Asymptotic analysis of complete fluid system on varying domain: from compressible to incompressible flow

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The asymptotic analysis of the complete fluid system on a varying domain: from the compressible to the incompressible flow

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Abstract

We will present the asymptotic analysis of solutions to the compressible Navier-Stokes-Fourier system, when the Mach number is small proportional to ε, a Froude number is proportional to $\sqrt{\varepsilon}$ and $\varepsilon \to 0$ and the domain containing the fluid varies with changing parameter ε. In particular, the fluid is driven by a gravitation generated by object(s) placed in the fluid of diameter converging to zero. As $\varepsilon \to 0$, we will show that the fluid velocity converges to a solenoidal vector field satisfying the Oberbeck-Boussinesq approximation on R^3 space with a concentric gravitation force. Our approach is based on weak solutions. In order to pass to the limit in a convective term we apply the spectral analysis of the associated wave propagator (Neumann Laplacian) governing the motion of acoustic waves.

Key words: Oberbeck-Boussinesq system, singular limit, unbounded domain.

1 Introduction

The Oberbeck-Boussinesq approximation is a mathematical model of a stratified flow, where the fluid is assumed to be incompressible and yet convecting a diffusive quantity creating positive and negative buoyancy force. The resulting system of equations reads:

$$\mathrm{div}_x \boldsymbol{U} = 0, \tag{OB^1}$$

$$\overline{\varrho} \left(\partial_t \boldsymbol{U} + \mathrm{div}_x \left(\boldsymbol{U} \otimes \boldsymbol{U} \right) \right) + \nabla_x P = \mu \Delta \boldsymbol{U} + r \nabla_x F, \tag{OB^2}$$

$$\overline{\varrho} c_p \left(\partial_t \Theta + \mathrm{div}_x \left(\boldsymbol{U} \Theta \right) \right) - \kappa(\overline{\vartheta}) \Delta \Theta - \overline{\varrho}\,\overline{\vartheta} \alpha \mathrm{div}_x \left(F \boldsymbol{U} \right) = 0, \tag{OB^3}$$

$$r + \overline{\varrho} \alpha \Theta = 0, \tag{OB^4}$$

where $\boldsymbol{U}$ denotes the velocity of the fluid, Θ stands for the deviation of the temperature, P is the pressure, μ is the viscosity coefficient, $\kappa > 0$ the heat conductivity coefficient, by $\overline{\varrho}$ we denote the constant density of the fluid, and $\overline{\vartheta} > 0$ the constant reference temperature. By c_p we mean specific heat at constant pressure, by $\alpha > 0$ we mean the coefficient of thermal expansion of the fluid, both are evaluated at the reference density $\overline{\varrho}$ and temperature $\overline{\vartheta}$. Here F stands for potential of a driving force (e.g. gravitational potential) acting on the fluid. Let us note that the density is constant in the Oberbeck-Boussinesq approximation except in the buoyancy force, where it is interrelated in the temperature deviation through Bousisinesq relation (OB^4), (see Zeytounian [32]). Moreover Θ is a deviation of temperature from the equilibrium rather then the temperature itself and the temperature deferences are not caused by the flow, but exists independent of flow.

Thy Oberbeck-Boussinesq approximation (OB) was also recently studied on the whole space $\Omega = \mathbb{R}^3$, with $\nabla_x F = g[0, 0, -1]$, see Brandolese, Schonbek [3], Danchin, Paicu [4]. Such an assumption is in fact reasonable and convenient for mathematical analysis and corresponds to problems occupying bounded domains or where gravitation field can be taken constant. But it is justified to consider also models where self gravitation of objects placed in the fluid is taken into account, see [13]. Formally we may have

$$-\Delta F = \sum_{i=1}^{N} \lambda_i \delta(x_i), \quad \sum_{i=1}^{N} \lambda_i = 1 \tag{1.1}$$

if the size of objects is negligible.

In our investigation we want to show that the Oberbeck-Boussinesq approximation (OB^1 - OB^4) can be derived as a singular limit of the full Navier-Stokes-Fourier (NSF, see section 2.2) (with suitable boundary conditions and with the Mach and the Froude numbers tending to zero and when the family of domains on which the primitive problems are stated depends on singular number and converge to the whole space $\mathbb{R}^3$ in certain sense. The novelty of this article is to allow the forcing term to take the form of (1.1) in the limit.

Models of fluid dynamics are wildly applied including prediction of weather patterns in meteorology, numerous engineering problems involving fluids in motion, understanding complicated dynamics of gaseous stars and the interstellar space in astrophysics, and even certain traffic problems. Most of these applications concern compressible or at least slightly compressible fluids, but the prevailing part of theoretical studies is devoted to mathematical models of idealised incompressible fluids.

The Oberbeck-Boussinesq approximation is widely used when when the density of the fluid is almost constant but there exists pressure differences due to temperature changes, what causes imbalance of the hydrostatic equilibrium. Such effect can be observed with many convection problems where temperature differences are independent of the flow dynamics. If we require the Mach number tends to zero, it allow the density ϱ approach a constant state $\bar{\varrho}$, while the gravitation force is rescaled by the Froude number. Then, the temperature differences are not caused by the flow, but exist independent of the flow (Zeytounian [32]).

In a series of seminal papers Ebin and Klainerman, Majda [16] presented a rigorous basis of a mathematical theory for singular limits, particularly in a low Mach number regime (the characteristic speed of the fluid is dominated by the speed of sound in a compressible medium and therefore the model is later driven to incompressibility). Klein et al. [18, 20, 17, 19] employing the same idea proposed several numerical methods for solving complex problems in the fluid dynamics in the singular limit regimes.

In our studies we consider a singular limit of compressible Navier-Stokes-Fourier with low Mach and Froude number to (OB) system. In contrast to approach based on strong solutions (Klinerman, Majda [15, 16]) ours is based on weak solutions as in the monograph [12]. It allow us not to impose any essential restrictions on the size of data and the length of the relevant time interval. On the other hand we may say that the major drawback of the method is its dependence on the energy estimate for the NSF system which is based on the control of the entropy production rate in the whole process of incompressible limit. In particular, we have to assume that the initial distribution of the density and the temperature should be close to the equilibrium sate.

There is wide range of results concerning the theory of existence of weak solutions to Navier-Stokes-Fourier system and consideration of this system in low Mach number regime. We have to mention here the monograph of E. Feireisl and A. Novotný [12], where authors summarise series of their papers.

We will show that the Oberbeck-Boussinesq approximation with gravitation force of centres placed in the fluid may be viewed as a singular limit of full Navier-Stokes-Fourier system considered on a family of sufficiently large domains being exterior to small hols where Mach and Froude numbers tend to zero.

In low Mach number regime the continuity equation should reduce to incompressibility condition. In this procedure the most delicate question is a convective term in the momentum equation. A priori our estimate do not provide any bound for the gradient part of time derivative of the velocity field (coming from Helmholtz decomposition) since in momentum equation singular term is present. Therefore the most difficult step is to show strong convergence of the velocity in order to control the convective term. Therefore verification of its weak compactness have to be based on subtle knowledge about possible oscillations in time and its mutual damping in acoustic wave described by a gradient part of Helmholtz decomposition of the momentum governed by acoustic equation.

If the (NSF) system is set on whole $\mathbb{R}^3$ the expected local decay of the acoustic energy follows form dispersive estimates. Desjardins and Grenier [5] employed this idea combined with the non-trivial Strichartz estimates for the acoustic equation to show strong (pointwise) convergence of the velocity field in the low Mach number limit for a barotropic fluid flow in the whole physical space $\mathbb{R}^3$. But Strichartz estimates become much more delicate when some influence of physical boundary has to be considered and then some restrictions on geometry of the domain has to be imposed. If the domain is exterior, then obstacle must be star-shaped or at least non-trapping, see [2], [24].

The problem of the low Mach number limit on varying domains with rough boundary of obstacle in barotropic case was considered in [11]. As in this article a delicate analysis of a weak compactness of the convective term is based on local decay of acoustic wave based on the spectral analysis of associated Neumann Laplace operator developed also in [9]. In this approach the influence of the perturbations of the domain is essential. Moreover the analysis of complete (heat conducting) fluid system driven to Oberbeck-Boussinesq approximation on large, unbounded domains can be found in [13, 9].

The organisation of the paper is as follows: In section 2 we formulate our problem, describe the primitive system with structural restrictions on thermodynamical functions and initial data, we recall some well known facts concerning the compressible Navier-Stokes-Fourier system, in particular available existence result. In section 2.3 the result of the paper is formulated. Section 3 is dedicated to the proof of the main result and consists of: uniform bounds based on total dissipation balance - section 3.2, preliminary convergence results - section 3.3, analysis of acoustic equation and its abstract form in terms of Neumann Laplace operator - section 3.4, in particular we provide local deca of acoustic wave in section 3.4.11 and compactness in time of the momentum is concluded in section 3.5, and finally in section 3.6 we summarise convergence procedure of NSF system to OB approximation.

2 Formulation of the Problem

Our aim is to study stability of the rescaled compressible Navier-Stokes-Fourier system in the regime that speed of sound dominates characteristic speed of fluid. Namely we take a Mach number proportional to a small parameter $Ma = \varepsilon$, where $\varepsilon \to 0$ and a Froude number $Fr = \sqrt{\varepsilon}$ (external sources of mechanical energy are small, and $\frac{Ma}{Fr} \to 0$, what corresponds to low stratification). About other characteristic numbers as Strouhal, Reynolds, Péclet number we assume they are equal one. Moreover, in our case we want our system to be driven by an exterior force related to gravitation of the rigid object(s) contain in the fluid domain. Namely our primitive system is posed on a family of domains Ω_ε, where Ω_ε poses small hole(s), being the rigid object(s) acting on the fluid by their gravitation, and $\Omega_\varepsilon \to \mathbb{R}^3$ in certain sense. Our goal is to study incompressible limits, with the Mach number $Ma = \varepsilon \to 0$, on a family of domain Ω_ε varying with $\varepsilon > 0$.

2.1 Notation

We denote by $\langle \cdot, \cdot \rangle$ duality pairing. By $L^p(B)$ we mean the space of Lebesgue measurable functions g, where $|g|^p$ is integrable over set B. The Sobolev space of functions which derivatives are integrable up to order k in L^p we denote by $W^{k,p}$. By $\mathcal{D}^{k,p}(B)$ we set homogenous Sobolev spaces i.e. $\mathcal{D}^{k,p}(B) = \{g \in L^1_{\text{loc}}(B) \,:\, D^\alpha g \in L^p(B),\ |\alpha| = k\}$, where $k \geq 0$ and $p \geq 1$. In the whole paper c will denote generic constant which may change from line to line.

2.2 Primitive system

To begin let us introduce "primitive system" - the rescaled Navier-Stokes-Fourier system with small Mach and Froude number which consists of: the continuity equation (conservation of mass), the momentum equation, the entropy balance and the total energy balance respectively

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x (\varrho_\varepsilon \boldsymbol{u}_\varepsilon) = 0, \tag{NSF$^1_\varepsilon$}$$

$$\partial_t(\varrho_\varepsilon \boldsymbol{u}_\varepsilon) + \operatorname{div}_x (\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) = \operatorname{div}_x \mathbb{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F_\varepsilon, \tag{NSF$^2_\varepsilon$}$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \operatorname{div}_x (\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \boldsymbol{u}_\varepsilon) + \operatorname{div}_x \left(\frac{\boldsymbol{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)}{\vartheta_\varepsilon} \right) = \sigma_\varepsilon, \tag{NSF$^3_\varepsilon$}$$

$$\frac{d}{dt} \int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{\varepsilon} \varrho_\varepsilon F_\varepsilon \right) dx = 0. \tag{NSF$^4_\varepsilon$}$$

The viscous stress tensor in (NSF$^2_\varepsilon$) is given by Newton's rheological law

$$\mathbb{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) = \mu(\vartheta_\varepsilon) \left(\nabla_x \boldsymbol{u}_\varepsilon + \nabla_x^T \boldsymbol{u}_\varepsilon - \frac{2}{3} \operatorname{div}_x \boldsymbol{u}_\varepsilon \mathbb{I}\mathrm{d} \right) + \eta(\vartheta_\varepsilon) \operatorname{div}_x \boldsymbol{u}_\varepsilon \mathbb{I}\mathrm{d} \tag{2.1}$$

and the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon} \left(\varepsilon^2 \mathbb{S}_\varepsilon(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{u}_\varepsilon - \frac{\boldsymbol{q}_\varepsilon(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right). \tag{2.2}$$

The heat flux $\boldsymbol{q}$ in (NSF$^3_\varepsilon$) is determined by Fourier's law

$$\boldsymbol{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) = -\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon, \tag{2.3}$$

where κ is a positive heat coefficient. The unknowns are the fluid mass density $\varrho_\varepsilon = \varrho_\varepsilon(x,t)$, the velocity field $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(t,x) : (0,T) \times \Omega_\varepsilon \to \mathbb{R}^3$ and absolute temperature $\vartheta_\varepsilon = \vartheta_\varepsilon(x,t) : (0,T) \times \Omega_\varepsilon \to \mathbb{R}$. The pressure p, the specific internal energy e and the specific entropy s are given scalar valued functions of ϱ and ϑ which are related through Gibbs' equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right). \tag{2.4}$$

Small parameter ε in the system (NSF_ε^1 - NSF_ε^4) results from dimensionless form of a Navier-Stokes-Fourier system and corresponds to small Mach and Froude number (Ma= ε, Fr= $\sqrt{\varepsilon}$), see [12], Klein at al. [20], Zeytounian [33]. Smallness of Mach number physically means that characteristic speed of the flow is dominated by the speed of the sound in the medium under consideration. A similar system can be obtained also by constitute scaling. Namely when rheological properties of the fluid are changing instead of characteristic geometrical parameters of the flow [25, 26].

We consider our "primitive" system on the following family of (bounded) domains which size depends on ε and is sufficiently large when $\varepsilon \to 0$ and moreover it contains holes which radiuses converge sufficiently "slow" to zero when $\varepsilon \to 0$. It can take the following form:

$$\Omega_\varepsilon = \frac{1}{\varepsilon^\delta} B_{R_0}(0) \setminus \left(\cup_{i=1,\dots,N}\, \varepsilon^\beta B_{r_i}(x_i)\right) \quad \text{where } \delta > 1 \text{ and } \beta < \frac{1}{4} \tag{2.5}$$

$$\text{and } B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset \text{ for } i,j = 1,\dots,N,\ i \neq j, \text{ and there exists an open set } B_R$$
$$\text{s.t. } \overline{B_R} \subset R_{R_0}(0) \text{ and } \left(\cup_{i=1,\dots,N} \varepsilon^\beta \overline{B_{r_i}(x_i)}\right) \subset B_R \text{ for all } \varepsilon \in (0,1].$$

Let us remark that such a very specific geometry of the domain in fact is not recurred by the analysis (it is choose for the brevity and simplicity). The result of the paper holds also for more general domains of sufficiently regular boundary where size of holes converge to zero sufficiently slow and outer boundary "escapes" to infinity sufficiently fast.

The system is supplemented with the following complete slip boundary conditions

$$\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0, \quad \text{and} \quad [\mathbf{S}(\vartheta_\varepsilon, \nabla_x \boldsymbol{u}_\varepsilon)\boldsymbol{n}] \times \boldsymbol{n} = 0 \tag{2.6}$$

and the boundary of physical space is thermally isolated, i.e.:

$$\boldsymbol{q} \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0. \tag{2.7}$$

Since $\delta > 0$ and speed of sound is proportional to $\frac{1}{\varepsilon}$, the outer boundary of Ω_ε becomes irrelevant when one considers the behaviour of acoustic waves, satisfied by the gradient component of the velocity, on some compact set of physical space. To provide dispersive estimates for the acoustic wave which allow to provide that the acoustic wave is not present in the limit $\varepsilon \to 0$, we need to assume that the inner perturbation of the domain in not too "fast", the similar restriction is assumed in [9, 11], in our case it means that $\beta < \frac{1}{4}$.

If we assume that self-gravitation of the fluid can be neglected, the origin of the gravitational force is an object (or are objects) surrounded by the domain Ω_ε and it may take the following form

$$F_\varepsilon(x) = \int_{\mathbb{R}^3} \frac{\sum_{i=1}^N m_{i,\varepsilon}(y)}{|x-y|} dy, \quad \text{with } m_{i,\varepsilon} \geq 0, \text{supp } m_{i,\varepsilon} \subset \varepsilon^\beta B_{r_i}(x_i),\ i = 1,\dots,N, \tag{2.8}$$

where m denotes mass density of rigid object(s) acting on the fluid by gravitation. Gravitation force can be written as harmonic function in Ω_ε

$$-\Delta F_\varepsilon = \sum_{i=1}^N m_{i,\varepsilon} \text{ in } \mathbb{R}^3, \quad m_{i,\varepsilon} \geq 0, \quad \int_{\mathbb{R}^3} \sum_{i=1}^N m_{i,\varepsilon} = 1 \text{ and } \nabla F_\varepsilon \in L^2(\mathbb{R}^3;\mathbb{R}^3) \text{ for each } \varepsilon \in (0,1]. \quad (2.9)$$

Then a gravitation force takes the following form $F \approx \frac{1}{|x|}$ far from rigid objects ($|x| \to \infty$) and consequently we have $\nabla_x F \approx -\frac{x}{|x|^3}$, $-\Delta F = \delta(0)$ (or if we have more then one object generating gravity, $-\Delta F = \sum_{i=1}^N \lambda_i \delta(x_i)$ in the limit, where $\sum_{i=1}^N \lambda_i = 1$ and δ denotes the Dirac delta). Then we may assume without loss of generality that for each $\varepsilon \in (0,1)$

$$\begin{aligned} &\|F_\varepsilon|_{B_R}\|_{L^p(B_R)} \leq c, \text{ for all } p \in [1,3) \text{ and } \|F_\varepsilon|_{B_R^c}\|_{L^p(B_R^c)} \leq c \text{ for all } p \in (3,\infty], \\ &\text{and } \|F_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq c\frac{1}{\varepsilon^\beta}. \end{aligned} \quad (2.10)$$

moreover

$$\begin{aligned} &\|\nabla F_\varepsilon|_{B_R}\|_{L^p(B_R)} \leq c \text{ for all } p \in [1,3/2) \text{ and } \|\nabla F_\varepsilon|_{B_R^c}\|_{L^p(B_R^c)} \leq c \text{ for all } p \in (3/2,\infty], \\ &\text{and } \|\nabla F_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq c\frac{1}{\varepsilon^{2\beta}}, \end{aligned} \quad (2.11)$$

where B_R contains (covers) all holes in Ω_ε and their neighbourhood (see (2.5)). Furthermore

$$\begin{aligned} &F_\varepsilon \to F \text{ in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for } p \in [1,3), \nabla F_\varepsilon \to \nabla F \text{ in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for } p \in [1,3/2) \text{ as } \varepsilon \to 0, \\ &F_\varepsilon \to F \text{ in } L^q(K) \text{ for } q \in [3,\infty), \nabla F_\varepsilon \to \nabla F \text{ in } L^q(K) \text{ for } q \in (3/2,\infty) \text{ as } \varepsilon \to 0, \end{aligned} \quad (2.12)$$

for any $K \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$.

2.2.1 Structural restrictions

In order to be able to use the existence result of [12], and later to build uniform estimates, we need to impose structural restrictions on the thermodynamical functions p, e, s as well as on the transport coefficients μ, η, κ. We set

$$p(\varrho_\varepsilon, \vartheta_\varepsilon) = \vartheta_\varepsilon^{5/2} P\left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}\right) + \frac{a}{3}\vartheta_\varepsilon^4, \ a > 0, \quad (2.13)$$

(the first component in (2.13) corresponds to the standard molecular pressure of a general monoatomic gas while the second one represents thermal radiation) where

$$P \in C^1[0,\infty) \cap C^2(0,\infty), \quad P(0) = 0, \quad P'(Z) > 0 \text{ for all } Z \geq 0 \quad (2.14)$$

and due to positive compressibility

$$\partial_\varrho p(\varrho, \vartheta) > 0. \quad (2.15)$$

Additionally to (2.14) we assume

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0. \quad (2.16)$$

The condition (2.16) means that the specific heat at constant volume is positive, namely $\partial_\vartheta e(\varrho,\vartheta)$ is positive and bounded. Since (2.16) is satisfied, $Z \to P(Z)/Z^{5/3}$ is decreasing function, and additionally we assume

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0. \tag{2.17}$$

Accordingly to Gibbs' relation (2.4), the specific internal energy and the specific entropy can be written in the following forms

$$e(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \qquad s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4}{3}a\frac{\vartheta^3}{\varrho} \tag{2.18}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{2.19}$$

The transport coefficients: μ - shear viscosity, η - bulk viscosity and κ - heat conductivity are assumed to be continuously differentiable functions of the temperature $\vartheta \in [0,\infty)$ satisfying the following growth conditions for all $\vartheta \geq 0$

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \underline{\mu}(1+\vartheta), \quad 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta), \quad 0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3), \tag{2.20}$$

where $\underline{\mu}$, $\overline{\mu}$, $\overline{\eta}$, $\underline{\kappa}$ and $\overline{\kappa}$ are positive constants. Let us remark that the above assumptions can be not optimal from point of view of the existence theory.

2.2.2 Equilibrium state

The so-called equilibrium state (static state) for each scaled NSF_ε system consist of static density $\tilde{\varrho}_\varepsilon$ and constant temperature distribution $\overline{\vartheta}$ satisfying

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon\tilde{\varrho}_\varepsilon \nabla_x F_\varepsilon \quad \text{ in } \Omega_\varepsilon.$$

(A'priori it is not known that the temperature has to be constant. This is a consequence that entropy production rate σ_ε has to be kept small and then $\nabla\vartheta_\varepsilon$ needs to vanish as $\varepsilon \to 0$). For a convenience we consider a static density $\tilde{\varrho}_\varepsilon$ defined on the whole space $\mathbb{R}^3$, i.e.

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon\tilde{\varrho}_\varepsilon \nabla_x F_\varepsilon \text{ in } \mathbb{R}^3 \qquad \text{where} \lim_{|x|\to\infty} \tilde{\varrho}_\varepsilon(x) = \overline{\varrho}. \tag{2.21}$$

Hence we have

$$\tilde{\varrho}_\varepsilon - \overline{\varrho} = \frac{\varepsilon}{P'(\overline{\varrho})} F_\varepsilon + \varepsilon^2 h_\varepsilon F_\varepsilon, \quad P'(\varrho) = \frac{1}{\varrho}\partial_\varrho p(\varrho,\overline{\vartheta}), \quad \text{ with } \|h_\varepsilon\|_{L^\infty(\mathbb{R}^3)} < c, \tag{2.22}$$

$$|\nabla_x \tilde{\varrho}_\varepsilon(x)| \leq \varepsilon c |\nabla_x F_\varepsilon(x)| \quad \text{ for } x \in \mathbb{R}^3. \tag{2.23}$$

The above gives closeness of the limit equilibrium state density to a constant state.

2.2.3 Ill-prepered initial data

Since we work with weak solutions based on energy estimates and control of entropy production rate we need to assume that initial data are close to equilibrium state. Namely initial density and initial temperature are of the following form

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \qquad \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)} \tag{2.24}$$

where $\overline{\vartheta} > 0$ is positive constants characterising the static distribution of the absolute temperature and $\varrho_{0,\varepsilon}^{(1)}$ and $\vartheta_{0,\varepsilon}^{(1)}$ are bounded measurable functions and satisfy

$$\begin{gathered} \|\varrho_{0,\varepsilon}^{(1)}\|_{L^\infty \cap L^2(\Omega_\varepsilon)} \leq c \quad \text{and} \quad \int \varrho_{0,\varepsilon}^{(1)} \,\mathrm{d}x = 0, \quad \|\vartheta_{0,\varepsilon}^{(1)}\|_{L^\infty \cap L^2(\Omega_\varepsilon)} \leq c \quad \text{and} \quad \int \vartheta_{0,\varepsilon}^{(1)} \,\mathrm{d}x = 0, \\ \|\sqrt{\varrho_{0,\varepsilon}}\boldsymbol{u}_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \leq c, \quad \|\boldsymbol{u}_{0,\varepsilon}\|_{L^\infty \cap L^2(\Omega_\varepsilon)} \leq c \end{gathered} \tag{2.25}$$

for all $\varepsilon \in (0,1]$. The above uniform bounds will allow to control right hand side of total dissipation balance which is a source of uniform estimates needed to perform the limit system. Nevertheless, such a choice allow to consider nontrivial dynamics but on the other hand it causes oscillations in acoustic equation. Those will be eliminated by dispersive estimates.

2.2.4 Weak solutions to primitive system

We recall following result from [12]:

Theorem 2.1 *Let Ω_ε be a bounded domain with boundary of class $C^{2+\nu}$ for $\nu > 0$. Assume that p, e, s satisfies Gibbs' relation* (2.4) *and structural hypothesis (2.13 - 2.19) and transport coefficients μ, η, κ enjoy growth conditions (2.20). Let initial data be given by (2.24) and $\varrho_{0,\varepsilon}^{(1)}$, $\boldsymbol{u}_\varepsilon(0)$, $\vartheta_{0,\varepsilon}^{(1)}$ are bounded measurable functions, and $F_\varepsilon \in W^{1,\infty}(\Omega_\varepsilon)$.*

Then for every $\varepsilon > 0$ so small that $\varrho_{0,\varepsilon}$, $\vartheta_{0,\varepsilon} > 0$, there exists a weak solution $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_\varepsilon$ to the Navier-Stokes-Fourier system (NSF_ε^1 - NSF_ε^4), (2.1 - 2.3) with boundary conditions (2.6 - 2.7) and initial condition (2.24) in the following sense

$$\int_0^T \int_{\Omega_\varepsilon} (\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi) \,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \varphi(0,\cdot) \,\mathrm{d}x \tag{2.26}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon)$;

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} \left(\varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon [\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon] : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho_\varepsilon, \vartheta_\varepsilon) \mathrm{div}_x \, \varphi \right) \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\Omega_\varepsilon} \left(\mathsf{S}_\varepsilon : \nabla_x \varphi - \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F_\varepsilon \cdot \varphi \right) \mathrm{d}x\mathrm{d}t - \int_{\Omega_\varepsilon} (\varrho_{0,\varepsilon} \boldsymbol{u}_{0,\varepsilon}) \cdot \varphi(0,\cdot) \,\mathrm{d}x \end{aligned} \tag{2.27}$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3)$, $\varphi \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$;

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{\varepsilon} \varrho_\varepsilon F_\varepsilon \right)(t) \,\mathrm{d}x \\ &= \left(\frac{1}{2} \varrho_{0,\varepsilon} |\boldsymbol{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \frac{1}{\varepsilon} \varrho_{0,\varepsilon} F_\varepsilon \right) \mathrm{d}x \end{aligned} \tag{2.28}$$

for a.a. $t \in (0,T)$*;*

$$\int_0^T \int_{\Omega_\varepsilon} (\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\partial_t \varphi + \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi) \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \frac{\boldsymbol{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \, \mathrm{d}x\mathrm{d}t$$
$$+ \langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega}_\varepsilon)} = -\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\varphi(0,\cdot)\, \mathrm{d}x \tag{2.29}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon)$*, with* $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega}_\varepsilon)$*.*

We always tacitly assume that all integrals in the definition of a weak solutions are well defined, i.e. the quantities are at least integrable on $(0,T) \times \Omega_\varepsilon$. Let us notice that the mappings $t \to \varrho_\varepsilon(t,\cdot)$ and $t \to (\varrho_\varepsilon \boldsymbol{u}_\varepsilon)(t,\cdot)$ are weakly continuous and $\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon}$, $(\varrho_\varepsilon \boldsymbol{u}_\varepsilon)(0,\cdot) = \varrho_{0,\varepsilon}\boldsymbol{u}_{0,\varepsilon}$.

The following regularity of solutions $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_\varepsilon$ can be obtained (see [12, Theorem 3.2 pp. 122])

$$\varrho_\varepsilon \in C_{\mathrm{weak}}(0,T; L^{5/3}(\Omega_\varepsilon)),\ \varrho_\varepsilon \in L^q((0,T)\times\Omega_\varepsilon) \text{ for a certain } q > \frac{5}{3}, \quad \boldsymbol{u}_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)).$$

The absolute temperature ϑ_ε is a measurable function $\vartheta_\varepsilon(t,x) > 0$ for a.a. $(t,x) \in (0,T)\times\Omega_\varepsilon$

$$\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon)) \cap L^\infty(0,T; L^4(\Omega_\varepsilon)), \quad \log\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega_\varepsilon)).$$

Since entropy rate can be a nonnegative measure (and can posses some jumps), the total entropy $\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)$ may not be weakly continuous in time. To avoid this problem, following [12], we introduce a time lifting Σ of the measure σ in the following form

$$\langle \Sigma_\varepsilon, \varphi \rangle = \langle \sigma_\varepsilon, I[\varphi] \rangle, \quad \text{where } I[\varphi](t,x) = \int_0^t \varphi(z,x)\, \mathrm{d}z \quad \text{for any } \varphi \in L^1(0,T; C(\overline{\Omega}_\varepsilon)). \tag{2.30}$$

It can be shown that Σ_ε can be identified with an abstract function $\Sigma_\varepsilon \in L^\infty_{\mathrm{weak}}(0,T; \mathcal{M}^+(\overline{\Omega}_\varepsilon))$, where

$$\langle \Sigma_\varepsilon(\tau), \varphi \rangle = \lim_{\delta\to 0^+} \langle \sigma_\varepsilon, \psi_\delta \varphi \rangle, \quad \text{with} \quad \psi_\delta(t) = \begin{cases} 0 & \text{for } t \in [0,\tau), \\ \frac{1}{\delta}(t-\tau) & \text{for } t \in (\tau, \tau+\delta), \\ 1 & \text{for } t \geq \tau+\delta, \end{cases}$$

particularly, the measure Σ_ε is well-defined for any $\tau \in [0,T]$, and mapping $\tau \to \Sigma_\varepsilon$ is non-increasing in the sense of measures. Here L^∞_{weak} stands for "weakly measurable".

Weak formulation of entropy balance can be rewritten now as

$$\int_{\Omega_\varepsilon} [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\varphi(0,\cdot)]\, \mathrm{d}x + \langle \Sigma_\varepsilon(\tau), \varphi(\tau,\cdot) \rangle - \langle \Sigma_\varepsilon(0), \varphi(0,\cdot) \rangle$$
$$= \int_0^\tau \langle \Sigma_\varepsilon, \partial_t \varphi \rangle \, \mathrm{d}t + \int_0^\tau \int_{\Omega_\varepsilon} \left(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\partial_t \varphi + \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi + \frac{\boldsymbol{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \right) \mathrm{d}x\mathrm{d}t \tag{2.31}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon)$ and the mapping $t \to \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(t,\cdot) + \Sigma_\varepsilon(t)$ is continuous with values in $\mathcal{M}^+(\overline{\Omega}_\varepsilon)$, provided that $\mathcal{M}^+$ is endowed with a weak* topology.

2.3 Main result

We say that functions $\boldsymbol{U}$, Θ and r are a weak solution to the Oberbeck-Boussinesq approximation if

$$
\begin{aligned}
&\boldsymbol{U} \in L^\infty(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)),\\
&\Theta \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^3))\\
&r \in L^\infty(0,T;L^{5/3}_{\mathrm{loc}}(\mathbb{R}^3))
\end{aligned}
$$

$$
\mathrm{div}_x \boldsymbol{U} = 0 \text{ a.e. on } (0,T)\times\mathbb{R}^3,
$$

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} (\overline{\varrho}(\boldsymbol{U}\cdot\partial_t\boldsymbol{\varphi} + (\boldsymbol{U}\otimes\boldsymbol{U}) : \nabla_x\boldsymbol{\varphi}))\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^3} \overline{\varrho}\boldsymbol{U}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x\\
+\int_0^T\int_{\mathbb{R}^3} \left(\mathbb{S} : \nabla_x\boldsymbol{\varphi} - \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta\nabla_x F\cdot\boldsymbol{\varphi}\right)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.32}
$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T);C_c^\infty(\mathbb{R}^3;\mathbb{R}^3))$, where $\mathrm{div}_x\,\boldsymbol{\varphi} = 0$ and $\mathbb{S} = \mu(\overline{\vartheta})(\nabla_x\boldsymbol{U} + \nabla_x\boldsymbol{U}^T)$.

Moreover

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\Theta\left(\partial_t\varphi + \boldsymbol{U}\cdot\nabla_x\varphi\right)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{R}^3}\left(\overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})F\boldsymbol{U} + \kappa(\overline{\vartheta})\nabla_x\Theta\right)\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\\
&= -\int_{\mathbb{R}^3}\overline{\varrho}c_p\Theta_0\varphi(0,\cdot)\,\mathrm{d}x
\end{aligned}
\tag{2.33}
$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^3)$. Furthermore the Boussinesq relation is satisfied in the following sense

$$
\int_0^T\int_{\mathbb{R}^3} r\mathrm{div}_x\,\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^3}\frac{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}\Theta\mathrm{div}_x\,\boldsymbol{\varphi} + \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}F\mathrm{div}_x\,\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t. \tag{2.34}
$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times\mathbb{R}^3)$ By c_p we mean specific heat at constant pressure and $c_p(\overline{\varrho},\overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho},\overline{\vartheta}) + \alpha(\overline{\varrho},\overline{\vartheta})\frac{\overline{\vartheta}}{\overline{\varrho}}\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$ by $\alpha > 0$ we mean the coefficient of thermal expansion of the fluid, $\alpha(\overline{\varrho},\overline{\vartheta}) = \frac{1}{\overline{\varrho}}\frac{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}$, both are evaluated at the reference density $\overline{\varrho}$ and temperature $\overline{\vartheta}$.

Then the main result reads as follows:

Theorem 2.2 *Let $\Omega_\varepsilon \subset \mathbb{R}^3$ be a family of domains defined by (2.5) with $\beta < \frac{1}{4}$ and $\delta > 1$. Assume that p, e, and s satisfy (2.13 - 2.17), the transport coefficients μ, η and κ satisfy growth conditions (2.20), and driving force is determined by a scalar potential F_ε defined by (2.8) satisfying (2.10 - 2.12). Let $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the scaled Navier-Stokes-Fourier system (NSF_ε^1 - NSF_ε^4), (2.1 - 2.3) on the sets $(0,T)\times\Omega_\varepsilon$, supplemented with boundary conditions (2.6), (2.7) and initial data (2.24), with $\tilde{\varrho}_\varepsilon > 0$, $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$ and satisfying (2.25) for all $\varepsilon \in (0,1)$. Moreover we assume that*

$$
\begin{aligned}
&\varrho_{0,\varepsilon}^{(1)} \rightharpoonup \varrho_0^{(1)} \text{ weakly in } L^2(\mathbb{R}^3), \quad \boldsymbol{u}_{0,\varepsilon} \rightharpoonup \boldsymbol{U}_0 \text{ weakly in } L^2(\mathbb{R}^3;\mathbb{R}^3),\\
&\vartheta_{0,\varepsilon}^{(1)} \rightharpoonup \vartheta_0^{(1)} \text{ weakly in } L^2(\mathbb{R}^3).
\end{aligned}
\tag{2.35}
$$

Then for suitable subsequence we obtain that

$$\varrho_\varepsilon \to \overline{\varrho} \text{ strongly in } L^\infty(0,T;L^{5/3}(K)), \qquad \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \rightharpoonup r \text{ weakly in } L^2(0,T;L^2(K)),$$
$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \rightharpoonup \Theta \text{ weakly in } L^2(0,T;W^{1,2}(K)),$$
$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{U} \text{ weakly in } L^2(0,T;W^{1,2}(K;\mathbb{R}^3)), \qquad \boldsymbol{u}_\varepsilon \to \boldsymbol{U} \text{ strongly in } L^2((0,T)\times K;\mathbb{R}^3)$$

for any compact set $K \subset \mathbb{R}^3$, where functions $\boldsymbol{U}$, Θ is a weak solution of the Oberbeck-Boussinesq approximation (OB^1 – OB^4) in $(0,T)\times\mathbb{R}^3$ in the sense specified in (2.32 - 2.34) with $\boldsymbol{U}(0,\cdot) = \boldsymbol{H}[\boldsymbol{U}_0]$ and $\Theta_0 = \vartheta_0^{(1)}$.

Here $\boldsymbol{H}[\cdot]$ denotes the projection on the space of divergence free functions of Helmholtz decomposition.

3 The proof of the main result

3.1 Notation and some tools

In a forthcoming section we present uniform estimates being a consequence of a total dissipation balance. To this end, similarly as in [12] we introduce the decomposition on essential and residual part of a measurable function h as

$$h = [h]_{\rm ess} + [h]_{\rm res}, \quad [h]_{\rm ess} = \chi(\varrho_\varepsilon,\vartheta_\varepsilon)h, \quad [h]_{\rm res} = (1-\chi(\varrho_\varepsilon,\vartheta_\varepsilon))h,$$

where $\chi \in C_c^\infty((0,\infty)\times(0,\infty))$, $0 \le \chi \le 1$, $\chi = 1$ on the set $\mathcal{O}_{\rm ess}$ and

$$\mathcal{O}_{\rm ess} = [\overline{\varrho}/2, 2\overline{\varrho}] \times [\overline{\vartheta}/2, 2\overline{\vartheta}], \quad \mathcal{O}_{\rm res} = (0,\infty)^2 \setminus \mathcal{O}_{\rm ess}$$

Hence we we obtain the following generalised Korn-Poincaré inequality (see [1, Proposition 4.1], [29])

Proposition 3.1 *Let $\boldsymbol{v} \in W^{1,2}(\Omega_\varepsilon \cap B;\mathbb{R}^3)$ and $M \subset \Omega_\varepsilon \cap B$, s.t. $|M| > m > 0$, where B is a bounded ball. Then*

$$\|\boldsymbol{v}\|_{W^{1,2}(\Omega_\varepsilon\cap B;\mathbb{R}^3)} \le c(m)\left(\left\|\nabla_x \boldsymbol{v} + \nabla_x^T \boldsymbol{v} - \frac{2}{3}{\rm div}_x\, \boldsymbol{v}\mathbb{I}\right\|_{L^2(\Omega_\varepsilon\cap B;\mathbb{R}^{3\times3})} + \int_M |\boldsymbol{v}|^2\,{\rm d}x\right)$$

where the constant c is independent of $\varepsilon \to 0$.

The result of Sauter and Warnke [30, Lemma 2.2] provide the extension property independent of ε, namely the following holds

Proposition 3.2 *Let Ω_ε defined by (2.5), then there exist extension operator $E_\varepsilon : W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3) \to W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)$ such that*

$$\|E_\varepsilon(\boldsymbol{v})\|_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} \le c\|\boldsymbol{v}\|_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)}$$

for all $\boldsymbol{v} \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$ and c is independent of sufficiently small $\varepsilon \to 0$.

3.2 Total dissipation balance and uniform estimates, part I

The total dissipation balance is here the main tool to obtain all uniform bounds for the family $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_\varepsilon$ of weak solutions to the primitive system. Introducing a ballistic free energy (see Eriksen [6, Chapter 13])

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho\left(e(\varrho,\vartheta) - \overline{\vartheta}s(\varrho,\vartheta)\right).$$

we may notice by (2.21) that

$$\frac{\partial H(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\partial\varrho} = \varepsilon F_\varepsilon + \text{ const in } \Omega_\varepsilon$$

provided $\tilde\varrho_\varepsilon > 0$ in Ω_ε (due to positive compressibility (2.15)). Since mass M_ε is conserved in time, i.e. for a.a. $\tau \in (0,T)$ $\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon}\,\mathrm{d}x = \int_{\Omega_\varepsilon} \varrho_\varepsilon(\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \tilde\varrho_\varepsilon\,\mathrm{d}x$, combining the total energy balance (2.28) and the entropy equation (2.29) we obtain the following form of a total dissipation balance

$$\begin{aligned}
&\int_{\Omega_\varepsilon}\left(\frac{1}{2}\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2\right)(t)\,\mathrm{d}x\\
&+\frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon-\tilde\varrho_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})\right)(t)\,\mathrm{d}x + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\left[[0,t]\times\overline{\Omega}_\varepsilon\right]\\
&=\int_{\Omega_\varepsilon}\left(\frac{1}{2}\varrho_{0,\varepsilon}|\boldsymbol{u}_{0,\varepsilon}|^2\right)\,\mathrm{d}x + \frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon}-\tilde\varrho_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})\right)\,\mathrm{d}x.
\end{aligned}\tag{3.1}$$

It is provided (see Lemma 5.1 in [12]) that $H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon-\tilde\varrho_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})$ is non-negative, strictly coercive, attain global minimum zero at point $(\tilde\varrho_\varepsilon,\overline{\vartheta})$, dominates internal energy ϱe and entropy s far from $(\tilde\varrho_\varepsilon,\overline{\vartheta})$.

Lemma 3.3 *(see [12, Lemma 5.1]) Let $\tilde\varrho_\varepsilon > 0$, $\overline{\vartheta} > 0$. Let e and s satisfies structural restrictions (2.13)-(2.17). Let $V \subset (0,\infty)^2$ be a compact set containing $(\tilde\varrho_\varepsilon,\overline{\vartheta})$. Then there exists positive constants $c_i(V)$, $i = 1,2,3$ such that*

$$\begin{aligned}
c_1\left(|\varrho_\varepsilon-\tilde\varrho_\varepsilon|^2 + |\vartheta_\varepsilon-\overline{\vartheta}|^2\right) &\le H_{\tilde\varrho_\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon-\tilde\varrho_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\varrho} - H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})\\
&\le c_2\left(|\varrho_\varepsilon-\tilde\varrho_\varepsilon|^2 + |\vartheta_\varepsilon-\overline{\vartheta}|^2\right)
\end{aligned}\tag{3.2}$$

for all $(\varrho_\varepsilon,\ \vartheta_\varepsilon) \in V$;

$$H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon-\tilde\varrho_\varepsilon)\frac{\partial H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\varrho} - H_{\overline{\vartheta}}(\tilde\varrho_\varepsilon,\overline{\vartheta}) \ge c_3\left(1 + \varrho_\varepsilon|e(\varrho_\varepsilon,\vartheta_\varepsilon)| + \varrho_\varepsilon|s(\varrho_\varepsilon,\vartheta_\varepsilon)|\right)\tag{3.3}$$

for all $(\varrho_\varepsilon,\ \vartheta_\varepsilon) \in (0,\infty)^2 \setminus V$.

With the above lemma and with assumptions on initial data (2.25) we are able to control right hand side of (3.1). Then from (3.1) we deduce that

$$\operatorname*{ess\ sup}_{t\in(0,T)}\int_{\Omega_\varepsilon}\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2(t,\cdot)\,\mathrm{d}x \le c\tag{3.4}$$

employing Lemma 3.3, particularly (3.2), relation (3.1) and assumptions (2.25) together with (2.22), (2.23) and due to uniform closeness of $\bar{\varrho}$ and $\tilde{\varrho}_\varepsilon$ as $\varepsilon \to 0$ (see (2.22) and (2.10)) we deduce

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{ess}} (t,\cdot) \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{3.5}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\mathrm{ess}} (t,\cdot) \right\|_{L^2(\Omega_\varepsilon)} \leq c. \tag{3.6}$$

Additionally

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \leq c \tag{3.7}$$

and

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\Omega_\varepsilon)} \leq \varepsilon^2 c. \tag{3.8}$$

Let us notice that measure of "residual" set is small, although the measure of Ω_ε tends to infinity as $\varepsilon \to 0$. According to (3.3) and structural assumptions (2.13), (2.14), (2.18), (2.16), (2.17) we infer

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} (|[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}| + |[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}| + |[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}| \,\mathrm{d}x) \leq \varepsilon^2 c \tag{3.9}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]^{5/3}_{\mathrm{res}}(t,\cdot)\,\mathrm{d}x \leq \varepsilon^2 c, \tag{3.10}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^{4}_{\mathrm{res}}(t,\cdot)\,\mathrm{d}x \leq \varepsilon^2 c. \tag{3.11}$$

Moreover directly by (3.3) the "residual" set is also small (even if $|\Omega_\varepsilon| \to \infty$ as $\varepsilon \to 0$), i.e.

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} \mathbb{1}_{\mathrm{res}}(t,\cdot) dx \leq \varepsilon^2 c. \tag{3.12}$$

Then, by (3.10), (3.12) we immediately have

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{res}} \right\|_{L^1(\Omega_\varepsilon)} \leq c\varepsilon. \tag{3.13}$$

Since (2.1), (2.2), (2.3) ($\mathbf{S}$ and $\boldsymbol{q}$ are linear functions of $\nabla_x \boldsymbol{u}_\varepsilon$ and $\nabla_x \vartheta_\varepsilon$ and we use (2.20), see [12, Chapter 2.2.3]), the estimate (3.8) provides

$$\int_0^T \|\mathbf{S}\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times3})} \,\mathrm{d}t \leq c, \tag{3.14}$$

$$\int_0^T \left\| \nabla_x \boldsymbol{u}_\varepsilon + \nabla_x^T \boldsymbol{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x\, \boldsymbol{u}_\varepsilon \mathbf{Id} \right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times3})} \,\mathrm{d}t \leq c \tag{3.15}$$

and

$$\int_0^T \left\| \nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon;R^3)} \mathrm{d}t + \int_0^T \left\| \nabla_x \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon;R^3)} \mathrm{d}t < c. \tag{3.16}$$

As (3.12) hols, we can apply the Poicaré inequality to infer

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \mathrm{d}t + \int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \mathrm{d}t < c. \tag{3.17}$$

Generalized version of the Korn inequality, see Proposition 3.1 provides

$$\int_0^T \|\boldsymbol{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \mathrm{d}t < c \tag{3.18}$$

as a consequence of (3.4), $[\varrho_\varepsilon]_{\mathrm{ess}} \geq \overline{\varrho}/2 > 0$ and (3.12).

3.3 Convergence with $\varepsilon \to 0$ to the limit system. Part I.

Let us, for the moment, choose a fixed test function φ in the formulation of the limit problem with a compact support K, $K \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$. Then we will find such ε_0, that $K \subset \Omega_\varepsilon$ for all $\varepsilon < \varepsilon_0$.

Now we want to pass to the limit with $\varepsilon \to 0$ in the sequence $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \vartheta_\varepsilon\}_\varepsilon$. First, let us notice that by (3.5), (3.10), (2.10), (2.22), (3.12) we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \tilde{\varrho}_\varepsilon\|_{(L^2+L^{5/3})(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0$$

and by (2.22)

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\tilde{\varrho}_\varepsilon(t,\cdot) - \overline{\varrho}\|_{L^q_{\mathrm{loc}}(\mathbb{R}^3)} \leq \varepsilon C \text{ for } q \in [1,3), \tag{3.19}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\tilde{\varrho}_\varepsilon(t,\cdot) - \overline{\varrho}\|_{(L^{5/3}+L^q)(\mathbb{R}^3)} \to 0 \text{ as } \varepsilon \to 0 \quad \text{for } q > 3 \tag{3.20}$$

(here $v \in L^{p_1} + L^{p_2}$ means that there exist $v_1 \in L^{p_1}$, $v_2 \in L^{p_2}$ s.t. $v = v_1 + v_2$). Then

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\|_{(L^2+L^{5/3})(\Omega_\varepsilon\cap K)} \to 0 \text{ as } \varepsilon \to 0$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\|_{(L^2+L^{5/3}+L^q)(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0 \quad \text{for } q > 3, \tag{3.21}$$

and finally

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\|_{L^{5/3}(\Omega_\varepsilon\cap K)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.22}$$

Therefore fluid density become constant since $\varepsilon \to 0$, i.e. as the Mach number tends to zero. By the same arguments we immediately have

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \overset{*}{\rightharpoonup} r \text{ weakly* in } L^\infty(0,T;L^{5/3}(K)) \text{ for any compact } K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}. \tag{3.23}$$

Extending properly ϱ_ε we may obtain the above convergence also locally on whole $\mathbb{R}^3$ space.

Next by (3.6), (3.11), (3.12) we have

$$\operatorname{ess}\sup_{t\in(0,T)} \|\vartheta_\varepsilon(t,\cdot)-\overline{\vartheta}\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon\to 0.$$

Due to uniform extension property (Proposition 3.2) and by (3.16), (3.17) one gets

$$\Theta_\varepsilon = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \rightharpoonup \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3)) \tag{3.24}$$

and by (3.18)

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{U} \text{ weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3)). \tag{3.25}$$

Then continuity equations provides by (3.20), (3.21) that

$$\operatorname{div}_x \boldsymbol{U} = 0 \text{ a.a. in } (0,T)\times\mathbb{R}^3.$$

To pass to the limit in rescaled NSF$_\varepsilon$ system one of the most difficult steps it to provide strong convergence of the velocity field the velocity field in order to control the limit of convective term. Namely we need to show that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{U} \quad \text{strongly in } L^2((0,T)\times K) \text{ for any compact } K\subset\mathbb{R}^3\setminus\{x_1,\dots,x_N\}. \tag{3.26}$$

One can observe that it is sufficient to provide that

$$\varrho_\varepsilon\boldsymbol{u}_\varepsilon \to \overline{\varrho}\boldsymbol{U} \quad \text{in } L^2(0,T;W^{-1,2}(K)). \tag{3.27}$$

To see it let us notice (or we refer to [13]) that for any $\varphi\in C^\infty(\mathbb{R}^3)$ where $\operatorname{supp}\varphi\subset K$

$$\overline{\varrho}\int_0^T\int_{\mathbb{R}^3}\varphi|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^3}\varphi(\overline{\varrho}-\varrho_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}\varphi\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

By (3.22) and by embedding $W^{1,2}\subset L^6$ we have

$$\int_0^T\int_{\mathbb{R}^3}\varphi(\overline{\varrho}-\varrho_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\to 0$$

and by (3.25) and (3.27)

$$\int_0^T\int_{\mathbb{R}^3}\varphi\varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\to\overline{\varrho}\int_0^T\int_{\mathbb{R}^3}\varphi|\boldsymbol{U}|^2\,\mathrm{d}x\,\mathrm{d}t$$

what provides convergence in convective term. Then it is enough to prove, instead of (3.27), that

$$\left\{t\to\int_{\mathbb{R}^3}(\varrho_\varepsilon\boldsymbol{u}_\varepsilon)(t,\cdot)\boldsymbol{\varphi}\,\mathrm{d}x\right\} \quad \text{is precompact in } L^2(0,T) \tag{3.28}$$

and

$$\left\{t\to\int_{\mathbb{R}^3}\varrho_\varepsilon\boldsymbol{u}_\varepsilon(\cdot,t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\right\} \quad\to\quad \left\{t\to\overline{\varrho}\int_{\mathbb{R}^3}\boldsymbol{U}(\cdot,t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\right\} \text{ in } L^2(0,T) \tag{3.29}$$

for any fixed $\varphi \in C^\infty(\mathbb{R}^3)$ where $\operatorname{supp}\varphi \subset K$ and for sufficiently small ε. Indeed by (3.4), (3.5), (3.10) we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon \boldsymbol{u}_\varepsilon\|_{L^{5/4}(K;\mathbb{R}^3)} \leq c(K) \quad \text{for any compact } K \subset \mathbb{R}^3 \setminus \{x_1, \ldots, x_N\},$$

then by the compact embedding $L^{5/4}(K) \subset W^{-1,2}(K)$ we get (3.27). Now we can concentrate only on proving (3.28).

3.4 Analysis of the acoustic equation

3.4.1 Reformulation to wave equation

As it was already emphasised, our aim is to show (3.28) for any fixed $\varphi \in C^\infty(\mathbb{R}^3)$, $\operatorname{supp}\varphi \subset K$. To this end we can rewrite our primitive NSF_ε system in the form of Lighthill's acoustic analogy (see [23]). Since it is more convenient to use the time lifting introduced in (2.30) (due to presence of σ_ε which may cause discontinuities in time) NSF_ε may be rewritten in the following form

$$\begin{aligned} \varepsilon\partial_t S_\varepsilon + \omega \mathrm{div}_x \boldsymbol{V}_\varepsilon &= \varepsilon \tilde{f}^1_\varepsilon, \\ \varepsilon\partial_t \boldsymbol{V}_\varepsilon + \nabla_x S_\varepsilon &= \varepsilon \tilde{\boldsymbol{f}}^2_\varepsilon, \end{aligned} \tag{3.30}$$

with homogenous Neumann boundary condition $\boldsymbol{V}_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$, where

$$S_\varepsilon = A\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho}s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\right) - \overline{\varrho}F_\varepsilon + \frac{B}{\varepsilon}\Sigma_\varepsilon, \qquad \boldsymbol{V}_\varepsilon = \varrho_\varepsilon \boldsymbol{u}_\varepsilon, \tag{3.31}$$

$$\tilde{f}^1_\varepsilon = \left[\mathrm{div}_x\, B \underbrace{\left(\varrho_\varepsilon \frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \boldsymbol{u}_\varepsilon\right)}_{H^1_\varepsilon} + \mathrm{div}_x\, B \underbrace{\left(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon}\right)}_{H^2_\varepsilon} \right], \tag{3.32}$$

$$\begin{aligned} \tilde{\boldsymbol{f}}^2_\varepsilon = &\underbrace{B\frac{1}{\varepsilon^2}\nabla_x \Sigma_\varepsilon}_{\nabla_x G^1_\varepsilon} + \mathrm{div}_x \underbrace{\mathbb{S}_\varepsilon}_{G^{2,1}_\varepsilon} - \mathrm{div}_x \underbrace{(\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon)}_{G^{2,2}_\varepsilon} + \underbrace{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\nabla_x F_\varepsilon}_{G^4_\varepsilon} \\ &+ \nabla_x \underbrace{\frac{1}{\varepsilon}\left[A\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho}s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\right)\right]}_{G^3_\varepsilon}. \end{aligned} \tag{3.33}$$

Here the constants A, B, ω are chosen s.t.

$$B\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) = \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}), \quad A + B\partial_\varrho(\varrho s)(\overline{\varrho}, \overline{\vartheta}) = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}), \quad \omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})} > 0. \tag{3.34}$$

Let us notice that ω is bounded due to assumption on thermodynamical function p and s and due to positive compressibility (2.15) and positive specific heat at constant volume.

The system (3.30) should be understand in a weak sense, namely the following identities are satisfied

$$\begin{aligned}&\varepsilon\int_0^T\langle S_\varepsilon(t,\cdot),\partial_t\varphi\rangle\,\mathrm{d}t+\omega\int_0^T\int_{\Omega_\varepsilon}\boldsymbol{V}_\varepsilon\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\\&=-\varepsilon\,\langle S_{0,\varepsilon},\varphi(0,\cdot)\rangle+\varepsilon\int_0^T\int_{\Omega_\varepsilon}(H^1_\varepsilon+H^2_\varepsilon)\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\end{aligned}\tag{3.35}$$

for any $\varphi\in C^1_c([0,T)\times\overline{\Omega}_\varepsilon)$ and

$$\begin{aligned}&\varepsilon\int_0^T\int_{\Omega_\varepsilon}\boldsymbol{V}_\varepsilon\cdot\partial_t\varphi\,\mathrm{d}t+\int_0^T\langle S_\varepsilon,\mathrm{div}_x\,\varphi\rangle\,\mathrm{d}t\\&=-\varepsilon\int_{\Omega_\varepsilon}\boldsymbol{V}_{0,\varepsilon}\cdot\varphi(0,\cdot)\,\mathrm{d}x+\varepsilon\int_0^T\langle G^1_\varepsilon(t,\cdot),\mathrm{div}_x\,\varphi\rangle\,\mathrm{d}t+\varepsilon\int_0^T\int_{\Omega_\varepsilon}G^{2,1}_\varepsilon:\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\\&-\varepsilon\int_0^T\int_{\Omega_\varepsilon}G^{2,2}_\varepsilon:\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t+\varepsilon\int_0^T\int_{\Omega_\varepsilon}G^3_\varepsilon\mathrm{div}_x\,\varphi\,\mathrm{d}x\mathrm{d}t+\varepsilon\int_0^T\int_{\Omega_\varepsilon}G^4_\varepsilon\cdot\varphi\,\mathrm{d}x\mathrm{d}t\end{aligned}\tag{3.36}$$

for any $\varphi\in C^1_c([0,T)\times\overline{\Omega}_\varepsilon;\mathbb{R}^3)$, $\varphi\cdot\boldsymbol{n}|_{\partial\Omega_\varepsilon}=0$.

In the forthcoming part of the paper we will concentrate on a gradient part of the momentum $\boldsymbol{V}_\varepsilon$ coming from the Helmholtz decomposition. In particular we provide its local decay. We will need to reformulate the above acoustic equation in terms of Laplace operator with Neumann boundary conditions on $\partial\Omega_\varepsilon$. To this end we will estimate forcing term in and initial data in terms of suitable power of the operator $\Delta_{\varepsilon,\mathrm{N}}$. And finally we will evaluate rate of a local decay to the aquatic wave.

Let us emphasise that bounds on a forcing term and initial data in terms of $\Delta_{\varepsilon,\mathrm{N}}$ of acoustic equation generally may depend on shape of Ω_ε. As a result the forcing term becomes unbounded as ε and this defect have to be compensated by uniform dispersive estimates of order $\sqrt{\varepsilon}$ which is obtained in spirit of a result due to Kato [14] as in [9].

In the following part of the paper we will discuss some basic properties of the acoustic equation which allow as to provide the above result and consequently to prove (3.28).

3.4.2 Uniform estimates, part II: S_ε, $\boldsymbol{V}_\varepsilon$ and initial data

In the following section we concentrate on building uniform bounds on S_ε, $\boldsymbol{V}_\varepsilon$. To this end we adopt steps from [10, 13] and use uniform bounds showed in Section 3.2. Let us notice that

$$\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}=\frac{\varrho_\varepsilon-\tilde{\varrho}_\varepsilon}{\varepsilon}+\frac{\tilde{\varrho}_\varepsilon-\overline{\varrho}}{\varepsilon}=\left[\frac{\varrho_\varepsilon-\tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\mathrm{ess}}+\left[\frac{\varrho_\varepsilon-\tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\mathrm{res}}+\frac{\tilde{\varrho}_\varepsilon-\overline{\varrho}}{\varepsilon},\tag{3.37}$$

where we use decomposition introduced in Section 3.1. To deal with the above let us recall (3.5) and (3.13)

$$\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\left[\frac{\varrho_\varepsilon-\tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\mathrm{ess}}\right\|_{L^2(\Omega_\varepsilon)}\le c,\quad\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\left[\frac{\varrho_\varepsilon-\tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\mathrm{res}}\right\|_{L^1(\Omega_\varepsilon)}\le\varepsilon c.\tag{3.38}$$

By (2.10), (2.11) and and (2.22), (2.23) we obtain

$$\left\| \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|_{B_R} \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{\varepsilon^\beta}, \qquad \left\| \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|_{B_R^c} \right\|_{L^\infty(\mathbb{R}^3)} \leq c, \tag{3.39}$$

$$\left\| \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|_{B_R} \right\|_{L^p(\mathbb{R}^3)} \leq c \text{ for any } p < 3, \qquad \left\| \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|_{B_R^c} \right\|_{L^q(\mathbb{R}^3)} \leq c \text{ for any } q > 3, \tag{3.40}$$

$$\left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \leq \frac{1}{\varepsilon^{2\beta}}, \tag{3.41}$$

$$\left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \Big|_{B_R} \right\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \leq c \text{ with } p \in [1, 3/2) \quad \left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \Big|_{B_R^c} \right\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \leq c \text{ with } q > \frac{3}{2}. \tag{3.42}$$

Next we rewrite the second term in S_ε (see (3.31)) as follows

$$\begin{aligned} &\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} = \frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} + \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \\ &= \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm ess} + \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} + \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}. \end{aligned} \tag{3.43}$$

In accordance with the uniform estimates (3.5), (3.6), structural assumptions on s, [12, Proposition 5.2] we have

$$\text{ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c. \tag{3.44}$$

By (3.9), (3.12) and since $\tilde{\varrho}_\varepsilon$ is uniformly close to $\overline{\varrho}$ we get

$$\text{ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} \right\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon c. \tag{3.45}$$

Due to (2.22), (2.23), (2.10), (2.11) and structural restrictions on s we obtain

$$\left\| \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big|_{B_R} \right\|_{L^q(\mathbb{R}^3)} \leq c \text{ for } p \in [1, 3),$$

$$\left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \Big|_{B_R^c} \right\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \leq c \text{ for any } p > 3/2.$$

The estimate (3.8) provides

$$\text{ess} \sup_{t \in (0,T)} \left\| \frac{\Sigma_\varepsilon(t, \cdot)}{\varepsilon} \right\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \leq \varepsilon c. \tag{3.46}$$

Setting

$$S_\varepsilon = S_\varepsilon^1 + S_\varepsilon^2 + S_\varepsilon^3 + S_\varepsilon^4,$$

by the above estimates and (2.10), (2.11) we get

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|S^1_\varepsilon\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \le \varepsilon c, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|S^2_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le c, \tag{3.47}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|S^3_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)} \le c, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|S^4_\varepsilon\|_{L^1(\Omega)} \le c\varepsilon. \tag{3.48}$$

Moreover

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,T)} \|S^1_{0,\varepsilon}\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \le \varepsilon c, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|S^2_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \le c, \\ &\operatorname*{ess\,sup}_{t\in(0,T)} \|S^3_{0,\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)} \le c, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|S^4_{0,\varepsilon}\|_{L^1(\Omega)} \le c\varepsilon \end{aligned} \tag{3.49}$$

and

$$S_\varepsilon \in C_{\text{weak}-*}([0,T];\mathcal{M}^+(\overline{\Omega}_\varepsilon)). \tag{3.50}$$

Next let us concentrate on the momentum $\boldsymbol{V}$. Rewriting

$$\boldsymbol{V}_\varepsilon = [\varrho_\varepsilon \boldsymbol{u}_\varepsilon]_{\text{ess}} + [\varrho_\varepsilon \boldsymbol{u}_\varepsilon]_{\text{res}}, \tag{3.51}$$

by (3.4), (3.12), (3.10) we have that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]_{\text{ess}}\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]_{\text{res}}\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le \varepsilon c. \tag{3.52}$$

Moreover by (3.4), (3.10), (3.12) we get

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]_{\text{res}}\|_{L^q(\Omega_\varepsilon;\mathbb{R}^3)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|[\sqrt{\varrho_\varepsilon}]_{\text{res}}\sqrt{\varrho_\varepsilon}\boldsymbol{u}_\varepsilon\|_{L^q(\Omega_\varepsilon;\mathbb{R}^3)} \le c\varepsilon^{\frac{3}{5}} \quad \text{with } q = \frac{5}{4}. \tag{3.53}$$

Moreover by assumption on initial data (2.25), (2.24) (with uniform closeness of $\overline{\varrho}$ and $\tilde{\varrho}_\varepsilon$)

$$\|\boldsymbol{V}_{0,\varepsilon}\|_{L^\infty\cap L^2(\mathbb{R}^3;\mathbb{R}^3)} \le c \tag{3.54}$$

and consequently $\boldsymbol{V}_\varepsilon \in C_{\text{weak}}([0,T];L^1(\Omega_\varepsilon))$, $\boldsymbol{V}_\varepsilon \in C_{\text{weak}}([0,T];L^{\frac{5}{4}}(\Omega_\varepsilon))$.

3.4.3 Uniform estimates, part II: forcing terms of the acoustic equation

Next we focus on forcing terms $\tilde{f}^1_\varepsilon$ and $\tilde{\boldsymbol{f}}^2_\varepsilon$ of the acoustic equation (3.30) defined by (3.32), (3.33). We start with

$$H^1_\varepsilon = B[\varrho_\varepsilon]_{\text{ess}} \left(\frac{s(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} + \frac{s(\overline{\varrho},\overline{\vartheta}) - s(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\varepsilon} \right) \boldsymbol{u}_\varepsilon + B[\varrho_\varepsilon]_{\text{res}} \frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \boldsymbol{u}_\varepsilon.$$

Applying the same procedure as in [12] (see Chapter 8.4.1, 5.3.2 and Proposition 5.2) together with estimates (3.18), (3.5), (3.6) we obtain

$$\int_0^T \left\| [\varrho_\varepsilon]_{\text{ess}} \frac{s(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \boldsymbol{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le c. \tag{3.55}$$

By the closeness of $\overline{\varrho}$ and $\tilde{\varrho}_\varepsilon$ (see (3.40) with $p = 2$) the same arguments give

$$\int_0^T \left\| [\varrho_\varepsilon]_{\mathrm{ess}} \left[\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\varepsilon} \right] \mathbb{1}_{B_R} \boldsymbol{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le c. \tag{3.56}$$

Again closeness of $\overline{\varrho}$ and $\tilde{\varrho}_\varepsilon$, i.e. (2.22), (2.11) with structural properties of s and (3.18) provide

$$\int_0^T \left\| [\varrho_\varepsilon]_{\mathrm{ess}} \left[\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\varepsilon} \right] \mathbb{1}_{B_R^c} \boldsymbol{u}_\varepsilon \right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c. \tag{3.57}$$

Following exactly [12, Chapter 8.4.1] by structural restrictions on s and uniform estimates obtained in section 3.2 we have

$$\int_0^T \left\| [\varrho_\varepsilon]_{\mathrm{res}} \frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \boldsymbol{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \mathrm{d}t < c. \tag{3.58}$$

Then we conclude summarising (3.55), (3.56), (3.57), (3.58) that

$$\int_0^T \|H^{1,1}\|^2_{L^1(\Omega_\varepsilon)} + \|H^{1,2}\|^2_{L^2(\Omega_\varepsilon)} \,\mathrm{d}t < c. \tag{3.59}$$

Setting

$$H^2_\varepsilon = H^{2,1}_\varepsilon + H^{2,2}_\varepsilon = B \frac{[\kappa(\vartheta_\varepsilon)]_{\mathrm{ess}}}{\vartheta_\varepsilon} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} + B \frac{[\kappa(\vartheta_\varepsilon)]_{\mathrm{res}}}{\vartheta_\varepsilon} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon}$$

we have by (2.2), (2.3) and (3.8), (3.11) (for more details see [12, Chapter 8.4.1]) that

$$\int_0^T \left\|H^{2,1}_\varepsilon\right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} dt \le c, \quad \int_0^T \left\|H^{2,2}_\varepsilon\right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} dt \le c. \tag{3.60}$$

Next we concentrate on terms in (3.33). In the same way as (3.46) one gets

$$\int_0^T \|G^1_\varepsilon\|^2_{\mathcal{M}_+(\Omega_\varepsilon)} = \int_0^T \left\| B \frac{\Sigma_\varepsilon}{\varepsilon^2} \right\|^2_{\mathcal{M}_+(\Omega_\varepsilon)} \le c. \tag{3.61}$$

Directly by (3.14) and by (3.4) we infer respectively that

$$\int_0^T \|G^{2,1}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \,\mathrm{d}t = \int_0^T \|\mathbf{S}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \,\mathrm{d}t < c, \tag{3.62}$$

$$\int_0^T \|G^{2,2}_\varepsilon\|^2_{L^1(\Omega_\varepsilon)} \,\mathrm{d}t = \int_0^T \|\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon\|^2_{L^1(\Omega_\varepsilon)} \,\mathrm{d}t < c. \tag{3.63}$$

In order to provide estimates on G^3_ε we follow [13] and we notice that

$$\begin{aligned} G^3_\varepsilon =& A\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) \\ =& A\left(\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^2}\right) + B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right) \\ &+ A\left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + B\left(\frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon,\vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\tilde{\varrho}_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) \end{aligned}$$

Since A and B are such that (3.34) is satisfied, the quantity

$$A\left(\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon^2}\right)+B\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)-\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)-\left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\tilde\varrho_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)=G_\varepsilon^{3,1,1}+G_\varepsilon^{3,1,2}$$

posses only quadratic terms proportional to $\varrho_\varepsilon-\tilde\varrho_\varepsilon$ and $\vartheta_\varepsilon-\overline{\vartheta}$ and as a such may be estimated in terms of (3.5 - 3.13), i.e.

$$\text{ess}\sup_{t\in(0,T)}\|G^{3,1,1}\|_{L^1(\Omega_\varepsilon)}\le c,\qquad \text{ess}\sup_{t\in(0,T)}\|G^{3,1,2}\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)}\le c. \tag{3.64}$$

In similar way by (2.22) and (2.10)

$$\|G_\varepsilon^{3,2,1}\|=\left\|\left[A\left(\frac{\tilde\varrho_\varepsilon-\overline{\varrho}}{\varepsilon^2}\right)+B\left(\frac{\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\vartheta_\varepsilon)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)-\left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)\right]\Big|_{B_R^c}\right\|_{L^\infty\cap L^p(\mathbb{R}^3)}\le c$$

for any $p>3/2$ and

$$\|G_\varepsilon^{3,2,2}\|=\left\|\left[A\left(\frac{\tilde\varrho_\varepsilon-\overline{\varrho}}{\varepsilon^2}\right)+B\left(\frac{\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\vartheta_\varepsilon)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)-\left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)\right]\Big|_{B_R}\right\|_{L^p(\mathbb{R}^3)}\le c$$

for any $p<3/2$. Then summarising the above considerations we have

$$G_\varepsilon^3=G_\varepsilon^{3,1}+G_\varepsilon^{3,2}+G_\varepsilon^{3,3} \tag{3.65}$$

$$\text{where }\|G_\varepsilon^{3,1}\|_{L^\infty(0,T;L^{5/3}(\Omega_\varepsilon))}\le c,\quad \|G_\varepsilon^{3,2}\|_{L^\infty(0,T;L^1(\Omega_\varepsilon))}\le c,\quad \|G_\varepsilon^{3,3}\|_{L^\infty(0,T;\mathcal{M}^+(\overline{\Omega_\varepsilon}))}\le c. \tag{3.66}$$

Next we are going to have a look on G_ε^4. First let us set that

$$G_\varepsilon^4=\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_xF_\varepsilon=\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_xF_\varepsilon\right]\Big|_{B_R}+\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_xF_\varepsilon\right]\Big|_{B_R^c}. \tag{3.67}$$

By (3.37), (3.38), (3.39), (3.40), (2.11) we obtain

$$\text{ess}\sup_{t\in(0,T)}\left\|\left[\left[\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon}\right]_{\text{ess}}\nabla_xF_\varepsilon\right]\Big|_{B_R}\right\|_{L^2(\Omega_\varepsilon)}<\frac{1}{\varepsilon^{2\beta}}c,\quad \text{ess}\sup_{t\in(0,T)}\left\|\left[\left[\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon}\right]_{\text{res}}\nabla_xF_\varepsilon\right]\Big|_{B_R}\right\|_{L^1(\Omega_\varepsilon)}<\frac{\varepsilon}{\varepsilon^{2\beta}}c,$$

$$\text{ess}\sup_{t\in(0,T)}\left\|\left[\left[\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon}\right]_{\text{ess}}\nabla_xF_\varepsilon\right]\Big|_{B_R^c}\right\|_{L^2(\Omega_\varepsilon)}<c,\quad \text{ess}\sup_{t\in(0,T)}\left\|\left[\left[\frac{\varrho_\varepsilon-\tilde\varrho_\varepsilon}{\varepsilon}\right]_{\text{res}}\nabla_xF_\varepsilon\right]\Big|_{B_R^c}\right\|_{L^1(\Omega_\varepsilon)}<c$$

and

$$\left\|\left[\frac{\tilde\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_xF_\varepsilon\right]\Big|_{B_R}\right\|_{L^p(\Omega_\varepsilon)}<\frac{1}{\varepsilon^{2\beta}}c\text{ for }p\in[1,3),\qquad \left\|\left[\frac{\tilde\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_xF_\varepsilon\right]\Big|_{B_R^c}\right\|_{L^\infty\cap L^q(\Omega_\varepsilon)}<\ \text{for }q>\frac{3}{2}.$$

Consequently

$$G_\varepsilon^4=G_\varepsilon^{4,1}+G_\varepsilon^{4,2}+G_\varepsilon^{4,3}+G_\varepsilon^{4,4} \tag{3.68}$$

where

$$\begin{gathered}\text{ess}\sup_{t\in(0,T)}\left\|G_\varepsilon^{4,1}\right\|_{L^2(\Omega_\varepsilon)}<\frac{1}{\varepsilon^{2\beta}}c,\quad \text{ess}\sup_{t\in(0,T)}\left\|G_\varepsilon^{4,2}\right\|_{L^1(\Omega_\varepsilon)}<\frac{\varepsilon}{\varepsilon^{2\beta}}c,\quad \text{ess}\sup_{t\in(0,T)}\left\|G_\varepsilon^{4,3}\right\|_{L^2(\Omega_\varepsilon)}<c,\\ \text{ess}\sup_{t\in(0,T)}\left\|G_\varepsilon^{4,4}\right\|_{L^1(\Omega_\varepsilon)}<c\varepsilon.\end{gathered} \tag{3.69}$$

3.4.4 Regularisation. Reduction to a smooth data.

Next in order to facilitate future analysis, we regularise equations (3.35), (3.36) in x-variable and extend them to the exterior domain $\mathbb{R}^3 \setminus \left(\cup_{i=1,\dots,N}\, \varepsilon^\beta B_{r_i}(x_i)\right)$.

Without loss of generality we may assume that all quantities appearing in (3.35), (3.36) are smooth. For detailed justification we refer the reader to [13, Section 2.5.2] or [12, Section 8.4.2], [8]. The idea is to replace all quantitates and given data in (3.35), (3.36) by smooth compactly supported functions, its λ-approximations. Such regularisations needs to converge in spaces where particular term belongs to (according to bounds obtained in previous sections). Then one may consider the unique solution $S_{\varepsilon,\lambda}$, $\boldsymbol{V}_{\varepsilon,\lambda}$ to new initial-boundary problem. The crucial is to observe that keeping $\varepsilon > 0$ fixed and letting with regularising parameter to the limit ($\lambda \to 0$) one have

$$\operatorname{ess}\sup_{t\in(0,T)} \left| \int_{\mathbb{R}^3} (\boldsymbol{V}_\varepsilon - \boldsymbol{V}_{\varepsilon,\lambda})(t,\cdot)\cdot \boldsymbol{\varphi}\, \mathrm{d}x \right| \to 0 \text{ as } \lambda \to 0$$

for any $\boldsymbol{\varphi}$ as in (3.29), namely solutions to such regularised equation is sufficiently close to exact solutions, at least on the support of the test function $\boldsymbol{\varphi}$.

Due to the above discussion now it is enough to show (3.29) with $\boldsymbol{V}_\varepsilon$ replaced by $\boldsymbol{V}_{\varepsilon,\lambda(\varepsilon)}$. In what follows, we drop the subscript λ, but we need to keep in mind that since now we may replace the weak formulation of the acoustic equation (3.35), (3.36) by its classical counterpart in strong form supplemented with smooth data (belonging to $W^{k,2}$ with sufficiently large k) satisfying the bounds established in previous section uniformly for $\varepsilon \to 0$.

3.4.5 Propagation of acoustic waves. Extension to an exterior domain

We already stated that our interest is only to describe the local behaviour of solution, more precisely that

$$\left\{ t \to \int_{\mathbb{R}^3} \boldsymbol{V}_\varepsilon \cdot \boldsymbol{\varphi}\, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \text{ for any } \boldsymbol{\varphi} \in C_c(K;\mathbb{R}^3)$$

where K is compact set and $K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}$. Noticing that in the system (3.30) the speed of propagation is finite and proportional to $\sqrt{\omega}/\varepsilon$, we observe if the initial data coincides for two solutions on the set $B_{T\sqrt{\omega}/\varepsilon}(0) = \{x \in \Omega_\varepsilon \,|\, |x| < R + T\sqrt{\omega}/\varepsilon\} \subset B_{1/\varepsilon^\delta}$ with the same forcing terms on $(0,T)\times B_{T\sqrt{\omega}/\varepsilon}(0)$, then two solutions also coincides on the cone

$$\left\{ (t,x) \in (0,T)\times B_{T\sqrt{\omega}/\varepsilon}(0) \,|\, \operatorname{dist}[x, \partial B_{T\sqrt{\omega}/\varepsilon}(0)] > t\sqrt{\omega}/\varepsilon \right\}.$$

Since we need in our consideration only local behaviour of solutions and since in (2.5) $\delta > 1$, extending data in acoustic equations by zero outside of $B_{R_0/\varepsilon^\delta}$, in fact, we may solve the the acoustic system (3.30) on the set

$$\widetilde{\Omega}_\varepsilon = \mathbb{R}^3 \setminus \left(\cup_{i=1,\dots,N}\, \varepsilon^\beta B_{r_i}(x_i)\right) \tag{3.70}$$

which is an exterior domain with holes (obstacles) defined in (2.5) (varying with $\varepsilon \to 0$). This observation provides facilitating properties of considered the Neumann Laplacian operator which we concentrate on in next section.

3.4.6 Neumann Laplace operator

In order to provide the decay of acoustic wave, what provides (3.29), we introduce an abstract formulation of the acoustic equation in terms of an abstract differential operator Neumann Laplacian $\Delta_{\varepsilon,\mathrm{N}}$ which plays a crucial role in the analysis of local decay of acoustic wave governed by gradient part of momentum. The operator $-\Delta_{\varepsilon,\mathrm{N}}$ may be viewed as a non-negative self-adjoint operator on the space $L^2(\Omega_\varepsilon)$, with

$$\mathcal{D}(-\Delta_{\varepsilon,\mathrm{N}}) = \Big\{ w \in W^{1,2}(\Omega_\varepsilon) \mid \int_{\Omega_\varepsilon} \nabla_x w \cdot \nabla_x \varphi dx = \int_{\Omega_\varepsilon} g\varphi \,\mathrm{d}x \text{ for all } \varphi \in C_c^\infty(\overline{\Omega}_\varepsilon) \text{ an certain } g \in L^2(\Omega_\varepsilon) \Big\},$$

namely $-\Delta_{\varepsilon,\mathrm{N}}[w] = g$. Supplemented with the homogenous Neumann boundary condition $\nabla w \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$. Due to regularity of $\partial\Omega_\varepsilon$, standard elliptic theory gives

$$\mathcal{D}(-\Delta_{\varepsilon,\mathrm{N}}) = \Big\{ w \in W^{2,2}(\Omega_\varepsilon) \mid \nabla_x w \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0 \Big\}.$$

3.4.7 Elliptic estimates for a second gradient

Since we intend to control on forcing terms and initial data in the acoustic equation in terms of $-\Delta_{\varepsilon,\mathrm{N}}$, we need to provide bounds on $\nabla_x^2 \varphi$ in terms of $-\Delta_{\varepsilon,\mathrm{N}}[\varphi]$. As second derivatives of boundary mapping are proportional to $\frac{1}{\varepsilon^{2\beta}}$, $W^{2,p}$ bounds may blow up as $\varepsilon \to 0$. Let us notice that, if we consider the rescaled family of domains

$$\widehat{\Omega}_\varepsilon = \frac{1}{\varepsilon^\beta} \widetilde{\Omega}_\varepsilon \tag{3.71}$$

where $\widetilde{\Omega}$ is defined by (3.70) and boundaries of $\widehat{\Omega}_\varepsilon$ are uniformly regular (at least C^2 class). Then applying the standard elliptic estimates we infer

$$\|\nabla_x^2 \varphi\|_{L^p(\widehat{\Omega}_\varepsilon)} \le c(p) \left(\|\Delta_x \varphi\|_{L^p(\widehat{\Omega}_\varepsilon)} + \|\varphi\|_{L^p(\widehat{\Omega}_\varepsilon)} \right) \quad \text{for } 1 < p < \infty$$

and for any $\varphi \in C_c^\infty(\overline{\widehat{\Omega}_\varepsilon})$, $\nabla_x \varphi \cdot \boldsymbol{n}|_{\partial\widehat{\Omega}_\varepsilon} = 0$ with constant $c(p)$ independent of ε. After rescaling to the original domain Ω_ε, we obtain the following

$$\|\nabla_x^2 \varphi\|_{L^p(\Omega_\varepsilon)} \le c(p) \left(\|\Delta_x \varphi\|_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\varphi\|_{L^p(\Omega_\varepsilon)} \right) \tag{3.72}$$

for any $\varphi \in C_c^\infty(\overline{\Omega_\varepsilon})$ with $\nabla_x \varphi \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$ and with $1 < p < \infty$.

3.4.8 Helmholtz decomposition

Let $\boldsymbol{H}_\varepsilon[\boldsymbol{v}]$ denote Helmholtz projection of a function $\boldsymbol{v} \in L^p(\Omega_\varepsilon; \mathbb{R}^3)$ on the subspace of divergence-free functions defined by

$$\boldsymbol{v} = \boldsymbol{H}_\varepsilon[\boldsymbol{v}] + \nabla_x \Psi \tag{3.73}$$

with $\Psi \in D^{1,p}(\Omega_\varepsilon)$ (homogenous Sobolev space, i.e. a completion of $C_c^\infty(\overline{\Omega_\varepsilon})$ in L^p norm of gradients) - a unique solution of

$$\int_{\Omega_\varepsilon} \nabla_x \Psi \cdot \nabla \varphi \,\mathrm{d}x = \int_{\Omega_\varepsilon} \boldsymbol{v} \cdot \nabla_x \varphi \quad \text{for all } \varphi \in C_c^\infty(\overline{\Omega}_\varepsilon)$$

which formally means: $\Delta\Psi = \mathrm{div}_x\, \boldsymbol{v}$ and $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$.

Since $\{\Omega_\varepsilon\}_\varepsilon$ posses the uniform extension property, the following Sobolev inequality is satisfied with $c(p)$ independent of ε

$$\|\phi\|_{L^q(\Omega_\varepsilon)} \le c(p)\|\nabla_x\phi\|_{L^p(\Omega_\varepsilon;\mathbb{R}^3)} \quad \text{with } p = \frac{3p}{3-p},\ 1 \le p < 3 \text{ for } \phi \in D^{1,p}(\Omega_\varepsilon).$$

As above we consider $\widehat{\Omega}_\varepsilon$ defined by (3.71). By $\widetilde{\boldsymbol{H}}_\varepsilon[\boldsymbol{v}]$ we denote Helmholtz projection associated with vector field on $\widetilde{\Omega}_\varepsilon$. By the result of Farwig et al. [7] we get

$$\|\widehat{\boldsymbol{H}}_\varepsilon[\boldsymbol{v}]\|_{L^p\cap L^2(\widehat{\Omega}_\varepsilon;\mathbb{R}^3)} \le c(p)\|\boldsymbol{v}\|_{L^p\cap L^2(\widehat{\Omega}_\varepsilon;\mathbb{R}^3)} \quad \text{for any } 2 \le p < \infty, \tag{3.74}$$

where constant $c(p)$ is independent of ε (it depends only on r_i, $i = 1,\dots,N$ from definition domain (2.5)). If we go beck to the original domain Ω_ε, we obtain

$$\|\widetilde{\boldsymbol{H}}_\varepsilon[\boldsymbol{v}]\|_{L^p\cap L^2(\widetilde{\Omega}_\varepsilon;\mathbb{R}^3)} \le \varepsilon^{-\beta\left(\frac{3}{2}-\frac{3}{p}\right)}c(p)\|\boldsymbol{v}\|_{L^p\cap L^2(\widetilde{\Omega}_\varepsilon;\mathbb{R}^3)} \text{ for any } 2 \le p < \infty \tag{3.75}$$

for $\varepsilon \to 0$. And by duality argument it follows that

$$\|\widetilde{\boldsymbol{H}}_\varepsilon[\boldsymbol{v}]\|_{L^p+L^2(\widetilde{\Omega}_\varepsilon;\mathbb{R}^3)} \le \varepsilon^{-\beta\left(\frac{3}{p}-\frac{3}{2}\right)}c(p)\|\boldsymbol{v}\|_{L^p+L^2(\widetilde{\Omega}_\varepsilon;\mathbb{R}^3)} \text{ for any } 1 < p < 2. \tag{3.76}$$

Moreover let us notice that (3.75), (3.76) holds also for $\widetilde{\Omega}_\varepsilon$ replaced by Ω_ε and $\widetilde{\boldsymbol{H}}_\varepsilon$ by $\boldsymbol{H}_\varepsilon$ – Helmholtz projection associated with vector field on Ω_ε.

3.4.9 New formulation

Our aim next will be to rewrite the acoustic equations in terms of the operator $-\Delta_{\varepsilon,\mathrm{N}}$ and functions ranging in the Hilbert space L^2 (see also [11, 13]). To this end we need to take $\nabla_x\Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]$ as a test function in weak formulation of (3.36), where $\Delta_{\varepsilon,\mathrm{N}}$ is the Neumann Laplacian in Ω_ε. Let us notice that $\nabla_x\Delta^{-1}_{\varepsilon,\mathrm{N}}[\varphi]$ is continuously differentiable on $\overline{\Omega_\varepsilon}$ and belongs to the space $C^\infty_c([0,T);W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$ for $\varphi \in C^\infty_c([0,T)\times\overline{\Omega}_\varepsilon)$.

Let $\nabla_x\Phi_\varepsilon$ denote acoustic potential, i.e.

$$\boldsymbol{V}_\varepsilon = \boldsymbol{H}_\varepsilon[\boldsymbol{V}_\varepsilon] + \nabla_x\Phi_\varepsilon. \tag{3.77}$$

According to (3.77) we rewrite (3.35) in the following form

$$\begin{aligned} &\varepsilon\int_0^T \langle S_\varepsilon(t,\cdot),\partial_t\varphi\rangle\,\mathrm{d}t + \omega\int_0^T\int_{\Omega_\varepsilon}\nabla_x\Phi_\varepsilon\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t \\ &= \varepsilon\,\langle S_{0,\varepsilon},\varphi(0,\cdot)\rangle + \varepsilon\int_0^T\int_{\Omega_\varepsilon}(H^1_\varepsilon + H^2_\varepsilon)\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.78}$$

for all $\varphi \in C_c^\infty([0,T] \times \overline{\Omega}_\varepsilon)$. Next since $\varphi = \nabla_x \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]$ is an admissible test function in (3.36) (due to slip boundary condition on $\boldsymbol{u}_\varepsilon$) we obtain by integration by parts that

$$\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Phi_\varepsilon \cdot \partial_t \varphi \,\mathrm{d}t - \int_0^T \langle S_\varepsilon, \varphi \rangle_{[\mathcal{M},C]} \,\mathrm{d}t \\
&= -\varepsilon \int_{\Omega_\varepsilon} V_{0,\varepsilon} \cdot \nabla_x \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi(0,\cdot)] \,\mathrm{d}x - \varepsilon \Big\{ \int_0^T \langle G_\varepsilon^1(t,\cdot), \varphi \rangle \,\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^{2,1} : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi] \,\mathrm{d}x\mathrm{d}t \qquad (3.79)\\
&+ \int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^{2,2} : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi] \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^3 \varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^4 \cdot \nabla_x \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi] \,\mathrm{d}x\mathrm{d}t \Big\}.
\end{aligned}$$

The above equations represent a weak formulation of the acoustic equation for the potential of the gradient part of the momentum with Neumann boundary condition. Let us notice that due to relations (3.52), (3.53) and (3.76) the acoustic potential Φ_ε can be rewritten as follows

$$\Phi_\varepsilon = \Phi_\varepsilon^1 + \Phi_\varepsilon^2,$$

where

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\Phi_\varepsilon^1\|_{D^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} < c, \qquad \operatorname*{ess\,sup}_{t\in(0,T)} \|\Phi_\varepsilon^2\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \to 0 \text{ as } \varepsilon \to 0. \qquad (3.80)$$

3.4.10 Uniform estimates, part III

Now we are ready to concentrate on forcing terms in (3.79) and to show how to control them in terms of $\Delta_{\varepsilon,\mathrm{N}}$.

Let us start with G_ε^1. By (3.61)

$$\left| \int_0^T \langle G_\varepsilon^1(t,\cdot), \varphi \rangle \,\mathrm{d}t \right| \le \int_0^T \|G_\varepsilon^1\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \|\varphi\|_{C(\Omega_\varepsilon)} \,\mathrm{d}t.$$

Note that

$$\|\varphi\|_{C(\Omega_\varepsilon)} \le c_1 \left(\|\varphi\|_{L^6(\Omega_\varepsilon)} + \|\nabla_x \varphi\|_{L^6(\Omega_\varepsilon)} \right).$$

Then by the Sobolev imbedding and (3.72) we have

$$\|\nabla_x \varphi\|_{L^6(\Omega_\varepsilon)} \le c_2 \|\nabla_x^2 \varphi\|_{L^2(\Omega_\varepsilon)} \le c_3 \left(\|\Delta_{\varepsilon,\mathrm{N}}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\varphi\|_{L^2(\Omega_\varepsilon)} \right), \qquad (3.81)$$

$$\|\varphi\|_{L^6(\Omega_\varepsilon)} \le c_4 \|\nabla_x \varphi\|_{L^2(\Omega_\varepsilon)} \quad \text{and} \quad \|\nabla_x \varphi\|_{L^2(\Omega_\varepsilon)} = \|(\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)}.$$

Hence the Riesz theorem provides that there exist $J_\varepsilon^{1,i} \in L^2((0,T) \times \Omega_\varepsilon)$, $i = 1,2,3$ s.t.

$$\int_0^T \langle G_\varepsilon^1(t,\cdot), \varphi(t,\cdot) \rangle \,\mathrm{d}t = \frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^{1,1} \varphi + J_\varepsilon^{1,2} (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J_\varepsilon^{1,3} (-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \,\mathrm{d}x \,\mathrm{d}t$$

and $\|J_\varepsilon^{1,i}\|_{L^2((0,T)\times\Omega_\varepsilon)} < c$ for $i = 1,2,3$ and for $\varepsilon \to 0$. Again by (3.72) and (3.62) we obtain that

$$\left| \int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^{2,1} : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi] \,\mathrm{d}x \,\mathrm{d}t \right| \le c \int_0^T \|G_\varepsilon^{2,1}\|_{L^2(\Omega_\varepsilon)} \left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)} \right) \mathrm{d}t.$$

By Riesz theorem we can write

$$\int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^{2,1} : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t = \frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^{2,1,1}\varphi + J_\varepsilon^{2,1,2}(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t,$$

where $\|J_\varepsilon^{2,1,i}\|_{L^2((0,T)\times\Omega_\varepsilon)} < c$ for $i = 1,2$ and for $\varepsilon \to 0$. Setting

$$G_\varepsilon^{2,2} = [\varrho_\varepsilon]_{\mathrm{ess}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon + [\varrho_\varepsilon]_{\mathrm{res}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon = G_\varepsilon^{2,2,1} + G_\varepsilon^{2,2,2},$$

where by estimates (3.10), (3.18), definition of $[\cdot]_{\mathrm{ess}}$ and (3.10)

$$\|G_\varepsilon^{2,2,1}\|_{L^{3/2}(\Omega_\varepsilon)} \le \|[\varrho_\varepsilon]_{\mathrm{ess}} \boldsymbol{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\boldsymbol{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} \le c,$$

$$\|G_\varepsilon^{2,2,2}\|_{L^{15/14}(\Omega_\varepsilon)} \le \|[\varrho_\varepsilon]_{\mathrm{res}}\|_{L^{5/3}(\Omega_\varepsilon)} \|\boldsymbol{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} \|\boldsymbol{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} \le c\varepsilon^{\frac{6}{5}} \|\boldsymbol{u}_\varepsilon\|^2_{L^6(\Omega_\varepsilon)}.$$

Then by (3.72) we have

$$\begin{aligned}\left| \int_{\Omega_\varepsilon} [\varrho_\varepsilon]_{\mathrm{ess}} \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x \right| &\le \|G_\varepsilon^{2,2,1}\|_{L^{3/2}(\Omega_\varepsilon)} \|\nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^3(\Omega_\varepsilon)} \\ &\le c\|G_\varepsilon^{2,2,1}\|_{L^{3/2}(\Omega_\varepsilon)} \left(\|\varphi\|_{L^3(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^3(\Omega_\varepsilon)} \right).\end{aligned}$$

Employing interpolation, uniform extension property and the Sobolev inequality we obtain

$$\begin{aligned}\|\varphi\|_{L^3(\Omega_\varepsilon)} &\le c_1(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \|\varphi\|_{L^6(\Omega_\varepsilon)}) \le c_2(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \|\nabla_x \varphi\|_{L^2(\Omega_\varepsilon)}) \\ &= c_2(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \|(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)})\end{aligned}$$

and in the same way

$$\|\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^3(\Omega_\varepsilon)} \le c(\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \|(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)}).$$

Consequently there exist $J_\varepsilon^{2,2,i} \in L^2((0,T)\times\Omega_\varepsilon)$, $i = 3,4,5,6$, $\|J_\varepsilon^{2,2,i}\|_{L^2((0,T)\times\Omega_\varepsilon)} < c$ s.t.

$$\begin{aligned}&\int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^{2,2,1} : \nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t = \\ &\frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^{2,2,3}\varphi + J_\varepsilon^{2,2,4}(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi] + J_\varepsilon^{2,2,5}(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J_\varepsilon^{2,2,6}(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Let us now concentrate on the term associated with $G_\varepsilon^{2,2,2}$. Employing (3.72) we have

$$\|\nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^{15}(\Omega_\varepsilon)} \le c\left(\|\varphi\|_{L^{15}(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^{15}(\Omega_\varepsilon)} \right).$$

Due to interpolation arguments, the Sobolev inequality, (3.72) it follows that

$$\|\varphi\|_{L^{15}(\Omega_\varepsilon)} \le c\left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \|(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \|(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|\varphi\|_{L^2(\Omega_\varepsilon)} \right)$$

and

$$\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^{15}(\Omega_\varepsilon)}$$
$$\leq c\left(\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}+\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)}+\|\varphi\|_{L^2(\Omega_\varepsilon)}+\frac{1}{\varepsilon^{2\beta}}\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}\right).$$

Noticing that for $\beta\leq\frac{3}{5}$ we can summarise that there exist $J_\varepsilon^{2,2,i}\in L^2((0,T)\times\Omega_\varepsilon)$, for $i=7,\dots,11$ s.t. $\|J_\varepsilon^{2,2,i}\|_{L^2((0,T)\times\Omega_\varepsilon}<c$ and

$$\int_0^T\int_{\Omega_\varepsilon}G_\varepsilon^{2,2,2}:\nabla_x^2\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t=$$
$$\frac{1}{\varepsilon^{2\beta}}\int_0^T\int_{\Omega_\varepsilon}J_\varepsilon^{2,2,7}\varphi+J_\varepsilon^{2,2,8}(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi]+J_\varepsilon^{2,2,9}(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi]+J_\varepsilon^{2,2,10}(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi]$$
$$+J_\varepsilon^{2,2,11}(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t.$$

Next by (3.65), (3.66) we get

$$\left|\int_0^T\int_{\Omega_\varepsilon}G_\varepsilon^3\varphi\,\mathrm{d}x\mathrm{d}t\right|\leq\int_0^T\|G_\varepsilon^{3,1}\|_{L^{5/3}(\Omega_\varepsilon)}\|\varphi\|_{L^{5/2}(\Omega_\varepsilon)}+(\|G_\varepsilon^{3,2}\|_{L^1(\Omega_\varepsilon))}+\|G_\varepsilon^{3,3}\|_{\mathcal{M}^+(\overline{\Omega_\varepsilon}))})\|\varphi\|_{C(\Omega_\varepsilon)}\,\mathrm{d}t.$$

Thus similar arguments as for previous terms, the Riesz theorem yield that there exist $J_\varepsilon^{3,i}\in L^2((0,T)\times\Omega_\varepsilon)$ such that

$$\int_0^T G_\varepsilon^3(t,\cdot)\varphi\,\mathrm{d}t=\frac{1}{\varepsilon^{2\beta}}\int_0^T\int_{\Omega_\varepsilon}J_\varepsilon^{3,1}\varphi+J_\varepsilon^{3,2}(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi]+J_\varepsilon^{3,3}(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\,\mathrm{d}x\,\mathrm{d}t,$$

where $\|J_\varepsilon^{3,i}\|_{L^2((0,T)\times\Omega_\varepsilon)}<c$ for $i=1,2,3$ and $\varepsilon\to 0$.

Let us consider now the term with G_ε^4 given by (3.67), (3.68). According to (3.69) it follows that

$$\left|\int_0^T\int_{\Omega_\varepsilon}G_\varepsilon^4\cdot\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t\right|\leq\int_0^T\|G_\varepsilon^{4,1}+G_\varepsilon^{4,3}\|_{L^2(\Omega_\varepsilon)}\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}$$
$$+\|G_\varepsilon^{4,2}+G_\varepsilon^{4,4}\|_{L^1(\Omega_\varepsilon)}\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^\infty(\Omega_\varepsilon)}\,\mathrm{d}t$$

where

$$\|G_\varepsilon^{4,1}+G_\varepsilon^{4,3}\|_{L^2(\Omega_\varepsilon)}<\frac{1}{\varepsilon^{2\beta}}c\quad\|G_\varepsilon^{4,2}+G_\varepsilon^{4,4}\|_{L^1(\Omega_\varepsilon)}<\frac{\varepsilon}{\varepsilon^{2\beta}}c.$$

Let us notice that the following holds

$$\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}=\|\Delta_{\varepsilon,\mathrm{N}}^{1/2}\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}=\|\Delta_{\varepsilon,\mathrm{N}}^{-1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)}$$

and by the Sobolev imbedding

$$\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^\infty(\Omega_\varepsilon)}\leq c\left(\|\nabla_x^2\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)}+\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)}\right).$$

We observe then (by (3.72)) that

$$
\begin{aligned}
&\|\nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)} \le c\left(\|\varphi\|_{L^6(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)}\right)\\
&\le c\left(\|\nabla_x\varphi\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}\right) = c\left(\|\Delta_{\varepsilon,\mathrm{N}}^{1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\Delta_{\varepsilon,\mathrm{N}}^{-1/2}[\varphi]\|_{L^2(\Omega_\varepsilon)}\right)
\end{aligned}
$$

and

$$
\|\nabla_x \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)} \le \|\nabla_x^2 \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\|_{L^6(\Omega_\varepsilon)} \le c\left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\|_{L^2(\Omega_\varepsilon)}\right).
$$

Consequently for $\beta < \frac{1}{2}$ we obtain that there exist $J_\varepsilon^{4,i} \in L^2$, for $i = 1, 2, 3, 4$

$$
\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} G_\varepsilon^4 \cdot \nabla_x \Delta_{\varepsilon,\mathrm{N}}^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t =\\
&\frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^{4,1}\varphi + J_\varepsilon^{4,2}(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi] + J_\varepsilon^{4,3}(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J_\varepsilon^{4,4}(-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi]\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

where $\|J_\varepsilon^{4,i}\|_{L^2((0,T)\times\Omega_\varepsilon)} < c$ for $i = 1, 2, 3, 4$.

Now let us concentrate on the right-hand side of the equation (3.78). Let us notice that by (3.59), (3.60) we obtain

$$
\begin{aligned}
&\Big|\int_0^T \int_{\Omega_\varepsilon} (H_\varepsilon^1 + H_\varepsilon^2) \cdot \nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le \int_0^T \left\{\|H_\varepsilon^{1,1} + H_\varepsilon^{2,2}\|_{L^1(\Omega_\varepsilon)}\|\nabla_x\varphi\|_{L^\infty(\Omega_\varepsilon)} + \|H_\varepsilon^{1,1} + H_\varepsilon^{2,1}\|_{L^2(\Omega_\varepsilon)}\|\nabla_x\varphi\|_{L^2(\Omega_\varepsilon)}\right\}\mathrm{d}t.
\end{aligned}
$$

Then interpolation and by the Sobolev imbedding theorem we obtain

$$
\|\nabla_x\varphi\|_{L^\infty(\Omega_\varepsilon)} \le c\left(\|\nabla_x^2\varphi\|_{L^6(\Omega_\varepsilon)} + \|\nabla_x\varphi\|_{L^6(\Omega_\varepsilon)}\right)
$$

and by (3.72) we have

$$
\begin{aligned}
&\|\nabla_x^2\varphi\|_{L^6(\Omega_\varepsilon)} \le c_1\left(\|\Delta_{\varepsilon,\mathrm{N}}[\varphi]\|_{L^6(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\varphi\|_{L^6(\Omega_\varepsilon)}\right)\\
&\le c_2\left(\|\nabla_x\Delta_{\varepsilon,\mathrm{N}}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\nabla_x\varphi\|_{L^2(\Omega_\varepsilon)}\right) = c_2\left(\|\Delta_{\varepsilon,\mathrm{N}}^{3/2}[\varphi]\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}}\|\Delta_{\varepsilon,\mathrm{N}}^{1/2}\varphi\|_{L^2(\Omega_\varepsilon)}\right).
\end{aligned}
$$

Therefore the above and (3.81) provide by the Riesz theorem that

$$
\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} (H_\varepsilon^1 + H_\varepsilon^2) \cdot \nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\\
&= \frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^{5,1}\varphi + J_\varepsilon^{5,2}(-\Delta_{\varepsilon,\mathrm{N}})^{3/2}[\varphi] + J_\varepsilon^{5,3}(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J_\varepsilon^{5,4}(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

for some $J_\varepsilon^{5,i}$, where $\|J_\varepsilon^{5,i}\|_{L^2((0,T)\times\Omega_\varepsilon)} \le c$, for $i = 1, \ldots, 4$.

3.4.11 Abstract formulation of the acoustic equation and local decay of acoustic waves

Summarising computation from previous section equation (3.78) and (3.79) can be rewritten in the following more conscious form

$$\begin{aligned}\varepsilon \int_0^T \langle S_\varepsilon(t,\cdot), \partial_t \varphi \rangle \,\mathrm{d}t + \omega \int_0^T \int_{\Omega_\varepsilon} \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t = \varepsilon \,\langle S_{0,\varepsilon}, \varphi(0,\cdot)\rangle \\ + \frac{\varepsilon}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} J_\varepsilon^1 \varphi + J_\varepsilon^2 (-\Delta_{\varepsilon,\mathrm{N}})^{3/2}[\varphi] + J_\varepsilon^3 (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] + J_\varepsilon^4 (-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.82}$$

for all $\varphi \in C_c^\infty([0,T] \times \overline{\Omega}_\varepsilon)$ and

$$\begin{aligned}&\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Phi_\varepsilon \cdot \partial_t \varphi \,\mathrm{d}t - \int_0^T \langle S_\varepsilon, \varphi \rangle \,\mathrm{d}t \\ &= -\varepsilon \int_{\Omega_\varepsilon} \Phi_{0,\varepsilon} \varphi(0,\cdot) \,\mathrm{d}x - \frac{\varepsilon}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} \Big\{ \tilde{J}_\varepsilon^1 \varphi + \tilde{J}_\varepsilon^2 (-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\varphi] + \tilde{J}_\varepsilon^3 (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\varphi] \\ &+ \tilde{J}_\varepsilon^4 (-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\varphi] + \tilde{J}_\varepsilon^5 (-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \Big\} \,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.83}$$

for any $\varphi \in C_c^\infty([0,T) \times K)$, K compact subset of $\mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$, $\nabla_x \varphi \cdot \boldsymbol{n}|_{\partial\Omega_\varepsilon} = 0$, where

$$\|J_\varepsilon^i\|_{L^2((0,T)\times\Omega_\varepsilon)} < c \text{ for } i = 1, \dots, 4 \text{ and } \|\tilde{J}_\varepsilon^j\|_{L^2((0,T)\times\Omega_\varepsilon)} < c \text{ for } j = 1, \dots, 5$$

and for sufficiently small ε and supplemented with the following initial data

$$S_{0,\varepsilon} = (-\Delta_{\varepsilon,\mathrm{N}})[\tilde{S}_{0,\varepsilon}^1] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\tilde{S}_{0,\varepsilon}^2] + \tilde{S}_{0,\varepsilon}^3, \quad \text{with } \|\tilde{S}_{0,\varepsilon}^i\|_{L^2(\Omega_\varepsilon)} \leq c$$

and

$$\Phi_{0,\varepsilon} = (-\Delta_{\varepsilon,\mathrm{N}})^{-1} \mathrm{div}_x \, V_{0,\varepsilon}, \quad \text{where } \|(-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\Phi_{0,\varepsilon}]\|_{L^2(\Omega_\varepsilon)} \leq c.$$

Duhamel's formula gives us an explicit formulation for acoustic potential, i.e.:

$$\begin{aligned}\Phi_\varepsilon(t,\cdot) &= \frac{1}{2} \exp\left(i\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}} \frac{t}{\varepsilon} \right) \left[\Phi_{0,\varepsilon} + \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} [S_{0,\varepsilon}] \right] \\ &+ \frac{1}{2} \exp\left(-i\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}} \frac{t}{\varepsilon} \right) \left[\Phi_{0,\varepsilon} - \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} [S_{0,\varepsilon}] \right] \\ &+ \varepsilon^{-2\beta} \frac{1}{2} \int_0^T \exp\left(i\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}} \frac{t-s}{\varepsilon} \right) \left[\tilde{F}_{2,\varepsilon}(s) + \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} \tilde{F}_{1,\varepsilon}(s) \right] \mathrm{d}s \\ &+ \varepsilon^{-2\beta} \frac{1}{2} \int_0^T \exp\left(-i\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}} \frac{t-s}{\varepsilon} \right) \left[\tilde{F}_{2,\varepsilon}(s) - \frac{i}{\sqrt{-\omega\Delta_{\varepsilon,\mathrm{N}}}} \tilde{F}_{1,\varepsilon}(s) \right] \mathrm{d}s.\end{aligned} \tag{3.84}$$

Where

$$\tilde{F}_{1,\varepsilon} = J_\varepsilon^1 + (-\Delta_{\varepsilon,\mathrm{N}})^{3/2}[J_\varepsilon^2] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[J_\varepsilon^3] + (-\Delta_{\varepsilon,\mathrm{N}})[J_\varepsilon^4]$$

and

$$\tilde{F}_{2,\varepsilon} = \tilde{J}^1_\varepsilon + (-\Delta_{\varepsilon,\mathrm{N}})^{-1/2}[\tilde{J}^2_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\tilde{J}^3_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})^{-1}[\tilde{J}^4_\varepsilon] + (-\Delta_{\varepsilon,\mathrm{N}})[\tilde{J}^5_\varepsilon]$$

(see (3.82), (3.83)). With above formulation at hand and by methods developed in [9] we are able to provide local decay of acoustic wave and consequently to show that

$$\left\{ t \to \int_{\Omega_\varepsilon} \Phi_\varepsilon G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\,\mathrm{d}x \right\} \to 0 \quad \text{in } L^2(0,T) \tag{3.85}$$

for any fixed $G \in C^\infty_c(0,\infty)$, $\varphi \in C^\infty_c(\Omega_\varepsilon)$. This eliminates possible oscillations in time of momentum while $\varepsilon \to 0$. To justify expression $G(-\Delta_{\varepsilon,\mathrm{N}})$ let us denote $\{P_\lambda\}_{\lambda>0}$ the spectral resolution associated to $-\Delta_{\varepsilon,\mathrm{N}}$ - a system of orthogonal projections in $L^2(\Omega_\varepsilon)$ s.t.

$$\int_{\Omega_\varepsilon} (-\Delta_{\varepsilon,\mathrm{N}})[v]w\,\mathrm{d}x = \int_0^\infty \int_{\Omega_\varepsilon} \lambda\,\mathrm{d}P_\lambda[v]w\,\mathrm{d}x$$

for any $v \in \mathcal{D}(-\Delta_{\varepsilon,\mathrm{N}})$ and $w \in L^2(\Omega_\varepsilon)$. Using $\{P_\lambda\}_\lambda$ we can define $G(-\Delta_{\varepsilon,\mathrm{N}})$ for (possibly complex valued) Borel function through the following formula

$$\int_{\Omega_\varepsilon} G(-\Delta_{\varepsilon,\mathrm{N}})[v]w\,\mathrm{d}x = \int_0^\infty \int_{\Omega_\varepsilon} G(\lambda)\,\mathrm{d}P_\lambda[v]w\,\mathrm{d}x.$$

The following lemma gives a local decay of acoustic waves, in particular, the rate of the decay is independent of the scaling parameter ε and stable w.r.t. perturbations of the domain of $\widetilde{\Omega}_\varepsilon$ (see (3.70)).

Lemma 3.4 *We have*

$$\int_0^T \left| \left\langle \exp\left(i\sqrt{-\Delta_{\varepsilon,\mathrm{N}}}\frac{t}{\varepsilon}[\Psi], G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \right) \right\rangle_{\widetilde{\Omega}_\varepsilon} \right|^2 \mathrm{d}t \le \varepsilon c(\varphi, G)\|\Psi\|^2_{L^2(\widetilde{\Omega}_\varepsilon)} \tag{3.86}$$

for any $\varphi \in C^\infty_c(K)$, $\Psi \in L^2(\widetilde{\Omega}_\varepsilon)$, and any $G \in C^\infty_c(0,\infty)$, where is s.t. $\overline{K} \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$.

A detailed proof of Lemma 3.4 the reader can find in [9, Theorem 2.3 and Section 3.2] or [11, Lemma 5.2] (see also [27]). Here we recall only some main steps. Since the boundary of $\partial\widetilde{\Omega}_\varepsilon$ is regular, then the standard elliptic theory provides that $\mathcal{D}(-\Delta_{\varepsilon,\mathrm{N}}) = \{w \in W^{2,2}(\widetilde{\Omega}_\varepsilon) \mid \nabla_x w \cdot \boldsymbol{n}|_{\partial\widetilde{\Omega}_\varepsilon} = 0\}$. Then we may reformulate the problem in terms of spectral measures associated to the function φ defined on positive real line – the spectrum of $-\Delta_{\varepsilon,\mathrm{N}}$ – which may be given by Stone's formula, i.e.

$$\mu_\varphi(a,b) = \lim_{\delta\to0^+}\lim_{\eta\to0^+}\int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{-\Delta_{\varepsilon,\mathrm{N}} - \lambda - i\eta} - \frac{1}{-\Delta_{\varepsilon,\mathrm{N}} - \lambda + i\eta} \right)\varphi, \varphi \right\rangle d\lambda \tag{3.87}$$

(see [28]). Let us recall that for a local analysis the outer boundary of Ω_ε is irrelevant since the the outer boundary "escape" sufficiently fast to infinity with $\varepsilon \to 0$ (i.e. radius of outer sphere is proportional to $\varepsilon^{-\delta}$, $\delta > 0$), see section 3.4.5. Therefor it is justified to consider the operator $-\Delta_{\varepsilon,\mathrm{N}}$ on the set $\widetilde{\Omega}_\varepsilon$ introduced by (3.70) instead of Ω_ε. Next by the spectral theorem (see [28]) we have

$$\left\langle \exp\left(i\sqrt{-\Delta_{\varepsilon,\mathrm{N}}}\frac{t}{\varepsilon}\right)[\Psi], G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \right\rangle = \int_0^\infty \exp\left(i\sqrt{\lambda}\frac{t}{\varepsilon}\right) G(\lambda)\tilde{\Psi}(\lambda)d\mu_{\varepsilon,\varphi}(\lambda) \tag{3.88}$$

for any $\tilde{\Psi} \in L^2(\Omega_\varepsilon, d\mu_\varphi)$, $\|\tilde{\Psi}\|_{L^2,\mu_\varphi} \leq \|\Psi\|_{L^2}$. Then crucial is to observe that one can pass to the limit in (3.87) with $\eta \to 0$, which holds when $-\Delta_{\varepsilon,\mathrm{N}}$ satisfy the Limiting Absorption Principle (LAP) (see [22, 31]). Namely operators

$$(1+|x|^2)^{-s/2} \circ (-\Delta_{\varepsilon,\mathrm{N}} - \lambda \pm i\eta)^{-1} \circ (1+|x|^2)^{-s/2} : L^2(\widetilde{\Omega}_\varepsilon) \to L^2(\widetilde{\Omega}_\varepsilon),$$

are bounded for any $s > 1$ uniformly for $\lambda \in [a,b]$, $0 < a < b$, $\eta > 0$. Thus, the spectral measure (3.87) is absolutely continuous and its formula reduces to

$$\mu_\varphi(a,b) = \int_a^b \langle (v^-_{\varepsilon,\lambda} - v^+_{\varepsilon,\lambda}); \varphi \rangle_{\widetilde{\Omega}_\varepsilon} \,\mathrm{d}\lambda, \tag{3.89}$$

for $0 < a < b$, where $v^\pm_{\varepsilon,\lambda}$ are solutions to the Neumann problem

$$\nabla v^\pm_{\varepsilon,\lambda} + \lambda v^\pm_{\varepsilon,\lambda} \text{ in } \widetilde{\Omega}_\varepsilon, \quad \nabla_x v^\pm_{\varepsilon,\lambda} \cdot \boldsymbol{n}|_{\partial\widetilde{\Omega}_\varepsilon} = 0$$

with Sommerfeld's radiation condition (see [31]) $\lim_{r\to\infty} r\left(\partial_r \pm i\sqrt{\lambda}\right) v^\pm_{\varepsilon,\lambda}$, $r = |x|$. Then one needs to provide uniform estimates on $v^\pm_{\varepsilon,\lambda}$ s.t.

$$0 \leq \langle (v^-_{\varepsilon,\lambda} - v^+_{\varepsilon,\lambda}), \varphi \rangle_{\widetilde{\Omega}_\varepsilon} \leq c(a,b,\varphi)$$

uniformly for $\varepsilon \to 0$, which in fact is a uniform bound on spectral measures (3.89). Next the rate of the decay in (3.86) (the coefficient ε on the right-hand side) is a consequence of the above property and suitable manipulations with right-hand side of (3.88) (see also [21]). The main advantage of the result of Lemma 3.4 is resistance of the rate of the decay on suitable perturbations of the domain.

Lemma 3.4 applied to Φ_ε given by formula (3.84) provides (3.85), if $\beta < \frac{1}{4}$, see [9] for details. Namely, the explicitly given rate of the decay in Lemma 3.4 allow to compensate exploding coefficient $\varepsilon^{-2\beta}$ which reflects the influence of perturbations of the domain and possible unboundedness of the gravity force in forcing terms in acoustic equation.

3.5 Convergence with $\varepsilon \to 0$ to the limit system. Part II. Compactness in time of the momentum.

Now we are ready to prove (3.29). To this end we employ the Helmholtz decomposition introduced in (3.73) and we infer the following

$$\begin{aligned}
\int_{\Omega_\varepsilon} \boldsymbol{V}_\varepsilon \cdot \boldsymbol{\phi} \,\mathrm{d}x &= \int_{\Omega_\varepsilon} \boldsymbol{H}_\varepsilon[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \cdot \boldsymbol{\phi} \,\mathrm{d}x - \int_{\Omega_\varepsilon} \Phi_\varepsilon \mathrm{div}_x\, \boldsymbol{\phi} \,\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] \,\mathrm{d}x - \int_{\Omega_\varepsilon} \Phi_\varepsilon \mathrm{div}_x\, \boldsymbol{\phi} \,\mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{H}[\boldsymbol{\phi}] \,\mathrm{d}x + \int_{\Omega_\varepsilon} \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}]) \,\mathrm{d}x - \int_{\Omega_\varepsilon} \Phi_\varepsilon \mathrm{div}_x\, \boldsymbol{\phi} \,\mathrm{d}x
\end{aligned} \tag{3.90}$$

for all $\boldsymbol{\phi} \in C_c^\infty(\Omega_\varepsilon)$ with support on K being a compact set in $\mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$. Let us consider each term of the right-hand side of (3.90). Accordingly to the Aubin-Lions argument and (3.36) by estimates provided in Section 3.4.3 we obtain

$$\left\{t \to \int_{\Omega_\varepsilon} \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{H}[\boldsymbol{\phi}]\,\mathrm{d}x\right\} \quad \to \quad \left\{t \to \overline{\varrho}\int_{\mathbb{R}^3} \boldsymbol{U}(\cdot,t)\cdot\boldsymbol{\phi}\,\mathrm{d}x\right\} \text{ in } L^2(0,T) \text{ for } \boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3), \tag{3.91}$$

where $\operatorname{supp}\boldsymbol{\phi} \subset K$ and $K \subset \Omega_\varepsilon$ for all sufficiently small ε. Let us remark that here we localise our investigation on sets where ∇F_ε is of sufficiently high integrability to provide boundedness of the forcing (gravity) term in the momentum equation. Moreover, we extend here $\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}]$ by zero outside Ω_ε (here $\boldsymbol{H}$ is Helmholtz projection defined on $\mathbb{R}^3$). Then let us consider the second term on the right-hand side of (3.90)

$$\begin{aligned}\int_{\Omega_\varepsilon} \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}])\,\mathrm{d}x = &\int_{\Omega_\varepsilon} (\varrho_\varepsilon - \overline{\varrho})\boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}])\,\mathrm{d}x\\ &+ \int_{\Omega_\varepsilon} \overline{\varrho}\boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}])\,\mathrm{d}x.\end{aligned}$$

By estimates (3.38), (3.10), (3.12), (3.39), (3.40), (3.18) and the Sobolev inequality, and by (3.75), (3.76) for $\boldsymbol{H}$, $\boldsymbol{H}_\varepsilon$ we obtain

$$\left\{t \to \int_{\Omega_\varepsilon} (\varrho_\varepsilon - \overline{\varrho})\boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}])\,\mathrm{d}x\right\} \to 0 \text{ in } L^2(0,T) \text{ for } \boldsymbol{\phi} \in C_c^\infty(K).$$

Then by (3.75)

$$\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] \to \boldsymbol{H}[\boldsymbol{\phi}] \text{ weakly in } L^2(K;\mathbb{R}^3)$$

holds and according to (3.18) we get

$$\left\{t \to \int_{\Omega_\varepsilon} \boldsymbol{u}_\varepsilon \cdot (\boldsymbol{H}_\varepsilon[\boldsymbol{\phi}] - \boldsymbol{H}[\boldsymbol{\phi}])\,\mathrm{d}x\right\} \to 0 \text{ in } L^2(0,T) \text{ for } \boldsymbol{\phi} \in C_c^\infty(K).$$

Now we have to deal with the last term in (3.90), i.e. with $\int_{\Omega_\varepsilon} \Phi_\varepsilon \mathrm{div}_x\,\boldsymbol{\phi}\,\mathrm{d}x$. Let us rewrite it as follows

$$\int_{\Omega_\varepsilon} \Phi_\varepsilon \mathrm{div}_x\,\boldsymbol{\phi}\,\mathrm{d}x = \int_{\Omega_\varepsilon} \Phi_\varepsilon G(-\Delta_{\varepsilon,\mathrm{N}})[\mathrm{div}_x\,\boldsymbol{\phi}]\,\mathrm{d}x + \int_{\Omega_\varepsilon} \Phi_\varepsilon (1 - G(-\Delta_{\varepsilon,\mathrm{N}}))[\mathrm{div}_x\,\boldsymbol{\phi}]\,\mathrm{d}x.$$

The relation (3.85) provides for any $G \in C_c^\infty(0,\infty)$, $\varphi \in C_c^\infty(K)$ with compact $K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}$

$$\left\{t \to \int_{\Omega_\varepsilon} \Phi_\varepsilon G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi]\,\mathrm{d}x\right\} \to 0 \quad \text{in } L^2(0,T). \tag{3.92}$$

Rewriting $\Phi_\varepsilon = \Phi_\varepsilon^1 + \Phi_\varepsilon^2$, where Φ_ε^1, Φ_ε^2 satisfies (3.80), for the second term we have

$$\left\{t \to \int_{\Omega_\varepsilon} \Phi_\varepsilon^2 (1 - G(-\Delta_{\varepsilon,\mathrm{N}}))[\varphi]\,\mathrm{d}x\right\} \to 0 \quad \text{in } L^2(0,T). \tag{3.93}$$

Noticing that

$$\mathbb{1}_{\Omega_\varepsilon} G(-\Delta_{\varepsilon,\mathrm{N}})[\varphi] \to \mathbb{1}_{\mathbb{R}^3\setminus\{x_1,\dots,x_N\}} G(-\Delta_N)[\varphi] \text{ in } L^2(\mathbb{R}^3) \quad \text{as } \varepsilon \to 0 \tag{3.94}$$

for any $G \in C_c^\infty(0,\infty)$, $\varphi \in C_c^\infty(K)$, $K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}$ and compact, we get

$$\left\{ t \to \int_{\Omega_\varepsilon} \Phi_\varepsilon^1 (1 - G(-\Delta_{\varepsilon,\mathrm{N}}))[\varphi]\,\mathrm{d}x \right\} \to \left\{ t \to \int_{\mathbb{R}^3 \setminus \{x_1,\dots,x_N\}} \Phi^1 (1 - G(-\Delta_N))[\varphi]\,\mathrm{d}x \right\} \text{ in } L^2(0,T). \tag{3.95}$$

The observation that

$$\int_\Omega \Phi^1 (1 - G(-\Delta_{\varepsilon,\mathrm{N}}))[\mathrm{div}_x\, \boldsymbol{\phi}]\,\mathrm{d}x = \int_\Omega (-\Delta_{\varepsilon,\mathrm{N}})^{1/2} \Phi^1 \frac{1}{(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}} (1 - G(-\Delta_{\varepsilon,\mathrm{N}}))[\mathrm{div}_x\, \boldsymbol{\phi}]\,\mathrm{d}x$$

where

$$\|(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}[\Phi^1]\|_{L^2(\Omega)} = \|\nabla_x \Phi^1\|_{L^2(\Omega)}$$

and

$$\begin{aligned} \left\| \frac{1}{(-\Delta_{\varepsilon,\mathrm{N}})^{1/2}} [\mathrm{div}_x\, \boldsymbol{\phi}] \right\|_{L^2(\Omega)} &= \left\| (-\Delta_{\varepsilon,\mathrm{N}})^{1/2} (-\Delta_{\varepsilon,\mathrm{N}})^{-1} [\mathrm{div}_x\, \boldsymbol{\phi}] \right\|_{L^2(\Omega)} = \left\| \nabla_x (-\Delta_{\varepsilon,\mathrm{N}})^{-1} [\mathrm{div}_x\, \boldsymbol{\phi}] \right\|_{L^2(\Omega)} \\ &\le \|\boldsymbol{\phi}\|_{L^2(\Omega)} \end{aligned}$$

gives us smallness of the last expression in (3.95) if only $G \approx \mathbb{1}_{[0,\infty)}$.

Relations above imply then (3.29) what provides convergence in the convective term in the momentum equation.

3.6 Convergence with $\varepsilon \to 0$ to the limit system. Part III.

Let us recall that in the limit $\varepsilon \to 0$ continuity equation reduces to the condition $\mathrm{div}_x \boldsymbol{U} = 0$ (see section (3.3)).

According to continuity equation (2.26) the entropy balance (2.29) may take the following form

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) (\partial_t \varphi + \boldsymbol{u}_\varepsilon \cdot \nabla_x \varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon} \right) \cdot \nabla_x \varphi\,\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{\varepsilon} \langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega}_\varepsilon)} = - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \left(\frac{s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \varphi(0,\cdot)\,\mathrm{d}x \end{aligned}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega_\varepsilon)$ with $\mathrm{supp}\,\varphi \subset K$ and compact $K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}$. Then due to limits obtained in Section 3.3 adapting steps from [12, Section 5.3.2] we may pass to the limit with $\varepsilon \to 0$ to obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \overline{\varrho} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} r + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \right) (\partial_t \varphi + \boldsymbol{U} \cdot \nabla_x \varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \nabla_x \Theta \cdot \nabla_x \varphi\,\mathrm{d}x\mathrm{d}t \\ &= - \int_{\mathbb{R}^3} \overline{\varrho} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} r_0 + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta_0 \right) \varphi(0,\cdot)\,\mathrm{d}x \end{aligned}$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\mathrm{supp}\,\varphi \subset K$, with compact $K \subset \mathbb{R}^3 \setminus \{x_1,\dots,x_N\}$ and where by (3.19), (2.35)

$$\frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} \rightharpoonup r_0 \text{ weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^3), \qquad \frac{\vartheta_{0,\varepsilon} - \overline{\vartheta}}{\varepsilon} \rightharpoonup \Theta_0 \text{ weakly in } L^2(\mathbb{R}^3).$$

In order to obtain relation between Θ and r we consider again the momentum equation multiplied by ε. Then after letting $\varepsilon \to 0$ we get

$$\int_0^T \int_{\mathbb{R}^3} \left(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} r + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \right) \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = - \int_0^T \int_{\mathbb{R}^3} \overline{\varrho} \nabla_x F \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\mathrm{supp}\, \boldsymbol{\varphi} \subset K$ and compact $K \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$. Consequently

$$\int_0^T \int_{\mathbb{R}^3} r \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = - \int_0^T \int_{\mathbb{R}^3} \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} \Theta \mathrm{div}_x \boldsymbol{\varphi} + \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} F \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t. \tag{3.96}$$

Finally we may conclude by (3.96) and by Gibbs' relation that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Theta \left(\partial_t \varphi + \boldsymbol{U} \cdot \nabla_x \varphi \right) \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} \left(\overline{\varrho}\overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F \boldsymbol{U} \cdot \nabla_x \varphi + \kappa(\overline{\vartheta}) \nabla_x \Theta \cdot \nabla_x \varphi \right) \mathrm{d}x\mathrm{d}t \\ &= - \int_{\mathbb{R}^3} \overline{\varrho}\overline{\vartheta} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} r_0 + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta_0 + \alpha(\overline{\varrho}, \overline{\vartheta}) F \right) \varphi(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{3.97}$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\mathrm{supp}\, \varphi \subset K$ and compact $K \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$, and where

$$c_p(\overline{\varrho}, \overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho}, \overline{\vartheta}) + \alpha(\overline{\varrho}, \overline{\vartheta}) \frac{\overline{\vartheta}}{\overline{\varrho}} \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}), \qquad \alpha(\overline{\varrho}, \overline{\vartheta}) = \frac{1}{\overline{\varrho}} \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})}.$$

Moreover, one deduce also the Boussinesq relation (in a sense of (3.96))

$$\tilde{r} + \overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta = 0, \qquad \text{where} \qquad \tilde{r} = r + \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} F. \tag{3.98}$$

As soon as $\boldsymbol{\varphi}$ is div-free recalling the result of the limit procedure from Section 3.3 the momentum equation in the limit reads

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} \left(\overline{\varrho} \boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{U} \otimes \boldsymbol{U} : \nabla_x \boldsymbol{\varphi} \right) \mathrm{d}x\mathrm{d}t = &\int_0^T \int_{\mathbb{R}^3} \left(\mu(\overline{\vartheta}) (\nabla_x \boldsymbol{U} + \nabla_x^T \boldsymbol{U}) : \nabla_x \boldsymbol{\varphi} - r \nabla F \cdot \boldsymbol{\varphi} \right) \mathrm{d}x\mathrm{d}t \\ &- \int_{\mathbb{R}^3} (\overline{\varrho} \boldsymbol{U}_0) \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{3.99}$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\mathrm{supp}\, \varphi \subset K$ and compact $K \subset \mathbb{R}^3 \setminus \{x_1, \dots, x_N\}$ (more details the reader can find in [12, Section 5.3.3]). Now let us notice that by $(3.98)_2$ in (3.99) r can be replaced by $\tilde{r}$ since weak formulation is based on solenoidal functions. Then due to $(3.98)_1$ one can replace r again by $\overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta$. This observation allow us to rewrite (3.99) in the following formulation

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \left(\overline{\varrho} \boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{U} \otimes \boldsymbol{U} : \nabla_x \boldsymbol{\varphi} \right) \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^3} \left(\mu(\overline{\vartheta}) (\nabla_x \boldsymbol{U} + \nabla_x^T \boldsymbol{U}) : \nabla_x \boldsymbol{\varphi} - \overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta \nabla F \cdot \boldsymbol{\varphi} \right) \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^3} (\overline{\varrho} \boldsymbol{U}_0) \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x, \end{aligned} \tag{3.100}$$

which is satisfied, due to density property for all $\boldsymbol{\varphi} \in C_c^\infty([0,T); W^{1,p}(\mathbb{R}^3 \setminus \{x_1,\dots,x_N\})$ with any $p \in (2,3)$ but solenoidal and with compact support in $\mathbb{R}^3 \setminus \{x_1,\dots,x_N\})$ (recall that $\Theta \in L^2(W^{1,2})$). Moreover density arguments allow us to use $\varphi \in C_c^\infty([0,T); W^{1,p}(\mathbb{R}^3 \setminus \{x_1,\dots,x_N\})$ for any $p \in (2,3)$ but with compact support in $\mathbb{R}^3 \setminus \{x_1,\dots,x_N\})$ also in variational formulations (3.97), (3.96). Since the $p-$capacity of finite number of points is zero, i.e. $\mathrm{Cap}_p(\{x_1,\dots,x_N\}) = 0$, the $W^{1,p}(\mathbb{R}^3 \setminus \{x_1,\dots,x_N\})$ is equivalent with the space $W^{1,p}(\mathbb{R}^3)$ if $p \in (2,3)$. Then, again, we may restrict the set of test functions for variational formulation for the momentum equation (3.100) to $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\mathrm{div}_x \boldsymbol{\varphi} = 0$ and for (3.97) to $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^3)$.

Moreover the initial condition for the velocity is determined by

$$\boldsymbol{u}_{0,\varepsilon} \rightharpoonup \boldsymbol{U}_0 \text{ weakly in } L^2(\mathbb{R}^3; \mathbb{R}^3)$$

and for the Θ_0

$$\Theta_0 = \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\overline{\vartheta})} \left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} r_0 + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + \alpha(\overline{\varrho},\overline{\vartheta}) F \right) \tag{3.101}$$

where we used that $\varrho_{0,\varepsilon}^{(1)} \rightharpoonup \varrho_0^{(1)}$ in $L^2(\mathbb{R}^3)$, closeness of $\tilde{\varrho}$ to $\overline{\varrho}$ and $\vartheta_{0,\varepsilon}^{(1)} \rightharpoonup \vartheta_0^{(1)}$ in $L^2(\mathbb{R}^3)$. The following compatibility condition for r_0, $\vartheta_0^{(1)}$ is satisfied $\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} = \overline{\varrho} F$. On the left-hand side we have linearisation of the pressure p at the point $(\overline{\varrho},\overline{\vartheta})$ applied to the vector $[r_0, \vartheta_0^{(1)}]$. Relation (3.101) is reduced to $\Theta_0 = \vartheta_0^{(1)}$.

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