rangle|\,\mathrm{d}t \\ &\leq \int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^k\varphi\rangle|\,\mathrm{d}t + \int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^l\varphi\rangle|\,\mathrm{d}t,\end{aligned} \tag{2.25}$$

where the equality results from orthogonality of subspaces $\mathrm{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), \ldots, \boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}$ and $\mathrm{lin}\{\boldsymbol{\zeta}^k_1, \ldots, \boldsymbol{\zeta}^k_l\}$. Then

$$\begin{aligned}\int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \varphi\rangle|\,\mathrm{d}t &\leq \int_0^T |\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})P^k\varphi\,\mathrm{d}x|\,\mathrm{d}t \\ &\quad + \int_0^T |\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})P^l\varphi\,\mathrm{d}x|\,\mathrm{d}t \\ &\leq \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\varphi\|_{L^\infty(\Omega)}\,\mathrm{d}t \\ &\quad + \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^l\varphi\|_{L^\infty(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{2.26}$$

Hence $s > \frac{3}{2}$ and by Sobolev inequality $\|P^l\varphi\|_{L^\infty(\Omega)} \leq \tilde{c}\|P^l\varphi\|_{H^s(\Omega)}$ and $\|P^k\varphi\|_{L^\infty(\Omega)} \leq \tilde{c}\|P^k\varphi\|_{H^s(\Omega)}$, where $\tilde{c}$ is an optimal embedding constant. Then

$$
\begin{aligned}
\int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\varphi\rangle|\,\mathrm{d}t &\le \tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\varphi\|_{H^s(\Omega)}\,\mathrm{d}t \\
&\quad + \tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^l\varphi\|_{H^s(\Omega)}\,\mathrm{d}t \\
&\le c\tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\varphi\|_{H^s(\Omega)}\,\mathrm{d}t \\
&\quad + \tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\varphi\|_{H^s(\Omega)}\,\mathrm{d}t \\
&\le (1+c)\tilde{c}\|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(Q)}\|\varphi\|_{L^\infty(0,T,H^s(\Omega))}.
\end{aligned}
\tag{2.27}
$$

It is obvious that $\|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(Q)}$ is bounded. Hence there exists $C>0$ such that

$$
\sup_{\substack{\varphi\in L^\infty(0,T,H^s(\Omega))\\ \|\varphi\|_{L^\infty(0,T,H^s(\Omega))}\le 1}} \int_0^T |\langle(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\varphi\rangle|\,\mathrm{d}t \le C
\tag{2.28}
$$

and hence the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^1(0,T,(H^s(\Omega,\mathcal{S}^3))')$. □

The remaining part is to consider the internal energy of the system. Two following lemmas come from [18].

Lemma 5. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ with respect to k and l.*

Lemma 6. *There exists a constant C, depending on domain Ω and time interval $(0,T)$, such that for every $k\in\mathbb{N}$*

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\theta_{k,l}(t)\|^2_{L^2(\Omega)}+\|\theta_{k,l}\|^2_{L^2(0,T,W^{1,2}(\Omega))}+\|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \\
&\qquad \le C\Big(\|\mathcal{T}_k\Big((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\Big)\|^2_{L^2(0,T,L^2(\Omega))}+\|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\Big).
\end{aligned}
\tag{2.29}
$$

Estimates in Lemma 6 depend on k. To complete this Section we observe that the uniform boundedness of solutions implies the global existence of approximate solutions. For each $n=1,...,k$ and $m=1,...,l$ the solutions $\{\alpha^n_{k,l}(t),\beta^m_{k,l}(t),\gamma^n_{k,l}(t),\delta^m_{k,l}(t)\}$ exist on the whole time interval $[0,T]$.

2.4. **Limit passage $l\to\infty$ and uniform estimates.** Multiplying (2.15) by time dependent test functions $\varphi_1(t),\varphi_2(t),\varphi_3(t)\in C^\infty([0,T])$ and $\varphi_4(t)\in C^\infty_c([-\infty,T])$ and then after integration over time interval $[0,T]$, we obtain the following system of equations

$$
\begin{aligned}
&\int_0^T\int_\Omega \boldsymbol{T}_{k,l}:\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t=0 \\
&\boldsymbol{T}_{k,l}=\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}), \\
&\int_0^T\int_\Omega(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t:\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\int_0^T\int_\Omega(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t:\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T\int_\Omega\theta_{k,l}v_m\varphi_4'(t)\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\theta_{0,k,l}(x)v_m\varphi_4(0)\,\mathrm{d}x+\int_0^T\int_\Omega\nabla\theta_{k,l}\cdot\nabla v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad=\int_0^T\int_\Omega\mathcal{T}_k((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.30}
$$

where the fist and the third equation holds for $n=1,...,k$ and the fourth and the fifth holds for $m=1,...,l$.

Firstly, we pass to the limit with $l \to \infty$ - Galerkin approximation of temperature. From the previous section we get uniform boundedness with respect to l for appropriate sequences. Using appropriate subsequence, but still denoted by the indexes k and l, we get the following convergences

$$
\begin{array}{ll}
\boldsymbol{T}_{k,l} \rightharpoonup \boldsymbol{T}_k & \text{weakly in } L^2(Q,\mathcal{S}^3),\\
\boldsymbol{T}^d_{k,l} \rightharpoonup^* \boldsymbol{T}^d_k & \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\
\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) \rightharpoonup^* \boldsymbol{\chi}_k & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\
\theta_{k,l} \rightharpoonup \theta_k & \text{weakly in } L^2(0,T,W^{1,2}(\Omega)),\\
\theta_{k,l} \to \theta_k & \text{a.e. in } \Omega\times(0,T),\\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t \rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t & \text{weakly in } L^1(0,T,(H^s(\Omega,\mathcal{S}^3))').
\end{array}
\tag{2.31}
$$

We pass to the limit with $l \to \infty$ in (2.30) using convergences from (2.31). For $n = 1,...,k$ and $m \in \mathbb{N}$ the following equations hold

$$
\begin{aligned}
\int_0^T\int_\Omega \boldsymbol{T}_k : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t &= 0,\\
\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t,\\
\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.32}
$$

Moreover, $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_n),\boldsymbol{\zeta}_m\}_{n=1,..,k;m=1,..,\infty}$ is a base of whole space $H^s(\Omega,\mathcal{S}^3)$ then $(2.32)_{(2)}$ and $(2.32)_{(3)}$ equations can be replaced by

$$
\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\zeta}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\zeta}\,\mathrm{d}x\,\mathrm{d}t
\tag{2.33}
$$

for $\boldsymbol{\zeta} \in L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))$. To show that the (2.33) holds also for all $\boldsymbol{\zeta} \in L_M(Q,\mathcal{S}^3)$ we proceed similar as in [22, 23].

To complete the limit passage in heat equation $(2.30)_{(5)}$ we encounter the same problem as in [18], but here we should use different technique. It holds due to utilization the generalized Orlicz spaces instead of Lebesgue spaces. As previously, we may precisely consider the right hand side of $(2.30)_{(5)}$ and this reasoning consists of three steps. The first step is to show the inequality in the Lemma 7. The second step is to identify the weak limit of nonlinear term by using Minty-Browder trick. For Minty-Browder trick in nonreflexive spaces we refer the reader to [38]. And finally, the last step is to show the convergence of right hand side of heat equation. We present detailed calculation for these steps below.

Step 1. *Limiting inequality.*

Lemma 7. *The following inequality holds for the solution of approximate systems.*

$$
\limsup_{l\to\infty}\int_0^t\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^t\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k\,\mathrm{d}x\,\mathrm{d}t.
\tag{2.34}
$$

Proof. Let us start with the definition of function $\psi_{\mu,\tau} : \mathbb{R}_+ \to \mathbb{R}_+$. For each $\mu > 0, \tau \le T-\mu, s \ge 0$, the function $\psi_{\mu,\tau}$ is defined by

$$
\psi_{\mu,\tau}(s) = \begin{cases} 1 & \text{for } \ s \in [0,\tau),\\ -\frac{1}{\mu}s+\frac{1}{\mu}\tau+1 & \text{for } \ s\in[\tau,\tau+\mu),\\ 0 & \text{for } \ s \ge \tau+\mu. \end{cases}
\tag{2.35}
$$

We use $\psi_{\mu,\tau}(t)$ as a test function in (2.24), then after integration over time interval $(0,T)$ we get

$$
\int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,\mathrm{d}t \ = -\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t.
\tag{2.36}
$$

Integrating by parts the left hand side of (2.36) we obtain

$$\int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,\mathrm{d}t = \frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t)\,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0). \tag{2.37}$$

Passing to the limit with $l\to\infty$ and using the lower semicontinuity in $L^2(0,T,L^2(\Omega;\mathcal{S}^3))$ we get

$$\begin{aligned}
&\liminf_{l\to\infty}\int_0^T \frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,\mathrm{d}t\\
&\qquad = \liminf_{l\to\infty}\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t)\,\mathrm{d}t - \lim_{l\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0)\\
&\qquad \geq \frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(t)\,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(0).
\end{aligned} \tag{2.38}$$

Comparing (2.38) and (2.36) we obtain

$$\begin{aligned}
&\liminf_{l\to\infty}\left(-\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\qquad = \liminf_{l\to\infty}\int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,\mathrm{d}t\\
&\qquad \geq \frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(t)\,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(0),
\end{aligned} \tag{2.39}$$

which is equivalent to

$$\begin{aligned}
&\limsup_{l\to\infty}\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\;\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \leq -\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(t)\,\mathrm{d}t + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k})(0).
\end{aligned} \tag{2.40}$$

Since $(\alpha_k^n)_t$ is not regular enough to use as a test function in $(2.15)_{(1)}$ we may use the mollifier to improve its regularity. Thus let η_ϵ be a standard mollifier and we mollify with respect to time. Then let us choose $\varphi_1(t) = ((\alpha_k^n)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in $(2.32)_{(1)}$ and $\boldsymbol{\zeta} = (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (2.33), then

$$\begin{aligned}
&\int_Q \boldsymbol{T}_k : \boldsymbol{\varepsilon}(((\alpha_k^n)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon \boldsymbol{w}_n)\,\mathrm{d}x\,\mathrm{d}t = 0,\\
&\int_Q (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{\chi}_k : (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.41}$$

for $n = 1,...,k$. Summing $(2.41)_{(1)}$ over $n = 1,...,k$ we obtain

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.42}$$

Moreover, using properties of traceless matrices

$$\int_{t_1}^{t_2}\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k * \eta_\epsilon)_t : \boldsymbol{T}_k * \eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}_k * \eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t, \tag{2.43}$$

and products in (2.43) are well defined. Since for the matrices $\boldsymbol{A}\in\mathcal{S}^3_d$ and $\boldsymbol{B}\in\mathcal{S}^3$ the equivalence $\boldsymbol{A}:\boldsymbol{B}^d = \boldsymbol{A}:\boldsymbol{B}$ holds and the sequence $\{\boldsymbol{T}^d_k\}$ is uniformly bounded in $L_M(Q,\mathcal{S}^3_d)$. Subtracting (2.42) and (2.43) we get

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon : ((\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon)_t\,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}^d_k * \eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{2.44}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k$ belongs to $L^2(Q,\mathcal{S}^3)$ we may pass to the limit with $\epsilon\to 0$ in the left hand side of equation (2.44).

To make a limit passage with $\epsilon \to 0$ on the right hand side of (2.44) we use the lemmas presented in Appendix B. From Lemma 22, we know that sequences $\{M(x, \boldsymbol{T}_k^d * \eta_\epsilon)\}$ and $\{M^*(x, \boldsymbol{\chi}_k * \eta_\epsilon)\}$ are uniformly integrable. Moreover, $\{\boldsymbol{T}_k^d * \eta_\epsilon\}_\epsilon$ converge in measure to $\boldsymbol{T}_k^d$ and $\{\boldsymbol{\chi}_k * \eta_\epsilon\}_\epsilon$ convergence in measure to $\boldsymbol{\chi}_k$ (by Lemma 21) as ϵ goes to 0. Uniform integrability of the sequence and convergence in measure of this sequence imply (by Lemma 18) that

$$\begin{aligned} \boldsymbol{T}_k^d * \eta_\epsilon &\xrightarrow{M} \boldsymbol{T}_k^d \qquad \text{modularly in } L_M(Q), \\ \boldsymbol{\chi}_k * \eta_\epsilon &\xrightarrow{M^*} \boldsymbol{\chi}_k \qquad \text{modularly in } L_{M^*}(Q), \end{aligned} \tag{2.45}$$

as $\epsilon \to 0$. On the basis of Lemma 20 we complete the limit passage on the right hand side of (2.44). Then we obtain the following equality

$$\frac{1}{2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}})\,\mathrm{d}x\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \,\mathrm{d}x\,\mathrm{d}t. \tag{2.46}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}} \in C_w([0,T], L^2(\Omega, \mathcal{S}^3))$, then we may pass with $t_1 \to 0$ and conclude

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t_2) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) = -\int_0^{t_2}\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \,\mathrm{d}x\,\mathrm{d}t. \tag{2.47}$$

To complete the proof let us multiply (2.47) by $\frac{1}{\mu}$ and integrate over the interval $(\tau, \tau+\mu)$.

$$\frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(s)\,\mathrm{d}s - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) = -\frac{1}{\mu}\int_\tau^{\tau+\mu}\int_0^s\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{2.48}$$

For conciseness let use define the function

$$F(t) := \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \,\mathrm{d}x \tag{2.49}$$

which belongs to $L^1(0,T)$. Then applying the Fubini theorem we obtain

$$\begin{aligned} \frac{1}{\mu}\int_\tau^{\tau+\mu}\int_0^s F(t)\,\mathrm{d}t\,\mathrm{d}s &= \frac{1}{\mu}\int_{\mathbb{R}^2} \mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le \tau+\mu\}}(s)F(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\left(\frac{1}{\mu}\int_{\mathbb{R}} \mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le \tau+\mu\}}(s)\,\mathrm{d}s\right)F(t)\,\mathrm{d}t. \end{aligned} \tag{2.50}$$

Using the definition of function $\psi_{\mu,\tau}$ we observe that

$$\psi_{\mu,\tau}(t) = \frac{1}{\mu}\int_{\mathbb{R}} \mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le \tau+\mu\}}(s)\,\mathrm{d}s. \tag{2.51}$$

Hence, comparing (2.40) and (2.48) we obtain

$$\limsup_{l\to\infty}\int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\ \psi_{\mu,\tau}(t)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d\ \psi_{\mu,\tau}(t)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.52}$$

To complete the proof of Lemma 7 let us show the following estimates

$$
\begin{aligned}
&\limsup_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \limsup_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \limsup_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \limsup_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\;\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad+\lim_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\;\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \int_0^{t_2+\mu}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad+\lim_{l\to\infty}\int_{t_2}^{t_2+\mu}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^{t_2+\mu}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\;\psi_{\mu,t_2}(t)\,\mathrm{d}x\,\mathrm{d}t+\lim_{l\to\infty}\int_{t_2}^{t_2+\mu}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d_k\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.53}
$$

Passing with $\mu\to 0$ yields (2.34). The proof is complete. □

Step 2. *Minty-Browder trick.*

We use the Minty-Browder trick to identify the weak limits $\boldsymbol{\chi}_k$. For $s\in(0,T]$ let us define $Q^s=\Omega\times(0,s)$. From the monotonicity of function $\boldsymbol{G}(\theta,\cdot)$ we obtain

$$
\int_{Q^s}\Big(\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\Big):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)\,\mathrm{d}x\,\mathrm{d}t\ge 0 \qquad \forall\ \boldsymbol{W}^d\in L^\infty(Q,\mathcal{S}^3_d).
\tag{2.54}
$$

Repeating the procedure from the proof of Lemma 7 we obtain

$$
\limsup_{n\to\infty}\int_{Q^s}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,\mathrm{d}x\,\mathrm{d}t\le\int_{Q^s}\boldsymbol{\chi}_k:\boldsymbol{T}^d_k\,\mathrm{d}x\,\mathrm{d}t.
\tag{2.55}
$$

Moreover, using (2.31) we get

$$
\lim_{n\to\infty}\int_{Q^s}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{W}^d\,\mathrm{d}x\,\mathrm{d}t=\int_{Q^s}\boldsymbol{\chi}_k:\boldsymbol{W}^d\,\mathrm{d}x\,\mathrm{d}t.
\tag{2.56}
$$

Pointwise convergence of $\{\theta_{k,l}\}$ implies the pointwise convergence of $\{\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\}$. Furthermore, from the Assumption 1 and non-negativity of N-functions we get

$$
|\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d|\ge c\frac{M^*(x,\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d))}{|\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)|}.
\tag{2.57}
$$

Since $\tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d$ belongs to $L^\infty(Q,\mathcal{S}^3_d)$ and M^* is an N-function then the sequence $\{\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\}$ belongs to $L^\infty(Q,\mathcal{S}^3_d)$. By Lemma 18 we obtain

$$\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\xrightarrow{M^*}\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d), \tag{2.58}$$

modularly in $L_{M^*}(Q)$. Then

$$\begin{aligned}\int_Q &|\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W})-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k}-\boldsymbol{W}^d)|\,\mathrm{d}x\,\mathrm{d}t\\ &\le\int_Q|(\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)|\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_Q|\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_{k})|\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{2.59}$$

Using Hölder inequality (Lemma 17) we get

$$\begin{aligned}\int_Q &|\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W})-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k}-\boldsymbol{W}^d)|\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2\|\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\|_{L,M^*}\|\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d\|_{L,M}\\ &\quad+\int_Q|\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_{k})|\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{2.60}$$

Since $\|\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d\|_{L,M}$ is uniformly bounded, $\|\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\|_{L,M^*}\to 0$ and $\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_{k}\rightharpoonup 0$ in $L_M(Q,\mathcal{S}^3_d)$ as l goes to ∞, then the right hand side of (2.60) goes to zero as l goes to ∞. Hence

$$\lim_{l\to\infty}\int_{Q^s}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)\,\mathrm{d}x\,\mathrm{d}t=\int_{Q^s}\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k}-\boldsymbol{W}^d)\,\mathrm{d}x\,\mathrm{d}t\tag{2.61}$$

Summing up, passing to the limit with $l\to\infty$ in (2.54), we get

$$\int_{Q^s}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\Big):(\boldsymbol{T}^d_{k}-\boldsymbol{W}^d)\,\mathrm{d}x\,\mathrm{d}t\ge 0\qquad\forall\boldsymbol{W}^d\in L^\infty(Q^s,\mathcal{S}^3_d).\tag{2.62}$$

For $i>0$ let us define the set

$$Q_i=\{(t,x)\in Q^s:\ |\boldsymbol{T}^d_k|\le i\text{ a.e. in }Q^s\}.\tag{2.63}$$

Then for $0<j<i$ and for arbitrary $h>0$ we define the function

$$\boldsymbol{W}^d=-\tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}+\boldsymbol{T}^d_k\mathbf{1}_{Q_i}+h\boldsymbol{U}^d\mathbf{1}_{Q_j}\tag{2.64}$$

where $\boldsymbol{U}^d\in L^\infty(Q,\mathcal{S}^3_d)$ and $\mathbf{1}_H$ is a characteristic function of set H. Using the function defined in (2.64) as a test function in (2.62) we obtain

$$\begin{aligned}\int_{Q^s}&\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d-\tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}+\boldsymbol{T}^d_k\mathbf{1}_{Q_i}+h\boldsymbol{U}^d\mathbf{1}_{Q_j})\Big):\\ &(\boldsymbol{T}^d_k+\tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}-\boldsymbol{T}^d_k\mathbf{1}_{Q_i}-h\boldsymbol{U}^d\mathbf{1}_{Q_j})\,\mathrm{d}x\,\mathrm{d}t\ge 0\end{aligned}\tag{2.65}$$

Since $Q_j\subset Q_i\subset Q^s$ we get

$$\begin{aligned}-h&\int_{Q_j}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q_i\setminus Q_j}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):(\boldsymbol{T}^d_k-\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{Q^s\setminus Q_i}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_{k},\mathbf{0})\Big):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t\ge 0.\end{aligned}\tag{2.66}$$

Using Assumption 1 we obtain

$$M^*(x,\boldsymbol{G}(\tilde\theta+\theta_k,\mathbf{0}))=0\tag{2.67}$$

and then, using Definition 1, we get that $\boldsymbol{G}(\tilde\theta+\theta_k,\mathbf{0})=0$. Hence

$$-h\int_{Q_j}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t+\int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t\ge 0. \tag{2.68}$$

Moreover, from the definition of characteristic function

$$\int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\boldsymbol{T}^d_k+\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t=\int_Q\Big(\boldsymbol{\chi}_k:(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big)\mathbf{1}_{Q^s\setminus Q_i}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.69}$$

Since $\int_Q\boldsymbol{\chi}_k:(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)<\infty$ and $\Big(\boldsymbol{\chi}_k:(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big)\mathbf{1}_{Q^s\setminus Q_i}\to 0$ a.e. in Q as i goes to ∞, the Lebesgue dominated convergence theorem implies that

$$\lim_{i\to\infty}\int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{2.70}$$

Passing to the limit with i going to ∞ in (2.68) and dividing by h we obtain

$$\int_{Q_j}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t\le 0. \tag{2.71}$$

Since $\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d$ goes to $\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k$ a.e. in Q when $h\to 0^+$, $\{\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\}_{h>0}$ is uniformly bounded in $L_{M^*}(Q_j,\mathcal{S}^3)$, we conclude that

$$\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\rightharpoonup^*\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) \tag{2.72}$$

in $L_{M^*}(Q_j,\mathcal{S}^3)$ as h goes to 0^+. Consequently, passing to the limit with h going to ∞ in (2.71) we obtain

$$\int_{Q_j}\Big(\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t\le 0, \tag{2.73}$$

for all $\boldsymbol{U}^d\in L^\infty(Q,\mathcal{S}^3_d)$, so taking

$$\boldsymbol{U}^d=\begin{cases}\frac{\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)}{|\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)|} & \text{when } \boldsymbol{\chi}_k\neq\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k),\\ 0 & \text{when } \boldsymbol{\chi}_k=\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k),\end{cases} \tag{2.74}$$

we obtain

$$\int_{Q_j}|\boldsymbol{\chi}_k-\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)|\,\mathrm{d}x\,\mathrm{d}t\le 0, \tag{2.75}$$

i.e. $\boldsymbol{\chi}_k=\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ a.e. in Q^s. From the arbitrary choices of $j>0$ and $0\le s\le T$ we get $\boldsymbol{\chi}_k=\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ a.e. in Q.

Step 3. *Limit of the right hand side of the heat equations.*

The idea for third step came from paper of Gwiazda et al. [26]. Let us start from the formulation of auxiliary lemmas which may be found with the proof in [26]. We denote by $\xrightarrow{b}$ the biting limit used in Chacon's, cf. [4].

Definition 8. *Biting limit*

Let $\{f^\nu\}$ be a bounded sequence in $L^1(Q)$. We say that $f\in L^1(Q)$ is a biting limit of subsequence $\{f^\nu\}$ if there exist nonincreasing sequence $\{E_k\}$ with $E_k\subset Q$ and $\lim_{k\to\infty}|E_k|=0$, such that f^ν convergence weakly to f in $L^1(Q\setminus E_k)$ for fixed k.

Lemma 8. *Let $a_n\in L^1(Q)$ and let $0\le a_0\in L^1(Q)$ and*

$$a_n\ge -a_0,\quad a_n\xrightarrow{b}a\quad\text{and}\quad\limsup_{n\to\infty}\int_Q a_n\,\mathrm{d}x\,\mathrm{d}t\le\int_Q a\,\mathrm{d}x\,\mathrm{d}t \tag{2.76}$$

then

$$a_n\rightharpoonup a\quad\text{weakly in } L^1(Q). \tag{2.77}$$

Lemma 9. *For each $k \in \mathbb{N}$ the sequence $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}_{l=1}^{\infty}$ converges weakly to $\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ in $L^1(Q)$.*

Proof. To characterized the limit of right hand side we use the same argumentation as in [26]. Using the Assumption 1, Frechet-Young inequality and convexity of N-functions, we get

$$\begin{aligned} &c(M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)+M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)))\\ &\quad\le \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\\ &\quad\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))+M^*(x,\frac{d}{2}\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)))\\ &\quad\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))+\frac{d}{2}M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))), \end{aligned} \tag{2.78}$$

where $d=\min(c,1)$. And finally

$$cM(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)+\frac{2c-d}{2}M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)). \tag{2.79}$$

Hence the sequence $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\}_{l=1}^{\infty}$ is uniformly bounded in $L_{M^*}(Q)$. Using the monotonicity of function $\boldsymbol{G}(\cdot,\cdot)$ with respect to the second variable, we get

$$0\le\Big(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k). \tag{2.80}$$

The right hand side of abovmentioned inequality is uniformly bounded in $L^1(Q)$. Thus, there exists a Young measure denoted by $\mu_{x,t}(\cdot,\cdot)$, see [31, Theorem 3.1], such that the following convergence holds

$$\begin{aligned} &(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\\ &\quad\xrightarrow{b}\int_{\mathbb{R}\times\mathbb{R}^{3\times3}}\Big(\boldsymbol{G}(s,\boldsymbol{\lambda})-\boldsymbol{G}(s,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\mu_{x,t}(s,\boldsymbol{\lambda}). \end{aligned} \tag{2.81}$$

as $l\to\infty$. Using Lemma 24 we obtain that the measure $\mu_{x,t}(s,\boldsymbol{\lambda})$ can be presented in the form of $\delta_{\tilde{\theta}+\theta_k}\otimes\nu_{x,t}(\boldsymbol{\lambda})$. Then

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}^{3\times3}}\Big(\boldsymbol{G}(s,\boldsymbol{\lambda})-\boldsymbol{G}(s,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\mu_{x,t}(s,\boldsymbol{\lambda})\\ &\quad=\int_{\mathbb{R}^{3\times3}}\Big(\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda})-\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\nu_{x,t}(\boldsymbol{\lambda})\\ &\quad=\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\nu_{x,t}(\boldsymbol{\lambda})\\ &\qquad-\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\nu_{x,t}(\boldsymbol{\lambda}). \end{aligned} \tag{2.82}$$

Since $\int_{\mathbb{R}^{3\times3}}\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})=\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k$ a.e., the second term in abovementioned equation disappears. Indeed,

$$\begin{aligned} &-\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))d\nu_{x,t}(\boldsymbol{\lambda})\\ &\qquad=-\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\left(\int_{\mathbb{R}^{3\times3}}\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\right). \end{aligned} \tag{2.83}$$

Moreover, the uniform boundedness of the sequence $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}_{l=1}^{\infty}$ in $L^1(Q)$ implies that

$$\begin{aligned} \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})&\xrightarrow{b}\int_{\mathbb{R}\times\mathbb{R}^{3\times3}}\boldsymbol{G}(s,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\mu_{x,t}(s,\boldsymbol{\lambda})\\ &=\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda}) \end{aligned} \tag{2.84}$$

Hence, by the positivity of $\boldsymbol{G}(\tilde\theta+\cdot,\tilde{\boldsymbol{T}}^d+\cdot):(\tilde{\boldsymbol{T}}^d+\cdot)$ and using Lemma 23 we get

$$\liminf_{l\to\infty}\int_Q \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \ge \int_Q\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde\theta+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})\,\mathrm{d}x\,\mathrm{d}t. \tag{2.85}$$

Using Lemma 7 and knowing that $\boldsymbol{\chi}_k=\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ a.e. in Q, we get

$$\int_Q \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}+\boldsymbol{T}_k):(\tilde{\boldsymbol{T}}+\boldsymbol{T}_k)\,\mathrm{d}x\,\mathrm{d}t\ge\int_Q\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde\theta+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})\,\mathrm{d}x\,\mathrm{d}t. \tag{2.86}$$

Since $\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)=\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde\theta+\theta_k,\boldsymbol{\lambda})d\nu_{x,t}(\boldsymbol{\lambda})$ and (2.86) holds we obtain that the right hand side of (2.80) is not positive. Hence

$$\Big(\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k})\Big):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\xrightarrow{b}0. \tag{2.87}$$

Using again biting limit we get

$$\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k})):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\xrightarrow{b}0. \tag{2.88}$$

Hence

$$\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\boldsymbol{T}^d_{k}+\tilde{\boldsymbol{T}}^d)\xrightarrow{b}\boldsymbol{G}(\tilde\theta+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k}):(\boldsymbol{T}^d_{k}+\tilde{\boldsymbol{T}}^d). \tag{2.89}$$

We use Lemma 8 to complete the proof. □

Now, we can pass to the limit with $l\to\infty$ in (2.30) with $\varphi_4(t)\in C^\infty([0,T]\times\Omega)$.

$$\begin{aligned}-\int_0^T\int_\Omega\theta_k(\varphi_4(t))_t\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\theta_k(x,0)\varphi_4(x,0)\,\mathrm{d}x+\int_0^T\int_\Omega\nabla\theta_k\cdot\nabla v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t\\ =\int_0^T\int_\Omega\mathcal{T}_k((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\boldsymbol{G}(\theta_k+\tilde\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.90}$$

We finish this section with two lemmas. We prove the uniform boundedness of the sequences $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ and $\{\boldsymbol{u}_k\}$ in proper spaces. This allows us to make the limit passage with second parameter in the next section.

Lemma 10. *The sequence $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ is uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$. Moreover, the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t\}$ is also uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$.*

Proof. Let us consider the equation for the evolution of visco-elastic strain tensor

$$(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t=\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k).$$

Moreover

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t)=\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,\mathrm{d}s.$$

Integrating the value of $M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))$ over cylinder Q and using Δ_2-condition of N-function M^* (1.8) we get

$$\begin{aligned}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t&\le c\int_Q M^*(x,\frac12\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\&=c\int_Q M^*(x,\frac12\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\frac12\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,\mathrm{d}s))\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x.\end{aligned}$$

Using the convexity of M^* we obtain

$$\begin{aligned}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t&\le\frac c2\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0))\,\mathrm{d}x\,\mathrm{d}t\\&+\frac c2\int_Q M^*(x,\int_0^t\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,\mathrm{d}s))\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x.\end{aligned} \tag{2.91}$$

Let us focus on the middle term on the right hand side of abovementioned equation. Changing the variable $\tau = \frac{t}{T}$ we obtain

$$\int_0^T \int_\Omega M^*(x, \int_0^t \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)(x,s)\,\mathrm{d}s))\,\mathrm{d}x\,\mathrm{d}t$$
$$= T \int_0^1 \int_\Omega M^*(x, \int_0^{\tau T} \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)(x,s)\,\mathrm{d}s))\,\mathrm{d}x\,\mathrm{d}\tau.$$

By Jensen inequality we get

$$\begin{aligned}
T \int_0^1 \int_\Omega & M^*(x, \int_0^t \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)(x,s)\,\mathrm{d}s))\,\mathrm{d}x\,\mathrm{d}t \\
&\le T \int_0^1 \int_\Omega \frac{1}{\tau T} \int_0^{\tau T} M^*(x, \tau T \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le T \int_0^1 \int_\Omega \frac{1}{\tau T} \int_0^{\tau T} \tau M^*(x, T \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^1 \int_\Omega \int_0^{\tau T} M^*(x, T \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

There exists $d \in \mathbb{R}$ such that $2^d \ge T$. Then using the Δ_2-condition, coming back to original variable and using the Fubini theorem we get

$$\begin{aligned}
\int_0^1 \int_\Omega \int_0^{\tau T} & M^*(x, T \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \int_0^1 \int_\Omega \int_0^{\tau T} M^*(x, 2^d \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le c^d \int_0^1 \int_\Omega \int_0^{\tau T} M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau + C(d) \int_\Omega h(x)\,\mathrm{d}x \\
&= \frac{c^d}{T} \int_0^T \int_\Omega \int_0^t M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + C(d) \int_\Omega h(x)\,\mathrm{d}x \\
&\le c^d \int_0^T \int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}x\,\mathrm{d}t + C(d) \int_\Omega h(x)\,\mathrm{d}x.
\end{aligned} \tag{2.92}$$

Coming back to (2.91) we get

$$\begin{aligned}
\int_Q M^*(x, \boldsymbol{\varepsilon}_k^{\mathbf{p}}(x,t))\,\mathrm{d}x\,\mathrm{d}t \le \frac{cT}{2} & \int_\Omega M^*(x, \boldsymbol{\varepsilon}_k^{\mathbf{p}}(x,0))\,\mathrm{d}x \\
&+ c^d \int_0^T \int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)))\,\mathrm{d}x\,\mathrm{d}t + C(d) \int_\Omega h(x)\,\mathrm{d}x.
\end{aligned}$$

Lemma 3 and initial condition in $L_{M^*}(\Omega, \mathcal{S}^3_d)$ complete the proof. □

Lemma 11. *The sequence $\{\boldsymbol{u}_k\}$ is uniformly bounded in $BD_{M^*}(\Omega, \mathbb{R}^3)$.*

Proof. Let us start with showing the uniform boundedness of the sequence $\{\varepsilon(\boldsymbol{u}_k)\}$ in the space $L_{M^*}(Q)$. Using Δ_2-condition, convexity of N-function and Assumption 1 we obtain

$$
\begin{aligned}
\int_Q M^*(x,\varepsilon(\boldsymbol{u}_k))\,\mathrm{d}x\,\mathrm{d}t &\le c\int_Q M^*(x,\frac{1}{2}\varepsilon(\boldsymbol{u}_k))\,\mathrm{d}x\,\mathrm{d}t+\int_Q h(x)\,\mathrm{d}x\,\mathrm{d}t\\
&= c\int_Q M^*(x,\frac{1}{2}(\varepsilon(\boldsymbol{u}_k)-\varepsilon^{\mathbf{p}}_k)+\frac{1}{2}\varepsilon^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\
&\le \frac{c}{2}\int_Q M^*(x,\varepsilon(\boldsymbol{u}_k)-\varepsilon^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\varepsilon^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\
&\le \frac{c}{2}\int_Q |\varepsilon(\boldsymbol{u}_k)-\varepsilon^{\mathbf{p}}_k|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\varepsilon^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\
&\le \frac{c}{2}\int_Q |\boldsymbol{T}_k|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\varepsilon^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x.
\end{aligned}
\tag{2.93}
$$

Following Anzellotti and Giaquinta [3, Preposition 1.2 a)] we get the inequality

$$\|\boldsymbol{u}_k\|_{L^1(Q)}\le C\|\varepsilon(\boldsymbol{u}_k)\|_{L^1(Q)},$$

where C is a constant depending on Ω. Finally we get the estimates

$$\|\boldsymbol{u}_k\|_{L^1(Q)}\le C\int_Q M^*(x,\varepsilon(\boldsymbol{u}_k))\,\mathrm{d}x\,\mathrm{d}t,$$

which completes the proof. □

2.5. **Limit passage $k\to\infty$.** The considerations over the second limit passage we start from discussing the existence of heat equation solution. In the Appendix A, we prove the existence of renormalised solution to parabolic equation with Neumann boundary condition, which is an extension of results presented by Blanchard and Murat in series of papers. In [6, 7] the existence and uniqueness of renormalised solution is proved in the case of Dirichlet boundary condition.

Uniform boundedness presented in previous sections gives us the following convergences

$$
\begin{array}{ll}
\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} & \text{weakly in } BD_{M^*}(Q,\mathbb{R}^3),\\
\boldsymbol{T}_k \rightharpoonup \boldsymbol{T} & \text{weakly in } L^2(Q,\mathcal{S}^3),\\
\boldsymbol{T}^d_k \rightharpoonup^* \boldsymbol{T}^d & \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\
\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) \rightharpoonup^* \boldsymbol{\chi} & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\
(\varepsilon^{\mathbf{p}}_k)_t \rightharpoonup^* (\varepsilon^{\mathbf{p}})_t & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d).
\end{array}
\tag{2.94}
$$

Then as described in Appendix A we also have convergences

$$
\begin{array}{ll}
\mathcal{T}_K(\theta_k)\to\mathcal{T}_K(\theta) & \text{in } L^2(0,T,W^{1,2}(\Omega)),\\
\theta_k\to\theta & \text{a.e. in } \Omega\times(0,T),
\end{array}
\tag{2.95}
$$

for every $K>0$. Using these convergences in $(2.32)_{(1)}$ and (2.33), we get

$$
\begin{aligned}
&\int_Q \boldsymbol{T}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=0\\
&\int_Q (\varepsilon^{\mathbf{p}})_t:\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t=\int_Q \boldsymbol{\chi}:\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.96}
$$

for $\boldsymbol{\varphi}\in C^\infty([0,T],L^2(\Omega,\mathbb{R}^3))$ and $\boldsymbol{\psi}\in L_M(Q,\mathcal{S}^3)$.

To complete the limit passage we deal with the same problem as in the previous step, i.e. we have to identify the limit of the right hand side of heat equation. Once again, the identification of this limit contains of three steps.

In the proof of the following lemma we proceed similarly as in the proof of Lemma 7.

Lemma 12. *The following inequality holds for the solution of approximate systems.*

$$\limsup_{k\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d):\boldsymbol{T}_k^d\,\mathrm{d}x\,\mathrm{d}t\le\int_0^{t_2}\int_\Omega\boldsymbol{\chi}:\boldsymbol{T}^d\,\mathrm{d}x\,\mathrm{d}t. \tag{2.97}$$

Proof. Using the lower semicontinuity in $L^2(Q)$ we get

$$\begin{aligned}&\liminf_{k\to\infty}\int_0^T\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})\,\psi_{\mu,\tau}\,\mathrm{d}t\\&\qquad=\liminf_{k\to\infty}\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,\mathrm{d}t-\lim_{k\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(0)\\&\qquad\ge\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,\mathrm{d}t-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}^{\mathbf{p}})(0).\end{aligned} \tag{2.98}$$

We use $\boldsymbol{\varphi}_1=((\boldsymbol{\varepsilon}(\boldsymbol{u}_k)*\eta_\epsilon)_t\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon$, where η_ϵ is a standard mollifier with respect to time,5 as a test function in (2.96) then

$$\int_{t_1}^{t_2}\int_\Omega\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})*\eta_\epsilon:(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)*\eta_\epsilon)_t\,\mathrm{d}x\,\mathrm{d}t=0. \tag{2.99}$$

Moreover, we use $\boldsymbol{\psi}=(\boldsymbol{T}^d*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon$ as a test function in (2.33). Then

$$\int_{t_1}^{t_2}\int_\Omega(\boldsymbol{\varepsilon}_k^{\mathbf{p}}*\eta_\epsilon)_t:\boldsymbol{T}*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=\int_{t_1}^{t_2}\int_\Omega\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)*\eta_\epsilon:\boldsymbol{T}*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{2.100}$$

Products in (2.100) are well defined. Subtracting these two equations we get

$$\int_{t_1}^{t_2}\int_\Omega\boldsymbol{T}*\eta_\epsilon:(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}_k^{\mathbf{p}})_t*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=-\int_{t_1}^{t_2}\int_\Omega\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)*\eta_\epsilon:\boldsymbol{T}^d*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{2.101}$$

For every $\epsilon>0$ the sequence $\{(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}_k^{\mathbf{p}})_t*\eta_\epsilon\}$ belongs to $L^2(Q,\mathcal{S}^3)$ and is uniformly bounded in $L^2(Q,\mathcal{S}^3)$ with respect to k, hence we pass to the limit with $k\to\infty$ and we obtain

$$\int_{t_1}^{t_2}\int_\Omega\boldsymbol{T}*\eta_\epsilon:(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})_t*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=-\int_{t_1}^{t_2}\int_\Omega\boldsymbol{\chi}*\eta_\epsilon:\boldsymbol{T}^d*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Using the properties of convolution we get

$$\int_\Omega\boldsymbol{T}*\eta_\epsilon:(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})*\eta_\epsilon\,\mathrm{d}x\Big|_{t_1}^{t_2}=-\int_{t_1}^{t_2}\int_\Omega\boldsymbol{\chi}*\eta_\epsilon:\boldsymbol{T}^d*\eta_\epsilon*\eta_\delta\,\mathrm{d}x\,\mathrm{d}t.$$

In the same way as in the previous section we pass to the limit with $\epsilon\to0$ and then with $t_1\to0$

$$\int_\Omega\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}):(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})\,\mathrm{d}x\Big|_0^{t_2}=-\int_0^{t_2}\int_\Omega\boldsymbol{\chi}:\boldsymbol{T}^d\,\mathrm{d}x\,\mathrm{d}t. \tag{2.102}$$

We multiply (2.102) by $\frac{1}{\mu}$ and integrate over $(\tau,\tau+\mu)$ and proceed now in the same manner as in the proof of Lemma 7. □

The second and the third steps are conducted in the same way as in the previous limit passage, hence we omit this calculation. Using the Minty-Browder trick we show that

$$\boldsymbol{\chi}=\boldsymbol{G}(\tilde\theta+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d) \tag{2.103}$$

a.e. in Q. Moreover using the Young measures tools we may pass to the limit in right hand side term of heat equation. Repeating the procedure from the previous limit passage we obtain

$$\mathcal{T}_k((\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d):\boldsymbol{G}(\theta_k+\tilde\theta,\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d))\rightharpoonup(\boldsymbol{T}^d+\tilde{\boldsymbol{T}}^d):\boldsymbol{G}(\theta+\tilde\theta,\boldsymbol{T}^d+\tilde{\boldsymbol{T}}^d)) \tag{2.104}$$

in $L^1(Q)$. Using the solution to problem (2.1) we obtain

$$\int_0^T\int_\Omega(\tilde{\boldsymbol{T}}+\boldsymbol{T}):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.105}$$

where

$$\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \tag{2.106}$$

and (2.105) holds for every test function $\boldsymbol{\varphi} \in C^\infty([0,T], C_c^\infty(\Omega,\mathbb{R}^3))$. To get the renormalised solution to heat equation let us take $S'(\theta)\phi$ as a test function in (2.90), where S is a $C^\infty(\mathbb{R})$ function, such that S' has a compact support. Then, by Appendix A, limit passage in heat equation is clear and

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta)|\nabla(\theta)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d S'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.107}$$

holds for every test function $\phi \in C_c^\infty([-\infty,T), C^\infty(\Omega))$ and for every function $S \in C^\infty(\mathbb{R})$ such that $S' \in C_0^\infty(\mathbb{R})$, which completes the proof of Theorem 1.

Appendix A. Renormalised solutions to heat equation

To deal with the heat equations we introduce the renormalised solutions. Renormalised solution for parabolic equation was presented in [6, 7], but only for the Dirichlet boundary conditions. Some proof from [6, 7] need a modification for the case of Neumann boundary conditions.

Let us consider the system of equations

$$\begin{cases} \frac{\partial\theta^\varepsilon}{\partial t} - \Delta\theta^\varepsilon = f^\varepsilon & \text{in } Q, \\ \frac{\partial\theta^\varepsilon}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega\times(0,T), \\ \theta^\varepsilon(t=0) = \theta_0^\varepsilon, & \text{in } \Omega \end{cases} \tag{A.1}$$

where for every positive ε the function f^ε belongs to $L^2(Q)$ and converges weakly to f in $L^1(Q)$ and θ_0^ε belongs to $L^2(\Omega)$ and converges strongly to θ_0 in $L^1(\Omega)$ as ε tends to 0.

In our case $\frac{1}{\varepsilon} = k$ and

$$\begin{cases} f^\varepsilon = \mathcal{T}_k((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)) & \text{in } Q, \\ \theta^\varepsilon(x,0) = \mathcal{T}_k(\theta_0) & \text{in } \Omega, \end{cases} \tag{A.2}$$

and moreover we know that sequence $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\}$ is uniformly bounded in $L^1(Q)$. Hence, there exists a weak limit of this sequence. Identification of this weak limit is discussed in Section 2.5.

Definition 9. *Renormalised solution to heat equation [7, Definition 2.2]*
Let f belong to $L^1(Q)$ and θ_0 belong to $L^1(\Omega)$. A real-valued function θ defined on Q is a renormalised solution of heat equation if

a) *θ belongs to $C([0,T], L^1(\Omega))$ and $\mathcal{T}_K(\theta)$ belongs to $L^2(0,T,W^{1,2}(\Omega))$ for all positive K;*
b) *for all positive c*

$$\mathcal{T}_{K+c}(\theta) - \mathcal{T}_k(\theta) \to 0 \tag{A.3}$$

in $L^2(0,T,W^{1,2}(\Omega))$ as K goes to ∞;

c) *and $\theta(t=0) = \theta_0$.*

Moreover, for all functions $S \in C^\infty(\mathbb{R})$, such that S' belongs to $C_0^\infty(\mathbb{R})$ (S' has a compact support), the following equality holds

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta)|\nabla\theta|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f S'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{A.4}$$

for all $\phi \in C_0^\infty(Q)$.

We use the notation $\lim_{\eta,\varepsilon\to 0}$ when the order in the passing to the limit is not relevant, i.e.

$$\lim_{\eta,\varepsilon\to0} F_{\eta,\varepsilon} = \lim_{\eta\to0}\lim_{\varepsilon\to0} F_{\eta,\varepsilon} = \lim_{\varepsilon\to0}\lim_{\eta\to0} F_{\eta,\varepsilon}.$$

Lemma 13. *There exists a subsequence of the sequence* $\{\theta^\varepsilon\}_\varepsilon$ *(still denoted by* ε*) and* $\theta \in C([0,T],L^1(\Omega))$*, such that when* ε *tends to* 0 *and for any fixed positive real number* K *the following conditions are satisfied*

a) θ^ε *converges almost everywhere in* Q *to a measurable function* θ*;*
b) θ^ε *converges to* θ *in* $C([0,T],L^1(\Omega))$*;*
c) $\mathcal{T}_K(\theta^\varepsilon)$ *converges weakly to* $\mathcal{T}_K(\theta)$ *in* $L^2(0,T,W^{1,2}(\Omega))$*;*
d) *there exists the following limit*

$$\lim_{\eta,\varepsilon\to\infty}\int_Q |\nabla\mathcal{T}_k(\theta^\varepsilon-\theta^\eta)| = 0. \tag{A.5}$$

Proof. Let us take $\mathcal{T}_K(\theta^\varepsilon)$ as a test function in (A.1). Then for $t\in(0,T)$

$$\int_0^t\int_\Omega \frac{\partial\theta^\varepsilon}{\partial t}\mathcal{T}_K(\theta^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega f^\varepsilon\mathcal{T}_k(\theta^\varepsilon), \tag{A.6}$$

and

$$\int_\Omega \tilde{\mathcal{T}}_K(\theta^\varepsilon)(t)\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega f^\varepsilon\mathcal{T}_k(\theta^\varepsilon) + \int_\Omega \tilde{\mathcal{T}}_K(\theta_0^\varepsilon)\,\mathrm{d}x, \tag{A.7}$$

where $\tilde{\mathcal{T}}_K(r) = \int_0^r \mathcal{T}_K(z)\,\mathrm{d}z$ is a positive real valued function. Using definition of the truncation and linear growth of function $\tilde{\mathcal{T}}_K(r)$ at infinity, the following estimate holds

$$\int_\Omega \tilde{\mathcal{T}}_K(\theta^\varepsilon)(t)\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t \le K\|f\|_{L^1(Q)} + C(K)\|\theta_0^\varepsilon\|_{L^1(\Omega)}. \tag{A.8}$$

To show that the sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$, it is enough to estimate $\|\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)}$ by $\|\tilde{\mathcal{T}}_K(\theta^\varepsilon)\|_{L^1(Q)}$ and $\|\nabla\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)}$. By Poincaré inequality we get

$$\begin{aligned}\|\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)} &\le \|\mathcal{T}_K(\theta^\varepsilon)-(\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)}\\ &\le \|\nabla\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)},\end{aligned} \tag{A.9}$$

where by $(\mathcal{T}_K(\theta^\varepsilon))_\Omega$ we denote the mean value. Using the definition of truncation operator we obtain

$$\tilde{\mathcal{T}}_K(\theta^\varepsilon) = \begin{cases} \frac{1}{2}(\theta^\varepsilon)^2 & |\theta^\varepsilon|\le K,\\ \frac{1}{2}K^2 + K(\theta^\varepsilon - K) & |\theta^\varepsilon| > K,\end{cases} \tag{A.10}$$

and then it remains to show the estimates for $(\mathcal{T}_K(\theta^\varepsilon))_\Omega$

$$\int_\Omega |\mathcal{T}_K(\theta^\varepsilon)|^2\,\mathrm{d}x = \int_{\{x\in\Omega:|\theta^\varepsilon|\le K\}} |\theta^\varepsilon|^2 + \int_{\{x\in\Omega:|\theta^\varepsilon|>K\}} K^2 \le 2\int_\Omega \tilde{\mathcal{T}}_K(\theta^\varepsilon)\,\mathrm{d}x. \tag{A.11}$$

The finite measure of Q implies that the sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$.

For $\delta>0$, let us test the difference of two approximate equations (A.1)

$$\frac{\partial}{\partial t}(\theta^\varepsilon-\theta^\eta) - \Delta(\theta^\varepsilon-\theta^\eta) = f^\varepsilon - f^\eta. \tag{A.12}$$

by function $\frac{1}{\delta}\mathcal{T}_\delta(\theta^\varepsilon-\theta^\eta)$. As a result, we get

$$\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta^\varepsilon-\theta^\eta)(t) + \frac{1}{\delta}\int_0^t\int_\Omega |\nabla\mathcal{T}_\delta(\theta^\varepsilon-\theta^\eta)|^2 = \frac{1}{\delta}\int_0^t\int_\Omega (f^\varepsilon-f^\eta)\mathcal{T}_\delta(\theta^\varepsilon-\theta^\eta) + \frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_0^\varepsilon-\theta_0^\eta). \tag{A.13}$$

Using the positivity of the second term of left hand side we obtain

$$\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta^\varepsilon-\theta^\eta)(t) \le \int_0^t\int_\Omega |f^\varepsilon-f^\eta| + \frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_0^\varepsilon-\theta_0^\eta). \tag{A.14}$$

Passing to the limit as δ which tends to 0 we obtain

$$\begin{aligned}\lim_{\delta\to0}\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta^\varepsilon-\theta^\eta)(t) &= \int_\Omega (\theta^\varepsilon-\theta^\eta)(t)\,\mathrm{d}x,\\ \lim_{\delta\to0}\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_0^\varepsilon-\theta_0^\eta) &= \int_\Omega (\theta_0^\varepsilon-\theta_0^\eta)\,\mathrm{d}x.\end{aligned} \tag{A.15}$$

Therefore,

$$\int_\Omega (\theta^\varepsilon - \theta^\eta)(t)\,\mathrm{d}x \le \int_0^t \int_\Omega |f^\varepsilon - f^\eta| + \int_\Omega (\theta_0^\varepsilon - \theta_0^\eta)\,\mathrm{d}x, \tag{A.16}$$

and we conclude that the sequence $\{\theta^\varepsilon\}$ is a Cauchy sequence in $C([0,T],L^1(\Omega))$, hence there exists $\theta \in C([0,T],L^1(\Omega))$, such that $\theta^\varepsilon \to \theta$ in $C([0,T],L^1(\Omega))$ as ε tends to 0.

Testing the equation (A.12) by $\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)$, we obtain

$$\int_\Omega \tilde{\mathcal{T}}_K(\theta^\varepsilon - \theta^\eta)(T) + \int_Q |\nabla \mathcal{T}_K(\theta^\varepsilon - \theta^\eta)|^2 = \int_Q (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta) + \int_\Omega \tilde{\mathcal{T}}_K(\theta_0^\varepsilon - \theta_0^\eta) \tag{A.17}$$

Positivity of the first term on the left hand side in abovementioned equation and the convergences of right hand side functions and initial conditions imply that

$$\lim_{\varepsilon,\eta\to 0} \int_Q |\nabla \mathcal{T}_K(\theta^\varepsilon - \theta^\eta)|^2 = 0 \tag{A.18}$$

which completes the proof. □

Lemma 14. *Let K be a fixed positive real number. The sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}$ converges strongly to $\mathcal{T}_K(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$.*

The proof of this lemma can be found in [6].

Multiplying (A.1) by $S'(\theta^\varepsilon)\phi$, where $S \in C^\infty(\mathbb{R})$ and S' has a compact support and $\phi \in C_0^\infty(Q)$, we get

$$\begin{aligned} -\int_Q S(\theta^\varepsilon)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0^\varepsilon)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta^\varepsilon)|\nabla\theta^\varepsilon|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f^\varepsilon S'(\theta^\varepsilon)\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{A.19}$$

S' has a compact support, hence there exist $0 < M < \infty$ such that $\mathrm{supp}(S') \subset [-M,M]$. This allows us to enter into equation (A.19) the truncations operator

$$\begin{aligned} -\int_Q S(\theta^\varepsilon)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0^\varepsilon)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\mathcal{T}_M(\theta^\varepsilon))\nabla\mathcal{T}_M(\theta^\varepsilon)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\mathcal{T}_M(\theta^\varepsilon))|\nabla\mathcal{T}_M(\theta^\varepsilon)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f^\varepsilon S'(\mathcal{T}_M(\theta^\varepsilon))\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{A.20}$$

Using the Egorov theorem applied to $S'(\theta^\varepsilon)$ or to $S''(\theta^\varepsilon)$ and using the bounded character of the remaining terms we can pass to the limit with ε going to 0 in (A.20) and we obtain

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\mathcal{T}_M(\theta))\nabla\mathcal{T}_M(\theta)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\mathcal{T}_M(\theta))|\nabla\mathcal{T}_M(\theta)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f S'(\mathcal{T}_M(\theta))\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{A.21}$$

And finally, using the compact support of S' we can omit the truncations in (A.21)

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta)|\nabla\theta|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f S'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{A.22}$$

which completes the proof of existence regarding renormalised solution to parabolic equation with Neumann boundary condition.

Lemma 15. *Assuming that $\theta_{0,1}$ and $\theta_{0,2}$ lie in $L^1(\Omega)$, f_1 and f_2 lie in $L^1(Q)$ and they satisfy*

$$\begin{cases} \theta_{0,1} \le \theta_{0,2} \\ f_1 \le f_2 \end{cases} \tag{A.23}$$

Then if θ_1 and θ_2 are two renormalised solutions respectively for date $(\theta_{0,1}, f_1)$ and $(\theta_{0,2}, f_2)$, we have

$$\theta_1 \leq \theta_2 \tag{A.24}$$

almost everywhere in Q.

Proof of this lemma can be found in [7].

Remark. *As a consequence of Lemma 15, the renormalised solution is unique.*

Appendix B. Orlicz spaces tools

Assumption 1 requires the use of basic tools regarding generalized Orlicz spaces. Here we present some basic lemmas, which have been used to prove the existence of thermo-visco-elastic model solution. Following lemmas with the proof can be found in [14, 22, 23, 25].

Lemma 16. *Fenchel-Young inequality*
Let M be an N-function and M^* be complementary to M. Then the following inequality is satisfied

$$|\boldsymbol{\xi} : \boldsymbol{\eta}| \leq M(x, \boldsymbol{\xi}) + M^*(x, \boldsymbol{\eta}) \tag{B.1}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{S}^3$ and for almost all $x \in \Omega$.

Lemma 17. *Hölder inequality*
Let M be an N-function and M^* be complementary to M. Then the following inequality is satisfied

$$|\int_Q \boldsymbol{\xi} : \boldsymbol{\eta} \, \mathrm{d}x \, \mathrm{d}t| \leq 2\|\boldsymbol{\xi}\|_{L,M} \|\boldsymbol{\eta}\|_{L,M^*}. \tag{B.2}$$

Lemma 18. *Let $\boldsymbol{\xi}_i : Q \to \mathbb{R}^d$ be a measurable sequence. Then $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ in $L_M(Q)$ modularity if and only if $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$ in measure and there exist some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda \boldsymbol{\xi}_i)\}$ is uniformly integrable, i.e.*

$$\lim_{R \to \infty} \left(\sup_{i \in \mathbb{N}} \int_{\{(t,x):\ |M(x,\lambda \boldsymbol{\xi}_i)| \geq R\}} M(x, \lambda \boldsymbol{\xi}_i) \, \mathrm{d}x \, \mathrm{d}t \right) = 0. \tag{B.3}$$

Lemma 19. *Let M be an N-function and for all $i \in \mathbb{N}$, let $\int_Q M(x, \boldsymbol{\xi}_i) \, \mathrm{d}x \, \mathrm{d}t \leq c$. Then the sequence $\{\boldsymbol{\xi}_i\}$ is uniformly integrable.*

Lemma 20. *Let M be an N-function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\Phi}_i : Q \to \mathcal{S}^3$ and $\boldsymbol{\Psi}_i : Q \to \mathcal{S}^3$ are uniformly bounded in $L_M(Q)$ and $L_{M^*}(Q)$, respectively. Moreover, $\boldsymbol{\Phi}_i \xrightarrow{M} \boldsymbol{\Phi}$ modularly in $L_M(Q)$ and $\boldsymbol{\Phi}_i \xrightarrow{M^*} \boldsymbol{\Phi}$ modularly in $L_{M^*}(Q)$. Then, $\boldsymbol{\Phi}_i : \boldsymbol{\Psi}_i \to \boldsymbol{\Phi} : \boldsymbol{\Psi}$ strongly in L^1.*

Lemma 21. *Let ρ_i be a standard mollifier, i.e. $\rho \in C^\infty(\mathbb{R})$, ρ has a compact support and $\int_{\mathbb{R}} \rho(\tau) \, \mathrm{d}\tau = 1$, $\rho(\tau) = \rho(-\tau)$. We define $\rho_i(\tau) = i\rho(i\tau)$. Moreover, let $*$ denote a convolution in the variable τ. Then for any function $\boldsymbol{\Phi} : Q \to \mathcal{S}^3$, such that $\boldsymbol{\Phi} \in L^1(Q, \mathcal{S}^3$, it holds*

$$\rho_i * \boldsymbol{\Phi} \to \boldsymbol{\Phi} \qquad \textit{in measure.} \tag{B.4}$$

Lemma 22. *Let ρ_i be a standard mollifier. Given an N-function M and a function $\boldsymbol{\Phi} : Q \to \mathcal{S}^3$ such that $\boldsymbol{\Phi} \in \mathcal{L}_M(Q)$, the sequence $\{M(x, \rho_i * \boldsymbol{\Phi})\}$ is uniformly integrable.*

Appendix C. Young measures tools

Right hand side term in the approximated heat equation is a product of elements of two sequences which converge weakly. To characterize the limit of this term we use Young measure theory. In this section, we present necessary lemmas. They come from [31, Corollaries 3.2-3.4]. Similar technique was also used in [21, 34].

Lemma 23. *Suppose that the sequence of maps $z_j : Q \to \mathbb{R}^d$ generates the Young measure $\nu : Q \to \mathcal{M}(\mathbb{R}^d)$. Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Carathéodory function (i.e. measurable in the first argument and continuous in the second). Let us also assume that the negative part $F^-(x, z_j(x,t))$ is weakly relatively compact in $L^1(Q)$. Then*

$$\liminf_{j\to\infty} \int_E F(x, z_j(x,t))\,\mathrm{d}x\,\mathrm{d}t \geq \int_E \int_{\mathbb{R}^d} F(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x\,\mathrm{d}t \tag{C.1}$$

If, in addition, the sequence of functions $x \to |F|(x, z_j(x,t))$ is weakly relatively compact in $L^1(Q)$, then

$$F(\cdot, z_j(\cdot,\cdot)) \rightharpoonup \int_{\mathbb{R}^d} F(\cdot,\lambda)\,\mathrm{d}\nu_x(\lambda) \qquad \text{in } L^1(Q). \tag{C.2}$$

Lemma 24. *Let $u_j : Q \to \mathbb{R}^d$, $v_j : Q \to \mathbb{R}^{d'}$ be measurable and suppose that $u_j \to u$ a.e. while v_j generates the Young measure ν. Then the sequence of pairs $(u_j, v_j) : Q \to \mathbb{R}^{d+d'}$ generates the Young measure $x \to \delta_{u(x)} \otimes \nu_x$.*

Lemma 25. *Suppose that a sequence z_j of measurable functions from Q to $\mathbb{R}^d$ generates the Young measure $\nu : Q \to \mathcal{M}(\mathbb{R}^d)$. Then $z_j \to z$ in measure if and only if $\nu_x = \delta_{z(x)}$ a.e..*

Acknowledgement F.K. is a PhD student of the International PhD Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (PhD Programme: Mathematical Methods in Natural Sciences).

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, 1975.

[2] Hans-Dieter Alber and Krzysztof Chełmiński. Quasistatic problems in viscoplasticity theory ii: Models with nonlinear hardening. *Mathematical Models and Methods in Applied Sciences*, 17(02):189–213, 2007.

[3] G. Anzellotti and M. Giaquinta. Existence of the displacements field for an elasto-plastic body subject to Hencky's law and von mises yield condition. *manuscripta mathematica*, 32(1-2):101–136, 1980.

[4] J. M. Ball and F. Murat. Remarks on Chacon's biting lemma. *Proceedings of the American Mathematical Society*, 107(3), Nov., 1989.

[5] Leszek Bartczak. Mathematical analysis of a thermo-visco-plastic model with Bodnera-Partom constitutive equations. *Journal of Mathematical Analysis and Applications*, 385(2):961 – 974, 2012.

[6] D. Blanchard. Truncations and monotonicity methods for parabolic equations. *Nonlinear Analysis: Theory, Methods and Applications*, 21(10):725 – 743, 1993.

[7] D. Blanchard and F. Murat. Renormalised solutions of nonlinear parabolic problems with l1 data: existence and uniqueness. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 127:1137–1152, 1997.

[8] L. Boccardo and T. Gallouet. Non-linear elliptic and parabolic equations involving measure data. *Journal of Functional Analysis*, 87(1):149 – 169, 1989.

[9] M. Brokate, P. Krejčí, and D. Rachinskii. Some analytical properties of the multidimensional continuous Mróz model of plasticity.

[10] K. Chełmiński. On large solutions for the quasistatic problem in non-linear viscoelasticity with the constitutive equations of Bodner–Partom. *Mathematical Methods in the Applied Sciences*, 19(12):933–942, 1996.

[11] K. Chełmiński and P. Gwiazda. On the model of Bodner – Partom with nonhomogeneous boundary data. *Mathematische Nachrichten*, 214(1):5–23, 2000.

[12] K. Chełmiński and P. Gwiazda. Convergence of coercive approximations for strictly monotone quasistatic models in inelastic deformation theory. *Math. Methods Appl. Sci.*, 30(12):1357–1374, 2007.

[13] K. Chełmiński and R. Racke. Mathematical analysis of a model from thermoplasticity with kinematic hardening. *J. Appl. Anal.*, 12:37–57, 2006.

[14] E. Emmrich and A. Wróblewska-Kamińska. Convergence of a full discretization of quasi-linear parabolic equations in isotropic and anisotropic Orlicz spaces. *SIAM J. NUMER. ANAL.*, 51(2):1163 – 1184, 2013.

[15] L. C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[16] A. E. Green and P. M. Naghdi. A general theory of an elastic-plastic continuum. *Archive for Rational Mechanics and Analysis*, 18:251–281, 1965.

[17] P. Gwiazda and volume = 32 number = 1 pages = 103 – 114 year = 2008 A. Świerczewska Gwiazda), journal = Topol. Methods Nonlinear Anal. On steady non-newtonian flows with growth conditions in generalized orlicz spaces.

[18] P. Gwiazda, F. Z. Klawe, and A. Świerczewska Gwiazda. Thermo-visco-elasticity for norton-hoff-type models. *preprint: http://arxiv.org/abs/1404.1671*, 2014.

[19] P. Gwiazda, F.Z. Klawe, and A. Świerczewska-Gwiazda. Thermo-visco-elasticity for the Mróz model in the framework of thermodynamically complete systems. *Discrete and Continuous Dynamical Systems - Series S*, 7:981 – 991, 2014.

[20] P. Gwiazda, P. Minakowski, and A. Świerczewska Gwiazda. On the anisotropic Orlicz spaces applied in the problems of continuum mechanics. *Discrete and Continuous Dynamical Systems - Series S*, 6(5):1291 – 1306, 2013.

[21] P. Gwiazda and A. Świerczewska. Large eddy simulation turbulence model with Young measures. *Applied Mathematics Letters*, 18(8):923 – 929, 2005.

[22] P. Gwiazda and A. Świerczewska Gwiazda. On non-newtonian fluids with the property of rapid thickening. *Math. Models Methods Appl. Sci.*, 18(7):1073 – 1092, 2008.

[23] P. Gwiazda, A. Świerczewska Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.*, 33(2):125 – 248, 2010.

[24] P. Gwiazda, A. Świerczewska Gwiazda, and Aneta Wróblewska. Generalized Stokes system in Orlicz spaces. *Discrete and continuous dynamical systems*, 32(6), June 2012.

[25] P. Gwiazda, P. Wittbold, A. Wróblewska, and A. Zimmermann. Renormalized solutions of nonlinear elliptic problems in generalized orlicz spaces. *Journal Differential Equations*, 253:635 – 666, 2012.

[26] P. Gwiazda, P. Wittbold, A. Wróblewska-Kamińska, and A. Zimmermann. Renormalized solutions of nonlinear parabolic problems in generalized Musielak-Orlicz spaces.

[27] Dietmar Hömberg. A mathematical model for induction hardening including mechanical effects. *Nonlinear Analysis: Real World Applications*, 5(1):55 – 90, 2004.

[28] M. A. Krasnoselskij and Y. B. Rutitskij. *Convex Functions and Orlicz Spaces*. Noordhoff,Groningen, Netherlands, 1961.

[29] L.D. Landau and E.M. Lifshitz. *Theory of Elasticity*. Pergamon Press, 1970 (7th edition).

[30] J. Málek, J. Nečas, M. Rokyta, and M. Růžicka. *Weak and measure-valued solutions to evolutionary PDEs*. Chapman & Hall, London, 1996.

[31] S. Müller. *Variational models for microstructure and phase transitions*, volume 1713 of *Lecture Notes in Math.: Calculus of Variations and Geometric Evolution Problems*. Springer-Verlag, Berlin Heidelberg, 1999.

[32] J. Musielak. *Orlicz Spaces and Modular Spaces*. Springer Berlin Heidelberg, 1983.

[33] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*. Pure and Applied Math.Dekker, New York, 1991.

[34] A. Świerczewska. A dynamical approach to large eddy simulation of turbulent flows: existence of weak solutions. *Math. Methods Appl. Sci.*, 29(1):99 – 121, 2006.

[35] A. Świerczewska Gwiazda. Anisotropic parabolic problems with slowly or rapidly growing terms. *Colloq. Math.*, 134(1):113 – 130, 2014.

[36] A. Świerczewska Gwiazda. Nonlinear parabolic problems in musielak-orlicz spaces. *Nonlinear Analysis: TMA*, 98:48 – 65, 2014.

[37] Tullio Valent. *Boundary Value Problems of Finite Elasticity: Local Theorems on Existence, Uniqueness, and Analytic Dependence on Data*. Springer Publishing Company, Incorporated, 1st edition, 2011.

[38] Aneta Wróblewska. Steady flow of non-newtonian fluids — monotonicity methods in generalized orlicz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 72(11):4136 – 4147, 2010.

[39] E. Zeidler. *Nonlinear Functional Analysis II/B – Nonlinear Monotone Operators*. Springer–Verlag, Berlin–Heidelberg–New York, 1990.

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