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Abstract

We prove the existence of global-in-time weak solutions to a model of chemically reacting mixture. We consider a coupling between the compressible Navier-Stokes system and the reaction diffusion equations for chemical species when the thermal effects are neglected. We first prove the existence of weak solutions to the semi-discretization in time. Based of this, the existence of solutions to the evolutionary system is proved.

1 Introduction

We consider the model of motion for the n-component gaseous mixture undergoing an isothermal, chemical reaction. We focus on the Fick approximation of diffusion fluxes, which is often used to model the lean one-reaction flow [9]

$$\nu_F F + \nu_O O \to \nu_P P,$$

where F denotes the fuel, O denotes the oxidant, P denotes the products and ν_F, ν_O, ν_P denote stoichiometric coefficients. When the reaction takes place in the presence of dilutant denoted by N, and when the oxidant and dilutant are in excess, one may ignore the cross-effects in the diffusion fluxes and simply assume that they are proportional to the gradients of species concentrations Y_i, $i \in \{F, P\}$. Such model was investigated e.g. by Feireisl, Petzeltová and Trivisa in [8]. They proved the existence of weak variational entropy solutions to a system with arbitrary large number of reversible reactions and diffusion determined by the Fick law:

$$\mathbf{F} = -D\nabla Y_k, \quad k \in \{1, \ldots, n\}.$$

The analysis performed in this paper was motivated by previous studies of Klein et al. [10] in which the authors assumed that the pressure does not depend on the chemical composition of the mixture. Another application of such result is to model the mixtures of isotopes. Then the molar masses of species m_k are almost the same, and so the mean molar mass $\overline{m}$ is close to a constant

$$\frac{1}{\overline{m}} = \sum_{k=1}^{n} \frac{Y_k}{m_k} \approx c.$$

In the present work we aim at extending this result to more general equation of state like, for example, the Boyl law describing the pressure p of a mixture of ideal gasses

$$p = \sum_{k=1}^{n} \frac{R\varrho Y_k}{m_k}, \tag{1}$$

where R is the ideal gas constant and ϱ denotes the density of the mixture. This leads to a stronger coupling between the fluid equations and the mass balances of species. We investigate the model in which the flow of mixture is modeled by the compressible, viscous Navier-Stokes equations. We prove the global-in-time existence of weak solutions by semi-discretizaton in time. Our approach relies on an existence result for the stationary Navier-Stokes-like model of 4-component reactive mixture, due to Zatorska [24].

As far as the weak solutions with large data are concerned, the first existence result for the steady as well as the non-steady barotropic Navier-Stokes system is due to Lions [11]. He was essentially using the properties of the so-called *effective viscous flux*. A compactness of this quantity was studied already by Novotný [15] using the method of decomposition from [16]. Later on, this approach was extended by Feireisl [4]. He established a tool for studying density oscillations, which allowed to treat the case when density is not a-priori bounded in L^2. This technique was later on adopted by Novo and Novotný [14] to treat the steady case. The comparison of these methods together with complete approximation scheme can be found in the book of Novotný and Straškraba [19], mostly for the Dirichlet boundary conditions. For the steady problem with slip boundary conditions we refer to the papers of Mucha and Pokorný [12, 20], where also a new idea of construction of approximate solution has been introduced. For completeness, let us also mention the recent generalization of these results to the full Navier-Stokes-Fourier system in the evolutionary [5, 6, 7] and the stationary case [13, 17, 18].

The paper is organized as follows. In Section 2 we formulate the model, we specify the constitutive relations and the assumptions on the transport coefficients. Further, we introduce the notion of weak solution and we state the first main result of the paper given in Theorem 1. In Section 3 we introduce the discretized system and at the end, in Theorem 2, we give the second main result of this paper – the existence result for the fixed time step $\triangle t$. Then, in Section 4 we present the proof of this result by several regularizations and subsequent limit passages. Finally, in Section 5 we show a convergence to the continuous system when $\triangle t \to 0$.

2 Presentation of the continuous model

$$\left.\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) &= 0 \\ \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla\pi &= \varrho\mathbf{f} \\ \partial_t\varrho_k + \operatorname{div}(\varrho_k\mathbf{u}) + \operatorname{div}\mathbf{F}_k = \varrho\omega_k, &\quad k\in\{1,...,n\} \end{aligned}\right\} \quad \text{in } (0,T)\times\Omega. \tag{2}$$

This model is characterized by the state variables: the density of the mixture $\varrho = \varrho(t,x)$, the velocity vector field $\mathbf{u} = \mathbf{u}(t,x)$ and the species mass fractions Y_k for $k \in \{1,\dots,n\}$.

In system (2) quantity $\mathbf{S}$ stands for the viscous stress tensor, π denotes the internal pressure of the fluid, $\mathbf{f}$ – external force, ω_k stands for the production rate of the k-th species and by $\mathbf{F}_k$ we denote the diffusion flux.

The mass fractions Y_k $k \in \{1,\dots,n\}$ are defined by

$$Y_k = \frac{\varrho_k}{\varrho},$$

where m_k are the molar masses of the species.

The diffusion fluxes and the species production rates satisfy

$$\sum_{k=1}^n \mathbf{F}_k = 0, \quad \sum_{k=1}^n \omega_k = 0. \tag{3}$$

The system is supplemented by the following initial conditions

$$\varrho(x)|_{t=0} = \varrho^0(x), \quad Y_k(x)|_{t=0} = Y_k^0(x), \quad \mathbf{u}(x)|_{t=0} = \mathbf{u}^0(x). \tag{4}$$

We assume that

$$0 \le Y_k^0 \le 1, \quad \sum_{k=1}^n Y_k^0 = 1, \quad 0 < \underline{\varrho^0} \le \varrho^0 \le \overline{\varrho^0} < \infty$$

and that the total mass is given

$$\int_\Omega \varrho^0 \, \mathrm{d}x = M > 0.$$

We have set $\Omega \subset \mathbb{R}^3$ bounded with sufficiently smooth boundary, and we impose the following boundary conditions the impermeability conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{5}$$

together with the no-slip boundary condition

$$\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \tag{6}$$

The internal pressure is in the form

$$\pi(\varrho, Y) = \varrho^\gamma + \varrho \sum_{k=1}^n \frac{Y_k}{m_k}, \quad \gamma > 1. \tag{7}$$

The first term describes the barotropic pressure while the latter summand represents the thermodynamic pressure for the mixture of n species given by the Boyle law (1) (with R=1).

The fluxes $\mathbf{F}_i$ are given by

$$\mathbf{F}_i = -D\nabla Y_i, \quad D > 0. \tag{8}$$

The form of viscous stress tensor $\mathbf{S}$ is determined by the Newton rheological law

$$\mathbf{S} = 2\mu\mathbf{D}(\mathbf{u}) + \nu \operatorname{div} \mathbf{u}\mathbf{I}, \tag{9}$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$ and μ, ν are constant viscosity coefficients satisfying

$$\mu > 0, \quad 2\mu + 3\nu \ge 0. \tag{10}$$

The production rates ω_k are defined as

$$\omega_k = \omega_k(Y_1, \ldots, Y_n) = \omega_k^p(Y_1, \ldots, Y_n) - Y_k\omega_k^r(Y_1, \ldots, Y_n), \tag{11}$$

where ω_k^p, $Y_k\omega_k^r$ denote the rate of production and reduction of species k, respectively. We assume that ω_k^p, ω_k^r are bonded on $[0,1]^n$ and that

$$\omega_k^p(Y_1, \ldots, Y_n) \ge 0, \quad \omega_k^r(Y_1, \ldots, Y_n) \ge 0 \quad \text{for all } 0 \le Y_i \le 1, \tag{12}$$

thus, in particular

$$\omega_k(Y_1, \ldots, Y_n) \ge 0 \quad \text{whenever} \quad Y_k = 0.$$

For the above system we will look for a global in time week solution in the following sense.

Definition 1 *We say that $(\varrho, \mathbf{u}, Y_1, \ldots, n)$ is a weak solution to the problem (2-7), (8-12) provided $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$, $\mathbf{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$, $Y \in L^\infty((0,T) \times \Omega) \cap L^2(0,T; W^{1,2}(\Omega))$, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$, $Y_k, \varrho \ge 0$, $\int_\Omega \varrho(x,t)dx = M$ and $\sum_{k=1}^n Y_k = 1$ a.e. in Ω, and the system (2) is fulfilled in the distributional sense in $(0,T) \times \Omega$.*

The main theorem of this work reads.

Theorem 1 *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mu > 0$, $\nu + \frac{2}{3}\mu > 0$, $\gamma \ge 2$, $M > 0$ and $\rho_0 \ge 0$, $u_0 \in L^2(\Omega)$, $\rho_0 \in L^\gamma(\Omega)$ and $0 \le Y_k \le 1$. Then there exists a weak solution to (2-7), (8–12) in the sense of Definition 1.*

3 The time-discretized model

We will prove the existence of solutions to system (2) by a relevant time discretization and by letting the length of the time step go to 0. We also introduce the first approximation parameter $\delta > 0$ in front of the artificial pressure ϱ^Γ.

Below, we define uniform partitions of the time interval $[0, T]$, $0 = t_0 < t_1 < \cdots < t_{\tilde{N}} = T$ such that $(\triangle t) = t_j - t_{j-1} = const.$ Then the discretized system reads

$$
\begin{gathered}
(\triangle t)^{-1}\left(\varrho^j - \varrho^{j-1}\right) + \operatorname{div}(\varrho^j \mathbf{u}^j) = 0 \\
(\triangle t)^{-1}\left(\varrho^j \mathbf{u}^j - \varrho^{j-1}\mathbf{u}^{j-1}\right) + \operatorname{div}(\varrho^j \mathbf{u}^j \otimes \mathbf{u}^j) - \operatorname{div}\mathbf{S}(\mathbf{u}^j) + \nabla\pi(\varrho^j, Y^j) + \nabla\delta(\varrho^j)^\Gamma = \varrho^j \mathbf{f}^j \\
(\triangle t)^{-1}\left(\varrho^j Y_k^j - \varrho^{j-1} Y_k^{j-1}\right) + \operatorname{div}(\varrho^j Y_k^j \mathbf{u}^j) + \operatorname{div}\mathbf{F}_k(Y_k^j) = \varrho^j \omega_k(Y^j), \quad k \in \{1, ..., n\}
\end{gathered}
\tag{13}
$$

with boundary conditions

$$
\mathbf{u}^j|_{\partial\Omega} = \mathbf{F}_k(Y^j)\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{14}
$$

where for abbreviation we denoted $Y^j = \{Y_1^j, \ldots, Y_n^j\}$.

For the purposes of this part of the paper we introduce the following definition of a weak solution.

Definition 2 *We say $(\varrho^j, \mathbf{u}^j, Y^j)$ is a weak solution to the problem (13-14) provided $\varrho^j \in L^\gamma(\Omega)$, $\mathbf{u}^j \in W_0^{1,2}(\Omega)$, $Y^j \in W^{1,2}(\Omega)$, $\mathbf{F}_k(\varrho^j, Y^j)\cdot\mathbf{n}|_{\partial\Omega} = 0$, $Y_k^j, \varrho^j \geq 0$ and $\sum_{k=1}^n Y_k^j = 1$ a.e. in Ω, and the following integral equalities hold*

$$
(\triangle t)^{-1}\int_\Omega (\varrho^j - \varrho^{j-1})\xi \, \mathrm{d}x - \int_\Omega \varrho^j \mathbf{u}^j \cdot \nabla\xi \, \mathrm{d}x = 0, \tag{15}
$$

for all $\xi \in C^\infty(\overline{\Omega})$,

$$
\begin{aligned}
(\triangle t)^{-1}\int_\Omega (\varrho^j \mathbf{u}^j - \varrho^{j-1}\mathbf{u}^{j-1})\boldsymbol{\varphi} \, \mathrm{d}x - \int_\Omega \left(\varrho^j \left(\mathbf{u}^j \otimes \mathbf{u}^j\right) : \nabla\boldsymbol{\varphi}\right) \, \mathrm{d}x - \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla\boldsymbol{\varphi} \, \mathrm{d}x \\
- \int_\Omega \left(\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma\right)\operatorname{div}\boldsymbol{\varphi} \, \mathrm{d}x = \int_\Omega \varrho^j \mathbf{f}^j \cdot \boldsymbol{\varphi} \, \mathrm{d}x,
\end{aligned}
$$

for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$,

$$
\begin{aligned}
(\triangle t)^{-1}\int_\Omega (\varrho^j Y_k^j - \varrho^{j-1} Y_k^{j-1})\phi \, \mathrm{d}x - \int_\Omega \varrho^j Y_k^j \mathbf{u}^j \cdot \nabla\phi \, \mathrm{d}x \\
- \int_\Omega \mathbf{F}_k(Y_k^j)\nabla Y_k \cdot \nabla\phi \, \mathrm{d}x = \int_\Omega \varrho^j \omega_k(Y^j)\phi \, \mathrm{d}x,
\end{aligned}
$$

for all $\phi \in C^\infty(\overline{\Omega})$ and for $k \in \{1, \ldots, n\}$.

We will also use the notion of the renormalized solution to the continuity equation

Definition 3 *Let $\mathbf{u}^j \in W_{loc}^{1,2}(\mathbb{R}^3)$ and ϱ^{j-1}, $\varrho^j \in L_{loc}^{6/5}(\mathbb{R}^3)$ solve*

$$
(\triangle t)^{-1}\left(\varrho^j - \varrho^{j-1}\right) + \operatorname{div}(\varrho^j \mathbf{u}^j) = 0
$$

in the sense of distributions on $\mathbb{R}^3$, then the pair $(\varrho^j, \mathbf{u}^j)$ is called a renormalized solution to the continuity equation, if

$$
(\triangle t)^{-1}\left(\varrho^j - \varrho^{j-1}\right)b'(\varrho^j) + \operatorname{div}\left(b(\varrho^j)\mathbf{u}^j\right) + \left(\varrho^j b'(\varrho^j) - b(\varrho)j\right)\operatorname{div}\mathbf{u}^j \, \mathrm{d}x = 0, \tag{16}
$$

in the sense of distributions on $\mathbb{R}^3$, for all $b \in W^{1,\infty}(0,\infty) \cap C^1([0,\infty))$, such that $sb'(s) \in L^\infty(0,\infty)$.

Our main result for this system states as follows.

Theorem 2 *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mu > 0$, $\nu + \frac{2}{3}\mu > 0$, $\gamma \geq 2$, $M > 0$ and let $(\triangle t)^{-1}$ be constant, $\delta > 0$, $\Gamma \geq 3$ be fixed. Let $(\varrho^{j-1}, \mathbf{u}^{j-1}, Y_k^{j-1}) \in L^\gamma(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ be given functions such that*

$$Y_k^{j-1},\ \varrho^{j-1} \geq 0, \quad \sum_{k=1}^n Y_k^{j-1} = 1 \text{ a.e. in } \Omega, \quad \varrho^{j-1}\left(Y_k^{j-1}\right)^2,\ \varrho^{j-1}|\mathbf{u}^{j-1}|^2 \in L^1(\Omega),$$

$$\mathbf{F}_k(Y^{j-1}) \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{u}^{j-1}|_{\partial\Omega} = 0.$$

Then there exists a weak solution to (13–14) in the sense of Definition 2. Additionally, the pair $(\varrho^j, \mathbf{u}^j)$, extended by 0 outside Ω is a renormalized solution to the continuity equation in the sense of Definition 3.

4 Approximation

The purpose of this section is to prove Theorem 2. For this purpose we introduce a suitable regularization of system (13–14) indicated by presence of three parameters $\varepsilon > 0$ responsible for smoothing the solution to the continuity equation, the parameter of Galerkin approximation $N \in \mathbb{N}$. The artificial pressure parameter $\delta > 0$ was introduced already in the previous section.

For fixed ε, N, δ we will look for $(\varrho^j_{\delta,\varepsilon,N}, \mathbf{u}^j_{\delta,\varepsilon,N}, Y^j_{\delta,\varepsilon,N})$ (we will skip the subindexes when no confusion can arise) satisfying:

- the approximate continuity equation

$$\begin{aligned}(\triangle t)^{-1}\left(\varrho^j - \varrho^{j-1}\right) + \operatorname{div}(\varrho^j \mathbf{u}^j) - \varepsilon\Delta\varrho^j = 0,\\ \nabla\varrho^j \cdot \mathbf{n}|_{\partial\Omega} = 0,\end{aligned} \tag{17}$$

- the approximate momentum equation

$$\begin{aligned}(\triangle t)^{-1}\int_\Omega(\varrho^j\mathbf{u}^j - \varrho^{j-1}\mathbf{u}^{j-1})\cdot\boldsymbol{\varphi}\ \mathrm{d}x - \int_\Omega(\varrho^j\mathbf{u}^j \otimes \mathbf{u}^j) : \nabla\boldsymbol{\varphi}\ \mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}^j) : \boldsymbol{\varphi}\ \mathrm{d}x\\ - \int_\Omega\left(\pi(\varrho^j, \hat{Y}^j) + \delta(\varrho^j)^\Gamma\right)\operatorname{div}\boldsymbol{\varphi}\ \mathrm{d}x + \varepsilon\int_\Omega \nabla\varrho^j \cdot \nabla\mathbf{u}^j \cdot \boldsymbol{\varphi}\ \mathrm{d}x = \int_\Omega \varrho^j \mathbf{f}^j \boldsymbol{\varphi}\ \mathrm{d}x,\\ \mathbf{u}^j|_{\partial\Omega} = 0,\end{aligned} \tag{18}$$

is satisfied for each

$$\boldsymbol{\varphi} \in W_N = \operatorname{span}\{\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^N\} \subset W_0^{1,2}(\Omega)$$

i.e. the first N eigenfunctions of the Laplace operator with Dirichlet boundary conditions.

- the approximate species balance equations

$$\begin{aligned}(\triangle t)^{-1}(\varrho^j Y_k^j - \varrho^{j-1}Y_k^{j-1}) + \operatorname{div}(\varrho^j Y_k^j \mathbf{u}^j) + \operatorname{div}\mathbf{F}_k(Y_k^j) - \varepsilon\Delta\left(Y_k^j\varrho^j\right) = \varrho^j\omega_k(Y^j),\\ \mathbf{F}_k^\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad k \in \{1, \ldots, n\}.\end{aligned} \tag{19}$$

The aim of this section is to prove

Theorem 3 *Let $\varepsilon, \delta, \triangle t > 0$, $N \in \mathbb{N}$, $\Gamma \geq 3$ be fixed. Under assumptions of Theorem 2. There exists $(\varrho^j, \mathbf{u}^j, Y^j)$ being a regular solution to (17,18,19), such that $\varrho^j \in W^{2,p}(\Omega)$, $\mathbf{u}^j \in W_N$, $Y^j \in W^{2,p}(\Omega)$, $k \in \{1, \ldots, n\}$, for all $p < \infty$. Moreover, $\varrho^j \geq 0$ in Ω, $\int_\Omega \varrho^j\ \mathrm{d}x = \int_\Omega \varrho^{j-1}\ \mathrm{d}x = M$, $Y_k^j \geq 0$ and $\sum_{k=1}^n Y_k^{j-1} = 1$.*

The proof of this theorem is based on several auxiliary lemmas and it is presented in the next subsection.

4.1 Existence for fixed parameters

Step 1: We define the operator

$$\mathcal{S}\ :\ W_N \to W^{2,p}(\Omega),$$

$1 \le p < \infty$, $\mathcal{S}(\mathbf{u}^j) = \varrho^j$, where ϱ^j solves the approximate continuity equation (17) with the Neumann boundary condition. We then claim that the following result holds true

Lemma 4 *Let assumptions of Theorem 3 be satisfied. Then the operator $\mathcal{S}$ is well defined for all $p < \infty$. Moreover, if $\mathcal{S}(\mathbf{u}^j) = \varrho^j$, then $\varrho^j \ge 0$ in Ω and $\int_\Omega \varrho^j \,\mathrm{d}x = \int_\Omega \varrho^{j-1} \,\mathrm{d}x$. Additionally, if $\|\mathbf{u}^j\|_{W_N} \le L$, $L > 0$, then*

$$\|\varrho^j\|_{2,p} \le C(\varepsilon, p, \Omega)(1+L)\|\varrho^{j-1}\|_p, \quad 1 < p < \infty. \tag{20}$$

The above lemma is an analogue of Proposition 4.29 from [19], so we omit the proof.

Step 2: Our next aim is to show the non-negativity of the species concentrations under assumption that the solution to (17–19) is sufficiently smooth, i.e. $\varrho^j, \mathbf{u}^j$ and $Y_k^j \in W^{2,p}(\Omega)$, for any $p < \infty$, $k \in \{1, \ldots, n\}$ and $\varrho^j \ge 0$.

For the first $n-1$ species it will follow directly from the features of the species production terms. We test equation (19) by $Y^j_{k\,-} = \min\{Y_k^j, 0\}$. Note that this is a continuous function and that $Y_k^j Y^j_{k\,-} = (Y^j_{k\,-})^2$, $Y_k^j \nabla Y_k^j = Y^j_{k\,-} \nabla Y^j_{k\,-}$, thus integrating by parts we obtain

$$(\triangle t)^{-1} \int_\Omega \varrho^j (Y^j_{k\,-})^2 \,\mathrm{d}x - \int_\Omega \varrho^j Y_k^j \mathbf{u}^j \nabla Y^j_{k\,-} \,\mathrm{d}x + \int_\Omega (D + \varepsilon \varrho^j)|\nabla Y^j_{k\,-}|^2 \,\mathrm{d}x + \varepsilon \int_\Omega Y^j_{k\,-} \nabla \varrho^j \cdot \nabla Y^j_{k\,-} \,\mathrm{d}x$$
$$= \int_{\Omega_k^-} \varrho^j \omega_k(Y^j) \,\mathrm{d}x + (\triangle t)^{-1} \int_\Omega \varrho^{j-1} Y_k^{j-1} Y^j_{k\,-} \,\mathrm{d}x. \tag{21}$$

Note, that the first integral on the r.h.s. is non-positive due to assumptions imposed on ω_k (12). The second integral is non-positive due to assumptions on Y_k^{j-1}.

Next we multiply (17) by $\frac{1}{2}(Y^j_{k\,-})^2$ and we add the resulting expression to the above equality, we get

$$\frac{1}{2}(\triangle t)^{-1} \int_\Omega \varrho^j (Y^j_{k\,-})^2 + \varrho^{j-1}(Y^j_{k\,-})^2 \,\mathrm{d}x + \int_\Omega (D + \varepsilon \varrho^j)|\nabla Y^j_{k\,-}|^2 \,\mathrm{d}x \le 0.$$

By the fact that $\int_\Omega \varrho^j \,\mathrm{d}x = M > 0$, we can hanse conclude that $Y^j_{k\,-} = 0$,thus $Y_k^j \ge 0$.

Thus $Y_k^j \ge 0$ for $k \in \{1, \ldots, n-1\}$, however, so far we do not know if $Y_k^j \le 1$.

To show this we define $Y_n^j = 1 - \sum_{k=1}^{n-1} Y_k^j$, derive the equation for Y_n^j from the approximate continuity equation (17) and the first $n-1$ species equations (19) and repeat the above procedure to deduce that $Y_n^j \ge 0$ in Ω.

Remark 1 *Note that the lower and the upper bound for Y_i, $i \in \{1, \ldots, n\}$ do not depend on the approximation parameters, thus in the course of subsequent limit passages we will get that*

$$0 \le Y_k \le 1 \quad \textit{a.e. in } \Omega. \tag{22}$$

Step 3: We now prove the existence of solutions to the momentum and the species mass balance equations for $\mathbf{u}$, ϱ given. The main idea consists on applying the Leray-Schauder fixed point theorem to the mapping

$$\mathcal{T} : W_N \times [W^{2,p}]^n \to W_N \times [W^{2,p}]^n, \quad \mathcal{T}(\mathbf{u}^j, Y_k^j) \to (\mathbf{w}, X_k),$$

where $(\mathbf{w}, X_k)$ is a solution to the boundary-value problem

$$
\begin{aligned}
\int_\Omega \mathbf{S}(\mathbf{w}) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x =& (\triangle t)^{-1} \int_\Omega (\varrho^{j-1}\mathbf{u}^{j-1} - \varrho^j \mathbf{u}^j)\boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega (\varrho^j \mathbf{u}^j \otimes \mathbf{u}^j) \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x \\
&+ \int_\Omega \left(\pi(\varrho^j, \hat{Y}^j) + \delta (\varrho^j)^\Gamma \right) \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x - \varepsilon \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\
&+ \int_\Omega \varrho^j \mathbf{f}^j \boldsymbol{\varphi} \, \mathrm{d}x, \\
-\operatorname{div}((D + \varepsilon \varrho)\nabla X_k) =& \varrho^j \omega_k(Y^j) + (\triangle t)^{-1} \varrho^{j-1} Y_k^{j-1} - (\triangle t)^{-1} \varrho^j Y_k^j - \operatorname{div}\left(\varrho^j Y_k^j \mathbf{u}^j \right) + \varepsilon \operatorname{div}(Y_k^j \nabla \varrho^j), \\
& \nabla Y_k^j \cdot \mathbf{n}|_{\partial\Omega} = 0,
\end{aligned}
\tag{23}
$$

satisfied for $\boldsymbol{\varphi} \in W_N$ and $k \in \{1, \ldots, n\}$ and with $\hat{Y}_k^j = Y_k^j$ for $Y_k^j \le 1$, or 1, elsewhere.

We prove the following lemma.

Lemma 5 *Let assumptions of Theorem 3 be fulfilled and let ϱ^j be given by Lemma 4. Then, the operator $\mathcal{T}$ is continuous and compact from $W_N \times [W^{2,p}(\Omega)]^n$ into itself.*

Proof. The existence and uniqueness of solution to the system (23) is a consequence of Lax-Milgram theorem. Evidently, the mapping $\mathcal{T}$ is compact, since the r.h.s of (23) is sufficiently smooth and of lower order, it is also continuous. □

To conclude, we should show boundedness of possible fixed points to

$$
\lambda \mathcal{T}(\mathbf{u}^j, Y_k^j) = (\mathbf{u}^j, Y_k^j), \quad \lambda \in [0, 1]. \tag{24}
$$

We first prove the following lemma.

Lemma 6 *Let assumptions of Theorem 3 be satisfied. Then there exists $c > 0$ such that the solutions of (24) in the class $W_N \times [W^{2,p}(\Omega)]^n$ fulfil*

$$
\|\mathbf{u}^j\|_{W_N} + \|Y_k^j\|_{W^{2,p}(\Omega)} \le c,
$$

independently of t.

The second equality in (23) rewrites as

$$
\begin{aligned}
&-\operatorname{div}((D + \varepsilon \varrho^j)\nabla Y_k^j) \\
&\quad = \lambda \left(\varrho^j \omega_k + (\triangle t)^{-1} \varrho^{j-1} Y_k^{j-1} - (\triangle t)^{-1} \varrho^j Y_k^j - \operatorname{div}\left(\varrho^j Y_k^j \mathbf{u}^j \right) + \varepsilon \operatorname{div}(Y_k^j \nabla \varrho^j) \right).
\end{aligned}
\tag{25}
$$

Multiplying it by Y_k^j integrating by parts and using the boundary conditions we get

$$
\begin{aligned}
\int_\Omega (D + \varepsilon \varrho^j)|\nabla Y_k^j|^2 \, \mathrm{d}x =& \lambda (\triangle t)^{-1} \int_\Omega \left(\varrho^{j-1} Y_k^{j-1} Y_k^j - \varrho^j \left(Y_k^j \right)^2 \right) \, \mathrm{d}x - \frac{\lambda \varepsilon}{2} \int_\Omega \nabla \varrho^j \cdot \nabla \left(Y_k^j \right)^2 \, \mathrm{d}x \\
&- \frac{\lambda}{2} \int_\Omega \operatorname{div}(\varrho^j \mathbf{u}^j) \left(Y_k^j \right)^2 \, \mathrm{d}x + \lambda \int_\Omega \varrho^j \omega_k Y_k^j \, \mathrm{d}x.
\end{aligned}
$$

By the approximate continuity equation we obtain

$$
\begin{aligned}
\int_\Omega (D + \varepsilon \varrho^j)|\nabla Y_k^j|^2 \, \mathrm{d}x &+ \lambda (\triangle t)^{-1} \int_\Omega \left(\varrho^{j-1} \frac{(Y_k^j)^2}{2} + \varrho^j \frac{(Y_k^j)^2}{2} \right) \, \mathrm{d}x \\
&= \lambda \int_\Omega \varrho^j \omega_k Y_k^j \, \mathrm{d}x + \lambda (\triangle t)^{-1} \int_\Omega \varrho^{j-1} Y_k^{j-1} Y_k^j \, \mathrm{d}x.
\end{aligned}
\tag{26}
$$

Using in the fist equality in (23) $\phi = \mathbf{u}^j$ we obtain

$$\begin{aligned}\int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \mathbf{u}^j \ \mathrm{d}x =& \lambda \int_\Omega \left(\pi(\varrho^j, \hat{Y}^j) + \delta (\varrho^j)^\Gamma \right) \operatorname{div} \mathbf{u}^j \ \mathrm{d}x \\ &+ \lambda (\triangle t)^{-1} \int_\Omega \left(\varrho^{j-1} \mathbf{u}^{j-1} - \varrho^j \mathbf{u}^j \right) \cdot \mathbf{u}^j \ \mathrm{d}x - \lambda \int_\Omega \operatorname{div} (\varrho^j \mathbf{u}^j \otimes \mathbf{u}^j) \cdot \mathbf{u}^j \ \mathrm{d}x \\ &- \varepsilon \lambda \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \mathbf{u}^j \ \mathrm{d}x + \lambda \int_\Omega \varrho^j \mathbf{f}^j \cdot \mathbf{u}^j \ \mathrm{d}x. \end{aligned} \tag{27}$$

The first term on the l.h.s. can be used to control the norm of $\mathbf{u}$ in $W_0^{1,2}(\Omega)$. This is due to a simple generalization of the Korn inequality.

Lemma 7 *For $\mathbf{u} \in W_0^{1,2}(\Omega)$ and $\mathbf{S}$ satisfying (9) and (10), there exists a constant c depending on Ω and μ such that*

$$c \|\mathbf{u}\|^2_{W^{1,2}(\Omega)} \le \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \ \mathrm{d}x.$$

Proof. Rewriting the viscous part of the stress tensor in the form

$$\mathbf{S}(\mathbf{u}) = 2\mu \left(\mathbf{D}(\mathbf{u}) - \frac{1}{3} \operatorname{div} \mathbf{u} \mathbb{I} \right) + \xi \operatorname{div} \mathbf{u} \mathbb{I},$$

we can estimate

$$\int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \ \mathrm{d}x \ge \mu \int_\Omega \left(|\nabla \mathbf{u}|^2 + (\nabla \mathbf{u})^T : \nabla \mathbf{u} - \frac{2}{3} (\operatorname{div} \mathbf{u})^2 \right) \ \mathrm{d}x = \mu \int_\Omega \left(|\nabla \mathbf{u}|^2 + \frac{1}{3} (\operatorname{div} \mathbf{u})^2 \right) \ \mathrm{d}x,$$

and we conclude by application of the Poincaré inequality. □

Next, we use (17) with $\varrho = \mathcal{S}(\mathbf{u})$ to express

$$\begin{aligned} &\int_\Omega \nabla (\varrho^j)^\gamma \cdot \mathbf{u}^j \ \mathrm{d}x \\ &= \varepsilon \gamma \int_\Omega (\varrho^j)^{\gamma-2} |\nabla \varrho^j|^2 \ \mathrm{d}x + (\triangle t)^{-1} \frac{s}{s-1} \int_\Omega (\varrho^j)^\gamma \ \mathrm{d}x - (\triangle t)^{-1} \frac{\gamma}{\gamma - 1} \int_\Omega \varrho^{j-1} (\varrho^j)^{\gamma-1} \ \mathrm{d}x, \end{aligned}$$

and the same for artificial pressure $\delta(\varrho^j)^\Gamma$. Then, integrating (27) by parts we derive

$$\begin{aligned} &\int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \mathbf{u}^j \ \mathrm{d}x + \frac{t(\triangle t)^{-1}}{2} \int_\Omega \left(\varrho^{j-1} + \varrho^j \right) |\mathbf{u}^j|^2 \ \mathrm{d}x \\ &+ t\varepsilon\gamma \int_\Omega (\varrho^j)^{\gamma-2} |\nabla \varrho^j|^2 \ \mathrm{d}x + t\varepsilon\delta\Gamma \int_\Omega (\varrho^j)^{\Gamma-2} |\nabla \varrho^j|^2 \ \mathrm{d}x \\ &+ t(\triangle t)^{-1} \frac{\gamma}{\gamma-1} \int_\Omega (\varrho^j)^\gamma \ \mathrm{d}x + t(\triangle t)^{-1} \delta \frac{\Gamma}{\Gamma - 1} \int_\Omega (\varrho^j)^\Gamma \ \mathrm{d}x \\ =& t(\triangle t)^{-1} \frac{\gamma}{\gamma-1} \int_\Omega \varrho^{j-1} (\varrho^j)^{\gamma-1} \ \mathrm{d}x + t(\triangle t)^{-1} \frac{\delta\Gamma}{\Gamma-1} \int_\Omega \varrho^{j-1} (\varrho^j)^{\Gamma-1} \ \mathrm{d}x \\ &+ t \int_\Omega \sum_{k=1}^n \frac{\hat{Y}_k^j}{m_k} \varrho^j \operatorname{div} \mathbf{u}^j \ \mathrm{d}x + t(\triangle t)^{-1} \int_\Omega \varrho^{j-1} \mathbf{u}^{j-1} \cdot \mathbf{u}^j \ \mathrm{d}x + t \int_\Omega \varrho^j \mathbf{f}^j \cdot \mathbf{u}^j \ \mathrm{d}x. \end{aligned} \tag{28}$$

Summing up equations (26) and (28), using the Cauchy inequality, boundedness of ω_k and equivalency of norms on W_N, we show

$$\|\mathbf{u}^j\|_{W_N} + \|Y_k^j\|_{W^{1,2}(\Omega)} \le c, \tag{29}$$

with a constant c independent of t. Finally, we may estimate the norm of second gradient of Y_k directly from (25), we have

$$\begin{aligned} &-(D + \varepsilon \varrho^j) \Delta Y_k^j \\ &= -t \left(\varrho^j \omega_k + (\triangle t)^{-1} \varrho^{j-1} Y_k^{j-1} - (\triangle t)^{-1} \varrho^j Y_k^j - \operatorname{div} \left(\varrho^j Y_k^j \mathbf{u}^j \right) + \varepsilon \operatorname{div} \left(Y_k^j \nabla \varrho^j \right) \right) - \varepsilon \nabla \varrho^j \cdot \nabla Y_k^j. \end{aligned}$$

Due to regularity of $\varrho^j, \mathbf{u}^j$ and estimate (29), we first justify that $Y_k^j \in W^{2,2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega)$. Then, by the bootstrap procedure, we arrive at $\|Y_k\|_{W^{2,p}(\Omega)} \leq c$, which finishes the proof of Lemma 6.

To conclude, we observe that (22) is valid, in particular $\hat{Y_k}^j = Y_k^j$ in (23) and the proof of Theorem 3 is complete. □

4.2 Limit passage in the Galerkin approximation

First, we need estimates which are uniform with respect to N. They can be deduced easily from (26) and (28) taking $t = 1$. We have

$$\|\mathbf{u}_N^j\|_{W^{1,2}(\Omega)} + \sum_{k=1}^n \|Y_{k,N}^j\|_{W^{1,2}(\Omega)} + \|\varrho_N^j\|_{L^\Gamma(\Omega)} + \|\nabla \left(\varrho_N^j\right)^{\Gamma/2}\|_{L^2(\Omega)} \leq c. \tag{30}$$

Moreover, using standard elliptic theory, we can deduce that ϱ_N^j satisfies

$$\|\varrho_N^j\|_{W^{2,2}(\Omega)} \leq c. \tag{31}$$

Using (30) and (31) and the imbedding theorems, we may justify that there exists a subsequence (denoted by N) such that

$$\begin{array}{lll}
\varrho_N^j & \to \varrho^j, & \text{weakly in } W^{2,2}(\Omega) \quad \text{and strongly in } W^{1,q}(\Omega),\ q < 6, \\
\varrho_N^j & \to \varrho^j, & \text{weakly}^* \text{ in } L^\infty(\Omega) \\
\mathbf{u}_N^j & \to \mathbf{u}^j, & \text{weakly in } W^{1,2}(\Omega) \quad \text{and strongly in } L^q(\Omega),\ q < 6, \\
Y_{k,N}^j & \to Y_k^j, & \text{weakly in } W^{1,2}(\Omega) \quad \text{and strongly in } L^q(\Omega),\ q < 6, \\
Y_{k,N}^j & \to Y_k^j, & \text{weakly}^* \text{ in } L^\infty(\Omega).
\end{array}$$

Having this, justification of the limit in (17), (18) and (19) is an easy exercise, so we skip the details.

4.3 Uniform estimate of the pressure

We again start with deriving some uniform estimates. The uniform estimates resulting from (26) and (28) are the following

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon^j\|^2_{W^{1,2}(\Omega)} + \sum_{k=1}^n \|Y_{k,\varepsilon}^j\|^2_{W^{1,2}(\Omega)} + \delta(\triangle t)^{-1}\|\varrho_\varepsilon^j\|^\Gamma_{L^\Gamma(\Omega)} + (\triangle t)^{-1}\|\varrho_\varepsilon^j\|^\gamma_{L^\gamma(\Omega)} \\
&\qquad + \varepsilon\delta\|\nabla \left(\varrho_\varepsilon^j\right)^{\Gamma/2}\|^2_{L^2(\Omega)} + \varepsilon\|\nabla \left(\varrho_\varepsilon^j\right)^{\gamma/2}\|^2_{L^2(\Omega)} \leq c.
\end{aligned} \tag{32}$$

However, we still need a better estimate of the pressure. It can be derived from the Bogovskii-type of estimate. When $\gamma \geq 2$ and $\Gamma \geq 3$ this estimate allows to control the barotropic component of the pressure in a little better space then $L^1(\Omega)$, which becomes important in the course of all subsequent limit passages.

We test the approximate momentum equation (18) by a function

$$\Phi = \mathcal{B}\left(\left(\varrho^j\right)^\beta - \frac{1}{|\Omega|}\int_\Omega \left(\varrho^j\right)^\beta \, \mathrm{d}x\right),$$

where $\beta \in (0,1]$ and $\mathcal{B}$ is the Bogovskii operator for which we know in particular that

$$\|\nabla\Phi\|_p \leq c(p,\Omega)\|\left(\varrho^j\right)^\beta\|_p$$

and due to the Sobolev imbedding

$$\|\Phi\|_{\bar{p}} \leq c(p,\Omega)\|\left(\varrho^j\right)^\beta\|_p, \quad 1 < p < \infty, \qquad \bar{p} = \begin{cases} \frac{3p}{3-p} & \text{if} \quad p < 3, \\ \in [1,\infty) & \text{if} \quad p = 3, \\ \infty & \text{if} \quad p > 3. \end{cases}$$

For more details concerning Bogovskii operator we refer to [19]. This testing results in the following identity

$$
\begin{aligned}
&\int_\Omega \left(\left(\varrho^j\right)^{\gamma+\beta} + \delta \left(\varrho^j\right)^{\Gamma+\beta} \right) \, \mathrm{d}x \\
&\qquad = (\triangle t)^{-1} \int_\Omega \left(\varrho^j \mathbf{u}^j - \varrho^{j-1}\mathbf{u}^{j-1} \right) \cdot \Phi \, \mathrm{d}x - \int_\Omega \varrho^j \left(\mathbf{u}^j \otimes \mathbf{u}^j \right) : \nabla \Phi \, \mathrm{d}x \\
&\qquad + \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \Phi \, \mathrm{d}x + \varepsilon \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \Phi \, \mathrm{d}x + \int_\Omega \sum_{k=1}^n \frac{Y^j}{m_k} \left(\varrho^j\right)^{\beta+1} \, \mathrm{d}x \\
&\qquad + \frac{1}{|\Omega|} \int_\Omega \left(\pi(\varrho^j, Y^j) + \delta \left(\varrho^j\right)^{\Gamma} \right) \, \mathrm{d}x \int_\Omega \left(\varrho^j\right)^{\beta} \, \mathrm{d}x - \int_\Omega \varrho^j \mathbf{f}^j \cdot \Phi \, \mathrm{d}x \\
&\qquad = \sum_{i=1}^7 I_i.
\end{aligned}
\tag{33}
$$

We will estimate only the most restrictive terms. The first is the convective term, we have

$$
I_2 \leq \|\varrho^j\|_p \| |\mathbf{u}^j|^2 \|_3 \| \left(\varrho^j\right)^{\beta} \|_q \leq t \|\varrho^j\|_p \|\mathbf{u}^j\|_{1,2}^2 \|\varrho^j\|_{q\beta}^{\beta} \leq c \|\varrho^j\|_p \|\varrho\|_{q\beta}^{\beta} \tag{34}
$$

where $\frac{1}{p} + \frac{1}{q} = \frac{2}{3}$. In the last inequality we used (32) to estimate the norm of $\mathbf{u}^j$. Now we chose $p = q\beta = \frac{3(\beta+1)}{2}$ and we apply the interpolation inequality of the type $\|\varrho^j\|_p \leq \|\varrho^j\|_1^{\vartheta} \|\varrho^j\|_{\gamma+\beta}^{1-\vartheta}$ which leads to the restriction

$$
\beta \leq 2\gamma - 3.
$$

Next we handle I_4. Employing the Hölder inequality we get for $\frac{1}{\overline{p}'} + \frac{1}{\overline{p}} = \frac{1}{2}$

$$
I_4 = \varepsilon \int_\Omega \nabla \varrho^j \cdot \nabla \mathbf{u}^j \cdot \Phi \, \mathrm{d}x \leq c\varepsilon \|\nabla \varrho^j\|_{\overline{p}'} \|\mathbf{u}\|_{1,2} \|\Phi\|_{\overline{p}} \leq c\varepsilon \|\nabla \varrho^j\|_2 \|\varrho^j\|_{\beta p}^{\beta},
$$

for some $p > 3$. We choose p such that $\beta p = \beta + \Gamma$, so $\Gamma > 2\beta$.

To get the estimate for $\|\nabla\varrho\|_2$ we need to interpret the approximate continuity equation as a Neumann-boundary problem

$$
\begin{aligned}
-\varepsilon \Delta \varrho &= \operatorname{div} \mathbf{b} \quad \mathit{in}\ \Omega \\
\tfrac{\partial \varrho}{\partial \mathbf{n}} &= b \cdot \mathbf{n} \quad \mathit{at}\ \partial\Omega,
\end{aligned}
\tag{35}
$$

with the right hand side

$$
\mathbf{b} = (\triangle t)^{-1} \mathcal{B}(h - \varrho) - \varrho \mathbf{u}.
$$

¿From the classical theory we know that if $\partial\Omega$ is smooth enough and if $b \in L^q(\Omega)$, then there exists the unique $\varrho \in W^{1,q}(\Omega)$ satisfying (35) in the weak sense, such that $\int_\Omega \varrho \, \mathrm{d}x = \mathit{const}$. Moreover,

$$
\|\nabla \varrho\|_q \leq \frac{c(q,\Omega)}{\varepsilon} \|\mathbf{b}\|_q. \tag{36}
$$

In our case it is enough to see that the q-norm of b may be estimated as

$$
\|\mathbf{b}\|_2 \leq c(\triangle t)^{-1} \left(\|\varrho\|_\gamma + \|h\|_\gamma \right) + \|\varrho\|_3 \|\mathbf{u}\|_6. \tag{37}
$$

Thus observation (36) together with (32) and assumption $\Gamma \geq 3$ yields the following estimate of I_4

$$
I_4 \leq c(\delta) \|\varrho\|_{\beta+\Gamma}^{\beta}.
$$

Therefore, from (33) we deduce that independently of ε, we have

$$
\|\varrho\|_{(1+\beta)\gamma} + \delta \|\varrho\|_{(1+\beta)\Gamma} \leq c(\delta), \tag{38}
$$

for $\gamma \geq 2$, $\Gamma \geq 3$.

4.4 Limit passage $\varepsilon \to 0$

The estimates from the previous section (29), (32), (38) can be used to deduce that, at least for a suitable subsequences, we have

$$\mathbf{u}_\varepsilon^j \to \mathbf{u} \quad \text{weakly in } W^{1,2}(\Omega), \tag{39}$$

$$\varrho_\varepsilon^j \to \varrho \quad \text{weakly in } L^{(1+\beta)\gamma}(\Omega) \cup L^{(1+\beta)\Gamma}(\Omega), \tag{40}$$

$$\varepsilon \nabla \varrho_\varepsilon^j \to 0 \quad \text{strongly in } L^2(\Omega), \tag{41}$$

$$Y_{k,\varepsilon}^j \to Y_k \quad \text{weakly in } W^{1,2}(\Omega), \text{ strongly in } L^p(\Omega), \ p < 6, \tag{42}$$

$$Y_{k,\varepsilon}^j \to Y_k \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{43}$$

We are hence in a position to conclude that there exists $(\varrho^j, \mathbf{u}^j, Y^j)$ which satisfies the integral equalities:

$$(\triangle t)^{-1} \int_\Omega \left(\varrho^j - \varrho^{j-1}\right) \xi \, \mathrm{d}x - \int_\Omega \varrho^j \mathbf{u}^j \cdot \nabla \xi \, \mathrm{d}x = 0, \quad \forall \xi \in C^\infty(\overline{\Omega}), \tag{44}$$

$$\begin{aligned}(\triangle t)^{-1} \int_\Omega \left(\varrho^j \mathbf{u}^j - \varrho^{j-1} \mathbf{u}^{j-1}\right) \boldsymbol{\varphi} \, \mathrm{d}x - \int_\Omega \varrho \left(\mathbf{u}^j \otimes \mathbf{u}^j\right) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \\ - \int_\Omega \overline{\pi(\varrho, Y) + \delta \varrho^\Gamma} \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x = \int_\Omega \varrho^j \mathbf{f}^j \cdot \boldsymbol{\varphi} \, \mathrm{d}x, \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega),\end{aligned}$$

$$\begin{aligned}(\triangle t)^{-1} \int_\Omega \left(\varrho^j Y_k^j - \varrho^{j-1} Y_k^{j-1}\right) \phi \, \mathrm{d}x - \int_\Omega \varrho^j Y_k^j \mathbf{u}^j \cdot \nabla \phi \, \mathrm{d}x - \int_\Omega D \nabla Y_k^j \cdot \nabla \phi \, \mathrm{d}x \\ = \int_\Omega \varrho^j \omega_k(Y^j) \phi \, \mathrm{d}x \quad \forall \phi \in C^\infty(\overline{\Omega}),\end{aligned}$$

for $k \in \{1, \dots, n\}$. Here and in the sequel $\overline{g(\varrho^j, \mathbf{u}^j, Y^j)}$ denotes the weak limit of a sequence $g(\varrho_\varepsilon^j, \mathbf{u}_\varepsilon^j, Y_\varepsilon^j)$.

To conclude one needs to verify if $\overline{\pi(\varrho, Y) + \delta \varrho^\Gamma} = \pi(\varrho, Y) + \delta \varrho^\Gamma$. In view of the strong convergence of Y_k for $k \in K$, the positive answer to this question is in fact equivalent to the strong convergence of the density.

Since the strong convergence of the sequence approximating the density can not be deduced from the system directly, we will apply the technique introduced by Lions [11]. It is based on an observation that the missing compactness can be replaced by the compactness of the quantity called the effective viscous flux or the effective pressure.

To derive a key equality for this reasoning, we introduce the the inverse divergence operator $\mathcal{A} = \nabla \Delta^{-1}$ and the double Riesz transform $\mathcal{R} = \nabla \otimes \nabla \Delta^{-1}$ specified by (45) and (46) below

$$\mathcal{A}_j[v] = \left(\nabla \Delta^{-1}\right)_j v = -\mathcal{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2} \mathcal{F}(v)\right), \tag{45}$$

$$\mathcal{R}_{i,j}[v] = \partial_i \mathcal{A}_j[v] = \left(\nabla \otimes \nabla \Delta^{-1}\right)_{i,j} v = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F}(v)\right). \tag{46}$$

Here, the inverse Laplacian is identified through the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ as

$$(-\Delta)^{-1}(v) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2} \mathcal{F}(v)\right).$$

In what follows we recall some of basic properties of these operators.

Lemma 8 *The operator $\mathcal{R}$ is a continuous linear operator from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$. In particular, the following estimate holds true:*

$$\|\mathcal{R}[v]\|_{L^p(\mathbb{R}^3)} \le c(p) \|v\|_{L^p(\mathbb{R}^3)} \quad \textit{for all } v \in L^p(\mathbb{R}^3).$$

The operator $\mathcal{A}$ is a continuous linear operator from $L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and from $L^p(\mathbb{R}^3)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ for any $1 < p < 3$.

Moreover,

$$\|\nabla \mathcal{A}[v]\|_p \leq C(p)\|v\|_p, \quad 1 < p < \infty.$$

The proof of this lemma can be found e.g. in [6], Section 10.16. For more information about the operators defined by means of Fourier multiplier we refer the reader to [21], Chapters III and IV.

In what follows we present two important properties of commutators involving Riesz operator. The first result is a straightforward consequence of the *Div-Curl* lemma (see [22]), its proof can be found in [4], Lemma 5.1.

Lemma 9 *Let*

$$\mathbf{V}_\varepsilon \rightharpoonup \mathbf{V} \quad \text{weakly in } L^p(\mathbb{R}^3), \quad r_\varepsilon \rightharpoonup r \quad \text{weakly in } L^q(\mathbb{R}^3),$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$\mathbf{V}_\varepsilon \mathcal{R}(r_\varepsilon) - r_\varepsilon \mathcal{R}(\mathbf{V}_\varepsilon) \rightharpoonup \mathbf{V}\mathcal{R}(r) - r\mathcal{R}(\mathbf{V}) \quad \textit{weakly in } L^s(\mathbb{R}^3).$$

The next lemma can be deduced from the general results of Bajšanski and Coifman [1], and Coifman and Meyer [2].

Lemma 10 *Let $w \in W^{1,r}(\mathbb{R}^3)$ and $\mathbf{V} \in L^p(\mathbb{R}^3)$ be given, where $1 < r < 3$, $1 < p < \infty$, $\frac{1}{r} + \frac{1}{p} - \frac{1}{3} < \frac{1}{s} < 1$. Then for all such s we have*

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{\alpha,s}(\mathbb{R}^3)} \leq c(s,p,r)\|w\|_{W^{1,r}(\mathbb{R}^3)}\|\mathbf{V}\|_{L^p(\mathbb{R}^3)},$$

where α is given by $\frac{\alpha}{3} = \frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{r}$.

Here, $W^{\alpha,s}(\mathbb{R}^3)$ for $\alpha \in (0,\infty) \setminus \mathbb{N}$ denotes the Sobolev-Slobodeckii space (see e.g. [23]). The proof can be found in [6], Section 10.17.

In order to proceed we first observe that since $\varrho_\varepsilon \mathbf{u}_\varepsilon$ and $\nabla \varrho_\varepsilon$ possess zero normal traces, it is possible to extend the approximate continuity equation to the whole $\mathbb{R}^3$

$$(\triangle t)^{-1} 1_\Omega \varrho_\varepsilon^j + \operatorname{div}(1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j) = \varepsilon \operatorname{div}(1_\Omega \nabla \varrho_\varepsilon^j) + (\triangle t)^{-1} 1_\Omega \varrho_\varepsilon^{j-1}. \tag{47}$$

Next, we test the approximate momentum equation by the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla \Delta^{-1})[1_\Omega \varrho_\varepsilon^j], \ \zeta \in C_0^\infty(\Omega).$$

Note that this operation "gains" one derivative thus using only the $L^\Gamma(\Omega)$ integrability of ϱ_ε we justify that this is an admissible test function. Evidently $\sum_{i=1}^3 \mathcal{R}_{i,i}[v] = v$, thus integrating by parts we obtain the following equivalence

$$\begin{aligned}
&\int_\Omega \zeta \left((\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma)\varrho_\varepsilon^j - \mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \right) \, \mathrm{d}x \\
&\qquad = -(\triangle t)^{-1} \int_\Omega \zeta \left(\varrho_\varepsilon^j u_{\varepsilon,i}^j - \varrho_\varepsilon^{j-1} u_{\varepsilon,i}^{j-1} \right) \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x - \int_\Omega \zeta \varrho_\varepsilon^j u_{\varepsilon,i}^j u_{\varepsilon,k}^j \mathcal{R}_{i,k}[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x \\
&\qquad - \int_\Omega \varrho_\varepsilon^j u_{\varepsilon,i}^j u_{\varepsilon,k}^j \partial_k \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon] \, \mathrm{d}x + \int_\Omega S_{i,k} \partial_k \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x \\
&\qquad - \int_\Omega \left(\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma \right) \partial_i \zeta \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x + \varepsilon \int_\Omega \zeta \nabla \varrho_\varepsilon^j \cdot \nabla u_{\varepsilon,i}^j \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x \\
&\qquad - \int_\Omega \zeta \varrho^j f_i^j \mathcal{A}_i[1_\Omega \varrho_\varepsilon^j] \, \mathrm{d}x,
\end{aligned} \tag{48}$$

where we used the Einstein summation convention. Adding and subtracting $\int_\Omega \zeta \varrho_\varepsilon^j u_{\varepsilon,k}^j \mathcal{R}_{i,k}[1_\Omega \varrho_\varepsilon^j u_{\varepsilon,i}^j]\,\mathrm{d}x$ we may rewrite the r.h.s. in the form which lets the commutator appear. Then, using the fact that $R_{i,j}[v] = \partial_i \mathcal{A}_j[v]$, the basic properties of the Riesz operator:

$$\mathcal{R}_{i,j}[v] = \mathcal{R}_{j,i}[v], \quad \int_{\mathbb{R}^3} \mathcal{R}_{i,j}[v]u\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathcal{R}_{j,i}[v]u\,\mathrm{d}x, \quad v \in L^p(\mathbb{R}^3),\ u \in L^{p'}(\mathbb{R}^3)$$

and the approximate continuity equation, we transform (48) into

$$\begin{aligned}
&\int_\Omega \zeta \left((\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma)\varrho_\varepsilon^j - \mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \right)\,\mathrm{d}x \\
= &\int_\Omega \zeta \left(\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j] - \varrho_\varepsilon^j (\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \right)\,\mathrm{d}x - \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[\operatorname{div} 1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j]\,\mathrm{d}x \\
&+ (\triangle t)^{-1} \int_\Omega \zeta \left(\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j - \varrho_\varepsilon^{j-1} \mathbf{u}_\varepsilon^{j-1} \right) \cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon (\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x \\
&+ \int_\Omega \mathbf{S}(\mathbf{u}_\varepsilon^j) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x - \int_\Omega \left(\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma \right) \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x \\
&+ \varepsilon \int_\Omega \zeta \nabla\varrho_\varepsilon^j \cdot \nabla\mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x - \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{f}^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho_\varepsilon^j]\,\mathrm{d}x \\
= &\sum_{i=1}^8 I_i.
\end{aligned} \tag{49}$$

Finally, I_2 may be expressed by means of approximate (extended) continuity equation (47) in the following way

$$I_2 = -\varepsilon \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[\operatorname{div} 1_\Omega \nabla\varrho_\varepsilon^j]\,\mathrm{d}x + (\triangle t)^{-1} \int_\Omega \zeta \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \nabla\Delta^{-1}[1_\Omega(\varrho_\varepsilon^j - \varrho_\varepsilon^{j-1})]\,\mathrm{d}x. \tag{50}$$

We will compare (49) with a similar expression obtained by testing the limit momentum equation with the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega \varrho^j],\ \zeta \in C_0^\infty(\Omega),$$

note that this is still an admissible test function as $\Gamma \geq 3$. We have

$$\begin{aligned}
&\int_\Omega \zeta \left(\overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma}\varrho^j - \mathbf{S}(\mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right)\,\mathrm{d}x \\
= &\int_\Omega \zeta \left(\varrho^j \mathbf{u}^j \cdot \mathcal{R}[1_\Omega \varrho^j \mathbf{u}^j] - \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right)\,\mathrm{d}x - \int_\Omega \zeta \varrho^j \mathbf{u}^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho^{j-1}]\,\mathrm{d}x \\
&+ (\triangle t)^{-1} \int_\Omega \zeta \left(\varrho^j \mathbf{u}^j - \varrho^{j-1} \mathbf{u}^{j-1} \right) \cdot \nabla\Delta^{-1}[1_\Omega \varrho^j]\,\mathrm{d}x - \int_\Omega \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho^j]\,\mathrm{d}x \\
&+ \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla\zeta \nabla\Delta^{-1}[1_\Omega \varrho^j]\,\mathrm{d}x - \int_\Omega \overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma} \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho^j]\,\mathrm{d}x \\
&- \int_\Omega \zeta \varrho^j \mathbf{f}^j \cdot \nabla\Delta^{-1}[1_\Omega \varrho^j]\,\mathrm{d}x \\
= &\sum_{i=1}^7 I_i.
\end{aligned} \tag{51}$$

Then, we see that

$$(\nabla\Delta^{-1})[1_\Omega \varrho_\varepsilon^j] \to (\nabla\Delta^{-1})[1_\Omega \varrho^j] \quad \text{in } C(\overline{\Omega}), \tag{52}$$

which is the consequence of Lemma 8. Recalling (39-42) we show that the ε-dependent integral on the r.h.s. of (49) vanish, whence $I_2, \ldots, I_6$ converge to their counterparts in (51).

In what follows we give some more details of these limit passages. Firstly, due to the compact imbedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p < 6$, we have

$$\mathbf{u}_\varepsilon^j \to \mathbf{u}^j \quad \text{strongly in } L^p(\Omega), \quad 1 \leq p < 6, \tag{53}$$

taking into account also (40) we therefore get

$$\mathbf{u}_\varepsilon^j \varrho_\varepsilon^j \to \mathbf{u}^j \varrho^j \quad \text{weakly in } L^p(\Omega), \quad 1 \leq p < \frac{6(1+\beta)\gamma}{6+(1+\beta)\gamma}. \tag{54}$$

Since $\frac{6(1+\beta)\gamma}{6+(1+\beta)\gamma} > 2$ for $\gamma \geq 2$ and by virtue of (41) and Lemma 8, we check that the ε-dependent component in (50) tends to 0, i.e. $\varepsilon \int_\Omega \zeta \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta^{-1}[\operatorname{div} 1_\Omega \nabla \varrho_\varepsilon] \,\mathrm{d}x \to 0$. Next, due to (39) and (41) $\varepsilon(\nabla \varrho_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \to 0 \quad$ weakly in $L^1(\Omega)$, which coupled with (52), implies the zero limit of I_7.

Therefore, by letting ε to 0 in (49) and comparing the limit wit (51) we verify that

$$\begin{aligned}
&\int_\Omega \zeta \left(\left(\pi(\varrho_\varepsilon^j, Y_\varepsilon^j) + \delta(\varrho_\varepsilon^j)^\Gamma \right) \varrho_\varepsilon^j - \mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \right) \,\mathrm{d}x \\
&- \int_\Omega \zeta \left(\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j] - \varrho_\varepsilon^j (\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \right) \,\mathrm{d}x \\
&\qquad \to \int_\Omega \zeta \left(\overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma} \varrho^j - \mathbf{S}(\mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right) \,\mathrm{d}x \\
&\qquad - \int_\Omega \zeta \left(\varrho^j \mathbf{u}^j \cdot \mathcal{R}[1_\Omega \varrho^j \mathbf{u}^j] - \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j] \right) \,\mathrm{d}x.
\end{aligned} \tag{55}$$

Our next aim is to show that the last terms on both sides cancel.

For this purpose we take $\mathbf{V}_\varepsilon^j = \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j$, $r_\varepsilon^j = \varrho_\varepsilon^j$ and check that they fulfill assumptions of Lemma 9 with $p = \frac{6(1+\beta)\gamma}{(1+\beta)\gamma+6}$ and $q = (1+\beta)\gamma$, where by $\varrho_\varepsilon^j, \mathbf{u}_\varepsilon^j, \varrho^j, \mathbf{u}^j$ we mean the functions extended by 0 outside Ω. Thus, there is enough room to choose $s > 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and so Lemma 9 yields

$$\varrho_\varepsilon^j \mathbf{u}_\varepsilon^j \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon^j \mathbf{u}_\varepsilon^j] - \varrho_\varepsilon^j (\mathbf{u}_\varepsilon^j \otimes \mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \to \varrho^j \mathbf{u}^j \cdot \mathcal{R}[1_\Omega \varrho^j \mathbf{u}^j] - \varrho^j (\mathbf{u}^j \otimes \mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j],$$

weakly in $L^s(\Omega)$. Substituting this result into (55) we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_\Omega \zeta \left(\left(\pi(\varrho_\varepsilon, Y_\varepsilon) + \delta \varrho^\Gamma \right) \varrho_\varepsilon - \mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \right) \,\mathrm{d}x \\
&\qquad = \int_\Omega \zeta \left(\overline{\pi(\varrho, Y) + \delta \varrho^\Gamma} \varrho - \mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \,\mathrm{d}x.
\end{aligned} \tag{56}$$

We express $\mathbf{S}(\mathbf{u}_\varepsilon^j) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j]$ and $\mathbf{S}(\mathbf{u}^j) : \mathcal{R}[1_\Omega \varrho^j]$ in terms of divergence of $\mathbf{u}_\varepsilon^j$ and $\mathbf{u}^j$, respectively. For the second part of (9) we have

$$\nu \operatorname{div} \mathbf{u}_\varepsilon^j \mathbf{I} : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] = \nu \sum_{i=1}^3 \operatorname{div} \mathbf{u}_\varepsilon^j \mathcal{R}_{i,i}[1_\Omega \varrho_\varepsilon^j] = \nu 1_\Omega \operatorname{div} \mathbf{u}_\varepsilon^j \varrho_\varepsilon^j.$$

To handle the first part, we integrate by parts and we check that

$$\int_\Omega \zeta \mu \left(\nabla \mathbf{u}_\varepsilon^j + (\nabla \mathbf{u}_\varepsilon^j)^T \right) : \mathcal{R}[1_\Omega \varrho_\varepsilon^j] \,\mathrm{d}x = \int_\Omega \mathcal{R} : [\zeta \mu \left(\nabla \mathbf{u}_\varepsilon^j + (\nabla \mathbf{u}_\varepsilon^j)^T \right)] \varrho_\varepsilon^j \,\mathrm{d}x. \tag{57}$$

Observe that $\mathcal{R} : \left[\nabla \mathbf{u}_\varepsilon + (\nabla \mathbf{u}_\varepsilon)^T \right] = 2 \sum_{i,j=1}^3 \partial_i \mathcal{A}_j \partial_j u_{\varepsilon,i} = 2 \sum_i^3 \partial_i \sum_{j=1}^3 \mathcal{R}_{j,j} u_i = 2 \operatorname{div} \mathbf{u}_\varepsilon$, thus, the r.h.s. of (57) can be rewritten as

$$\int_\Omega \mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] \varrho_\varepsilon \,\mathrm{d}x = \int_\Omega \zeta 2\mu \operatorname{div} \mathbf{u}_\varepsilon \varrho_\varepsilon \,\mathrm{d}x + \int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] \right) \varrho_\varepsilon \,\mathrm{d}x.$$

Repeating the same procedure for the limit stress tensor $\mathbf{S}(\mathbf{u})$ we obtain from (56) the following expression

$$
\begin{aligned}
&\int_\Omega \zeta \left(\overline{(\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma)\,\varrho^j} - (2\mu+\nu)\,\overline{\operatorname{div}\mathbf{u}^j\,\varrho^j} \right)\,\mathrm{d}x \\
&\qquad = \int_\Omega \zeta \left(\overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma}\,\varrho^j - (2\mu+\nu)\operatorname{div}\mathbf{u}^j\,\varrho^j \right)\,\mathrm{d}x \\
&\qquad + \lim_{\varepsilon\to 0}\int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R} : [2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\right)\varrho^j_\varepsilon\,\mathrm{d}x \\
&\qquad - \int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}^j)] - \zeta\mathcal{R} : [2\mu\mathbf{D}(\mathbf{u}^j)]\right)\varrho^j\,\mathrm{d}x.
\end{aligned} \tag{58}
$$

In order to show that the two last integrals cancel we will apply Lemma 10 to each row of the matrix $\mathbf{D}(\mathbf{u}^j_\varepsilon)$, i.e. we take

$$
w = \zeta, \quad V^j_i = \partial_i u^j_{\varepsilon,k} + \partial_k u^j_{\varepsilon,i}, \; j = 1,2,3.
$$

By virtue of (32), $\mathbf{V}^j \in L^2(\mathbb{R}^3)$. Since ζ extended by 0 outside Ω belongs in particular to $W^{1,\infty}(\mathbb{R}^3)$ we can take any $s \in (1,6)$ and $\alpha = \frac{6-s}{2s}$ for which

$$
\|\mathcal{R} : [\zeta 2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R} : [2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\|_{W^{\alpha,s}(\mathbb{R}^3)} \le c.
$$

Next, we may use the fact that $W^{\alpha,s}(\mathbb{R}^3)$ is continuously embedded into $L^a(\mathbb{R}^3)$ for any $1 \le a \le 6$ and the embedding is compact for $a < 6$. Moreover, since $\frac{1}{q} = \frac{1}{a} + \frac{1}{(1+\beta)\gamma} < 1$, thus

$$
\left(\mathcal{R} : [\zeta 2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)] - \zeta\mathcal{R} : [2\mu\mathbf{D}(\mathbf{u}^j_\varepsilon)]\right)\varrho^j_\varepsilon \to \left(\mathcal{R} : [\zeta 2\mu\mathbf{D}(\mathbf{u}^j)] - \zeta\mathcal{R} : [2\mu\mathbf{D}(\mathbf{u}^j)]\right)\varrho^j
$$

weakly in $L^q(\mathbb{R}^3)$. Therefore (58) reduces to the following remarkable identity:

$$
\begin{aligned}
&\int_\Omega \zeta \left(\overline{\pi(\varrho^j, Y^j)\varrho^j + \delta(\varrho^j)^{\Gamma+1}} - (2\mu+\nu)\,\overline{\operatorname{div}\mathbf{u}^j\,\varrho^j} \right)\,\mathrm{d}x \\
&= \int_\Omega \zeta \left(\overline{\pi(\varrho^j, Y^j) + \delta(\varrho^j)^\Gamma}\,\varrho - (2\mu+\nu)\operatorname{div}\mathbf{u}^j\,\varrho^j \right)\,\mathrm{d}x.
\end{aligned} \tag{59}
$$

In what follows, we will exploit (59) by use of the renormalized continuity equation. The following result is a consequence of technique introduced and developed by DiPerna and Lions [3]. Applying it to the continuity equation (extended by 0 outside Ω) we obtain.

Lemma 11 *Let $\varrho^{j-1}, \varrho^j \in L^p(\mathbb{R}^3)$, $p \ge 2$, $\varrho \ge 0$, a. e. in Ω, $\mathbf{u} \in W^{1,2}_0(\mathbb{R}^3)$ satisfy the continuity equation*

$$
(\triangle t)^{-1}\varrho^j + \operatorname{div}(\varrho^j\mathbf{u}^j) = (\triangle t)^{-1}\varrho^{j-1}
$$

in the sense of distributions on $\mathbb{R}^3$.

Then the pair $(\varrho^j, \mathbf{u}^j)$ solves the renormalized continuity equation (16) in the sense of distributions on $\mathbb{R}^3$ where $b(\cdot)$ is specified as follows:

$$
\begin{gathered}
b \in C([0,\infty) \cap C^1((0,\infty)), \\
\lim_{s\to 0^+}(sb'(s) - b(s)) \in \mathbb{R}, \\
|b'(s)| \le Cs^\lambda, \quad s \in (1,\infty), \quad \lambda \le \tfrac{p}{2} - 1.
\end{gathered}
$$

The best general reference here is [6], Section 10.18, see also [19].

Applying Lemma 11 to the limit continuity equation we can verify that the pair of functions $(\varrho, \mathbf{u})$ extended by zero outside of Ω is a solution to the renormalized continuity equation, as specified in Definition 3. Moreover, taking $b(\varrho^j) = \varrho^j \log \varrho^j$ it can be deduced from (16) and from (44) with $\xi = 1$ that

$$
(\triangle t)^{-1}\int_\Omega \left(\varrho^j - \varrho^{j-1}\right)\log\varrho^j\,\mathrm{d}x + \int_\Omega \varrho^j \operatorname{div}\mathbf{u}^j\,\mathrm{d}x = 0.
$$

We now test the approximate continuity equation (17) with $\log(\varrho_\varepsilon^j+\eta)$, $\eta>0$, note that due to Lemma 4 this is an admissible test function

$$(\triangle t)^{-1}\int_\Omega(\varrho_\varepsilon^j-\varrho_\varepsilon^{j-1})\log(\varrho_\varepsilon^j+\eta)\ \mathrm{d}x-\int_\Omega\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j\cdot\frac{\nabla\varrho_\varepsilon^j}{\varrho_\varepsilon^j+\eta}\ \mathrm{d}x-\varepsilon\int_\Omega\Delta\varrho_\varepsilon^j\log(\varrho_\varepsilon^j+\eta)\ \mathrm{d}x=0.$$

Integrating by parts in the last integral, and using the boundary conditions we obtain the following expression

$$(\triangle t)^{-1}\int_\Omega(\varrho_\varepsilon^j-\varrho_\varepsilon^{j-1})\log(\varrho_\varepsilon^j+\eta)\ \mathrm{d}x-\int_\Omega\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j\cdot\frac{\nabla\varrho_\varepsilon^j}{\varrho_\varepsilon^j+\eta}\ \mathrm{d}x+\varepsilon\int_\Omega\frac{|\nabla\varrho_\varepsilon^j|^2}{\varrho_\varepsilon^j+\eta}\ \mathrm{d}x=0,$$

but the last term is non-negative, so we have

$$(\triangle t)^{-1}\int_\Omega(\varrho_\varepsilon^j-\varrho_\varepsilon^{j-1})\log(\varrho_\varepsilon^j+\eta)\ \mathrm{d}x-\int_\Omega\varrho_\varepsilon^j\mathbf{u}_\varepsilon^j\cdot\frac{\nabla\varrho_\varepsilon^j}{\varrho_\varepsilon^j+\eta}\ \mathrm{d}x\le 0.$$

Next, let $\eta\to 0$. Due to regularity of $\varrho_\varepsilon^j,\mathbf{u}_\varepsilon^j$, the only problematic term is $-\int_\Omega\varrho_\varepsilon^{j-1}\log(\varrho_\varepsilon^j+\eta)\ \mathrm{d}x$ for $\varrho_\varepsilon^j<1-\eta$, but as the above inequality has the right sign, we can handle it by use of the Lebesgue monotone convergence theorem. Integrating by parts once more, we end up with

$$(\triangle t)^{-1}\int_\Omega(\varrho_\varepsilon-\varrho_\varepsilon^{j-1})\log\varrho_\varepsilon^j\ \mathrm{d}x+\int_\Omega\operatorname{div}\mathbf{u}_\varepsilon^j\varrho_\varepsilon^j\ \mathrm{d}x\le 0,\tag{60}$$

so, after letting $\varepsilon\to 0$ we finally arrive at

$$(\triangle t)^{-1}\int_\Omega\overline{(\varrho^j-\varrho^{j-1})\log\varrho^j}\ \mathrm{d}x+\int_\Omega\overline{\operatorname{div}\mathbf{u}^j\varrho^j}\ \mathrm{d}x\le 0.$$

As a consequence, identity (59) may be transformed into:

$$\begin{aligned}\int_\Omega\overline{\pi(\varrho^j,Y^j)\varrho^j+\delta(\varrho^j)^{\Gamma+1}}\ \mathrm{d}x+(2\mu+\nu)(\triangle t)^{-1}\int_\Omega\overline{(\varrho^j-\varrho^{j-1})\log\varrho^j}\ \mathrm{d}x\\ \le\int_\Omega\overline{\pi(\varrho^j,Y^j)+\delta(\varrho^j)^\Gamma}\varrho^j\ \mathrm{d}x+(2\mu+\nu)(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\log\varrho^j\ \mathrm{d}x.\end{aligned}\tag{61}$$

The convexity of $\varrho^j\ln(\varrho^j)$ and $-\varrho^{j-1}\ln(\varrho^j)$ as functions of ϱ^j ensure lower semicontinuity of the functional $\int_\Omega(\varrho^j-\varrho^{j-1})\log\varrho^j\ \mathrm{d}x$, in other words

$$\int_\Omega(\varrho^j-\varrho^{j-1})\log\varrho^j\ \mathrm{d}x\le\int_\Omega\overline{(\varrho^j-\varrho^{j-1})\log\varrho^j}\ \mathrm{d}x.$$

Therefore, due to the definition of π, we have from (61)

$$\int_\Omega\left(\overline{(\varrho^j)^{\gamma+1}+\delta(\varrho^j)^{\Gamma+1}}+\overline{\varrho^j\sum_{k=1}^n\frac{Y_k^j}{m_k}\varrho^j}\right)\ \mathrm{d}x\le\int_\Omega\left(\overline{(\varrho^j)^\gamma+\delta(\varrho^j)^\Gamma}\varrho^j+\varrho^j\sum_{k=1}^n\overline{\frac{Y_k^j}{m_k}\varrho^j}\right)\ \mathrm{d}x.\tag{62}$$

This inequality can be used to show strong convergence of density as soon as one justifies

$$\overline{(\varrho^j)^\gamma}\varrho^j\le\overline{(\varrho^j)^{\gamma+1}},\quad\overline{(\varrho^j)^\Gamma}\varrho^j\le\overline{(\varrho^j)^{\Gamma+1}},\quad\overline{Y_k^j\varrho^j}\varrho^j\le\overline{Y_k^j(\varrho^j)^2}.\tag{63}$$

To do this we will use a well known result about weak convergence of monotone functions composed with weakly converging sequences, whose proof can be found e.g. in [6], Theorem 10.19.

Lemma 12 (Weak convergence of monotone functions) *Let Ω be a domain in $\mathbb{R}^N$ and $(P,G)\in C(\mathbb{R})\times C(\mathbb{R})$ be a couple of non-decreasing function. Assume that u_n is a sequence of functions from $L^1(\Omega)$ with values in $\mathbb{R}$ such that*

$$\left.\begin{aligned}P(u_n)&\to\overline{P(u)},\\ G(u_n)&\to\overline{G(u)},\\ P(u_n)G(u_n)&\to\overline{P(u)G(u)}\end{aligned}\right\}\text{weakly in }L^1(\Omega).$$

(i) Then

$$\overline{P(u)}\ \overline{G(u)} \leq \overline{P(u)G(u)} \quad \text{a.e. in } \Omega.$$

(ii) If, in addition

$$G(z) = z, \quad P \in C(\mathbb{R}), \quad P \text{ is non-decreasing}$$

and

$$\overline{P(u)G(u)} = \overline{P(u)}\ \overline{G(u)},$$

then

$$\overline{P(u)} = P(u).$$

Applying this lemma to (63), we see that the first inequality is evidently true since $P(\varrho_\varepsilon^j) = (\varrho_\varepsilon^j)^\gamma$ and $G(\varrho_\varepsilon^j) = \varrho_\varepsilon^j$ are increasing. Regarding the second inequality, by (42) we know that $\overline{Y_k^j \varrho^j} = Y_k^j \varrho^j$, thus $\varrho^j \overline{Y_k^j \varrho^j} = Y_k^j (\varrho^j)^2$, while the r.h.s. satisfies $\overline{\varrho^j Y_k^j \varrho^j} = Y_k^j \overline{(\varrho^j)^2} \geq Y_k^j (\varrho^j)^2$. Here we applied Lemma 12 with $P(\varrho_\varepsilon^j) = G(\varrho_\varepsilon^j) = \varrho_\varepsilon^j$. Hence, by comparison of (62) with (63) we obtain, using the statement (ii) of Lemma 12, that

$$\overline{(\varrho^j)^\gamma} = (\varrho^j)^\gamma \quad \text{a.e. in } \Omega.$$

This in turn implies the strong convergence of the density as $L^\gamma(\Omega)$ is a uniformly convex Banach space. This completes the proof of Theorem 2. □

5 Back to the continuous system

The aim of this section is to prove Theorem 1.The first part is devoted to derivation of estimates uniform with respect to $\triangle t$. Some of them can be deduced from the previous section after letting ε to 0. Then we let the time-discretization parameter $\triangle$ go to zero. Finally we discuss the limit passage with the last parameter δ.

5.1 Limit passage to a continuous system with artificial pressure

Before we let $\triangle t \to 0$, we turn back to (28) and rewrite it in a slightly changed form. First, we add and subtract $\frac{(\triangle t)^{-1}}{2} \int_\Omega \varrho^{j-1} |\mathbf{u}^{j-1}|^2 \ \mathrm{d}x$ and $\frac{(\triangle t)^{-1}}{\gamma - 1} \int_\Omega (\varrho^{j-1})^\gamma \ \mathrm{d}x + \frac{(\triangle t)^{-1}\delta}{\Gamma - 1} \int_\Omega (\varrho^{j-1})^\Gamma \ \mathrm{d}x$ to (28), we get

$$\begin{aligned}
&\frac{(\triangle t)^{-1}}{2} \int_\Omega \left(\varrho^j |\mathbf{u}^j|^2 - \varrho^{j-1} |\mathbf{u}^{j-1}|^2 \right) \ \mathrm{d}x + \frac{(\triangle t)^{-1}}{2} \int_\Omega \varrho^{j-1} |\mathbf{u}^j - \mathbf{u}^{j-1}|^2 \ \mathrm{d}x \\
&+ \int_\Omega \mathbf{S}(\mathbf{u}^j) : \nabla \mathbf{u}^j \ \mathrm{d}x + \frac{(\triangle t)^{-1}}{\gamma - 1} \int_\Omega \left((\varrho^j)^\gamma - (\varrho^{j-1})^\gamma \right) \ \mathrm{d}x + \frac{(\triangle t)^{-1}\delta}{\Gamma - 1} \int_\Omega \left((\varrho^j)^\Gamma - (\varrho^{j-1})^\Gamma \right) \ \mathrm{d}x \\
&+ \varepsilon\gamma \int_\Omega (\varrho^j)^{\gamma-2} |\nabla \varrho^j|^2 \ \mathrm{d}x + \varepsilon\delta\Gamma \int_\Omega (\varrho^j)^{\Gamma-2} |\nabla \varrho^j|^2 \ \mathrm{d}x \\
&+ \frac{(\triangle t)^{-1}}{\gamma - 1} \int_\Omega \left((\gamma - 1)\, (\varrho^j)^\gamma + (\varrho^{j-1})^\gamma - \gamma (\varrho^j)^{\gamma-1} \varrho^{j-1} \right) \ \mathrm{d}x \\
&+ \frac{(\triangle t)^{-1}\delta}{\Gamma - 1} \int_\Omega \left((\Gamma - 1)\, (\varrho^j)^\Gamma + (\varrho^{j-1})^\Gamma - \Gamma (\varrho^j)^{\Gamma-1} \varrho^{j-1} \right) \ \mathrm{d}x \\
= &\int_\Omega \varrho^j \mathbf{f}^j \cdot \mathbf{u}^j \ \mathrm{d}x + \sum_{k=1}^n \int_\Omega \frac{Y_k^j}{m_k} \varrho^j \operatorname{div} \mathbf{u}^j \ \mathrm{d}x.
\end{aligned} \tag{64}$$

Note, that since $\varrho^j,\ \varrho^{j-1} \geq 0,\ \gamma,\ \Gamma > 1$ the two last integrals from the l.h.s. are nonnegative. Let us introduce the following notation

$$\left. \begin{array}{l} \hat{\phi}(x,t) = \phi^k(x) \\ \tilde{\phi}(x,t) = \phi^k(x) + (t - t_k) \left(\frac{\phi^{k+1} - \phi^k}{\triangle t} \right)(x) \end{array} \right\} \quad \text{if } t_k \leq t < t_{k+1},\ k \in \{0, \ldots, \tilde{N}\} \tag{65}$$

and let us define the shift operator

$$\sigma \phi^k = \phi^{k-1}, \quad k \in \{1, \ldots, \tilde{N}\}.$$

We can then rewrite system as

$$\begin{aligned}
\partial_t \tilde{\varrho} + \operatorname{div}(\hat{\varrho}\hat{\mathbf{u}}) &= 0, \\
\partial_t \widetilde{\varrho \mathbf{u}} + \operatorname{div}(\hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) - \operatorname{div} \mathbf{S}(\hat{\mathbf{u}}) + \nabla \pi(\hat{\varrho}, \hat{Y}) + \delta \nabla \hat{\varrho}^\Gamma &= \mathbf{0}, \\
\partial_t \widetilde{\varrho Y_k} + \operatorname{div}(\hat{\varrho}\hat{Y}_k \hat{\mathbf{u}}) + \operatorname{div} \mathbf{F}_i(\hat{\varrho}, \hat{Y}) &= \hat{\varrho}\omega_k(\hat{Y}), \quad k \in \{1, \ldots, n\}
\end{aligned} \tag{66}$$

and keeping in mind (65) we can use (64) to deduce that

$$\hat{\varrho},\ \tilde{\varrho} \text{ are bounded in } L^\infty(0,T;L^\gamma(\Omega)), \quad \delta^{1/\Gamma}\hat{\varrho},\ \delta^{1/\gamma}\tilde{\varrho} \text{ are bounded in } L^\infty(0,T;L^\Gamma(\Omega)) \tag{67}$$

$$\hat{\varrho}|\hat{\mathbf{u}}|^2, \widetilde{\varrho|\mathbf{u}|^2} \text{ are bounded in } L^\infty(0,T;L^1(\Omega)) \tag{68}$$

$$\hat{\mathbf{u}},\ \tilde{\mathbf{u}} \text{ are bounded in } L^2(0,T;W^{1,2}(\Omega)) \tag{69}$$

$$\hat{\varrho}\hat{\mathbf{u}}, \widetilde{\varrho \mathbf{u}} \text{ are bounded in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}) \cup L^2(0,T;L^r(\Omega)) \tag{70}$$

for $1 \leq r \leq \frac{6\gamma}{6+\gamma}$, where the last one holds as

$$\|\hat{\varrho}\hat{\mathbf{u}}\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \leq \|\hat{\varrho}\|^{1/2}_{L^\gamma(\Omega)} \|\hat{\varrho}|\hat{\mathbf{u}}|^2\|^{1/2}_{L^1(\Omega)} \quad \text{and} \quad \|\hat{\varrho}\hat{\mathbf{u}}\|_{L^r(\Omega)} \leq \|\hat{\varrho}\|_{L^\gamma(\Omega)} \|\hat{\mathbf{u}}\|_{W^{1,2}(\Omega)}.$$

Furthermore (64) gives rise to two more estimates which are of crucial importance for the limit passage, namely to

$$\|\hat{\varrho} - \sigma\hat{\varrho}\|^\gamma_{L^\gamma(0,T;L^\gamma(\Omega))} + \delta \|\hat{\varrho} - \sigma\hat{\varrho}\|^\Gamma_{L^\Gamma(0,T;L^\Gamma(\Omega))} \leq c\triangle t, \tag{71}$$

and

$$\|\hat{\varrho}|\hat{\mathbf{u}} - \sigma\hat{\mathbf{u}}|^2\|_{L^1(0,T;L^1(\Omega))} \leq c\triangle t. \tag{72}$$

The first one is due to the fact that for $\gamma, \Gamma > 1$ there exists a positive constant c such that

$$(\gamma - 1)(\varrho^j)^\gamma + (\varrho^{j-1})^\gamma - \gamma(\varrho^j)^{\gamma-1}\varrho^{j-1} \geq c|\varrho^j - \varrho^{j-1}|^\gamma,$$

$$\delta\left((\Gamma - 1)(\varrho^j)^\Gamma + (\varrho^{j-1})^\Gamma - \Gamma(\varrho^j)^{\Gamma-1}\varrho^{j-1}\right) \geq c\delta|\varrho^j - \varrho^{j-1}|^\Gamma.$$

Repeating the steps leading to (29) one can verify that

$$\hat{\varrho}\hat{Y_k}^2, \widetilde{\varrho Y_k^2} \text{ are bounded in } L^\infty(0,T;L^1(\Omega)), \quad \hat{Y}_k,\ \tilde{Y}_k \text{ are bounded in } L^2(0,T;W^{1,2}(\Omega)), \tag{73}$$

also

$$\left\| \hat{\varrho}\left(\hat{Y}_k - \sigma\hat{Y}_k\right)^2 \right\|_{L^1(0,T;L^1(\Omega))} \leq c\triangle t. \tag{74}$$

And by (22) we deduce that

$$\hat{Y}_k,\ \tilde{Y}_k \text{ are bounded in } L^\infty((0,T)\times\Omega).$$

Finally, a similar estimate to (33) can be performed, so we get

$$\|\hat{\varrho}\|^{\gamma+\beta}_{L^{\gamma+\beta}((0,T)\times\Omega)} + \delta\|\hat{\varrho}\|^{\Gamma+\beta}_{L^{\Gamma+\beta}((0,T)\times\Omega)} \leq c(T,\Omega). \tag{75}$$

This is the last estimate needed to perform the limit passage $\triangle t \to 0$ in all the terms except the pressure. Indeed, passing to subsequence it can be shown, combining (67-74), that the following convergences hold

$$[\hat{\varrho} - \sigma\hat{\varrho}], [\hat{\varrho} - \tilde{\varrho}] \to 0 \quad \text{in } L^q(0,T;L^\gamma(\Omega)) \tag{76}$$

for $q \in [1, \infty)$,

$$[\hat{\varrho}\hat{\mathbf{u}} - \sigma\hat{\varrho}\hat{\mathbf{u}}],\ [\hat{\varrho}\hat{\mathbf{u}} - \widetilde{\varrho \mathbf{u}}] \to 0 \quad \text{in } L^q(0,T;L^r(\Omega)), \tag{77}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),\ r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$,

$$[\hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} - \widetilde{\varrho\mathbf{u}} \otimes \hat{\mathbf{u}}] \to 0 \quad \text{in } L^1(0,T;L^r(\Omega)) \cup L^q(0,T;L^1(\Omega)), \tag{78}$$

for $q \in [1,\infty)$ $r \in [1, \frac{3\Gamma}{3+\Gamma}]$,

$$[\hat{\varrho}\hat{Y}_k - \sigma\hat{\varrho}\hat{Y}_k],\ [\hat{\varrho}\hat{Y}_k - \widetilde{\varrho Y_k}] \to 0 \quad \text{in } L^q(0,T;L^r(\Omega)), \tag{79}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),\ r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$.

From what has already been written we deduce that

$$\hat{\varrho},\ \tilde{\varrho} \rightharpoonup \varrho \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma(\Omega)),\ \text{weakly in } L^{\Gamma+\beta}((0,T)\times\Omega), \tag{80}$$

$$\hat{\mathbf{u}} \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{81}$$

$$\hat{Y}_k \rightharpoonup Y_k \quad \text{weakly}^* \text{ in } L^\infty((0,T)\times\Omega),\ \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),. \tag{82}$$

Remark 2 *Since $\tilde{\varrho}$ $\hat{\varrho}$, $\hat{\mathbf{u}}$ satisfy continuity equation $\partial_t\tilde{\varrho} + \operatorname{div}(\hat{\varrho}\hat{\mathbf{u}}) = 0$, in the sense of distributions, thus the sequence of functions $f(t) = \left(\int_\Omega \tilde{\varrho}\phi \, \mathrm{d}x\right)(t)$ is bounded and equicontinuous in $C([0,T])$ for all $\phi \in C^\infty(\overline{\Omega})$, $\frac{\partial\phi}{\partial\mathbf{n}} = 0$ at $\partial\Omega$. Therefore, the Arzelà-Ascoli theorem, the density argument and the convergence established in (76) yield*

$$\tilde{\varrho} \to \varrho \quad \text{in } C_{weak}(0,T;L^\Gamma(\Omega)).$$

We now focus on the corresponding convergence of the products $\hat{\varrho}\hat{\mathbf{u}}$, $\hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}$, $\hat{\varrho}\hat{Y}_k$ and $\hat{\varrho}\hat{Y}_k\hat{\mathbf{u}}$. This can be done by repeated application of the following lemma.

Lemma 13 *Let g^n, h^n converge weakly to g, h respectively in $L^{p_1}(0,T;L^{p_2}(\Omega))$, $L^{q_1}(L^{q_2}(\Omega))$ where $1 \le p_1, p_2 \le \infty$ and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Let assume in addition that

$$\partial_t g^n \text{ is bounded in } L^1(0,T;W^{-m,1}) \text{ for some } m \ge 0 \text{ independent of } n \text{ and} \tag{83}$$

$$\|h^n - h^n(\cdot+\xi,t)\|_{L^{q_1}(0,T;L^{q_2}(\Omega))} \to 0 \text{ as } |\xi| \to 0, \text{ uniformly in } n. \tag{84}$$

Then $g^n h^n$ converges to gh in the sense of distributions on $\Omega \times (0,T)$.

For the proof we refer the reader to [11].

Since $\partial_t\tilde{\varrho}$ is bounded in $L^\infty(0,T;W^{-1,2\Gamma/(\Gamma+1)}(\Omega))$, $\partial_t\widetilde{\varrho\mathbf{u}}$ is bounded in $L^2(0,T;W^{-1,1+\beta})$ and $\partial_t\widetilde{\varrho Y_k}$ is bounded in $L^2(0,T;W^{-1,1})$, condition (83) is satisfied for $g^n = \tilde{\varrho}$, $\widetilde{\varrho\mathbf{u}}$, $\widetilde{\varrho Y_k}$ and $m = 1$ respectively. Additionally, $h^n = \hat{\mathbf{u}}$, $\hat{Y}_k$, which is bounded in $L^2(0,T;W^{1,2}(\Omega))$, satisfies condition (84).

Therefore, $\tilde{\varrho}\hat{\mathbf{u}}$ converges weakly* in $L^\infty(0,T;L^{2\Gamma/(\Gamma+1)}(\Omega))$ and weakly in $L^2(0,T;L^{6\Gamma/(\Gamma+6)}(\Omega))$ to $\varrho\mathbf{u}$, $\tilde{\varrho}\hat{Y}_k$ converges weakly* in $L^\infty(0,T;L^\Gamma(\Omega))$ to ϱY_k. And so, in view of (76), (77), (79)

$$\widetilde{\varrho\mathbf{u}},\ \hat{\varrho}\hat{\mathbf{u}} \rightharpoonup \varrho\mathbf{u} \quad \text{weakly in } L^q(0,T;L^r(\Omega)) \tag{85}$$

and

$$\widetilde{\varrho Y_k},\ \hat{\varrho}\hat{Y}_k \rightharpoonup \varrho Y_k \quad \text{weakly in } L^q(0,T;L^r(\Omega)) \tag{86}$$

for $\{q \in [1,\infty),\ r \in [1, \frac{2\Gamma}{\Gamma+1}]\} \cup \{q \in [1,2),\ r \in [1, \frac{6\Gamma}{6+\Gamma}]\}$.

Moreover, $\widetilde{\varrho\mathbf{u}} \otimes \hat{\mathbf{u}}$ converges weakly in $L^1(0,T;L^{3\Gamma/(\Gamma+3)}(\Omega))$ weakly* in $L^\infty(0,T;L^1(\Omega))$ to $\varrho\mathbf{u} \otimes \mathbf{u}$, and $\widetilde{\varrho Y_k}\hat{\mathbf{u}}$ converges weakly in $L^1(0,T;L^{3\Gamma/(\Gamma+3)}(\Omega))$ weakly* in $L^\infty(0,T;L^1(\Omega))$ to $\varrho Y_k\mathbf{u}$. Thus, again (78), (79) can be used to show that

$$\hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} \rightharpoonup \varrho\mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^1(0,T;L^r(\Omega)) \cup L^q(0,T;L^1(\Omega)), \tag{87}$$

and

$$\hat{\varrho}\hat{Y_k}\hat{\mathbf{u}} \rightharpoonup \varrho Y_k \mathbf{u} \quad \text{weakly in } L^1(0,T;L^r(\Omega)) \cup L^q(0,T;L^1(\Omega)), \tag{88}$$

for $q \in [1,\infty)$ $r \in [1, \frac{3\Gamma}{3+\Gamma}]$.

All these considerations allow us to let $\triangle t \to 0$ in the system (66) and we obtain

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S}(\mathbf{u}) + \nabla \overline{\pi(\varrho, Y)} + \delta \nabla \overline{\varrho^\Gamma} = \varrho \mathbf{f}, \\ \partial_t (\varrho Y_k) + \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k(\varrho, Y_k) = \overline{\varrho \omega_k(Y)}, \quad k \in \{1,\dots,n\}, \end{aligned} \tag{89}$$

which is now satisfied in the sense of distributions on $(0,T) \times \Omega$, together with boundary conditions (5) and (6). Regarding the initial conditions, we can repeat the argument from Remark 2 to verify that

$$\widetilde{\varrho \mathbf{u}} \to \varrho \mathbf{u} \quad \text{in } C_{weak}(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)), \qquad \widetilde{\varrho Y_k} \to \varrho Y_k \quad \text{in } C_{weak}(0,T;L^\Gamma(\Omega)). \tag{90}$$

The last part of the proof is devoted to the issue of strong convergence of the density, which is necessary to identify the limits in the nonlinear terms. As previously, we seek to derive the effective viscous flux equality.

Note that the functions $\tilde{\varrho}$ and $\hat{\varrho}\hat{\mathbf{u}}$ extended by 0 outside Ω satisfy the continuity equation in whole $\mathbb{R}^3$. Next, one can check that

$$\hat{\varphi}(t,x) = \psi(t)\zeta(x)\tilde{\phi}, \quad \tilde{\phi} = (\nabla\Delta^{-1})[1_\Omega \tilde{\varrho}], \quad \psi \in C_c^\infty((0,T)),\ \zeta \in C_c^\infty(\Omega),$$

is an admissible test function for the momentum equation, after straightforward manipulations we obtain

$$\begin{aligned} &\int_\Omega \psi\zeta \left(\left(\pi(\hat{\varrho}, \hat{Y}) + \delta \hat{\varrho}^\Gamma \right) \hat{\varrho} - \mathbf{S}(\hat{\mathbf{u}}) : \mathcal{R}[1_\Omega \tilde{\varrho}] \right) \,\mathrm{d}x \\ &= - \int_\Omega \psi\zeta \hat{\varrho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \mathcal{R}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x + \int_\Omega \psi\zeta \widetilde{\varrho \mathbf{u}} \cdot \mathcal{R}[1_\Omega \hat{\varrho}\hat{\mathbf{u}}] \,\mathrm{d}x - \int_\Omega \partial_t \psi\zeta \widetilde{\varrho \mathbf{u}} \cdot \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x \\ &\quad - \int_\Omega \psi \hat{\varrho}(\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x + \int_\Omega \psi \mathbf{S}(\hat{\mathbf{u}}) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x \\ &\quad - \int_\Omega \psi \left(\pi(\hat{\varrho}, \hat{Y}) + \delta \hat{\varrho}^\Gamma \right) \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x - \int_\Omega \zeta \varrho \mathbf{f} \cdot \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x = \sum_{i=1}^6 I_i. \end{aligned} \tag{91}$$

Where we have used the approximate continuity equation to write

$$\partial_t \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] = -\nabla\Delta^{-1}[\operatorname{div}(1_\Omega \hat{\varrho}\hat{\mathbf{u}})].$$

Since $\Gamma > 3$, we can use the analogous function to test the limit momentum equation

$$\varphi(t,x) = \psi(t)\zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega \varrho], \quad \psi \in C_c^\infty((0,T)),\ \zeta \in C_c^\infty(\Omega), \tag{92}$$

we obtain

$$\begin{aligned} &\int_\Omega \psi\zeta \left(\overline{\pi(\varrho, Y) + \delta \varrho^\Gamma} \varrho - \mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \,\mathrm{d}x \\ &= - \int_\Omega \psi\zeta \varrho \mathbf{u} \otimes \mathbf{u} : \mathcal{R}[1_\Omega \varrho] \,\mathrm{d}x + \int_\Omega \psi\zeta \varrho \mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho \mathbf{u}] \,\mathrm{d}x - \int_\Omega \partial_t \,\mathrm{d}x \psi\zeta \varrho \mathbf{u} \cdot \nabla\Delta^{-1}[1_\Omega \varrho] \\ &\quad - \int_\Omega \psi \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho] \,\mathrm{d}x + \int_\Omega \psi \mathbf{S}(\mathbf{u}) : \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho] \,\mathrm{d}x \\ &\quad - \int_\Omega \psi \overline{\pi(\varrho, Y) + \delta \varrho^\Gamma} \nabla\zeta \otimes \nabla\Delta^{-1}[1_\Omega \varrho] \,\mathrm{d}x - \int_\Omega \zeta \varrho \mathbf{f} \cdot \nabla\Delta^{-1}[1_\Omega \tilde{\varrho}] \,\mathrm{d}x = \sum_{i=1}^6 I_i. \end{aligned} \tag{93}$$

Again, it is not difficult to check that comparison of (93) with the limit of (91) gives rise to the following equality

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \psi\zeta\left(\left(\pi(\hat\varrho,\hat Y)+\delta\hat\varrho^\Gamma\right)\hat\varrho-\mathbf{S}(\hat{\mathbf{u}}):\mathcal{R}[1_\Omega\tilde\varrho]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\!\!\int_\Omega \psi\zeta\left(\hat\varrho\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}:\mathcal{R}[1_\Omega\tilde\varrho]-\widetilde{\varrho\mathbf{u}}\cdot\mathcal{R}[1_\Omega\hat\varrho\hat{\mathbf{u}}]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\to\int_0^T\!\!\int_\Omega \psi\zeta\left(\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\,\varrho-\mathbf{S}(\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\!\!\int_\Omega \psi\zeta\left(\varrho\mathbf{u}\otimes\mathbf{u}:\mathcal{R}[1_\Omega\varrho]-\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}]\right)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{94}
$$

To show that the terms involving commutators cancel we will again use Lemma 9 with $\mathbf{V}_\varepsilon=\hat\varrho\hat{\mathbf{u}}$, $r_\varepsilon=\hat\varrho$ and we first check that

$$
\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde\varrho]-\tilde\varrho\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}]\to\varrho\mathbf{u}\mathcal{R}[1_\Omega\varrho]-\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}]
$$

in the sense of distributions on Ω but for all $t\in[0,T]$. The second property follows from Remark 2.

Next, by (81), (85) and the density argument we check that this convergence can be extended to a weak convergence in $L^{2\Gamma/(\Gamma+3)}(\Omega)$. However we see that this space is compactly embedded into $W^{-1,2}(\Omega)$ only if $\Gamma>\frac{9}{2}$. This, together with (81) implies that

$$
\lim_{\Delta\to0}\int_0^T\!\!\int_\Omega \psi\zeta\hat{\mathbf{u}}\cdot\left(\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde\varrho]-\tilde\varrho\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}]\right)\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\!\!\int_\Omega \psi\zeta\mathbf{u}\cdot\left(\varrho\mathbf{u}\mathcal{R}[1_\Omega\varrho]-\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}]\right)\,\mathrm{d}x\,\mathrm{d}t. \tag{95}
$$

Equality (95) is then almost what we need. To prove that

$$
\begin{aligned}
&\lim_{\Delta\to0}\int_0^T\!\!\int_\Omega \psi\zeta\left(\hat\varrho\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}:\mathcal{R}[1_\Omega\tilde\varrho]-\widetilde{\varrho\mathbf{u}}\cdot\mathcal{R}[1_\Omega\hat\varrho\hat{\mathbf{u}}]\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\lim_{\Delta\to0}\int_0^T\!\!\int_\Omega \psi\zeta\hat{\mathbf{u}}\cdot\left(\widetilde{\varrho\mathbf{u}}\mathcal{R}[1_\Omega\tilde\varrho]-\tilde\varrho\mathcal{R}[1_\Omega\widetilde{\varrho\mathbf{u}}]\right)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

we use properties (76) and (77). Then, repeating the steps leading from (56) to (59), we can transform (94) to

$$
\overline{\left(\pi(\varrho,Y)+\delta\varrho^\Gamma\right)\varrho}-(2\mu+\nu)\overline{\operatorname{div}\mathbf{u}\varrho}=\overline{\pi(\varrho,Y)+\delta\varrho^\Gamma}\,\varrho-(2\mu+\nu)\operatorname{div}\mathbf{u}\varrho,\quad\text{a.e. in }\Omega. \tag{96}
$$

Next, we take $\eta>0$ and multiply the discrete version of the continuity equation by $\log(\varrho^j+\eta)$. After integrating by parts over Ω one get

$$
(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\log(\varrho^j+\eta)\,\mathrm{d}x-\int_\Omega\varrho^j\mathbf{u}^j\frac{\nabla\varrho^j}{\varrho^j+\eta}\,\mathrm{d}x=0.
$$

By the Lebesgue monotone convergence theorem we can pass with $\eta\to0^+$ and then integrate by parts once more to find

$$
(\triangle t)^{-1}\int_\Omega(\varrho^j-\varrho^{j-1})\log\varrho^j\,\mathrm{d}x+\int_\Omega\operatorname{div}\mathbf{u}^j\varrho^j\,\mathrm{d}x=0.
$$

Recall that $\int_\Omega\varrho^j\,\mathrm{d}x=\int_\Omega\varrho^{j-1}\,\mathrm{d}x$, thus whereas $x\log(x)$ is a convex function above equality may be changed into

$$
(\triangle t)^{-1}\int_\Omega\left(\varrho^j\log\varrho^j-\varrho^{j-1}\log\varrho^{j-1}\right)\,\mathrm{d}x+\int_\Omega\operatorname{div}\mathbf{u}^j\varrho^j\,\mathrm{d}x\le0. \tag{97}
$$

Now, we sum (97) from $j=1$ to $j=\tilde N$, multiply by $\triangle$ and pass to the limit to get

$$
\int_\Omega\overline{\varrho\log\varrho}(T)\,\mathrm{d}x+\int_0^T\!\!\int_\Omega\overline{\varrho\operatorname{div}\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\le\int_\Omega\varrho\log\varrho(0)\,\mathrm{d}x, \tag{98}
$$

For the limit continuity equation, we take advantage of the fact that it is satisfied in the whole space in the sense of distributions, thus the solution is automatically a renormalized solution, see for instance [6], i.e. by an appropriate renormalization we may get

$$\int_\Omega \varrho \log \varrho(T) \,\mathrm{d}x + \int_0^T\int_\Omega \varrho \operatorname{div} \mathbf{u} \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega \varrho \log \varrho(0) \,\mathrm{d}x. \tag{99}$$

Consequently, the two results (98) and (99) give rise to

$$\int_\Omega \overline{\varrho \log \varrho}(T) \,\mathrm{d}x + \int_0^T\int_\Omega \overline{\varrho \operatorname{div} \mathbf{u}} \,\mathrm{d}x \,\mathrm{d}t \le \int_\Omega \varrho \log \varrho(T) \,\mathrm{d}x + \int_0^T\int_\Omega \varrho \operatorname{div} \mathbf{u} \,\mathrm{d}x \,\mathrm{d}t.$$

which joined with (96) leads to desired conclusion

$$\int_0^T\int_\Omega \left(\overline{\varrho^{\gamma+1} + \delta\varrho^{\Gamma+1}} + \overline{\varrho \sum_{k\in K} \frac{Y_k}{m_k}\varrho} \right) \,\mathrm{d}x \,\mathrm{d}t \le \int_0^T\int_\Omega \left(\overline{\varrho^{\gamma} + \delta\varrho^{\Gamma}}\varrho + \varrho \sum_{k\in K} \frac{\overline{Y_k}}{m_k}\varrho \right) \,\mathrm{d}x \,\mathrm{d}t. \tag{100}$$

As in the stationary case, using Lemma 12 we easily verify that

$$\overline{\varrho^\gamma}\varrho \le \overline{\varrho^{\gamma+1}}, \quad \overline{\varrho^\Gamma}\varrho \le \overline{\varrho^{\Gamma+1}}, \tag{101}$$

so to deduce the strong convergence of the density one should only check that $\overline{Y_k\varrho}\varrho \le \overline{Y_k\varrho^2}$. It is easy to identify the l.h.s. due to (82), Remark 2 and the compact embedding of $L^\Gamma(\Omega)$ into $W^{-1,2}(\Omega)$. Identifying the limit from the r.h.s. is now a little more involved, however we can show that

$$\hat{Y}_k \to Y_k \quad \text{a.e. on } \{(x,t) : \varrho(x,t) > 0\}. \tag{102}$$

Indeed, it is a consequence of the weak convergence of $\hat{Y}_k$ and a following convergence of norms

$$\int_0^T\int_\Omega \varrho \hat{Y}_k^2 \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T\int_\Omega \varrho Y_k^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{103}$$

where ϱ is to be understood as a positive density. On account of (82) and (90) we can repeat the argument from Remark 2 to verify that

$$\widetilde{\varrho Y_k}\hat{Y}_k \to \varrho Y_k^2 \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma). \tag{104}$$

Note that also $\nabla \hat{Y}_k^2$ is uniformly bounded in $L^2(0,T;W^{1,2})$, therefore

$$(\tilde{\varrho} - \varrho)\hat{Y}_k^2 \to 0 \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^\Gamma). \tag{105}$$

Thus, convergence (103) follows from (79) combined with (104),(105) and a triangle inequality.

Having proven (102) we justify that $\overline{Y_k\varrho}\varrho = Y_k\varrho^2 \le \overline{Y_k\varrho^2} = Y_k\overline{\varrho^2}$ on a set $\{\varrho > 0\}$, it is obvious on account of the weak lower semicontinuity of convex functions. For the set $\{\varrho = 0\}$ the l.h.s. becomes equal to 0 while the r.h.s. is always nonnegative, so the inequality is valid.

Recapitulating, the above considerations leads to equality

$$\overline{\varrho^\gamma}\varrho = \overline{\varrho^{\gamma+1}} \quad \text{a.e. on } (0,T)\times\Omega,$$

hence the strong convergence follows.

The strong convergence of the density implies together with (82)

$$\overline{\varrho\omega(Y)} = \varrho\omega(Y).$$

5.2 Limit passage $\delta \to 0$

Passage to limit with the last approximation parameter differs only in one step in comparison to the analysis performed in the previous subsection. Namely, at this level we cannot derive the effective viscous flux equality in the same way. Instead of testing the momentum equation with the function ϕ specified in (92) we have to use

$$\varphi(t,x) = \psi(t)\zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega \varrho^\vartheta], \quad \psi \in C_c^\infty((0,T)),\ \zeta \in C_c^\infty(\Omega),$$

with $\vartheta < \frac{1}{6}$. Since we already know how to identify the limit in the molecular pressure term for the time-dependent case, the proof of strong convergence of the density would be just a repetition of the standard proof for the case of barotropic Navier-Stokes equations. Details of this procedure can be found e.g. in [6], Chapter 3 or in [19]. The proof of Theorem 1 is complete. □

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