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Thermo-visco-elasticity for the Mróz model in the framework of thermodynamically complete systems

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Abstract

We derive a thermomechanical model for the evolution of the visco-elastic body subject to the action of external forces. The presented framework captures elastic and visco-elastic deformation as well as thermal effects occurring in the material. Consequently we couple the momentum balance with the heat equation . The system is supplemented by the constitutive relation for the Cauchy stress tensor and visco-elastic strain tensor. As an example exhibiting the proposed approach one can mention the Mróz model. We establish existence of solutions to the quasi-static version of the derived system.

1 Introduction

The aim of this paper is to derive and discuss the physical meaning of the thermo-visco-elastic model. Using the standard physical principles, such as balance of linear momentum and balance of energy, we derive the system of equations. This system is supplemented by the constitutive relation for the Cauchy stress tensor and the constitutive equation for the evolution of visco-elastic strain tensor. Additionally, we assume an infinitesimal displacement of the body, hence we linearize the elastic relation.

We consider the body $\Omega \subset \mathbb{R}^3$, which is an open bounded set with a Lipschitz boundary. The body is subject to the action of external forces which causes spatial deformation and changes of the temperature and exhibits two types of displacements: elastic and visco-elastic. By the elastic deformation we mean the deformation which returns to the initial state after the action of external forces expires. The visco-elastic deformation is here understood as a deformation which is irreversible. The changes of the temperature affect the rate of visco-elastic stress tensor and inversely, the changes of the Cauchy stress tensor affect the temperature of the body. Due to this fact, the proposed system of equations is coupled - we cannot consider separately equations for each of the unknowns.

The theory of inelastic and infinitesimal deformations was widely studied in a case when no coupling with thermal effects appears and the nonlinear inelastic constitutive relation is of monotone type, see [1] and also [9–11] among others. Mathematical analysis of linear thermo-elasticity is a classical, well understood topic, cf. [18], contrary to an analysis of thermo-visco-elastic models used in the engineering practice which is until today very unsatisfactory. By thermo-visco-elastic models we mean the system consisting of visco-elastic equations including the contribution of thermal effects in the balance of momentum and an additional equation for the evolution of temperature. In the literature there are only few results for special models or for simplified models of this type [3,4,12]. In [3] the author considered a thermoplastic model with the so-called Bodner-Partom flow rule. In the article [12] a modification of the Prandtl-Reuss flow rule model with a yield function depending explicitly on the temperature is studied. Moreover, it is assumed that the considered material possesses the kinematic hardening property and the evolution equation for the backstress (new internal variable enlarging locally the set of admissible stresses) is linear.

The paper is organized as follows: Section 2 is dedicated to the derivation of the model. We start with the discussion on the balance of momentum, balance of energy and constitutive relations. All these considerations are summarized in Section 2.5. This part of the paper is completed with a section dedicated to the verification of the model. We concentrate here on the question of the thermodynamical completeness, in the meaning of conservation of energy and entropy inequality satisfied by the system. In Section 3 we concentrate on analysis of a simplified model, namely the quasi-static evolution. Such case of slow evolution where consequently the acceleration term in the equation for balance of momentum is neglected has often been studied, see Bartczak [3], Chełmiński [9], Chełmiński and Racke [12], Duvaut and J.L. Lions [15], Johnson [13,14], Suquet [21–23] and Temam [25,26] among others.

We denote by: $L^p(\Omega)$ a standard Lebesgue space, for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, by $W^{k,p}(\Omega)$ the Sobolev spaces, by $H^{\frac{1}{2}}(\partial\Omega)$ the fractional order Sobolev space and by $L^p(0,T,L^q(\Omega))$ Bochner space, by $C(K)$ continuous functions on K, by $C_c^\infty(K)$ compactly supported smooth functions on K. Additionally, $\boldsymbol{\varepsilon}$ denotes the symmetric part of the gradient of displacement $\boldsymbol{u}$, i.e. $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$. We use the notation $g|_{t_1}^{t_2} = g(t_2) - g(t_1)$, where g is an arbitrary function of time.

2 Derivation of the model

In the next few subsections we will formulate the thermo-visco-elastic system of equations, cf. Green and Naghdi [16] or Landau and Lifshitz [19]. For this purpose we shall use two conservation laws for linear momentum and energy. The constitutive relation for the Cauchy stress tensor shall be accompanied by the physical discussion. The last equation will be the constitutive relation for the visco-elastic deformation.

2.1 Balance of momentum

A linear momentum is a conserved quantity. Hence, changes of a linear momentum correspond to the action of external forces. We include two kinds of external forces: volume and surface forces. Let us consider an open subset $\mathcal{O}$ of the body Ω. Then the balance of momentum has the following form

$$\frac{d}{dt}\int_{\mathcal{O}} \rho\boldsymbol{u}_t \,\mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{f}\,\mathrm{d}x + \int_{\partial\mathcal{O}} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}s, \tag{1}$$

where ρ is the density of the body, $\boldsymbol{f}$ is the density of external volume force, $\boldsymbol{\sigma}$ stands for the Cauchy stress tensor and $\boldsymbol{n}$ is a unit outward normal vector to the boundary $\partial\mathcal{O}$ and $\boldsymbol{u}$ is a displacement. Symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$ follows from the principle of the conservation of angular momentum, cf. [19]. The term $\int_{\partial\mathcal{O}} \boldsymbol{\sigma}\cdot\boldsymbol{n}$ describes the surface forces. Using the Green theorem we obtain

$$\int_{\mathcal{O}} \rho\boldsymbol{u}_{tt}\,\mathrm{d}x - \int_{\mathcal{O}} \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{f}\,\mathrm{d}x. \tag{2}$$

Equation (2) is tantamount to the weak formulation of the following equation

$$\rho\boldsymbol{u}_{tt} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}. \tag{3}$$

2.2 Balance of energy

The changes of global energy of the closed system are equal to the work done on the system and the heat supplied to the system, namely

$$\frac{d}{dt}\mathcal{E} = P_{external} + \frac{d}{dt}Q, \tag{4}$$

where $\mathcal{E}$ is the global energy of the system, $P_{external}$ denotes the rate of work of external forces and Q is the heat. We assume that the density of energy is in the form

$$e = c(\theta-\theta_R) + \frac{1}{2}\boldsymbol{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p) + \frac{1}{2}\rho|\boldsymbol{u}_t|^2, \tag{5}$$

where $\boldsymbol{D}$ is a linear, positively defined and bounded operator from $\mathcal{S}^3$ to $\mathcal{S}^3$. By $\mathcal{S}^3$ we denote the set of symmetric 3×3–matrices with real entries, $\boldsymbol{\varepsilon}$ is a symmetric gradient of displacement $\boldsymbol{u}$, $\boldsymbol{\varepsilon}^p$ describes visco-elastic strain tensor, and θ_R is a constant reference temperature. Constant c stands for the heat capacity of the body. Symmetry of $\boldsymbol{\varepsilon}^p$ follows from the material objectivity and isotropy of the material. Tensor $\boldsymbol{D}$ describes the material behaviour. We define tensor $\boldsymbol{T}$ as follows

$$\boldsymbol{T} := \boldsymbol{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p), \tag{6}$$

hence the density of global energy can be reformulated as follows

$$e = c(\theta-\theta_R) + \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{T}:\boldsymbol{T} + \frac{1}{2}\rho|\boldsymbol{u}_t|^2. \tag{7}$$

The density of global energy consist of three parts: thermal, potential and kinetic. Let us consider again an open subset $\mathcal{O}$ of the body Ω. Changes of the global energy in the set $\mathcal{O}$ are prescribed as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{\mathcal{O}} &= \frac{d}{dt}\int_{\mathcal{O}} \big(c(\theta-\theta_R) + \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{T}:\boldsymbol{T} + \frac{1}{2}\rho|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \big(c\theta_t + \boldsymbol{T}:\nabla\boldsymbol{u}_t - \boldsymbol{T}:\boldsymbol{\varepsilon^p}_t + \frac{1}{2}\rho\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x.\end{aligned} \tag{8}$$

The rate of work of external forces acting on the set $\mathcal{O}$ is equal to the rate of work of surface and volume forces

$$\begin{aligned} P_{external} &= \int_{\partial\mathcal{O}} \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{u}_t \,\mathrm{d}s + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \mathrm{div}\,(\boldsymbol{\sigma}\boldsymbol{u}_t)\,\mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t \,\mathrm{d}x + \int_{\mathcal{O}} \mathrm{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{u}_t \,\mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t \,\mathrm{d}x + \int_{\mathcal{O}} (\mathrm{div}\,\boldsymbol{\sigma} + \boldsymbol{f})\cdot\boldsymbol{u}_t \,\mathrm{d}x. \end{aligned} \tag{9}$$

On the basis of the equation for the linear momentum (2) we conclude further

$$\begin{aligned} P_{external} &= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t \,\mathrm{d}x + \int_{\mathcal{O}} \rho\boldsymbol{u}_{tt}\cdot\boldsymbol{u}_t \,\mathrm{d}x \\ &= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t \,\mathrm{d}x + \int_{\mathcal{O}} \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2 \,\mathrm{d}x. \end{aligned} \tag{10}$$

The changes of the heat are equal to the heat produced by the heat sources in the body (in our case the density of the heat sources is denoted by r) and the heat flux through the boundary of considered $\mathcal{O}$

$$\frac{d}{dt}Q = \int_{\mathcal{O}} r\,\mathrm{d}x - \int_{\partial\mathcal{O}} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\mathcal{O}} r\,\mathrm{d}x - \int_{\mathcal{O}} \mathrm{div}\,\boldsymbol{q}\,\mathrm{d}x. \tag{11}$$

According to the Fourier law, the heat flux is proportional to the gradient of the temperature ($\boldsymbol{q} = -\kappa\nabla\theta$). Using this observation, we obtain

$$\frac{d}{dt}Q = \int_{\mathcal{O}} r\,\mathrm{d}x + \int_{\mathcal{O}} \kappa\Delta\theta\,\mathrm{d}x. \tag{12}$$

Collecting all the components of the energy, we get complete form of the energy balance

$$\int_{\mathcal{O}} \big(c\theta_t + \boldsymbol{T} : \nabla\boldsymbol{u}_t - \boldsymbol{T} : {\boldsymbol{\varepsilon}^{\boldsymbol{p}}}_t + \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x = \int_{\mathcal{O}} \big(\boldsymbol{\sigma} : \nabla\boldsymbol{u}_t + r + \kappa\Delta\theta + \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x.$$

This equation holds for arbitrary subset $\mathcal{O}$ of Ω, hence it is equivalent to

$$c\theta_t - \kappa\Delta\theta + \big(\boldsymbol{T} - \boldsymbol{\sigma}\big) : \nabla\boldsymbol{u}_t = \boldsymbol{T} : {\boldsymbol{\varepsilon}^{\boldsymbol{p}}}_t + r. \tag{13}$$

The system of equations (3) and (13) needs to be supplemented by the constitutive relation for the Cauchy stress tensor and the constitutive relation for the evolution of the visco-elastic strain tensor.

2.3 The Cauchy stress tensor

We assume that the Cauchy stress tensor is in the form (Hooke's law)

$$\boldsymbol{\sigma} = \underbrace{\boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)}_{=\boldsymbol{T}} - \alpha(\theta - \theta_R)\boldsymbol{I}. \tag{14}$$

Here θ_R is the reference temperature, α is a positive constant and $\boldsymbol{I}$ is an identity matrix from $\mathcal{S}^3$. Our interest is directed to three kinds of phenomena: mechanical effects (elastic and visco-elastic deformation) and thermal effects. Consequently, to describe the problem appropriately we intend to include the dependence on $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and θ in the Cauchy stress tensor.

Deformation of the body is split into visco-elastic part $\boldsymbol{\varepsilon}^p$ and elastic part $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$. If we neglect thermal effects, the Cauchy stress tensor is equal to the elastic part of the deformation. This kind of linear dependence holds for infinitesimal displacements and should be understood as a relation after neglecting the higher order terms in Taylor expansion. Similar approach was used in [1–3, 11–14, 21–24, 26]

As the changes of temperature may cause changes of the body tension, we include in the Cauchy stress tensor the thermal part, namely $\alpha(\theta - \theta_R)\boldsymbol{I}$.

2.4 Constitutive relation for the evolution of the visco-elastic strain tensor

Let us denote by $\boldsymbol{T}^d$ the deviatoric (traceless) part of the tensor $\boldsymbol{T}$, i.e.

$$\boldsymbol{T}^d = \boldsymbol{T} - \frac{1}{3}tr(\boldsymbol{T})\boldsymbol{I}. \tag{15}$$

Similarly, $\mathcal{S}^3_d$ is a subset of $\mathcal{S}^3$ containing only traceless matrices. A second constitutive relation in the model is the evolutionary equation for the visco-elastic strain tensor, namely

$$\boldsymbol{\varepsilon}^p_t = \boldsymbol{G}(\theta, \boldsymbol{T}^d), \tag{16}$$

where $\boldsymbol{G} : \mathbb{R}_+ \times \mathcal{S}^3_d \to \mathcal{S}^3_d$ is a Carathéodory function, i.e. measurable with respect to θ and continuous with respect to $\boldsymbol{T}^d$, consequently if $\boldsymbol{\varepsilon}^p_0 \in \mathcal{S}^3_d$ then the plastic strain tensor $\boldsymbol{\varepsilon}^p$ is traceless. We make the following assumptions on the function $\boldsymbol{G}(\cdot,\cdot)$. For all matrices $\boldsymbol{T}^d$ and $\boldsymbol{T}^d_1 \neq \boldsymbol{T}^d_2$ from $\mathcal{S}^3_d$ and almost all $\theta \in \mathbb{R}_+$, it holds:

a) $(\boldsymbol{G}(\theta, \boldsymbol{T}^d_1) - \boldsymbol{G}(\theta, \boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) > 0$;

b) $|\boldsymbol{G}(\theta, \boldsymbol{T}^d)| \leq C|\boldsymbol{T}^d|$, where C is a positive constant;

c) $\boldsymbol{G}(\theta, \boldsymbol{T}^d) : \boldsymbol{T}^d \geq \beta|\boldsymbol{T}^d|^2$, where β is a positive constant;

d) $\boldsymbol{G}(\cdot, \boldsymbol{T}^d)$ is bounded.

An example of a model which satisfies such conditions is the Mróz model, cf. [20], where the constitutive function for plastic strain tensor has the following form:

$$\boldsymbol{G}(\theta, \boldsymbol{T}^d) = G_1(\theta)\boldsymbol{T}^d, \tag{17}$$

and $G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded measurable function.

2.5 Full thermo-visco-elastic model

As a consequence of the considerations in the previous sections we obtain a full system of equations describing the thermo-visco-elastic problem. Let $T < \infty$. For given external forces $\boldsymbol{f} : \Omega \times (0,T) \to \mathbb{R}^3$ we are interested in the evolution of $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^3$, $\boldsymbol{\varepsilon}^p : \Omega \times (0,T) \to \mathcal{S}^3_d$ and $\theta : \Omega \times (0,T) \to \mathbb{R}_+$ captured by the system

$$\left\{\begin{array}{rcll} \rho \boldsymbol{u}_{tt} - \operatorname{div} \boldsymbol{\sigma} & = & \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \boldsymbol{\sigma} & = & \boldsymbol{T} - \alpha(\theta - \theta_R)\boldsymbol{I} & \text{in } \Omega \times (0,T), \\ \boldsymbol{T} & = & \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) & \text{in } \Omega \times (0,T), \\ \boldsymbol{\varepsilon}^p_t & = & \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T), \\ c\theta_t - \kappa\Delta\theta + \alpha(\theta - \theta_R)\operatorname{div} \boldsymbol{u}_{\mathrm{t}} & = & \boldsymbol{T}^d : \boldsymbol{\varepsilon}^p_t + r & \text{in } \Omega \times (0,T). \end{array}\right. \tag{18}$$

where $\boldsymbol{T}^d$ is a deviatoric part of $\boldsymbol{T}$ defined by (15), ρ is the density of the material, α is a positive constant, c is the heat capacity, θ_R is the reference temperature, κ is material conductivity and r is the density of heat sources. The system is completed by the boundary and initial conditions:

$$\left\{\begin{array}{rcll} \boldsymbol{u} & = & \boldsymbol{g}_{\boldsymbol{u}} & \text{on} \quad \partial\Omega \times (0,T), \\ \frac{\partial \theta}{\partial \boldsymbol{n}} & = & g_\theta & \text{on} \quad \partial\Omega \times (0,T), \end{array}\right. \qquad \left\{\begin{array}{rclll} \boldsymbol{u}(x,0) & = & \boldsymbol{u}_0(x) & \text{in} & \Omega, \\ \boldsymbol{u}_t(x,0) & = & \boldsymbol{u}_1(x) & \text{in} & \Omega, \\ \theta(x,0) & = & \theta_0(x) & \text{in} & \Omega, \\ \boldsymbol{\varepsilon}^p(x,0) & = & \boldsymbol{\varepsilon}^p_0(x) & \text{in} & \Omega. \end{array}\right. \tag{19}$$

This boundary conditions can be understood as follows: using the Dirichlet condition for the displacement means that we control the shape of the body, and using the Neumann condition for temperature means that we control the flow of the energy through the boundary. If the transfer of energy through the boundary is equal to zero then the energy of the system in conserved. In particular, it holds for the homogeneous Dirichlet boundary condition for displacement and for the homogeneous Neumann boundary condition for the temperature.

2.6 Thermodynamical completeness of the system

The current section is crucial for understanding the physical advantages of the considered model. We are testing whether the system fulfills thermodynamical aspects: conservation of the energy and the rate of entropy production is positive. If these properties are satisfied we shall say that the system is thermodynamically complete. To demonstrate that the considered system is thermodynamically complete we shall consider an isolated system, namely with $\boldsymbol{f} = 0$, homogeneous boundary value and without any heat sources ($r = 0$).
Conservation of total energy
In the first step we intend to show that the global energy is preserved. With help of (4), (9) and (12) the changes of the global energy of the system could be written in the form

$$\frac{d}{dt}\mathcal{E}_{\mathcal{O}}(t) = \int_{\partial\mathcal{O}} \big(\boldsymbol{T}\boldsymbol{u}_t - \alpha(\theta - \theta_R)\boldsymbol{u}_t + \nabla\theta\big) \cdot \boldsymbol{n} \,\mathrm{d}s. \tag{20}$$

If we consider problem with homogeneous boundary condition then $\boldsymbol{u}_t = 0$ and $\nabla\theta \cdot \boldsymbol{n} = 0$ on the boundary $\partial\Omega$. Therefore, we conclude that the global energy $\mathcal{E}_\Omega$ is constant in time.
Entropy inequality
Before we consider the second law of thermodynamics let us define $\tilde{\theta} := \theta - \theta_R$. Assume that the initial temperature θ_0 is greater than the reference temperature θ_R. We rewrite $(18)_5$ in terms of $\tilde{\theta}$

$$c\tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \alpha\tilde{\theta}\operatorname{div} \boldsymbol{u}_{\mathrm{t}} = \boldsymbol{G}(\tilde{\theta} + \theta_{\mathrm{R}}, \boldsymbol{T}^{\mathrm{d}}) : \boldsymbol{T}^{\mathrm{d}}. \tag{21}$$

Hence, from the assumption on function $\boldsymbol{G}(\cdot,\cdot)$ the right hand side of (21) is positive

$$c\tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \alpha\tilde{\theta}\mathrm{div}\,\boldsymbol{u}_{\mathrm{t}} \geq 0. \tag{22}$$

For $\tilde{\theta} = 0$, i.e. $\theta = \theta_R$, the term $\alpha\tilde{\theta}\mathrm{div}\,\boldsymbol{u}_{\mathrm{t}}$ vanishes. Hence, by standard methods we conclude that if the initial condition $\tilde{\theta}_0$ is positive then also $\tilde{\theta}$ is positive.

Multiplying (21) by $1/\tilde{\theta}$ and integrating over $\mathcal{O} \subset \Omega$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{O}} c\ln\tilde{\theta}\,\mathrm{d}x - \int_{\mathcal{O}} \kappa\mathrm{div}\,\frac{\nabla\tilde{\theta}}{\tilde{\theta}}\,\mathrm{d}x - \int_{\mathcal{O}} \kappa\frac{|\nabla\tilde{\theta}|^2}{\tilde{\theta}^2}\,\mathrm{d}x + \int_{\mathcal{O}} \alpha\mathrm{div}\,\boldsymbol{u}_{\mathrm{t}}\,\mathrm{dx}& \\ = \int_{\mathcal{O}} \frac{\boldsymbol{G}(\tilde{\theta}+\theta_R,\boldsymbol{T}^d):\boldsymbol{T}^d}{\tilde{\theta}}\,\mathrm{d}x.&\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{O}} \left(c\ln\tilde{\theta} + \alpha\mathrm{div}\,\boldsymbol{u}\right)\mathrm{d}x + \int_{\mathcal{O}} \mathrm{div}\left(\frac{\boldsymbol{q}}{\tilde{\theta}}\right)\mathrm{d}x& \\ = \int_{\mathcal{O}} \frac{\boldsymbol{G}(\tilde{\theta}+\theta_R,\boldsymbol{T}^d):\boldsymbol{T}^d}{\tilde{\theta}}\,\mathrm{d}x + \int_{\mathcal{O}} \kappa\frac{|\nabla\tilde{\theta}|^2}{\tilde{\theta}^2}\,\mathrm{d}x.&\end{aligned}$$

Positivity of the right hand side follows from the assumption on the function $\boldsymbol{G}(\cdot,\cdot)$ and positivity of $\tilde{\theta}$ and κ. Therefore, the inequality holds

$$\frac{d}{dt}\int_{\mathcal{O}} \left(c\ln\tilde{\theta} + \alpha\mathrm{div}\,\boldsymbol{u}\right)\mathrm{d}x + \int_{\mathcal{O}} \mathrm{div}\left(\frac{\boldsymbol{q}}{\tilde{\theta}}\right)\mathrm{d}x \geq 0. \tag{23}$$

Arbitrariness of the domain $\mathcal{O}$ implies that this inequality is tantamount to

$$\left(c\ln\tilde{\theta} + \alpha\mathrm{div}\,\boldsymbol{u}\right)_t + \mathrm{div}\left(\frac{\boldsymbol{q}}{\tilde{\theta}}\right) \geq 0. \tag{24}$$

The obtained relation is the so-called Clausius-Duhem inequality and it is one of the equivalent formulations of the second principle of thermodynamics. Hence, using the homogeneous boundary condition and the definition of the heat flux ($\boldsymbol{q} = -\kappa\nabla\theta$), we obtain

$$\frac{d}{dt}\int_{\Omega} \ln\tilde{\theta} \geq 0. \tag{25}$$

We complete this section with the discussion on different models appearing in the theory of thermo-visco-elasticity. One of the approaches arising from visco-elasticity (heat independent considerations) is the linearisation technique. This framework in meaningful as a way to apply linear semigroups theory. Transferring this idea to heat dependent models leads to the approximation of the term $\alpha(\theta-\theta_R)\mathrm{div}\,\boldsymbol{u}_t$ by the linear term term $\alpha_0\mathrm{div}\,\boldsymbol{u}_t$ arguing that the temperature is close to the reference temperature θ_R, cf. Bartczak [3], Chełmiński and Racke [12]. In these cases we observe the lack of thermodynamical completeness already on the level of conservation of the energy. Moreover, the reasoning for showing the entropy inequality does not hold for the linearized models.

3 Simplified model

In the current section we consider the simplified model and formulate an existence theorem. Then we point out the main difficulties in the proof and present the short sketch of the proof. Consider the system of equations in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$

$$\left\{\begin{array}{rcll} -\mathrm{div}\, \boldsymbol{T} & = & \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \boldsymbol{T} & = & \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) & \text{in } \Omega \times (0,T), \\ \boldsymbol{\varepsilon}^p_t & = & \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T), \\ \theta_t - \Delta\theta & = & \boldsymbol{T}^d : \boldsymbol{\varepsilon}^p_t & \text{in } \Omega \times (0,T), \end{array}\right. \tag{26}$$

with the boundary and initial conditions

$$\left\{\begin{array}{rcll} \boldsymbol{u} & = & \boldsymbol{g_u} & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial\theta}{\partial\boldsymbol{n}} & = & g_\theta & \text{on } \partial\Omega \times (0,T), \end{array}\right. \qquad \left\{\begin{array}{rcll} \theta(x,0) & = & \theta_0(x) & \text{in } \Omega, \\ \boldsymbol{\varepsilon}^p(x,0) & = & \boldsymbol{\varepsilon}^p_0(x) & \text{in } \Omega. \end{array}\right. \tag{27}$$

To solve this system of equations we have to find $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^3$ - displacement of the material, $\boldsymbol{\varepsilon}^p : \Omega \times (0,T) \to \mathcal{S}^3_d$ - plastic strain tensor and $\theta : \Omega \times (0,T) \to \mathbb{R}_+$ - temperature of the material.

The first simplification is omitting the term $\boldsymbol{u}_{tt}$, which is motivated twofolds. The first motivation arises from physics: we assume that the change of kinetic energy can be omitted - the body is of very small density. The second one comes from the area of engineering: we assume that displacement is very slow and we are interested in long-time behavior. Furthermore, the material conductivity κ is a positive constant, for simplicity is equal to one, there are no heat sources in the material ($r = 0$), the heat capacity c is equal to one and α is equal to zero. Before we formulate the theorem let us define the space $W^{1,2}_{\boldsymbol{g_u}}(\Omega) = \{u \in W^{1,2}(\Omega) : u|_{\partial\Omega} = \boldsymbol{g_u}\}$.

Theorem 1. *Let $q \in (1, \frac{5}{4})$ and $s \in \mathbb{R}$ be large enough. Let initial conditions satisfy $\theta_0 \in L^1(\Omega)$, $\boldsymbol{\varepsilon}^p_0 \in L^2(\Omega, \mathcal{S}^3_d)$, boundary conditions satisfy $\boldsymbol{g_u} \in L^2(0,T,H^{\frac{1}{2}}(\partial\Omega))^3$ and $g_\theta \in L^2(0,T,L^2(\partial\Omega))$, function $\boldsymbol{f} \in W^{1,2}(0,T,L^2(\Omega))^3$ and let function $\boldsymbol{G}(\cdot,\cdot)$ satisfy assumptions* a) - d) *from Section 2.4. Then there exists a solution to the system* (26)-(27)*, i.e.*

$$\begin{aligned} &\boldsymbol{u} \in L^2(0,T,W^{1,2}_{\boldsymbol{g_u}}(\Omega))^3 \cap C([0,T],L^2(\Omega))^3, \\ &\theta \in L^q(0,T,W^{1,q}(\Omega)) \cap C([0,T],W^{-s,2}(\Omega)), \end{aligned} \tag{28}$$

which satisfies:

$$\int_0^T \int_\Omega \boldsymbol{D}\Big(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p_0 - \int_0^t \boldsymbol{G}(\theta(x,\tau), \boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau\Big) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t, \tag{29}$$

and

$$\begin{aligned} -\int_0^T \int_\Omega \theta\phi_t\,\mathrm{d}x\mathrm{d}t - \int_\Omega \theta_0(x)\phi(0,x)\,\mathrm{d}x \\ + \int_0^T \int_\Omega \nabla\theta \cdot \nabla\phi\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\partial\Omega} g_\theta\phi\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{T}^d : \boldsymbol{G}(\theta, \boldsymbol{T}^d)\phi\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{30}$$

for every test function $\boldsymbol{\varphi} \in C^\infty([0,T], C^\infty_c(\Omega))^3$ and $\phi \in C^\infty_c([0,T), C^\infty(\Omega))$.

Remark. *Additionally, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.*

$$\boldsymbol{\varepsilon}^p(x,t) = \boldsymbol{\varepsilon}^p_0(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau), \boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau, \tag{31}$$

for a.e. $x \in \Omega$ and $t \in [0,T)$. Moreover

$$\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{32}$$

and

$$\begin{aligned} \boldsymbol{\varepsilon}^p &\in W^{1,2}(0,T,L^2(\Omega,\mathcal{S}^3_d)),\\ \boldsymbol{T} &\in L^2((0,T)\times\Omega,\mathcal{S}^3),\\ \boldsymbol{T}^d &\in L^2((0,T)\times\Omega,\mathcal{S}^3_d). \end{aligned} \tag{33}$$

Sketch of the proof.

In a wide class of models, including the one proposed in a current paper, the product $\boldsymbol{T}^d : \boldsymbol{\varepsilon}^p_t$ is only an integrable function. This property inhibits in general using the energy method and enforces the use of more delicate tools. Following Boccardo and Gallouet [5] we obtain the solution θ in the space $L^q(0,T,W^{1,q}(\Omega)) \cap C([0,T],W^{-s,2}(\Omega))$ for s large enough and for $1 < q < \frac{5}{4}$. We shall truncate the right hand side of the heat equation and the initial condition for the temperature. This approach requires the use of independent Galerkin approximations for displacement and temperature. Similar method has been used by Bulíček, Feireisl and Málek in [7], see also Bulíček [6] or Bulíček and Pustějovská in [8].

We may concentrate on a modified problem, namely define $\tilde{\boldsymbol{u}}$ such that $\tilde{\boldsymbol{u}}|_{\partial\Omega} = \boldsymbol{g}_{\boldsymbol{u}}$ and consider the problem for $\boldsymbol{u} - \tilde{\boldsymbol{u}}$, and hence the boundary conditions become homogeneous. The same procedure follows for the temperature. We construct the approximate problems using two independent bases: the eigenfunctions of the Laplace operator with the domain $\{v \in W^{1,2}(\Omega) : \frac{\partial v}{\partial \boldsymbol{n}} = 0\}$ for temperature and the eigenfunctions of the operator $-\mathrm{div}\,\boldsymbol{D}(\nabla + \nabla^{\mathrm{T}})$ with the domain $W^{1,2}_0(\Omega)$ for displacement. Using standard ODE techniques we get the existence of solutions for each approximation step.

Next, we show the boundedness of the sequences of solutions. We define the potential energy of the system

$$\mathcal{E}_{pot}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) = \frac{1}{2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p). \tag{34}$$

Using the potential energy and the thermo-elastic system of equations - system of equations without the visco-elastic effect with the same right hand side function (the same density of external forces), we obtain the uniform boundedness of the sequence $\{\boldsymbol{T}_{k,l}\}$ in $L^2(0,T,L^2(\Omega,\mathcal{S}^3))$. This uniform boundedness and the constitutive equation for the evolution of visco-elastic strain tensor imply the uniform boundedness of the sequence $\{\boldsymbol{\varepsilon}^p_{k,l}\}$ in $L^2(0,T,L^2(\Omega,\mathcal{S}^3_d))$. From these two bounds and the triangle inequality we conclude the uniform boundedness of the displacement $\{\boldsymbol{u}_{k,l}\}$ in $L^2(0,T,W^{1,2}_{\boldsymbol{g}_{\boldsymbol{u}}}(\Omega))^3$. The above estimates are uniform in k and l.

Using standard tools, we get the uniform boundedness of the sequence of the temperature $\{\theta_{k,l}\}$ with respect only to the index l, namely the parameter of the approximation in temperature. The low regularity of the right hand side of the heat equation prevents from getting the uniform bound with respect to the second parameter.

Let $\boldsymbol{\chi}_k$ be the limit of the sequence $\boldsymbol{G}_k(\theta_{k,l}, \boldsymbol{T}^d_{k,l})$ after passing to the limit with $l \to \infty$. Due to characterize $\boldsymbol{\chi}_k$ we shall use Theorem 1.2 from [17]. To provide that the assumptions are satisfied we need to show that the following inequality holds

$$\limsup_{l\to\infty} \int_0^{t_2} \int_\Omega \boldsymbol{G}(\theta_{k,l}, \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l} \,\mathrm{d}x\mathrm{d}t \le \int_0^{t_2} \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k \,\mathrm{d}x\mathrm{d}t. \tag{35}$$

For this purpose we want to test $(26)_1$ with $(\boldsymbol{u}_{k,l})_t$. Since it is not sufficiently regular, we use the standard mollifier η_ϵ in time and test with $\boldsymbol{\phi} = ((\boldsymbol{u}_{k,l})_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$. For the constitutive equation $(26)_3$ we use a similar test function $\boldsymbol{\Phi} = (\boldsymbol{T}_{k,l} * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$. After standard calculations and with ϵ tending to zero we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \boldsymbol{D}\big(\boldsymbol{\varepsilon}_{k,l} - \boldsymbol{\varepsilon}^p_{k,l}\big) : \big(\boldsymbol{\varepsilon}_{k,l} - \boldsymbol{\varepsilon}^p_{k,l}\big)\,\mathrm{d}x\Big|_{t_1}^{t_2} = \\ &\qquad -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{f}_t \cdot \boldsymbol{u}_{k,l}\,\mathrm{d}x\mathrm{d}t + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_{k,l}\,\mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}_k(\theta_{k,l}, \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{36}$$

for almost each time $t_1, t_2 \in (0,T)$. We repeat this calculation with test functions: $\boldsymbol{\phi} = ((\boldsymbol{u}_k)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ and $\boldsymbol{\Phi} = (\boldsymbol{T}_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$. Then we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \boldsymbol{D}\big(\boldsymbol{\varepsilon}_{k,l} - \boldsymbol{\varepsilon}^p_{k,l}\big) : \big(\boldsymbol{\varepsilon}_{k} - \boldsymbol{\varepsilon}^p_{k}\big)\,\mathrm{d}x\Big|_{t_1}^{t_2} = \\ &\qquad -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{f}_t \cdot \boldsymbol{u}_{k}\,\mathrm{d}x\mathrm{d}t + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_{k}\,\mathrm{d}x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}_k(\theta_{k,l}, \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{37}$$

The standard theory of maximal regularity for elliptic problems, see e.g. [27], allows to extend (36) and (37) to the initial time $t_1 = 0$. In the next step we pass to the limit with l in (36) and (37). Since $\int_\Omega \boldsymbol{D}\boldsymbol{X} : \boldsymbol{X}$ for any $\boldsymbol{X} \in \mathcal{S}^3$ defines a norm in the space $\mathcal{S}^3$, hence, using the lower semicontinuity of the norm we pass to the limit in the left hand side of (36). Consequently, (35) holds.

With help of Theorem 1.2 from [17] we get the strong convergence of the subsequences $\{\boldsymbol{T}_{k,l}\}$, $\{\boldsymbol{T}^d_{k,l}\}$ and $\{\boldsymbol{G}_k(\theta_{k,l}, \boldsymbol{T}_{k,l})\}$ as $l \to \infty$ and are able to characterize the limit of the sequence $\{\boldsymbol{G}_k(\theta_{k,l}, \boldsymbol{T}_{k,l})\}$, namely $\boldsymbol{\chi}_k = \boldsymbol{G}_k(\theta_k, \boldsymbol{T}_k)$ almost everywhere.

Finally, we pass to the limit with the second parameter - index of the Galerkin approximation in the displacement. The procedure with all terms related with the displacement follows the same way as above. The only difference appears in the convergence in temperature. The sequence $\{\theta_k\}$ is not uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$ because of the low regularity of the right hand side of the heat equation and we show the existence of θ only in the space $L^q(0,T,W^{1,q}(\Omega)) \cap C([0,T], W^{-s,2}(\Omega))$, where $q < \frac{5}{4}$ and s is large enough, what completes the proof.

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