Viscoelastic fluid model with nonhomogeneous boundary conditions

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Abstract

We deal with a viscoelastic fluid model in a rectangle domain with nontrivial velocity at the inflow and outflow area and slip boundary condition at the rest of the boundary. For this model we investigate the local in time existence of solutions. Then we present 2D simulations and we discuss the possibility of the derivation of 1D model by limit passage with the height of the rectangle to zero. Moreover, we present few 1D models that we support by simulations.

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1 Introduction

Mathematical modeling of biological fluids is not trivial due to their complex molecular structure. A good example is blood, consisting of particles like red and white cells suspended in the main component, namely in blood plasma. One can describe each component of blood separately. Plasma exhibits Newtonian response which can be covered by Navier-Stokes equations. The other components are described using transport equation. However, it leads to huge amount of equations which may not be easy to investigate. Then one has to perform simplifications. Usually the object of interest is not to investigate the motion of one particle, but general behaviour of the blood. In that case the blood is considered as a continuum [1], [2]. Many experiments [14], [15] where performed to investigate the blood flow. It appears that if the diameter of vessel is large, for example like in aorta, the influence of the largest component to the motion of fluid is small, and it is reasonable to treat blood as if it is made only from plasma. In this situation, it is possible to model flow by Navier-Stokes equations [11]. On the other hand, when the radius of vessel is small, the size of individual component of blood cannot be neglected. In this case, we observe non-Newtonian behaviour [1], [2], [14] such as shear thinning, elastic or stress relaxation.

We are interested to model flow in the 2D domain of small height which we consider to be the rectangle $\{\Omega_\epsilon\} = (0,1)\times(0,\epsilon)$ in $\mathbb{R}^2$, where the boundary of Ω_ϵ is a sum of $\Gamma = (0,1)\times\{0,\epsilon\}$, $B = B_1\cup B_2 = (\{0\}\times(0,\epsilon))\cup(\{1\}\times(0,\epsilon))$ and corners. As the model of viscoelastic fluid we use Oldroyd-type system [6] in $(t,x)\in(0;\infty)\times\Omega_\epsilon$

$$\begin{aligned}
&\partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \operatorname{div}(FF^T),\\
&\operatorname{div} v = 0,\\
&\partial_t F + v\cdot\nabla F = \nabla v F,
\end{aligned} \tag{1}$$

with initial data: $v(0,x) = v_0(x)$, $F(0,x) = F_0(x)$.
We denote the viscosity by μ and the pressure by p. Moreover, $v(t,x)$ is a 2-dimensional velocity field of fluid. The deformation gradient $F(t,x)$ is such a matrix that $\operatorname{div} F = 0$ and $\det F = 1$. Then, in 2D case we can introduce $\phi = (\phi_1,\phi_2)$ such that F is given by

$$F = \begin{pmatrix} -\partial_{x_2}\phi_1 & -\partial_{x_2}\phi_2, \\ \partial_{x_1}\phi_1 & \partial_{x_1}\phi_2. \end{pmatrix}.$$

We obtain the following system

$$\begin{aligned}
&\partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = -\textstyle\sum_{i=1}^2 \Delta\phi_i\nabla\phi_i,\\
&\operatorname{div} v = 0,\\
&\partial_t\phi + v\cdot\nabla\phi = 0,
\end{aligned} \tag{2}$$

with initial conditions

$$v(0,x) = v_0(x), \quad \phi(0,x) = \phi_0(x).$$

We suppose that at the part B_1 of boundary there is constant inflow of fluid and at the part B_2 there is constant outflow. Moreover, we assume that the fluid is not flowing through part Γ. The proper choice of boundary conditions is discussed in Section 2. We need to set conditions not only for velocity but also for deformation gradient. This is essential to obtain the energy inequality in Section 3 and estimates for transport equation. Moreover, in Section 2 we present the main result, Theorem 2.1, concerning existence of solutions.

As the proof is long and technical, we firstly present Section 4 containing 2D simulations showing behaviour of the fluid in rectangles of different height. Next, in Section 5, we discuss a limit passage with height of the rectangle to zero. However, the 1D model that is possible to obtain has a very simple structure. That is why, in Section 6, we generalise it assuming that the flux depends on the energy. Checking few different relations between flux and energy, we obtain different models. This result is supported by simulations showing the evolution of energy.

Finally we come back to the proof of existence results. Section 7 contains theorems about regularity of Stokes equation and transport equation. Moreover, we can find there estimates of the nonlinear terms. We use all this results in Section 8 to prove Theorem 2.1.

2 Statement of the main result

In this section, we discuss the choice of boundary and initial conditions for velocity and also for deformation gradient in the terms of $\nabla\phi$. In [5], [6] the whole space, periodic box or smooth domain with zero Dirichlet boundary conditions for velocity were considered. It caused that all boundary terms vanished, and it was not needed to give any boundary conditions for ϕ. However, we do not assume velocity to be zero at the boundary. Instead of this, we have constant inflow and outflow of fluid at two ends of domain, and a kind of slip boundary condition for the rest of boundary. In this case we have to describe boundary conditions for the deformation gradient. It is reasonable to consider it only in the inflow area, where we are able to control the data.

We start with precise description of boundary terms for velocity. Let us suppose that B_1 denotes the part of the boundary with constant inflow of fluid and B_2 the part with constant outflow. Moreover, we assume that fluid is not flowing through the part Γ. Then the boundary conditions are as follows

$$n \cdot v = 0, \quad n \cdot D(v) \cdot \tau = 0 \quad at \quad \Gamma \times (0,T), \quad v = v^p \quad at \quad B \times (0,T), \tag{3}$$

where $n = (n_1, n_2)$ is the outward normal vector and τ is the tangent vector. Moreover, $D(v) = \frac{1}{2}(\nabla v + \nabla v^T)$ is the symmetric velocity gradient. Function v^p denotes the basic flow, which in our case, is constant in time and space, however, it depends on the height of domain ϵ and the flux of blood, which we assume to be equal to one. Then, v^p is given by the formula $v^p = (v_1^p, 0) = (\frac{1}{\epsilon}, 0)$.

As velocity of fluid at the boundary is not equal to zero, we need to describe behaviour of ϕ at the boundary. However, we are allowed to consider it only in the inflow area B_1. Moreover, instead of ϕ, we are more interested to model its gradient. Then, we consider system of equations

$$\partial_t(\partial_{x_i}\phi_j) + v \cdot \nabla(\partial_{x_i}\phi_j) + \partial_{x_i}v_1\partial_{x_1}\phi_j + \partial_{x_i}v_2\partial_{x_2}\phi_j = 0, \tag{4}$$

for $i, j = 1, 2$, and we describe conditions for $\nabla\phi$ at the boundary B_1 as follows

$$\partial_{x_1}\phi_1 = \tfrac{1}{c}, \quad \partial_{x_1}\phi_2 = 0, \quad \partial_{x_2}\phi_1 = 0, \quad \partial_{x_2}\phi_2 = c, \tag{5}$$

where c is constant.

The fact that $v_1 \neq 0$ at the part of boundary B_1 gives us local existence of solution to problem (4) near inflow area via method of characteristic. Moreover, this solution is a C^2 function, thus ϕ is a C^3 function near the boundary B_1.

The assumption that at the inflow area $\partial_{x_1}\phi_2 = \partial_{x_2}\phi_1 = 0$ together with special choice of initial condition,

$$\partial_{x_1}\phi_2(0, x_1, x_2) = \partial_{x_2}\phi_1(0, x_1, x_2) = 0, \tag{6}$$

causes that we are able to obtain energy inequality (14). Further discussion about relaxation of these conditions is contained in Section 3.

Our choice of $\nabla\phi$ in the inflow area is compatible with the condition $\det F = 1$, what in the language of ϕ means that following equation is satisfied

$$\partial_{x_1}\phi_1\partial_{x_2}\phi_2 - \partial_{x_2}\phi_1\partial_{x_1}\phi_2 = 1. \tag{7}$$

Moreover, using condition (7), boundary terms (12) and transport equation (4) for $\nabla\phi$, we are able to calculate or at least estimate higher derivatives of ϕ at the inflow area. We need it in Section 7 to prove $L^\infty(0,T;W_2^3(\Omega))\cap W_2^1(0,T;W_2^2(\Omega))$ estimates of ϕ in Lemma 7.2. All second order space derivatives of ϕ at the boundary B_1 are equal to zero except of

$$\partial_{x_1}\partial_{x_1}\phi_2 = -\frac{c}{v_1}\partial_{x_1}v_2. \tag{8}$$

What is more, all third order derivatives of ϕ at the boundary B_1 are equal to zero except of

$$\partial_{x_1}^2\partial_{x_2}\phi_2 = -\tfrac{c}{v_1}\partial_{x_2}\partial_{x_1}v_2, \quad \partial_{x_1}^2\phi_1 = -\tfrac{1}{cv_1}\partial_{x_1}^2 v_1, \quad \partial_{x_1}^2\phi_2 = \tfrac{c}{v_1^2}\partial_t\partial_{x_1}v_2 - \tfrac{c}{v_1}\partial_{x_1}^2 v_2. \tag{9}$$

Now, we change system (2) by introducing a new function $w = v - v^p$ to obtain problem for velocity with zero Dirichlet boundary conditions at B.

The new system for w is given by

$$\begin{array}{l} \partial_t w + w\cdot\nabla w - \mu\Delta w + \nabla p + \frac{1}{\epsilon}\partial_{x_1}w = -\sum_{i=1}^2 \Delta\phi_i\nabla\phi_i, \\ \operatorname{div} w = 0, \\ \partial_t\phi + w\cdot\nabla\phi + \frac{1}{\epsilon}\partial_{x_1}\phi = 0, \end{array} \tag{10}$$

with boundary conditions

$$w_2 = 0, \quad \partial_{x_2}w_1 = 0 \quad \text{at} \quad \Gamma\times(0,T), \quad w = 0 \quad \text{at} \quad B\times(0,T), \tag{11}$$

$$\partial_{x_1}\phi_1 = \tfrac{1}{c}, \quad \partial_{x_1}\phi_2 = 0, \quad \partial_{x_2}\phi_1 = 0, \quad \partial_{x_2}\phi_2 = c, \quad \text{at} \quad B_1\times(0,T) \tag{12}$$

and initial conditions

$$w = w_0, \quad \phi = \phi_0 \quad \text{for} \quad t = 0. \tag{13}$$

We require that $\partial_{x_1}\phi_2(0,x_1,x_2) = \partial_{x_2}\phi_1(0,x_1,x_2) = 0$ at the part of boundary Γ.

For this problem we formulate the main theorem about the existence of local in time solutions.

Theorem 2.1 *Let us assume that $w_0 \in W_2^2(\Omega)$ and $\phi_0 \in W_2^3(\Omega)$. Then there exists time $T > 0$ such that there exists a solution to the problem (10-13) on $(0,T)$ satisfying $w \in W_2^{3,\frac{3}{2}}(\Omega\times(0,T))\cap L^\infty(0,T;W_2^2(\Omega))$, $\phi \in L^\infty(0,T;W_2^3(\Omega))\cap W_2^1(0,T;W_2^2(\Omega))$.*

In [6] the same regularity for the initial conditions was needed to obtain existence result. This result was improved in [5] where the lower regularity for $\phi_0 \in W_p^2$ were considered. The main difference in Theorem 2.1 is that for bounded domain we consider different boundary conditions then in [5] and [6] what complicates the proof.

We divide the proof into two parts. The first one, Section 7, contains regularity result for nonstationary Stokes problem and estimates for nonlinear parts. The last one, Section 8, contains the proper proof which is based on the Banach fixed point theorem.

3 Energy estimates

In this section, we show that for smooth solutions of problem (10), the energy inequality holds if we choose proper boundary and initial conditions for ϕ. The problem appears because the boundary conditions for velocity are not zero. This produces the additional part in the right-hand side of inequality, namely: $-\int_{(0,t)}\int_\Gamma \sum_{i=1}^2 (n\nabla\phi_i)(w\cdot\nabla\phi_i)d\sigma d\tau$. We, however, are able to remove it due to our choice of boundary conditions (12) and initial conditions (6). In fact, there is larger class of conditions that makes this part disappearing, as it is enough to assume that $\partial_{x_1}\phi_2(0,x_1,x_2) = \partial_{x_2}\phi_1(0,x_1,x_2) = 0$ at the part of boundary Γ, and $\partial_{x_1}\phi_2(t,x) = \partial_{x_2}\phi_1(t,x) = 0$ in the points $\bar\Gamma \cap \bar B_1$. Equivalently, instead of $\partial_{x_1}\phi_2$, $\partial_{x_2}\phi_1$ we could consider $\partial_{x_1}\phi_1$ and $\partial_{x_2}\phi_2$.

Theorem 3.1 *Let us assume that solution of problem (10) is sufficiently smooth with initial conditions (6) and boundary conditions (12), then it fulfils the following inequality*

$$\begin{aligned}\int_{\Omega_\epsilon} \tfrac{1}{2}|w(t)|^2 + \tfrac{1}{2}|\nabla\phi(t)|^2 + \int_{(0,t)}|\nabla w|^2 d\tau dx \quad &\leq \quad \int_{\Omega_\epsilon}\tfrac{1}{2}|w(0)|^2 + \tfrac{1}{2}|\nabla\phi(0)|^2 dx \\ +\tfrac{1}{2\epsilon}\int_{(0,t)}\int_{(0,\epsilon)} |\nabla\phi(\tau,0,x_2)|^2 dx_2 d\tau. &\end{aligned} \tag{14}$$

Proof. Firstly, we differentiate the last equation of problem (10) and obtain

$$\partial_t\nabla\phi + \nabla(w\cdot\nabla\phi) + \frac{1}{\epsilon}\partial_{x_1}\nabla\phi = 0. \tag{15}$$

Now, we take the first equation of problem (10), multiply it by w and integrate over space variables

$$\int_{\Omega_\epsilon} \frac{1}{2}\partial_t|w|^2 + \frac{1}{2}w\cdot\nabla|w|^2 + \mu|\nabla w|^2 + \nabla p w + \frac{1}{2\epsilon}\partial_{x_1}|w|^2 dx = -\int_{\Omega_\epsilon}\sum_{i=1}^2 \Delta\phi_i w\cdot\nabla\phi_i dx.$$

By integration by parts, boundary conditions, and divergence free condition for w, we obtain that the second, forth and fifth terms are equal to zero. At last, we integrate by parts the right-hand side what produces additionally boundary term which in general may not vanish.

$$-\int_{\Omega_\epsilon}\sum_{i=1}^2 \Delta\phi_i w\cdot\nabla\phi_i dx = \int_{\Omega_\epsilon}\sum_{i=1}^2 \nabla\phi_i\nabla\left(w\cdot\nabla\phi_i\right)dx - \int_{\partial\Omega_\epsilon}\sum_{i=1}^2 \left(n\nabla\phi_i\right)\left(w\cdot\nabla\phi_i\right)d\sigma.$$

Here, we use the transport equation to the gradient of ϕ. Then, the part

$$\begin{aligned}\int_{\Omega_\epsilon}\sum_{i=1}^2 \nabla\phi_i\nabla\left(w\cdot\nabla\phi_i\right)dx &= -\int_{\Omega_\epsilon}\sum_{i=1}^2 \nabla\phi_i\left(\partial_t\nabla\phi_i + \frac{1}{\epsilon}\partial_{x_1}\nabla\phi_i\right)dx \\ &= -\int_{\Omega_\epsilon}\sum_{i=1}^2\left(\frac{1}{2}\partial_t|\nabla\phi_i|^2 + \frac{1}{2\epsilon}\partial_{x_1}\left((\partial_{x_1}\phi_i)^2 + (\partial_{x_2}\phi_i)^2\right)\right)\end{aligned}$$

$$\begin{aligned}= -\int_{\Omega_\epsilon}\sum_{i=1}^2 \frac{1}{2}\partial_t|\nabla\phi_i|^2 dx - \frac{1}{2\epsilon}\Bigg(&\int_{(0,\epsilon)}\sum_{i=1}^2\left((\partial_{x_1}\phi_i(t,1,x_2))^2 + (\partial_{x_2}\phi_i(t,1,x_2))^2\right)dx_2 \\ &- \int_{(0,\epsilon)}\sum_{i=1}^2\left((\partial_{x_1}\phi_i(t,0,x_2))^2 + (\partial_{x_2}\phi_i(t,0,x_2))^2\right)dx_2\Bigg).\end{aligned}$$

At the end, we obtain

$$\int_{\Omega_\epsilon}\frac{1}{2}\partial_t|w|^2 + \frac{1}{2}\partial_t|\nabla\phi|^2 + |\nabla w|^2 dx + \frac{1}{2\epsilon}\left(\int_{(0,\epsilon)}|\nabla\phi(t,1,x_2)|^2 dx_2\right.$$

$$
-\int_{(0,\epsilon)} |\nabla\phi(t,0,x_2)|^2 dx_2\Big) = -\int_{\partial\Omega_\epsilon} \sum_{i=1}^{2} (n\nabla\phi_i)(w\cdot\nabla\phi_i)\, d\sigma
$$

$$
= -\int_{\Gamma} \sum_{i=1}^{2} (n\nabla\phi_i)(w\cdot\nabla\phi_i)\, d\sigma,
$$

what after integrating over time gives energy inequality

$$
\int_{\Omega_\epsilon} \frac{1}{2}|w(t)|^2 + \frac{1}{2}|\nabla\phi(t)|^2 + \int_{(0,t)} |\nabla w|^2 d\tau dx \le \int_{\Omega_\epsilon} \frac{1}{2}|w(0)|^2 + \frac{1}{2}|\nabla\phi(0)|^2 dx
$$

$$
+\frac{1}{2\epsilon}\int_{(0,t)}\int_{(0,\epsilon)} |\nabla\phi(t,0,x_2)|^2 dx_2 d\tau - \int_{(0,t)}\int_{\Gamma} \sum_{i=1}^{2} (n\nabla\phi_i)(w\cdot\nabla\phi_i)\, d\sigma d\tau.
$$

Now, we prove that our choice of boundary and initial conditions for ϕ makes the boundary term to vanish. We obtain this by method of characteristic. Firstly, we simplify that term using boundary conditions for velocity

$$
\int_{\Gamma} \sum_{i=1}^{2} (n\nabla\phi_i)(w\cdot\nabla\phi_i)\, d\sigma = \int_{\Gamma} \sum_{i=1}^{2} n_2 (\partial_{x_2}\phi_i)(w_1\partial_{x_1}\phi_i)\, d\sigma.
$$

Then, we consider the transport equations at the part of boundary Γ for ϕ

$$
\partial_t\phi + w\cdot\nabla\phi + v^p\cdot\nabla\phi = 0,
$$

and for the derivative of ϕ

$$
\partial_t\partial_{x_j}\phi + \partial_{x_j}w\cdot\nabla\phi + w\cdot\nabla\partial_{x_j}\phi + v^p\cdot\nabla\partial_{x_j}\phi = 0,
$$

for $j=1,2$.

As $x\in\Gamma$, then $w_2=0$, $\partial_{x_1}w_2=0$ and $\partial_{x_2}w_1=0$ thus the transport equations simplifies to

$$
\partial_t\phi + \left(w_1+\frac{1}{\epsilon}\right)\partial_{x_1}\phi = 0,
$$

$$
\partial_t\left(\partial_{x_j}\phi\right) + \left(w_1+\frac{1}{\epsilon}\right)\partial_{x_1}\left(\partial_{x_j}\phi\right) = -\partial_{x_j}w_j\left(\partial_{x_j}\phi\right),
$$

for $j=1,2$.

By the method of characteristic, the every characteristic line $\mathbf{x}(s)$ that has a point in the set $\mathbb{R}\times\Gamma\times\mathbb{R}$, lies in that set. Moreover, solution ϕ is constant on the characteristic line and the gradient of ϕ, along $\mathbf{x}$ satisfies following equations

$$
\partial_{x_j}\phi\left(\mathbf{x}(s)\right) = \partial_{x_j}\phi\left(\mathbf{x}(0)\right) e^{-\int_0^s \partial_{x_j}w_j(\mathbf{x}(\tau))d\tau},
$$

for $j=1,2$. We use it to calculate the part at the line $\mathbf{x}$

$$
\sum_{i=1}^{2} n_2(\partial_{x_2}\phi_i)(w_1\partial_{x_1}\phi_i)(\mathbf{x}(s))
$$

$$
= \sum_{i=1}^{2} n_2\left(\partial_{x_2}\phi_i(\mathbf{x}(0))e^{-\int_0^s \partial_{x_2}w_2(\mathbf{x}(\tau))d\tau}\right)\left(w_1(\mathbf{x}(s))\partial_{x_1}\phi_i(\mathbf{x}(0))e^{-\int_0^s \partial_{x_1}w_1(\mathbf{x}(\tau))d\tau}\right).
$$

Then, due to our choice of boundary conditions (12) and initial conditions (6) we have $\partial_{x_2}\phi_1(\mathbf{x}(0)) = \partial_{x_1}\phi_2(\mathbf{x}(0)) = 0$ and in consequence this part disappears.
□

4 2D simulations

Here we present 2D simulations to the viscous approximation of equation (1), it is to the model

$$\begin{aligned} &\partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = \operatorname{div}(FF^T), \\ &\operatorname{div} v = 0, \\ &\partial_t F - k\Delta F + v \cdot \nabla F = \nabla v F, \end{aligned} \tag{16}$$

for k being a small constant. Our point of interest is the change of the velocity and energy, given by the formula $S = F : F$, in domains of different height.

Firstly, we set the parameters and boundary conditions. We choose μ being equal to 0.01 and $k = 0.0001$. For the velocity we use boundary conditions (3) choosing $v_p = 1$. Moreover, we assign function F at the inflow area to be equal to $F_{11} = 2$, $F_{22} = 0.5$ and $F_{12} = F_{21} = 0$. As domains we use rectangles of width equal to 5 and height equal to 1, 0.6 and 0.3 respectively. We used program FEniCS to perform calculations on a mesh containing 32×160 triangles. Here we present energy S and velocity v for time $t = 0.5$ and $t = 2.5$.

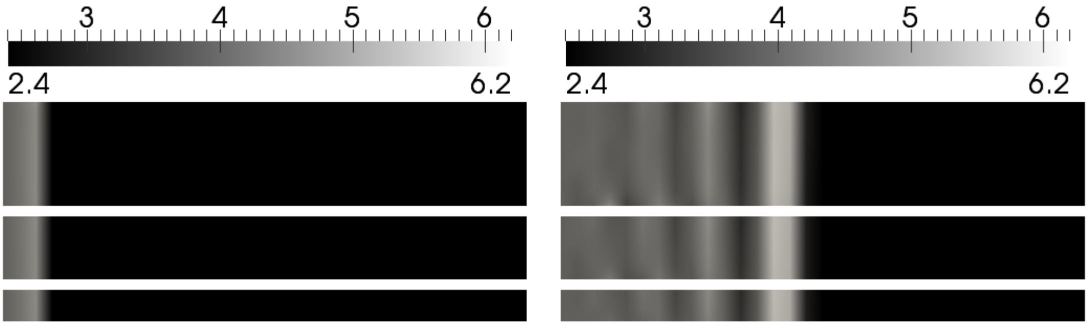

Figure 1: Energy S in time $t = 0.5$ and $t = 2.5$

Figure 1 shows energy S in time $t = 0.5$ and $t = 2.5$. We observe an increasing main wave transported in time followed by smaller waves. That phenomena occurs in all rectangles. Figure 2 shows visualization of first component of velocity vector v. In time

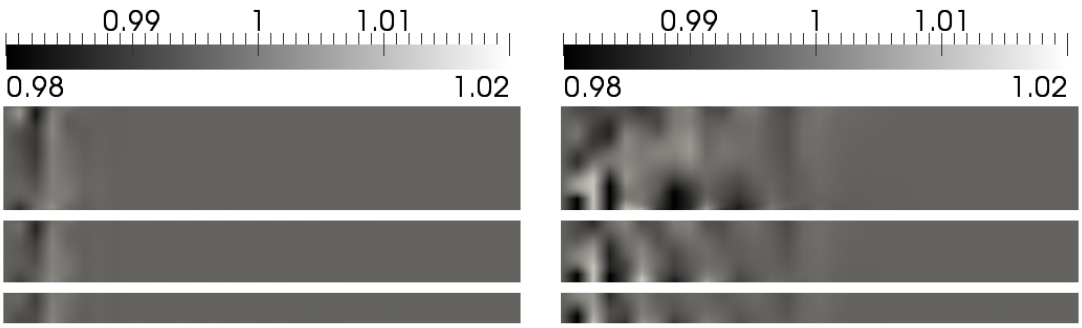

Figure 2: The first component of velocity v in time $t = 0.5$ and $t = 2.5$

$t = 0.5$ we observe small perturbations that are propagated with time. Moreover, the size of the perturbations increases. Here the first component of velocity changes from 0.98 to 1.02. The second component of velocity is presented at Figure 3. We observe

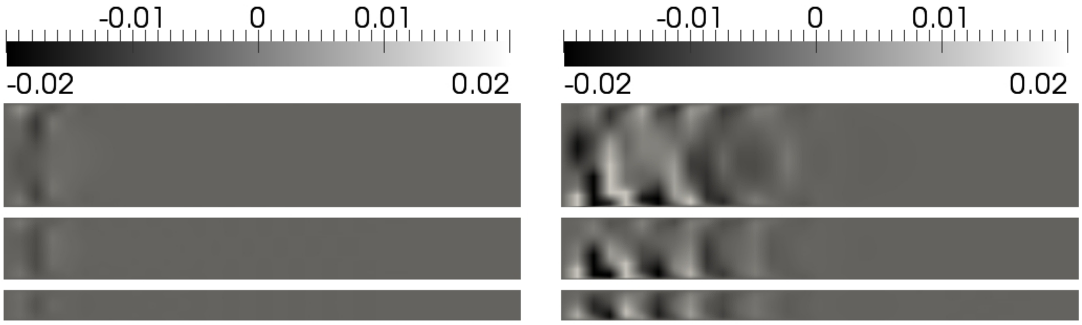

Figure 3: The second component of velocity v in time $t = 0.5$ and $t = 2.5$

here fluctuations that have similar sizes as the perturbations of first component of velocity. They occurs in all rectangles but we observe that there are smaller in the thinnest one.

Those simulations suggest that the fluid behaves in a similar way in the rectangles of different height. Thus there is a hope that after limit passage with the height of domain to zero, we would obtain a 1D model that is able to describe that fluid.

5 Limit model

In this section, we present a discussion about limit passage with the height of domain to zero. We start with 2D model (10)

$$\begin{array}{l} \partial_t w_\epsilon + w_\epsilon \cdot \nabla w_\epsilon - \mu \Delta w_\epsilon + \nabla p_\epsilon + v_{p,\epsilon} \partial_{x_1} w_\epsilon = -\sum_{i=1}^{2} \Delta \phi_{i,\epsilon} \nabla \phi_{i.\epsilon}, \\ \operatorname{div} w_\epsilon = 0, \\ \partial_t \phi_\epsilon + w_\epsilon \cdot \nabla \phi_\epsilon + v_{p,\epsilon} \partial_{x_1} \phi_\epsilon = 0, \end{array} \tag{17}$$

with boundary conditions satisfying (24) and (12). The index ϵ means here the dependence on the height of domain. We recall that $v_{p,\epsilon}$ denotes a constant flow in direction x_1 and is given by the formula $v_{p,\epsilon} = \frac{flux}{\epsilon}$. In contradiction to the situation in Section 2, we do not assume that the flux is equal to one, but that it depends on ϵ. Moreover, we assume that $v_{p,\epsilon} \to v_p$ as $\epsilon \to 0$.

For fixed ϵ, due to Theorem 2.1, system (17) possesses local in time solution. Unfortunately, the time T^*_ϵ of existence of solution depends on ϵ and it may happen that when $\epsilon \to 0$, the $T^*_\epsilon \to 0$. However, we assume that it is not the case, and there exists $T^* > 0$, such that $T^* < T^*_\epsilon$ for all ϵ.

From energy inequality (14), we obtain that L^2 norm of ∇w_ϵ is bounded by initial conditions and time. Now, we assume that the norm of initial conditions is bounded independently from ϵ. It gives us

$$\|\nabla w_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))} < K + \tilde{c} T^*, \tag{18}$$

for all ϵ. Then, we use the Poincaré inequality, to obtain

$$\|w_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))} \leq \epsilon \|\nabla w_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))} < \epsilon (K + \tilde{c} T^*).$$

Passing with ϵ to zero we obtain that $\|w_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))} \to 0$, and the limit velocity $\bar{w}$ is equal to zero.

Again, from the energy inequality (14) and estimates for transport equation, we obtain that the L^2 norm of $\nabla\phi_\epsilon$ and $\partial_t\phi_\epsilon$ is bounded independently on ϵ and, in consequence, there exists subsequence that $\nabla\phi_\epsilon$ and $\partial_t\phi_\epsilon$ converge weakly to $\nabla\phi$ and $\partial_t\phi$ respectively. Then, we consider the transport equation

$$\partial_t\phi_\epsilon + w_\epsilon \cdot \nabla\phi_\epsilon + v_{p,\epsilon}\partial_{x_1}\phi_\epsilon = 0.$$

The $\|w_\epsilon \cdot \nabla\phi_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))}$ is going to zero with ϵ proceeding to zero, due to the fact that $\|w_\epsilon\|_{L^2(0,T^*;L^2(\Omega_\epsilon))} \to 0$. The behaviour of the last part of this equation, however, depends on the $v_{p,\epsilon} = \frac{flux}{\epsilon} \to v_p$. It follows, that the limit model for transport equation is

$$\partial_t\phi + v_p\partial_{x_1}\phi = 0. \tag{19}$$

Equivalently, we consider the transport of energy S_ϵ, defined by

$$S_\epsilon = (\partial_{x_1}\phi_{\epsilon,1})^2 + (\partial_{x_1}\phi_{\epsilon,2})^2 + (\partial_{x_2}\phi_{\epsilon,1})^2 + (\partial_{x_2}\phi_{\epsilon,2})^2, \tag{20}$$

satisfying equation

$$\partial_t S_\epsilon + w_\epsilon \cdot \nabla S_\epsilon + v_{p,\epsilon}\partial_{x_1}S_\epsilon = -2\sum_{i,j=1}^{2} \partial_{x_j} w_\epsilon \nabla\phi_{i,\epsilon}\partial_{x_j}\phi_{i,\epsilon}.$$

The limit model for S is given by transport equation

$$\partial_t S + v_p\partial_{x_1}S = 0. \tag{21}$$

Models (19) and (21) have very simple structure and it is easy to find the solution by the method of characteristics.

6 Generalized 1D model and simulations

We would like to generalize 1D models obtained in Section 5. The main idea is to code in v_p the information about the shape of 2D domain. We assume that the 2D domain has the constant length, but its height I depends on the energy S. Moreover, we assume that blood flows only in the vertical direction, and the 1D velocity $v_p(t,x_1) \approx \frac{flux}{I}$ depends only on the height of 2D domain in the point (t,x_1). In fact, we obtain that v_p is the function of energy S. The most challenging task is to find the dependence between energy and the height of domain I. We postulate that when the height I goes to zero or to infinity then, the energy tends to infinity. Then, assuming the height I to be small, we generalize model (21) obtained in Section 5 by assumption that v_p is not constant, but depends on the energy S. We present simulations for different choice of function $v_p(S)$, namely, linear, quadratic or exponential function.

We consider the model

$$\partial_t S + v_p(S)\partial_{x_1}S = 0, \tag{22}$$

where $v_p(S)$ is a given function. Due to the method of characteristics, the solution S is constant along characteristic lines, which are straight and directed according to the initial value of S. That is why our point of interest is not to investigate the change of S, but to compare the evolution of domain under different choice of function v_p. Thus, we fix the initial shape to be given by the function $I_0(x) = 0.1 + 0.05x\sin(3\pi x)$ for $x \in (0,1)$. For different choice of v_p, namely $v_p(S) = S$, $v_p(S) = S^2$ and $v_p(S) = e^S$, we calculate the initial condition for S under assumption that $v_p(S) = \frac{1}{I}$. Then we solve in Octave the equation (22) and calculate the evolution of the shape I for the particular time steps. Following pictures illustrates the change of domain in time for different v_p. All the picture are drown in a rectangle $(0,2) \times (0,0.2)$. We observe that for each model, the shape of domain is not only transported in direction x, but it is also changing with time. The area, where function I is decreasing, is expanding in time. On the other hand, the area, where function I is increasing, decreases with time. This, together with the fact that maximal and minimal value of I is constant in time, lead to a steep hump.

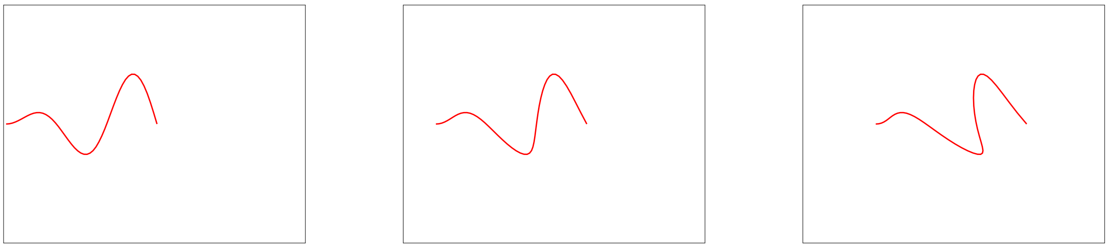

Figure 4: The shape of domain for $v_p = S$ in time $t = 2, 14, 30$

Figure 4 shows the shape of domain under assumption that $v_p = S$ for time step equal to 2, 14 and 30. We observe that for time 14 the graph of I begins to be vertical and in time 30 it even turns back for a while, what may be interpreted as the place, where the vessel ruptures or where an aneurism is created.

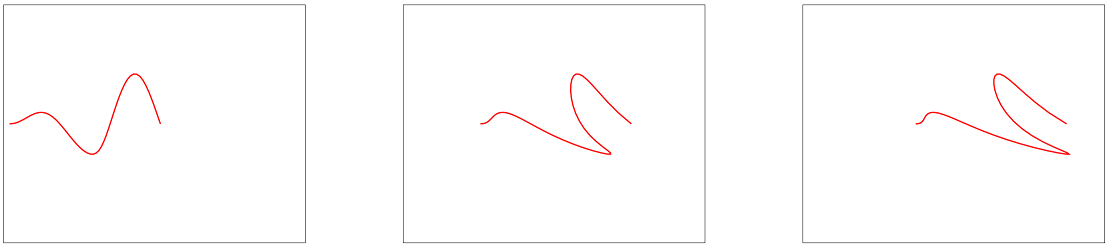

Figure 5: The shape of domain for $v_p = S^2$ in time $t = 2, 14, 20$

Figure 5 shows the shape of domain in case where $v_p = S^2$ for time step equal to 2, 14 and 20. Here we observe that the change of domain is much faster and in the time 14, graph is no longer a function.
Figure 6 shows the shape of domain under assumption that $v_p = e^S$ for time step equal to 2, 14 and 30. Here the evolution of the domain is similar to the one that is presented in the Figure 4.
Our future point of interest is to investigate the free boundary problem for viscoelastic fluid. We would like to check the existence of solutions and to do the limit passage with height of domain to zero. The interesting thing would be to compare the new model with the models presented in this section.

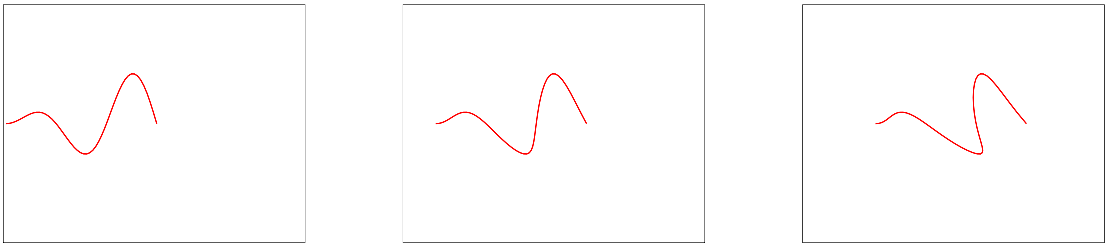

Figure 6: The shape of domain for $v_p = e^S$ in time $t = 2, 14, 30$

7 Regularity results for linear problems

In this section, we present regularity results and estimates that are needed to prove local in time existence of solutions to problem (10) in a box.

We start with the Stokes problem

$$\begin{aligned} &\partial_t w - \mu\Delta w + \nabla p = F, \\ &\operatorname{div} w = 0, \end{aligned} \tag{23}$$

in a rectangle Ω with boundary conditions

$$w_2 = 0, \quad \partial_{x_2} w_1 = 0 \quad \text{at} \quad \Gamma \times (0,T), \quad w = 0 \quad \text{at} \quad B \times (0,T). \tag{24}$$

The slip boundary conditions at the part Γ allow us to use the method of symmetry with respect to Γ. In this way we transform the problem in Ω to the system in the larger rectangle. We localise it and obtain the same regularity results as in the case of Stokes problem in infinite strip with zero Dirichlet boundary conditions. Then, we present well-known regularity result for nonstationary Stokes equation in a halfspace, that also holds in the case where domain is a strip. We consider

$$\begin{aligned} &\partial_t w - \mu\Delta w + \nabla p = F \\ &\operatorname{div} w = 0, \end{aligned} \tag{25}$$

in a halfspace with zero Dirichlet boundary conditions and w vanishing at the infinity.

Theorem 7.1 *If the pair (w,p) is the solution of problem (25) in $\mathbb{R}^2_+$ with initial condition $w_0(x) = w(0,x) \in W^1_2(\mathbb{R}^2_+)$, and $F \in L^2(0,T;L^2(\mathbb{R}^2_+))$, then $\nabla^2 w \in L^2(0,T;L^2(\mathbb{R}^2_+))$, $\nabla w \in L^\infty(0,T;L^2(\mathbb{R}^2_+))$, $\partial_t w \in L^2(0,T;L^2(\mathbb{R}^2_+))$, $\nabla p \in L^2(0,T;L^2(\mathbb{R}^2_+))$. Moreover, the following inequality holds true*

$$\begin{aligned} &\|\nabla^2 w\|_{L^2(0,T;L^2(\mathbb{R}^2_+))} + \|\partial_t w\|_{L^2(0,T;L^2(\mathbb{R}^2_+))} + \|\nabla p\|_{L^2(0,T;L^2(\mathbb{R}^2_+))} \\ &\leq c\left(\|F\|_{L^2(0,T;L^2(\mathbb{R}^2_+))} + \|w_0\|_{W^1_2(\mathbb{R}^2_+)}\right). \end{aligned}$$

Theorem 7.2 *If the pair (w,p) is the solution of problem (25) with initial condition $w_0(x) = w(0,x) \in W^2_2(\mathbb{R}^2_+)$, and $F \in W_2^{1,\frac{1}{2}}(\mathbb{R}^2_+ \times (0,T))$, then $v \in W_2^{3,\frac{3}{2}}(\mathbb{R}^2_+ \times (0,T))$, $\nabla^2 p \in W_2^{0,\frac{1}{2}}(\mathbb{R}^2_+ \times (0,T))$. Moreover, the following inequality holds true*

$$\|w\|_{W_2^{3,\frac{3}{2}}(\mathbb{R}^2_+\times(0,T))} + \|\nabla^2 p\|_{W_2^{0,\frac{1}{2}}(\mathbb{R}^2_+\times(0,T))} \leq c\left(\|F\|_{W_2^{1,\frac{1}{2}}(\mathbb{R}^2_+\times(0,T))} + \|w_0\|_{W^2_2(\mathbb{R}^2_+)}\right).$$

The following corollary shows estimates for convective therm that are needed in Section 8 to prove existence of solution to nonlinear problem.

Corollary 7.1 *Let us assume that* $u \in W_2^{3,\frac{3}{2}}(\Omega\times(0,T))\cap L^\infty(0,T;W_2^2(\Omega))$*, then* $u\cdot\nabla u \in W_2^{1,\frac{1}{2}}(\Omega\times(0,T))$*. Moreover, following inequality is obtained*

$$\|u\cdot\nabla u\|_{W_2^{1,\frac{1}{2}}(\Omega\times(0,T))} \le cT^\gamma \|u\|_{W_2^{3,\frac{3}{2}}(\Omega\times(0,T))} \|u\|_{L^\infty(0,T;W_2^2(\Omega))}$$

for some $\gamma > 0$.

Proof. Firstly, we estimate the $L^2(0,T;L^2(\Omega))$ norm of expression $u\cdot\nabla u$.

$$\|u\cdot\nabla u\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|u\|_{L^\infty(0,T;L^4)}\|\nabla u\|_{L^\infty(0,T;L^4)}.$$

Then, we calculate the $L^2(0,T;L^2(\Omega))$ norm of $\nabla(u\cdot\nabla u)$.

$$\begin{aligned}\|\nabla(u\cdot\nabla u)\|_{L^2(0,T;L^2)} &\le \|u\cdot\nabla\left(\nabla u\right)\|_{L^2(0,T;L^2)} + \|\nabla u\nabla u\|_{L^2(0,T;L^2)}\\ &\le \|\nabla u\|^2_{L^4(0,T;L^4)} + \|u\|_{L^\infty(0,T;L^\infty)}\|\nabla^2 u\|_{L^2(0,T;L^2)}\\ &\le T^{\frac{1}{2}}(c\|\nabla u\|^2_{L^\infty(0,T;W_2^1)} + \|u\|_{L^\infty(0,T;L^\infty)}\|\nabla^2 u\|_{L^\infty(0,T;L^2)}).\end{aligned}$$

At last, we estimate the half derivative of u with respect to time.

$$\begin{aligned}&\int_\Omega\int_0^T\int_0^T \frac{|u(t)\cdot\nabla u(t) - u(t')\cdot\nabla u(t')|^2}{|t-t'|^2}dtdt'dx\\ &= \int_\Omega\int_0^T\int_0^T \frac{|u(t)\cdot\nabla(u(t)-u(t')) - (u(t)-u(t'))\cdot\nabla u(t')|^2}{|t-t'|^2}dtdt'dx\\ &\le \|u\|^2_{L^\infty(0,T;L^\infty)}\int_\Omega\int_0^T\int_0^T \frac{|\nabla u(t)-\nabla u(t')|^2}{|t-t'|^2}dtdt'dx\\ &+\|\nabla u\|^2_{L^\infty(0,T;L^4)}\left(c\int_\Omega\int_0^T\int_0^T \frac{|\nabla u(t)-\nabla u(t')|^2}{|t-t'|^2}dtdt'dx\right.\\ &\left.+\delta\int_\Omega\int_0^T\int_0^T \frac{|u(t)-u(t')|^2}{|t-t'|^2}dtdt'dx\right).\end{aligned}$$

To get the final result we use Ladyzhenskaya's inequality. Thus, we indeed obtain the inequality

$$\|u\cdot\nabla u\|_{W_2^{1,\frac{1}{2}}(\Omega\times(0,T))} \le cT^\gamma \|u\|_{W_2^{3,\frac{3}{2}}(\Omega\times(0,T))} \|u\|_{L^\infty(0,T;W_2^2(\Omega))}$$

for some $\gamma > 0$.
□

Now, we present regularity results for the transport equation

$$\partial_t\phi + v\cdot\nabla\phi = 0 \tag{26}$$

in the rectangle Ω, with initial conditions (6) and boundary conditions (12) at B_1. Moreover, we assume that $v \in W_2^{3,\frac{3}{2}}(\Omega\times(0,T))$ is a divergence free function and at the boundary satisfies

$$v_2 = 0, \quad \partial_{x_2}v_1 = 0 \quad \text{at} \quad \Gamma, \quad v = v^p \quad \text{at} \quad B, \tag{27}$$

Theorem 7.3 *We consider equation (26) with initial condition $\phi_0 \in W_2^3(\Omega)$, then $\nabla\phi\Delta\phi \in W_2^{1,\frac{1}{2}}(\Omega \times (0,T))$. Moreover, for some $\gamma > 0$ the following inequality is satisfied*

$$\|\nabla\phi\Delta\phi\|_{W_2^{1,\frac{1}{2}}(\Omega\times(0,T))} \leq cT^\gamma \|\phi\|_{L^\infty(0,T;W_2^3(\Omega))} \|\partial_t\phi\|_{L^2(0,T;W_2^2(\Omega))}. \tag{28}$$

Before proving Theorem 7.3, we present regularity result for transport equation.

Corollary 7.2 *Let ϕ be a solution of equation (26) with initial condition $\phi_0 \in W_2^3(\Omega)$. Then the solution $\phi \in L^\infty(0,T;W_2^3(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))$.*

Proof. We present a priori estimates for transport equation (26).
Firstly, we test equation (26) by ϕ. Then, for $0 < T < \infty$, we have

$$\int_\Omega |\phi(T)|^2 dx \leq \int_\Omega |\phi(0)|^2 dx + c\|\nabla v\|_{L^2(0,T;L^\infty)} + \delta \int_0^T |\int_\Omega |\phi|^2 dx|^2 dt.$$

Thus, we obtain

$$\|\phi\|_{L^\infty(0,T;L^2)} \leq c\left(\|\phi(0)\|_{L^2(\Omega)}, \|\nabla v\|_{L^2(0,T;L^\infty)}\right). \tag{29}$$

Next, we take a gradient of equation (26), test by $\nabla\phi$

$$\frac{1}{2}\int_\Omega |\nabla\phi(T)|^2 dx = \frac{1}{2}\int_\Omega |\nabla\phi(0)|^2 dx - \int_0^T \int_\Omega \nabla v \cdot \nabla\phi\nabla\phi dxdt - \int_0^T \int_\Omega v \cdot \nabla\left(\nabla\phi\right)\nabla\phi dxdt,$$

and estimate it in the following way

$$\frac{1}{2}\int_\Omega |\nabla\phi(T)|^2 dx \leq \frac{1}{2}\int_\Omega |\nabla\phi(0)|^2 dx + c\|\nabla v\|^2_{L^2(0,T;L^\infty)} + \delta\|\nabla\phi\|^4_{L^4(0,T;L^2)} + \frac{1}{\epsilon}\|\nabla\phi\|^2_{L^2((0,T)\times B_1)}.$$

The last term appears as a result of integration by parts of the term $\int_0^T\int_\Omega v\cdot\nabla\left(\nabla\phi\right)\nabla\phi dxdt$, and, due to condition (12), the norm $\|\nabla\phi\|^2_{L^2((0,T)\times B_1)}$ is equal to $c_1\epsilon T$. Thus, we obtain

$$\|\nabla\phi\|_{L^\infty(0,T;L^2)} \leq c\left(\|\nabla\phi(0)\|_{L^2(\Omega)}, \|\nabla v\|_{L^2(0,T;L^\infty)}, T\right). \tag{30}$$

Next, we take a Laplacian of equation (26), and test it by $\Delta\phi$.

$$\frac{1}{2}\int_\Omega |\Delta\phi(T)|^2 dx - \frac{1}{2}\int_\Omega |\Delta\phi(0)|^2 dx$$

$$= -\int_0^T \int_\Omega \Delta v \cdot \nabla\phi\Delta\phi dxdt - 2\int_0^T \int_\Omega \nabla v \cdot \nabla\left(\nabla\phi\right)\Delta\phi dxdt - \int_0^T \int_\Omega v \cdot \nabla\left(\Delta\phi\right)\Delta\phi dxdt.$$

We estimate this as follows

$$\frac{1}{2}\int_\Omega |\Delta\phi(T)|^2 dx - \frac{1}{2}\int_\Omega |\Delta\phi(0)|^2 dx \leq c\|\Delta v\|^2_{L^2(0,T;L^4)} + \delta\int_0^T c\|\nabla\phi\|^2_{L^2(\Omega)} + \delta\|\Delta\phi\|^6_{L^2(\Omega)} dt$$

$$+c\|\nabla v\|^2_{L^2(0,T;L^\infty)} + \delta\|\Delta\phi\|^4_{L^4(0,T;L^2)} + \frac{1}{\epsilon}\|\nabla^2\phi\|^2_{L^2((0,T)\times B_1)}.$$

Again, we look closer at the boundary term using this time conditions (8) $\|\nabla^2\phi\|^2_{L^2((0,T)\times B_1)} = \|c\epsilon\partial_{x_1}v_2\|^2_{L^2((0,T)\times B_1)}$, what is bounded as $v \in W_2^{3,\frac{3}{2}}(\Omega \times (0,T))$. Thus, we obtain

$$\|\Delta\phi\|_{L^\infty(0,T;L^2)} \leq c\left(\|\Delta\phi(0)\|_{L^2(\Omega)}, \|v\|_{L^2(0,T;W_2^3)}, \|\nabla\phi\|_{L^2(0,T,L^2)}\right). \tag{31}$$

This time, we calculate the third derivative of equation (26), and we test it by $\nabla^3\phi$.

$$\frac{1}{2}\int_\Omega |\nabla^3\phi(T)|^2dx - \frac{1}{2}\int_\Omega |\nabla^3\phi(0)|^2dx$$

$$= -\int_0^T\int_\Omega \nabla^3 v\cdot\nabla\phi\nabla^3\phi dxdt - 3\int_0^T\int_\Omega \nabla v\cdot\nabla\left(\Delta\phi\right)\nabla^3\phi dxdt$$

$$-3\int_0^T\int_\Omega \Delta v\cdot\nabla\left(\nabla\phi\right)\nabla^3\phi dxdt - \int_0^T\int_\Omega v\cdot\nabla\left(\nabla^3\phi\right)\nabla^3\phi dxdt,$$

what we estimate by

$$r.h.s. \leq \delta\delta'\|\nabla\phi\|^4_{L^4(0,T;L^\infty)} + c\|\nabla^3 v\|^2_{L^2(0,T;L^2)} + c'\delta\|\nabla^3\phi\|^4_{L^4(0,T;L^2)}$$

$$+c\|\Delta v\|^2_{L^2(0,T;L^4)} + \delta\int_0^T c\|\Delta\phi\|^2_{L^2(\Omega)} + \delta\|\nabla^3\phi\|^6_{L^2(\Omega)}dt$$

$$+c\|\nabla v\|^2_{L^2(0,T;L^\infty)} + \delta\|\nabla^3\phi\|^4_{L^4(0,T;L^2)} + \frac{1}{\epsilon}\|\nabla^3\phi\|^2_{L^2((0,T)\times B_1)}.$$

Here, from condition (9), the fourth order weak derivatives of ϕ near the boundary B_1 are bounded by the $W_2^{3,\frac{3}{2}}(\Omega\times(0,T))$ norm of v. Thus, the third order space derivatives of ϕ at the boundary B_1 are well defined and we estimate them by

$$\frac{1}{\epsilon}\|\nabla^3\phi\|^2_{L^2((0,T)\times B_1)} \leq c\|\partial_{x_1}\partial_{x_2}v_2\|^2_{L^2((0,T)\times B_1)} + \|\partial_{x_1}\partial_{x_1}v_2\|^2_{L^2((0,T)\times B_1)} + \epsilon\|\partial_{x_1}\partial_t v_2\|^2_{L^2((0,T)\times B_1)},$$

what again is estimated by the $W_2^{3,\frac{3}{2}}(\Omega\times(0,T))$ norm of v.
Summing this up, we have

$$\|\nabla^3\phi\|_{L^\infty(0,T;L^2)} \leq c\left(\|\nabla^3\phi(0)\|_{L^2(\Omega)}, \|v\|_{L^2(0,T;W_2^3)}, \|\phi\|_{L^2(0,T,W_2^2)}, \frac{1}{\epsilon}\|v\|_{W_2^{3,\frac{3}{2}}(\Omega\times(0,T))}\right). \tag{32}$$

Now, we estimate the time derivative. We test equation (26) by $\partial_t\phi$ and we obtain

$$\int_0^T\int_\Omega |\partial_t\phi|^2dx = -\int_0^T\int_\Omega v\cdot\nabla\phi\partial_t\phi dxdt \leq c\|v\|^2_{L^2(0,T;L^\infty)}\|\nabla\phi\|^2_{L^\infty(0,T;L^2)} + \delta\|\partial_t\phi\|^2_{L^2(0,T;L^2)}.$$

$$\|\partial_t\phi\|_{L^2(0,T;L^2)} \leq c\left(\|v\|_{L^2(0,T;L^\infty)}, \|\nabla\phi\|_{L^\infty(0,T;L^2)}\right). \tag{33}$$

Then, again we take the gradient of equation (26), and testing it by $\partial_t\nabla\phi$ we obtain

$$\int_0^T\int_\Omega |\partial_t\nabla\phi|^2dxdt = -\int_0^T\int_\Omega \nabla v\cdot\nabla\phi\nabla\partial_t\phi dxdt - \int_0^T\int_\Omega v\cdot\nabla\left(\nabla\phi\right)\nabla\partial_t\phi dxdt$$

$$\leq c\|\nabla v\|^2_{L^2(0,T;L^\infty)}\|\nabla\phi\|^2_{L^\infty(0,T;L^2)} + \delta\|\nabla\partial_t\phi\|^2_{L^2(0,T;L^2)} + c\|v\|^2_{L^2(0,T;L^\infty)}\|\nabla^2\phi\|^2_{L^\infty(0,T;L^2)},$$

to have

$$\|\nabla\partial_t\phi\|_{L^2(0,T;L^2)} \leq c\left(\|v\|_{L^2(0,T;L^\infty)}, \|\nabla v\|_{L^2(0,T;L^\infty)}, \|\nabla^2\phi\|_{L^\infty(0,T;L^2)}, \|\nabla\phi\|^2_{L^\infty(0,T;L^2)}\right). \tag{34}$$

At last we take the Laplacian of equation (26) and test by $\Delta\partial_t\phi$.

$$\int_0^T\int_\Omega |\partial_t\Delta\phi|^2dxdt$$

$$= -\int_0^T \int_\Omega \Delta v \cdot \nabla \phi \Delta \partial_t \phi dx dt - \int_0^T \int_\Omega v \cdot \nabla (\Delta \phi) \Delta \partial_t \phi dx dt - 2 \int_0^T \int_\Omega \nabla v \cdot \nabla (\nabla \phi) \Delta \partial_t \phi dx dt$$

$$\leq c \|\Delta v\|^2_{L^2(0,T;L^2)} \|\nabla \phi\|^2_{L^\infty(0,T;L^\infty)} + \delta \|\Delta \partial_t \phi\|^2_{L^2(0,T;L^2)}$$

$$+ c \|\nabla^3 \phi\|^2_{L^\infty(0,T;L^2)} \|v\|^2_{L^2(0,T;L^\infty)} + c \|\Delta \phi\|^2_{L^\infty(0,T;L^2)} \|\nabla v\|^2_{L^2(0,T;L^\infty)}.$$

Thus, we have

$$\|\Delta \partial_t \phi\|_{L^2(0,T;L^2)} \leq c \left(\|v\|_{L^2(0,T;W_2^3)}, \|\phi\|_{L^\infty(0,T,W_2^3)} \right). \tag{35}$$

□

Now, we are ready to prove Theorem 7.3.

Proof of Theorem 7.3. We want to prove that expression $\nabla \phi \Delta \phi \in W_2^{1,\frac{1}{2}}(\Omega \times (0,T))$. In fact, we obtain better estimate, namely $\nabla \phi \Delta \phi \in W_2^{1,1}(\Omega \times (0,T))$.
Firstly, we estimate the $L^2(0,T;L^2(\Omega))$ norm of $\nabla \phi \Delta \phi$.

$$\|\nabla \phi \Delta \phi\|_{L^2(0,T;L^2)} \leq T^{\frac{1}{2}} \|\nabla \phi\|_{L^\infty(0,T;L^\infty)} \|\Delta \phi\|_{L^\infty(0,T;L^2)}.$$

Then, we calculate the $L^2(0,T;L^2(\Omega))$ norm of $\nabla (\nabla \phi \Delta \phi)$.

$$\|\nabla (\nabla \phi \Delta \phi)\|_{L^2(0,T;L^2)} \leq \|\nabla \phi \nabla \Delta \phi\|_{L^2(0,T;L^2)} + \|\nabla \nabla \phi \Delta \phi\|_{L^2(0,T;L^2)}$$

$$\leq \|\nabla^2 \phi\|^2_{L^4(0,T;L^4)} + \|\nabla \phi\|_{L^\infty(0,T;L^\infty)} \|\nabla^3 \phi\|_{L^2(0,T;L^2)}$$

$$\leq T^{\frac{1}{2}} \left(c \|\nabla^2 \phi\|^2_{L^\infty(0,T;W_2^1)} + \|\nabla \phi\|_{L^\infty(0,T;L^\infty)} \|\nabla^3 \phi\|_{L^\infty(0,T;L^2)} \right).$$

At last, we estimate the time derivative of $\nabla \phi \Delta \phi$.

$$\|\partial_t (\nabla \phi \Delta \phi)\|^2_{L^2(0,T;L^2)} \leq \|\partial_t \nabla \phi \Delta \phi\|^2_{L^2(0,T;L^2)} + \|\nabla \phi \partial_t \Delta \phi\|^2_{L^2(0,T;L^2)}$$

$$\leq \|\partial_t \nabla \phi\|^2_{L^2(0,T;L^4)} \|\Delta \phi\|^2_{L^\infty(0,T;L^4)} + \|\nabla \phi\|^2_{L^\infty(0,T;L^\infty)} \|\partial_t \Delta \phi\|^2_{L^2(0,T;L^2)}.$$

Moreover, we obtain following inequality

$$\|\nabla \phi \Delta \phi\|_{W_2^{1,\frac{1}{2}}(\Omega \times (0,T))} \leq c T^\gamma \|\phi\|_{L^\infty(0,T;W_2^3(\Omega))} \|\partial_t \phi\|_{L^2(0,T;W_2^2(\Omega))}$$

for some $\gamma > 0$.

□

Following remark is the immediate consequence of Theorem 7.3 and Corollary 7.2.

Remark 7.1 *For $v \in W_2^{3,\frac{3}{2}}(\Omega \times (0,T))$ we obtain*

$$\|\nabla \phi \Delta \phi\|_{W_2^{1,\frac{1}{2}}(\Omega \times (0,T))} \leq c T^\gamma \|v\|_{W_2^{3,\frac{3}{2}}(\Omega \times (0,T))} \|\phi(0)\|_{W_2^3(\Omega)}$$

for some $\gamma > 0$.

8 Local in time existence for fixed ϵ

This section shows existence of local in time solution to model (2) in the rectangle of height ϵ. The prove is based on a Banach fixed point theorem and regularity results for linear problems, namely Stokes and transport equations, that was presented in Section 7.

Instead of model (2), we consider equivalent problem (10), which we obtain by transformation of equations (2) in such a way, that the zero Dirichlet boundary conditions for velocity at the inflow and outflow area B are obtained.

$$\begin{aligned} &\partial_t w + w \cdot \nabla w - \mu \Delta w + \nabla p + \tfrac{1}{\epsilon}\partial_{x_1} w = -\textstyle\sum_{i=1}^2 \Delta\phi_i \nabla\phi_i, \\ &\operatorname{div} w = 0, \\ &\partial_t \phi + w \cdot \nabla\phi + \tfrac{1}{\epsilon}\partial_{x_1}\phi = 0, \end{aligned} \tag{36}$$

with boundary conditions for velocity satisfying

$$w_2 = 0, \quad \partial_{x_2} w_1 = 0 \quad \text{at} \quad \Gamma \times (0,T), \quad w = 0 \quad \text{at} \quad B \times (0,T), \tag{37}$$

and for ϕ, we consider boundary conditions (12). Moreover, we require that the initial conditions satisfy (6).

Theorem 8.1 *Let us assume that* $w_0 \in W_2^2(\Omega)$ *and* $\phi_0 \in W_2^3(\Omega)$. *Then there exists local in time solution to problem (36) such that* $w \in W_2^{3,\frac{3}{2}}(\Omega \times (0,T)) \cap L^\infty(0,T; W_2^2(\Omega))$, $\phi \in L^\infty(0,T; W_2^3(\Omega)) \cap W_2^1(0,T; W_2^2(\Omega))$.

Proof. Firstly, following [5], we present a version of the Banach fixed point theorem.

Theorem 8.2 *Let* X *be a reflexive Banach space or let* X *have a separable pre-dual. Let* H *be a convex, closed and bounded subset of* X *and let* $X \hookrightarrow Y$, *where* Y *is a Banach space.*
Let $A : X \to X$ *maps* H *into* H *and let*

$$\|Au - Av\|_Y < \delta \|u - v\|_Y,$$

where $u, v \in H$ *and* $\delta < 1$. *Then there exists unique fixed point of* A *in* H.

We use Theorem 8.2 with the space $X = W_2^{3,\frac{3}{2}}(\Omega \times (0,T)) \cap L^\infty(0,T; W_2^2(\Omega))$ and the space $Y = L^2(0,T; W_2^2(\Omega)) \cap L^\infty(0,T; W_2^1(\Omega))$. For given $u \in X$ we consider the operator $A : X \to X$ such that $A[u] = w$, where w is the solution of problem

$$\begin{aligned} &\partial_t w - \mu\Delta w + \nabla p = -\tfrac{1}{\epsilon}\partial_{x_1} u - u \cdot \nabla u - \textstyle\sum_{i=1}^2 \Delta T_i[u] \nabla T_i[u], \\ &\operatorname{div} w = 0, \end{aligned} \tag{38}$$

with boundary conditions

$$w_2 = 0, \quad \partial_{x_2} w_1 = 0 \quad \text{at} \quad \Gamma \times (0,T), \quad w = 0 \quad \text{at} \quad B \times (0,T). \tag{39}$$

Here, the operator $T[u] = \phi$ is the solution operator to the transport equation

$$\partial_t \phi + u \cdot \nabla\phi + \tfrac{1}{\epsilon}\partial_{x_1}\phi = 0. \tag{40}$$

Using Theorem 7.3 and Lemma 7.1 we obtain that

$$-\frac{1}{\epsilon}\partial_{x_1} u - u \cdot \nabla u - \sum_{i=1}^2 \Delta T_i[u] \nabla T_i[u] \in W_2^{1,\frac{1}{2}}(\Omega \times (0,T)).$$

Then the assumptions of Theorem 7.2 are fulfilled and, indeed, $w \in X$. Moreover, due to Remark 7.1 and Corollary 7.1, we are able to choose time T and the radius R of the ball in X, such that solution w stays in that ball.

It remains to prove that operator A is a contraction.

Let as define $w, \bar{w}, \phi, \bar{\phi}$ as follows: $A[u] = w$, $A[\bar{u}] = \bar{w}$, $T[u] = \phi$, $A[\bar{u}] = \bar{\phi}$. Moreover, we define $\tilde{u} = u - \bar{u}$, $\tilde{\phi} = \phi - \bar{\phi}$ and $\tilde{w} = w - \bar{w}$.
As nonstationary Stokes problem is a system of linear equations, then $\tilde{w}$ fulfils the following Stokes problem

$$\partial_t \tilde{w} - \mu \Delta \tilde{w} + \nabla \tilde{p} = F,$$

with the right-hand side

$$F = -\frac{1}{\epsilon}\partial_{x_1}\tilde{u} + \bar{u}\cdot\nabla\bar{u} + \sum_{i=1}^{2}\Delta\bar{\phi}_i\nabla\bar{\phi}_i - u\cdot\nabla u - \sum_{i=1}^{2}\Delta\phi_i\nabla\phi_i \in W_2^{1,\frac{1}{2}}(\Omega\times(0,T)).$$

Using Theorem 7.1 we obtain estimate $\|\nabla^2\tilde{w}\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t\tilde{w}\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\tilde{p}\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\tilde{w}\|_{L^\infty(0,T;L^2(\Omega))} \le c\|F\|_{L^2(0,T;L^2(\Omega))}$.
To finish the proof we need to show that $\|F\|_{L^2(0,T;L^2(\Omega))} \le \delta(\|\tilde{u}\|_{L^2(0,T;W_2^2(\Omega))}, \|\tilde{u}\|_{L^\infty(0,T;W_2^1(\Omega))})$ for sufficiently small δ.

We start with estimates of convective terms

$$\|\bar{u}\cdot\nabla\bar{u} - u\cdot\nabla u\|_{L^2(0,T;L^2)} = \|\bar{u}\cdot\nabla\tilde{u} - \tilde{u}\cdot\nabla u\|_{L^2(0,T;L^2)} \le \|\bar{u}\cdot\nabla\tilde{u}\|_{L^2(0,T;L^2)} + \|\tilde{u}\cdot\nabla u\|_{L^2(0,T;L^2)}.$$

Then, we deal with the particular parts in the following way

$$\|\tilde{u}\cdot\nabla u\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\tilde{u}\|_{L^\infty(0,T;L^p)}\|\nabla u\|_{L^\infty(0,T;L^q)},$$

and

$$\|\bar{u}\cdot\nabla\tilde{u}\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\bar{u}\|_{L^\infty(0,T;L^\infty)}\|\nabla\tilde{u}\|_{L^\infty(0,T;L^2)},$$

where $p, q > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$. Using Sobolev imbedding theorem we arrive at the inequality

$$\|\bar{u}\cdot\nabla\bar{u} - u\cdot\nabla u\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\tilde{u}\|_{L^\infty(0,T;W_2^1)}\left(\|\bar{u}\|_{L^\infty(0,T;L^\infty)} + c\|\nabla u\|_{L^\infty(0,T;L^q)}\right).$$

Now, we look closer at the part

$$\nabla\phi\Delta\phi - \nabla\bar{\phi}\Delta\bar{\phi} = \Delta\phi\nabla\tilde{\phi} + \nabla\bar{\phi}\Delta\tilde{\phi}.$$

We realise that we are able to estimate

$$\|\nabla\tilde{\phi}\Delta\phi\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\nabla\tilde{\phi}\|_{L^\infty(0,T;L^p)}\|\Delta\phi\|_{L^\infty(0,T;L^q)},$$

and

$$\|\nabla\bar{\phi}\Delta\tilde{\phi}\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\nabla\bar{\phi}\|_{L^\infty(0,T;L^\infty)}\|\Delta\tilde{\phi}\|_{L^\infty(0,T;L^2)},$$

where $p, q > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$. Again, using Sobolev imbedding theorem we arrive at the following inequality

$$\|\nabla\phi\Delta\phi - \nabla\bar{\phi}\Delta\bar{\phi}\|_{L^2(0,T;L^2)} \le T^{\frac{1}{2}}\|\tilde{\phi}\|_{L^\infty(0,T;W_2^2)}\|\phi\|_{L^\infty(0,T;W_2^3)} + T^{\frac{1}{2}}\|\tilde{\phi}\|_{L^\infty(0,T;W_2^2)}\|\bar{\phi}\|_{L^\infty(0,T;W_2^3)}.$$

We need estimates for the transport equation of $\tilde{\phi}$

$$\partial_t\tilde{\phi} + \frac{1}{\epsilon}\partial_{x_1}\tilde{\phi} + \tilde{u}\cdot\nabla\bar{\phi} + u\cdot\nabla\tilde{\phi} = 0, \tag{41}$$

with initial condition $\tilde{\phi}_0 = 0$.
We differentiate equation (41) and obtain

$$\partial_t\nabla\tilde{\phi} + \frac{1}{\epsilon}\nabla\partial_{x_1}\tilde{\phi} + \nabla\tilde{u}\cdot\nabla\bar{\phi} + \nabla u\cdot\nabla\tilde{\phi} + \tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right) + u\cdot\nabla\left(\nabla\tilde{\phi}\right) = 0.$$

We multiply it by $\nabla\tilde{\phi}$ and integrate over Ω

$$\frac{1}{2}\partial_t\|\nabla\tilde{\phi}\|_2^2 \leq -\int_\Omega \nabla\tilde{u}\cdot\nabla\bar{\phi}\nabla\tilde{\phi} + \nabla u\cdot\nabla\tilde{\phi}\nabla\tilde{\phi} + \tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right)\nabla\tilde{\phi} + u\cdot\nabla\left(\nabla\tilde{\phi}\right)\nabla\tilde{\phi}$$

$$\leq \|\nabla\tilde{\phi}\|_2^2\left(2\|\nabla u\|_\infty + 2c\right) + \delta\left(\|\nabla\tilde{u}\cdot\nabla\bar{\phi}\|_2^2 + \|\tilde{u}\cdot\nabla\nabla\bar{\phi}\|_2^2\right).$$

Using Gronwall's inequality we obtain

$$\|\nabla\tilde{\phi}\|_2^2(T) \leq \delta e^{cT+\int_0^T\|\nabla u\|_\infty(t)dt}\int_0^T \|\nabla\tilde{u}\cdot\nabla\bar{\phi}\|_2^2 + \|\tilde{u}\cdot\nabla\nabla\bar{\phi}\|_2^2 dt$$

$$\leq c\delta e^{cT+\int_0^T\|\nabla u\|_\infty(t)dt}\|\tilde{u}\|^2_{L^2(0,T;W_2^1)}\|\nabla\bar{\phi}\|^2_{L^\infty(0,T;W_2^2)}.$$

Then, we consider Laplacian of equation (41)

$$\partial_t\Delta\tilde{\phi} + \frac{1}{\epsilon}\Delta\partial_{x_1}\tilde{\phi} + \Delta\tilde{u}\cdot\nabla\bar{\phi} + \Delta u\cdot\nabla\tilde{\phi}$$

$$+2\nabla\tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right) + 2\nabla u\cdot\nabla\left(\nabla\tilde{\phi}\right) + \tilde{u}\cdot\nabla\left(\Delta\bar{\phi}\right) + u\cdot\nabla\left(\Delta\tilde{\phi}\right) = 0.$$

We multiply it by $\Delta\tilde{\phi}$ and integrate over Ω

$$\partial_t\|\Delta\tilde{\phi}\|_2^2 \leq -\int_\Omega \Delta\tilde{u}\cdot\nabla\bar{\phi}\Delta\tilde{\phi} + \Delta u\cdot\nabla\tilde{\phi}\Delta\tilde{\phi} + 2\nabla\tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right)\Delta\tilde{\phi}$$

$$+2\nabla u\cdot\nabla\left(\nabla\tilde{\phi}\right)\Delta\tilde{\phi} + \tilde{u}\cdot\nabla\left(\Delta\bar{\phi}\right)\Delta\tilde{\phi} + u\cdot\nabla\left(\Delta\tilde{\phi}\right)\Delta\tilde{\phi} \leq \|\Delta\tilde{\phi}\|_2^2\left(\tilde{c}\|u\|^2_{W_2^3} + 5c\right)$$

$$+\delta\left(\|\Delta\tilde{u}\cdot\nabla\bar{\phi}\|_2^2 + \bar{c}\|u\|^2_{W_2^3}\|\nabla\tilde{\phi}\|_2^2 + \|2\nabla\tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right)\|_2^2 + \|\tilde{u}\cdot\nabla\left(\Delta\bar{\phi}\right)\|_2^2\right).$$

Using Gronwall's inequality we obtain

$$\|\Delta\tilde{\phi}\|_2^2(T) \leq \delta e^{5cT+\tilde{c}\int_0^T\|u\|^2_{W_2^3}dt}\int_0^T \|\Delta\tilde{u}\cdot\nabla\bar{\phi}\|_2^2 + \bar{c}\|u\|^2_{W_2^3}\|\nabla\tilde{\phi}\|_2^2 + \|2\nabla\tilde{u}\cdot\nabla\left(\nabla\bar{\phi}\right)\|_2^2$$

$$+\|\tilde{u}\cdot\nabla\left(\Delta\bar{\phi}\right)\|_2^2 dt \leq c\delta e^{5cT+3\int_0^T\|\nabla u\|_\infty dt}\|\tilde{u}\|^2_{L^2(0,T;W_2^2)}\|\nabla\bar{\phi}\|^2_{L^\infty(0,T;W_2^2)}.$$

Hence for sufficiently small time T the operator A is a contraction.
□

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