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Ph.D. Programme: Mathematical Methods in Natural Sciences (MMNS)
e-mail: mmns@mimuw.edu.pl
http://mmns.mimuw.edu.pl

# ON UNSTEADY FLOWS OF IMPLICITLY CONSTITUTED INCOMPRESSIBLE FLUIDS

MIROSLAV BULÍČEK[†‡¶], PIOTR GWIAZDA[§**], JOSEF MÁLEK[†‖‡‡], AND AGNIESZKA ŚWIERCZEWSKA-GWIAZDA[§‡††]

**Abstract.** We consider unsteady flows of incompressible fluids with a general implicit constitutive equation relating the deviatoric part of the Cauchy stress $\mathbf{S}$ and the symmetric part of the velocity gradient $\mathbf{D}$ in such a way that it leads to a maximal monotone (possibly multivalued) graph and the rate of dissipation is characterized by the sum of a Young function depending on $\mathbf{D}$ and its conjugate being a function of $\mathbf{S}$. Such a framework is very robust and includes, among others, classical power-law fluids, stress power-law fluids, fluids with activation criteria of Bingham or Herschel-Bulkley type, and shear-rate dependent fluids with discontinuous viscosities as special cases. The appearance of $\mathbf{S}$ and $\mathbf{D}$ in all the assumptions characterizing the implicit relationship $\mathbf{G}(\mathbf{D},\mathbf{S}) = \mathbf{0}$ is fully symmetric. We establish long-time and large-data existence of weak solution to such a system completed by the initial and Navier's slip boundary conditions in both subcritical and supercritical cases. We use tools such as Orlicz functions, properties of spatially dependent maximal monotone operators and Lipschitz approximations of Bochner functions taking values in Orlicz-Sobolev spaces.

**Key words.** implicit constitutive theory, unsteady flow, weak solution, long-time and large-data existence, maximal monotone graph, Lipschitz approximation of Bochner functions, Orlicz-Sobolev spaces

**AMS subject classifications.** 35D05, 35Q35, 46E30, 76D03, 76Z99

**1. Introduction.** In continuum thermodynamics, which we understand as a powerful framework to describe responses of materials, the fundamental system of partial differential equations is a consequence of balance equations (for mass, linear and angular momentum, energy) and the formulation of the second law of thermodynamics. This system of equations includes the physical quantities such as the density, the velocity, the internal energy (or temperature), the heat flux, the Cauchy stress, and is then completed by constitutive relations that characterize the response of a given material to applied external loading. For fluids, the Cauchy stress is related to the velocity gradient (its symmetric part) and the heat flux to the temperature gradient, and these relations may depend on other quantities.

In a purely mechanical setting restricted to incompressible homogeneous fluids that flow at uniform temperature, this fundamental system of governing equations reduces to

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{and} \quad \varrho\,(\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})) - \operatorname{div}\mathbf{S} = -\nabla p + \varrho \boldsymbol{b}\,, \tag{1.1}$$

[†]Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Prague 8, Czech Republic.

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[§]University of Warsaw, Institute of Applied Mathematics and Mechanics, Banacha 2, 02-097 Warsaw, Poland. This work was partially supported by National Science Centre and by International Ph.D. Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 (Ph.D. Programme: Mathematical Methods in Natural Sciences).

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[¶]mbul8060@karlin.mff.cuni.cz

[‖]malek@karlin.mff.cuni.cz

[**]pgwiazda@mimuw.edu.pl

[††]aswiercz@mimuw.edu.pl

where $\varrho \in (0,\infty)$ is the constant density, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity, $p$ is the mean normal stress and $\mathbf{S}$, a part of the Cauchy stress $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, is the only quantity that specifies material properties of a given fluid. We suppose that $\mathbf{S}$ is symmetric.

In our simplified setting, the second law of thermodynamics takes the form

$$\mathbf{T} \cdot \mathbf{D} = \mathbf{S} \cdot \mathbf{D} \geq 0\,, \tag{1.2}$$

where $\mathbf{D} = \mathbf{D}(\boldsymbol{v})$ is the symmetric part of the velocity gradient. The quantity $\mathbf{S} \cdot \mathbf{D}$ appears in the mathematical considerations very naturally. Indeed, taking the scalar product of $(1.1)_2$ and $\boldsymbol{v}$, we end up with the equation

$$\left(\tfrac{1}{2}\varrho|\boldsymbol{v}|^2\right)_{,t} + \operatorname{div}\left((p + \tfrac{1}{2}\varrho|\boldsymbol{v}|^2)\boldsymbol{v}\right) - \operatorname{div}(\mathbf{S}\boldsymbol{v}) + \mathbf{S} \cdot \mathbf{D} = \varrho \boldsymbol{b} \cdot \boldsymbol{v}\,. \tag{1.3}$$

The integration over $\Omega$, a three-dimensional domain occupied by the material, together with the Gauss theorem then lead to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho|\boldsymbol{v}|^2\,dx + \int_\Omega \mathbf{S} \cdot \mathbf{D}\,dx\,dt \leq \int_\Omega \varrho \boldsymbol{b} \cdot \boldsymbol{v}\,dx \tag{1.4}$$

provided that the boundary terms satisfy

$$\int_{\partial\Omega} \left((p + \tfrac{1}{2}\varrho|\boldsymbol{v}|^2)\boldsymbol{v} \cdot \boldsymbol{n} - \mathbf{S}\boldsymbol{v} \cdot \boldsymbol{n}\right)\,dS \geq 0\,, \tag{1.5}$$

which is for example the case of no-slip boundary conditions when

$$\boldsymbol{v}(t,x) = \mathbf{0} \text{ for } t \in [0,T] \text{ and } x \in \partial\Omega\,, \tag{1.6}$$

where $T \in (0,\infty)$. Navier's slip boundary conditions combined with the impermeability of the boundary is another type of boundary conditions fulfilling (1.5): if $\boldsymbol{n} = \boldsymbol{n}(x)$ is an outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and $\boldsymbol{z}_\tau := \boldsymbol{z} - (\boldsymbol{z} \cdot \boldsymbol{n})\boldsymbol{n}$ denotes the projection of a vector $\boldsymbol{z}$ defined on $\partial\Omega$ to the tangent plane located at $x \in \partial\Omega$, then the fluid exhibits Navier's slip on the impermeable boundary if

$$\boldsymbol{v} \cdot \boldsymbol{n} = \mathbf{0} \quad \text{ and } \quad (\mathbf{S}\boldsymbol{n})_\tau = -\gamma_* \boldsymbol{v}_\tau \quad \text{ on } (0,T) \times \partial\Omega\,, \tag{1.7}$$

where $\gamma_* > 0$. Note that in our setting $(\mathbf{S}\boldsymbol{v})_\tau = (\mathbf{T}\boldsymbol{v})_\tau$. If $\gamma_* = 0$ in (1.7) then the fluid slips along the boundary. The no-slip condition (1.6) can be viewed as the limit of (1.7) if $\gamma_* \to \infty$. Since (for $\mathbf{S}$ symmetric and $\boldsymbol{v}$ fulfilling (1.7))

$$(\mathbf{S}\boldsymbol{v}) \cdot \boldsymbol{n} = (\mathbf{S}\boldsymbol{n}) \cdot \boldsymbol{v} = (((\mathbf{S}\boldsymbol{n}) \cdot \boldsymbol{n})\boldsymbol{n} + (\mathbf{S}\boldsymbol{n})_\tau) \cdot \boldsymbol{v}_\tau = (\mathbf{S}\boldsymbol{n})_\tau \cdot \boldsymbol{v}_\tau = -\gamma_*|\boldsymbol{v}_\tau|^2\,,$$

we observe that (1.7) fulfills (1.5) as well. We complete the considered problem by formulating the initial condition:

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x) \quad \text{in } \Omega, \tag{1.8}$$

where $\boldsymbol{v}_0$ is a given function fulfilling the compatibility conditions $\operatorname{div}\boldsymbol{v}_0 = 0$ in $\Omega$ and $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Let us return to the quantity $\mathbf{S} \cdot \mathbf{D}$. If the initial velocity $\boldsymbol{v}_0$ and $\boldsymbol{b}$ are given $L^2$-integrable functions then (1.4) implies that

$$\sup_{t\in[0,T]} \int_\Omega |\boldsymbol{v}|^2\,dx + \int_0^T \int_\Omega \mathbf{S} \cdot \mathbf{D}\,dx\,dt < \infty\,. \tag{1.9}$$

From the point of view of mathematical analysis it seems natural to address the question whether this type of a priori large-data information suffices to establish the existence of a long-time and large-data solution to relevant initial and boundary value problems driven by (1.1), for a general class of fluid models. Here, we focus on implicitly constituted fluids.

**1.1. Implicitly constituted incompressible fluids.** Newton's statement [47] "The resistance arising from the want of lubricity in parts of the fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another." is mostly interpreted as to give rise to the linear relationship between the shear stress and the shear rate, in which the constant of the proportionality is the viscosity, which is then generalized to the formula

$$\mathsf{S} = 2\mu_*\mathsf{D} \qquad \mu_* \in (0,\infty)\,. \tag{1.10}$$

One can however perceive Newton's statement more generally, namely, as the fact that the shear stress and the shear rate are related, and then one ends up with the implicit relation

$$\mathsf{G}(\mathsf{D},\mathsf{S}) = \mathbf{0}\,, \tag{1.11}$$

or even more generally

$$\tilde{\mathsf{G}}(\mathsf{D},\mathsf{T}) = \mathbf{0}\,. \tag{1.12}$$

There are fundamentally new discoveries and far-reaching consequences that come from this general viewpoint, in particular, if one investigates them in a systematic way as it is done in the original works by Rajagopal [49, 50] and Rajagopal and Srinivasa [51]. We summarize those relevant to incompressible fluids next.

Obviously, in comparison with traditional models, in which $\mathsf{S}$ (or $\mathsf{T}$) is a function of $\mathsf{D}$, the implicit equations (1.11) or (1.12) can describe much more complicated responses whereas the number of involved quantities is unchanged. The class (1.12) is capable of capturing several non-Newtonian phenomena such as shear-thinning, shear-thickening and pressure thickening and includes combinations of these effects with various activation and deactivation criteria. (In addition, such models can be developed within a unifying thermodynamic framework, see [51] and [46]). To give a simple example that falls to the class given by (1.11), let us consider the equation

$$2\nu(|\mathsf{D}|^2)\left(\tau_* + (|\mathsf{S}| - \tau_*)^+\right)\,\mathsf{D} = (|\mathsf{S}| - \tau_*)^+\,\mathsf{S} \quad \text{with } \tau_* > 0\,, \tag{1.13}$$

where $x^+$ denotes the positive part of $x$: $x^+ = \max\{x, 0\}$. Setting

$$\mathsf{G}(\mathsf{D},\mathsf{S}) = 2\nu(|\mathsf{D}|^2)\left(\tau_* + (|\mathsf{S}| - \tau_*)^+\right)\,\mathsf{D} - (|\mathsf{S}| - \tau_*)^+\,\mathsf{S}, \tag{1.14}$$

we see that (1.13) is of the form (1.11). More interestingly, one can easily observe that (1.13) is equivalent to the traditional description of fluids of a Bingham or Herschel-Bulkley type [20]:

$$|\mathsf{S}| \le \tau_* \Leftrightarrow \mathsf{D} = \mathbf{0} \quad \text{and} \quad |\mathsf{S}| > \tau_* \Leftrightarrow \mathsf{S} = \frac{\tau_*\mathsf{D}}{|\mathsf{D}|} + 2\nu(|\mathsf{D}|^2)\mathsf{D}. \tag{1.15}$$

Model (1.13) covers as a special case (by setting $\tau_* = 0$) the fluids with shear dependent viscosity

$$\mathsf{S} = 2\nu(|\mathsf{D}|^2)\mathsf{D} \quad \text{with} \quad \nu : \mathbb{R}^+ \to \mathbb{R}^+\,, \tag{1.16}$$

including the classical power-law fluids

$$\mathsf{S} = 2\mu_*|\mathsf{D}|^{r-2}\mathsf{D} \quad \text{with} \quad 1 \le r < \infty, \mu_* \in (0,\infty)\,, \tag{1.17}$$

and their various generalizations such as

$$\mathbf{S} = 2\mu_*(\alpha_* + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D} \quad \text{with} \quad r \in \mathbb{R}, \mu_*, \alpha_* \in (0,\infty). \tag{1.18}$$

The Navier-Stokes model (1.10) is achieved by taking $r = 2$ in (1.17).

The form (1.15), in which the response of fluids with the activation criterion is mostly written, motivated several researchers to include tools such as variational inequalities, multi-valued function analysis, functions with discontinuities into the theoretical investigation of relevant boundary value problems. On the other hand, the reformulation (1.13) with *continuous* function $\mathbf{G}$ enables us to avoid such tools and technical difficulties connected with them.

Another interesting class belonging to (1.11) are the stress power-law fluids (see [45] for a more detailed exposition focused on identifying different features between (1.18) and (1.19) and on solving several special problems in simple geometries) characterized through the relation

$$\mathbf{D} = \frac{1}{2\mu_*}(\beta_* + |\mathbf{S}|^2)^{\frac{s-2}{2}}\mathbf{S} \quad \text{with} \quad s \in \mathbb{R}, \beta_* \in (0,\infty), \tag{1.19}$$

that reduces to the Navier-Stokes fluid (1.10) for $s = 2$. Thus, we observe that the constitutive relations (1.11) and (1.12) contain two explicit subclasses as special cases, namely,

$$\mathbf{T} = \tilde{\mathbf{T}}(\mathbf{D}) \quad \text{and} \quad \mathbf{D} = \tilde{\mathbf{D}}(\mathbf{T})\,, \tag{1.20}$$

$$\mathbf{S} = \tilde{\mathbf{S}}(\mathbf{D}) \quad \text{and} \quad \mathbf{D} = \tilde{\mathbf{D}}(\mathbf{S})\,. \tag{1.21}$$

While the first subclass, in which the stress is a nonlinear function of $\mathbf{D}$, has been experimentally observed and systematically applied to modeling since the end of nineteenth century[1] and mathematically analyzed since the sixties[2], the significance of the second subclass, in which $\mathbf{D}$ is a nonlinear function of the stress, has been addressed quite recently, see [49] and [50], although such models were introduced before in geophysics (see for example [25]), chemical enegineering (see for example [55]), etc. (see also [16]).

From the point of view of continuum physics, Rajagopal [49], [50] and [48] provides several convincing arguments why the latter class should be preferable. Not only the equations $(1.20)_2$ and $(1.21)_2$ reflect naturally the fact that the force (per unit area) is the cause and the velocity gradient (or its symmetric part) is its effect, but the framework given by $(1.20)_2$ (and more generally by (1.12)) provides also a natural setting to incorporate the constraint of incompressibility into the constitutive equation and to justify incompressible fluid models with the viscosity depending on the mean normal stress (pressure)[3].

[1] See Schwedoff [54], Troutan [61] and further references in books on non-Newtonian fluids, such as Bird, Amstrong, Hassager [9], Huilgol [31], Schowalter [53], or in the survey paper [43].

[2] Theoretical analysis initiated by Ladyzhenskaya [35, 36], see also Lions [37], develops extensively during last decades, see for example studies of different type [6, 7, 10, 19, 18, 28, 40, 38, 39, 62].

[3] Such models are important in many applications such as elastohydrodynamic lubrication (see Szeri [59]). The fact that viscosity should depend on the pressure has been questioned by Stokes [58], experimentally first observed by Barus [5], and well documented in the book by Bridgman [12], see [30] and [15] for more details and further references. This class of incompressible materials that fits to implicitly constituted fluids (1.12) or $(1.20)_2$ is however not subject of the investigation in this study. We refer to [15] for the most recent results concerning mathematical analysis of incompressible fluids with the pressure and the shear rate dependent viscosity.

There are also mathematical reasons that make the class of implicitly constituted fluid attracting. The fact that we deal with ten first order equations instead of four second order equations (as it is the case of the Navier-Stokes equation) corresponds well to the approaches developed in the analysis of nonlinear partial differential equations if one deals with the concept of weak solution (a nice reference towards this direction is the classical book by Lions [37]). In spite of enlarged number of unknowns, such a framework is also promising from the point of view of finite element discretization and subsequent computer simulations as this approach does not introduce redundant differentiation. When well developed, such an approach could be also a good starting point for the analysis of rate type and integral type fluid models.

Observing that for the power-law fluid (1.17) with $r > 1$ (and $2\mu_* = 1$ for simplicity)

$$\mathbf{S} = |\mathbf{D}|^{r-2}\mathbf{D} \qquad \Longleftrightarrow \qquad \mathbf{D} = |\mathbf{S}|^{\frac{r-2}{r-2}}\mathbf{S}\,, \tag{1.22}$$

and consequently $\mathbf{S}$ is a monotone function of $\mathbf{D}$ (in the sense of the definition below) and vice versa, and the quantity $\xi = \mathbf{S}\cdot\mathbf{D}$ that enters the energy estimates (1.9) takes the form ($r' = (r-1)/r$)

$$\begin{aligned}\mathbf{S}\cdot\mathbf{D} &= |\mathbf{D}|^r = |\mathbf{S}|^{r'}\\ &= \frac{1}{r}\mathbf{S}\cdot\mathbf{D} + \frac{1}{r'}\mathbf{S}\cdot\mathbf{D} = \frac{1}{r}|\mathbf{D}|^r + \frac{1}{r'}|\mathbf{S}|^{r'}\end{aligned} \tag{1.23}$$

we have given motivation for the following assumptions on the structure of the implicit constitutive relations (1.11).

Introducing a natural identification

$$(\mathbf{D},\mathbf{S}) \in \mathcal{A} \qquad \Longleftrightarrow \qquad \mathbf{G}(\mathbf{D},\mathbf{S}) = \mathbf{0}\,, \tag{1.24}$$

we put the following assumptions on $\mathcal{A}$:

(i) *$\mathcal{A}$ comes through the origin.* $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$.

(ii) *$\mathcal{A}$ is a monotone graph.*

$$(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) \geq 0 \qquad \text{for all } (\mathbf{D}_1,\mathbf{S}_1),(\mathbf{D}_2,\mathbf{S}_2) \in \mathcal{A}.$$

(iii) *$\mathcal{A}$ is a maximal monotone graph.* Let $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{3\times3}_{\text{sym}} \times \mathbb{R}^{3\times3}_{\text{sym}}$ be given.

$$\text{If} \quad (\bar{\mathbf{S}} - \mathbf{S})\cdot(\bar{\mathbf{D}} - \mathbf{D}) \geq 0 \quad \text{for all } (\bar{\mathbf{D}},\bar{\mathbf{S}}) \in \mathcal{A} \quad \text{then } (\mathbf{D},\mathbf{S}) \in \mathcal{A}.$$

(iv) *$\mathcal{A}$ is a $\psi$-graph.* There are non-negative $m \in L^1(Q)$, $c_* > 0$ and $N$-function $\psi$ such that

$$\mathbf{S}\cdot\mathbf{D} \geq -m + c_*(\psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}|)) \quad \text{for all } (\mathbf{D},\mathbf{S}) \in \mathcal{A}\,.$$

Here, $\psi^*$ denotes the conjugate (dual) function to $\psi$. We provide the definition of $N$-functions (or Young functions) together with a brief summary of their properties, and the definition of Orlicz spaces in Subsect. 1.2 and 2.1. We notice that the choice $\psi(s) = \frac{1}{r}s^r$ covers the case discussed in (1.23) and there are further important constitutive relations that call for the setting given by the assumption *(iv)*. Using the symbol $f \sim g$ to denote "$f$ is equivalent to $g$ at $\infty$" [4], the framework delineated by the assumptions *(i)-(iv)* is suitable to describe fluids with non-polynomial growth

$$\mathbf{S} \sim (1+|\mathbf{D}|^2)^{\frac{r-2}{2}} \ln(1+|\mathbf{D}|)\mathbf{D} \implies \psi(\mathbf{D}) \sim |\mathbf{D}|^r \ln(1+|\mathbf{D}|)\,,$$

or fluids, in which the experimental data are reflected by a convex function $\psi$ with different polynomial upper and lower growth; in such a case $\psi(\mathbf{D}) := \psi(|\mathbf{D}|)$ fulfills for certain $1 < q \le r < \infty$ and positive constants $c_1$, $c_2$, $c_3$ and $c_4$ the condition[5]

$$c_1 s^q - c_2 \le \psi(s) \le c_3 s^r + c_4 \qquad s \in [0,\infty)\,. \tag{1.25}$$

For the sake of completeness we shall show in Lemma 1.1 below that (1.14) with $\nu(|\mathbf{D}|^2) = |\mathbf{D}|^{r-2}\mathbf{D}$ and $r \in [1,\infty)$ fulfills all the assumptions *(i)-(iv)*. Since any pair $(\psi,\psi^*)$ of $N$-functions fulfil the Young inequality

$$\mathbf{S}\cdot\mathbf{D} \le \psi(|\mathbf{S}|) + \psi^*(|\mathbf{D}|),$$

the framework characterized by the condition *(iv)* for some $N$-function $\psi$ seems to be optimal. We wish to emphasize that the role of $\mathbf{S}$ and $\mathbf{D}$ in the assumptions *(i)-(iv)* is equipollent and merely monotone property *(ii)* is required here. We are thus able to cover a broader class of implicitly constituted fluids in comparison to our previous study [14] where we analyzed steady flows and we required instead of *(ii)* strict monotone property either in $\mathbf{D}$ or $\mathbf{S}$. We also refer the reader to the introductory part of [14] where complementary information on implicitly constituted fluids are provided, including figures and other examples.

The framework considered here should not be confused with a complementary but different setting introduced by Minty [44] and generalized for $x-$dependent graphs by Francfort et al. [23]. Here, we start with the implicit constitutive equation (1.11) and through (1.24) introduce a maximal monotone graph. In [23], the authors start with a maximal monotone graph and observe that to every maximal monotone graph one can associate 1-Lipschitz function $\varphi$, such that: $(\mathbf{D},\mathbf{S}) \in \mathcal{A} \iff \mathbf{D}-\mathbf{S} = \boldsymbol{\varphi}(\mathbf{S}+\mathbf{D})$.

Note that it follows from (1.9) and the assumption *(iv)* that

$$\sup_{t\in[0,T]} \int_\Omega |\boldsymbol{v}|^2\,dx + \int_0^T \psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}|)\,dx\,dt < \infty\,. \tag{1.26}$$

The objective of this paper is to develop a mathematical theory for a class of initial and boundary value problems described by (1.1), (1.7), (1.8) and (1.11) and denoted as *Problem* $\mathcal{P}$ in what follows. *Problem* $\mathcal{P}$ includes two nonlinear terms: the implicit relation (1.11) and the quadratic nonlinearity $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$. In order to identify the limit in the latter term we need the compactness of the velocity in $L^2(0,T;L^2(\Omega)^3)$. Having this in mind we state the result established in this study in the following way:

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[4] More precisely, $f \sim g$ means that $0 < \liminf_{|r|\to\infty} \frac{|f(r)|}{|g(r)|} = \limsup_{|r|\to\infty} \frac{|f(r)|}{|g(r)|} < \infty$.

[5] Using the Lebesgue space setting generated by the lower and upper bound in (1.25), mathematical analysts have developed (see for example [1, 8, 22]) a theory for problems involving elliptic operators with non-standard growth based on the gradient estimates in $L^q(\Omega)$ but with $r$-growth that leads to an (artificial) condition relating $q$ and $r$. Such a condition is not needed if one directly works with the condition *(iv)*.

*For arbitrary set of data involving* $\Omega \subset \mathbb{R}^3$ *with smooth boundary* $\partial\Omega$, $T \in (0,\infty)$, $\boldsymbol{v}_0 \in L^2(\Omega)^3$, $\boldsymbol{b} \in L^2(0,T;L^2(\Omega)^3)$ *and* $\gamma_* > 0$, *there is long-time and large-data weak solution to Problem* $\mathcal{P}$ *provided that the graph* $\mathcal{A}$ *generated by* $\mathbf{G}$ *via the identification* (1.24) *fulfills the assumptions (i)-(iv) and the function spaces generated by* (1.26) *and the equation* $(1.1)_2$ *are compactly embedded into* $L^2(0,T;L^2(\Omega)^3)$.

In fact, since we aim to include into our theory as large class of constitutive relations as possible, we consider the following generalization of (1.11), namely

$$\mathbf{G}(t,x,\mathbf{D}(t,x),\mathbf{S}(t,x)) = \mathbf{0}\,. \qquad t \in [0,T], x \in \Omega, \tag{1.27}$$

that is able to capture the response of materials, changing the properties at each time $t$ and each spatial position $x$. We call the initial and boundary value problem (1.1), (1.7), (1.8) and (1.27) *Problem* $\mathcal{P}_{(t,x)}$. The generalization (1.27) requires to add one more assumption concerning the measurability of a selection function $\mathbf{S}^* = \mathbf{S}^*(\mathbf{D})$. The complete list of assumptions, the definition of weak solution and a precise formulation of the main theorem are given in the next subsection, where we also discuss why and in what sense this result generalizes previous studies, and we summarize the tools used in the proof, underlining their novel features. We aim to present a simple proof. Some of the key tools, in particular Orlicz spaces, regularization of maximal monotone graphs and Lipschitz approximations of the Bochner spaces with values in the Orlicz spaces are studied in detail in Section 2. Section 3 contains the complete proof of the theorem. Finally, we shall make several concluding remarks before Appendix that is split into three parts. The first one summarizes several lemmas related to Lipschitz approximations of Bochner-Sobolev functions. In the second part, we establish the global second derivatives regularity for Neumann problem to Poisson problem in Orlicz space setting. Finally, the third part contains details concerning the existence of pressure introduced within the proof in Section 3.

We finish this section by showing that (1.14) with the power-law viscosity fulfills all the assumptions *(i)-(iv)*.

LEMMA 1.1. *Let* $1 \le r < \infty$. *Assume that* $\mathbf{G} : \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}^{3\times3}_{\mathrm{sym}} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ *is given by the formula*

$$\mathbf{G}(\mathbf{D},\mathbf{S}) = |\mathbf{D}|^{r-2}\left(\tau_* + (|\mathbf{S}| - \tau_*)^+\right)\mathbf{D} - (|\mathbf{S}| - \tau_*)^+\,\mathbf{S} \qquad \text{with } \tau_* > 0\,. \tag{1.28}$$

*Then* $\mathcal{A}$ *defined by* (1.27) *fulfills the conditions (i)-(iv) above.*

*Proof.* Obviously, $\mathbf{G}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ and *(i)* holds. Since $\mathbf{G}(\mathbf{D},\mathbf{S}) = \mathbf{0}$ implies that

$$\begin{cases} \mathbf{D} = \mathbf{0} & \Leftrightarrow |\mathbf{S}| \le \tau_*\,, \\ \mathbf{D} \neq \mathbf{0} & \Leftrightarrow |\mathbf{S}| > \tau_* \Leftrightarrow \mathbf{D} = (|\mathbf{S}| - \tau_*)^{\frac{1}{r-1}}\frac{\mathbf{S}}{|\mathbf{S}|} \Leftrightarrow \mathbf{S} = \frac{\tau_*\mathbf{D}}{|\mathbf{D}|} + |\mathbf{D}|^{r-2}\mathbf{D}\,, \end{cases} \tag{1.29}$$

we distinguish three different cases to verify the monotone property *(ii)*. First, if $|\mathbf{S}_1| < |\mathbf{S}_2| \le \tau_*$ then $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{0}$ and *(ii)* is trivial. Next, if $|\mathbf{S}_1| \le \tau_* < |\mathbf{S}_2|$ then $\mathbf{D}_1 = \mathbf{0}$ and

$$\begin{aligned}(\mathbf{S}_2 - \mathbf{S}_1)\cdot(\mathbf{D}_2 - \mathbf{D}_1) &= (\mathbf{S}_2 - \mathbf{S}_1)\cdot\left((|\mathbf{S}_2| - \tau_*)^{\frac{1}{r-1}}\frac{\mathbf{S}_2}{|\mathbf{S}_2|}\right) \\ &= (|\mathbf{S}_2| - \tau_*)^{\frac{1}{r-1}}\left(|\mathbf{S}_2| - \frac{\mathbf{S}_1\cdot\mathbf{S}_2}{|\mathbf{S}_2|}\right) \ge (|\mathbf{S}_2| - \tau_*)^{\frac{1}{r-1}}\left(|\mathbf{S}_2| - |\mathbf{S}_1|\right) > 0\,.\end{aligned}$$

Last, if $\tau_* < |\mathbf{S}_1| \le |\mathbf{S}_2|$ then the monotone property *(ii)* follows from the observation that the function $\mu(s) := (s-\tau_*)^{\frac{1}{r-1}}/s$ is positive and increasing[6] on $(\tau_*, \infty)$. Maximal monotone property *(iii)* follows from the continuity of $\mathbf{G}$. Finally, we observe that for $|\mathbf{S}| > \tau_*$ we have on one hand side (by inserting the formula for $\mathbf{D}$)

$$\begin{aligned}\mathbf{S}\cdot\mathbf{D} &= (|\mathbf{S}|-\tau_*)^{\frac{1}{r-1}}|\mathbf{S}| = (|\mathbf{S}|-\tau_*)^{\frac{r}{r-1}} - (|\mathbf{S}|-\tau_*)^{\frac{1}{r-1}}\tau_* \\ &\ge (|\mathbf{S}|-\tau_*)^{\frac{r}{r-1}} - c(r,\tau_*) \ge \frac{1}{r}|\mathbf{S}|^{\frac{r}{r-1}} - c(r,\tau_*)\,,\end{aligned}$$

and on the other hand (by inserting the formula for $\mathbf{S}$)

$$\mathbf{S}\cdot\mathbf{D} = \tau_*|\mathbf{D}| + |\mathbf{D}|^r \ge |\mathbf{D}|^r\,.$$

As $\mathbf{S}\cdot\mathbf{D} = 0$ for $|\mathbf{S}| \le \tau_*$ we conclude easily from these observations that there are $c_* > 0$ and $c(r,\tau_*) > 0$ such that for all $\mathbf{D}$, $\mathbf{S}$ fulfilling $\mathbf{G}(\mathbf{D},\mathbf{S}) = \mathbf{0}$ we have

$$\mathbf{S}\cdot\mathbf{D} \ge c_*\left(\frac{|\mathbf{D}|^r}{r} + \frac{|\mathbf{S}|^{r'}}{r'}\right) - c(r,\tau_*)\,,$$

which is the condition *(iv)*. □

**1.2. Main result.** Before introducing weak solution to *Problem* $\mathcal{P}_{(t,x)}$ and stating the result concerning its existence, we fix notation and provide needful definitions.

Let $T \in (0,\infty)$ denote the length of the time interval and $\Omega \subset \mathbb{R}^d$, $d > 1$, be a bounded domain with $\mathcal{C}^{1,1}$-boundary $\partial\Omega$; then we write $\Omega \in \mathcal{C}^{1,1}$. We also set $Q = (0,T)\times\Omega$ and $\Gamma = (0,T)\times\partial\Omega$.

For $q \in [1,\infty]$ we define the Lebesgue spaces $L^q(\Omega)$ and the Sobolev spaces $W^{1,q}(\Omega)$ in a standard way, and we denote the trace of a Sobolev function $u$, if it exists, through $\operatorname{tr} u$. If $X$, $Y$ are Banach spaces, then $X^d := X \times \cdots \times X$ and we use $X^*$ for dual space to $X$ and $L^q(0,T;Y)$ to denote the Bochner spaces. For (scalar, vector- or tensor-valued) functions $g$ and $h$ we shall write

$$\begin{aligned}(f,g) &:= \int_\Omega f(x)g(x)\,dx && \text{if } fg \in L^1(\Omega)\,,\\ (f,g)_Q &:= \int_Q f(t,x)g(t,x)\,dx\,dt && \text{if } fg \in L^1(Q)\,,\\ (f,g)_{\partial\Omega} &:= \int_{\partial\Omega} f(S)g(S)\,dS && \text{if } fg \in L^1(\partial\Omega)\\ (f,g)_\Gamma &:= \int_\Gamma f(t,S)g(t,S)\,dS\,dt && \text{if } fg \in L^1(\Gamma)\,,\\ \langle g,f\rangle &:= \langle g,f\rangle_{X^*,X} && \text{if } f \in X \text{ and } g \in X^*\,.\end{aligned}$$

We also use the space $\mathcal{C}_{\text{weak}}(0,T;L^q(\Omega))$ consisting of all $u \in L^\infty(0,T;L^q(\Omega))$, satisfying $(u(t),\varphi) \in \mathcal{C}([0,T])$ for all $\varphi \in \mathcal{C}(\overline{\Omega})$.

We introduce the subspaces (and their duals) of vector-valued Sobolev functions from $W^{1,q}(\Omega)^d$ which have zero normal component on the boundary. First, we define in a standard way for any $p \in [1,\infty)$

$$L^q_{\boldsymbol{n},\text{div}} := \overline{\{\boldsymbol{v} \in \mathcal{D}(\Omega)^d;\, \operatorname{div}\boldsymbol{v} = 0\}}^{\|\cdot\|_q}.$$

[6] One observes that for $s > \tau_*$: $\mu'(s) = \frac{1}{s^2}\left(\frac{2-r}{r-1}s + \tau_*\right)(s-\tau_*)^{-\frac{r}{r-1}} \ge \frac{1}{s^2}\frac{\tau_*}{r-1}(s-\tau_*)^{-\frac{r}{r-1}} > 0$.

Then by $\mathcal{V}$ and $\mathcal{V}_{\mathrm{div}}$ we denote

$$\mathcal{V} := \{\boldsymbol{v} \in W^{d+2,2}(\Omega)^d;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \quad \mathcal{V}_{\mathrm{div}} := \mathcal{V} \cap L^2_{\boldsymbol{n},\mathrm{div}}.$$

Note, that $\mathcal{V} \subset W^{1,\infty}(\Omega)^d$ and therefore we can finally for any $q \in [1,\infty)$ introduce the following spaces

$$\begin{aligned} W^{1,q}_{\boldsymbol{n}} &:= \overline{\mathcal{V}}^{\|\cdot\|_{1,q}},\ W^{-1,q'}_{\boldsymbol{n}} := \left(W^{1,q}_{\boldsymbol{n}}\right)^* \quad (q' = q/(q-1)) \\ W^{1,q}_{\boldsymbol{n},\mathrm{div}} &:= \overline{\mathcal{V}_{\mathrm{div}}}^{\|\cdot\|_{1,q}}, W^{-1,q'}_{\boldsymbol{n},\mathrm{div}} := \left(W^{1,q}_{\boldsymbol{n},\mathrm{div}}\right)^*. \end{aligned}$$

We say that $\psi : \mathbb{R} \to \mathbb{R}_+$ is an $N$-function if $\psi$ is an even continuous convex function such that

$$\lim_{s\to 0+} \frac{\psi(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{\psi(s)}{s} = \infty\,. \tag{1.30}$$

We also define a complementary $N$-function $\psi^*$ as the Legendre transform of $\psi$, i.e.,

$$\psi^*(s) := \sup_{\ell\in\mathbb{R}} \left(s\cdot\ell - \psi(\ell)\right)\,. \tag{1.31}$$

An $N$-function $\psi$ satisfies $\Delta_2$-condition if there exist $C_1 > 0$ and $C_2 > 0$ such that for all $s \in \mathbb{R}$ we have

$$\psi(2s) \le C_1\psi(s) + C_2 \tag{1.32}$$

and $\psi$ satisfies $\nabla_2$-condition if there is $\beta > 0$ such that for all $s \ge 1$ we have

$$\psi(s/2) \le \frac{\psi(s)}{2^{(1+\beta)}}. \tag{1.33}$$

The statements (i) $\psi$ satisfies $\nabla_2$-condition, and (ii) $\psi^*$ satisfies $\Delta_2$-condition are equivalent, see [52, Chapter II, Thm. 3]. From $\Delta_2$- and $\nabla_2$-conditions for $\psi$ it follows that for certain $1 < q \le r < \infty$ and positive constants $c_1$, $c_1^*$, $c_2$, $c_2^*$, $c_3$, $c_3^*$, $c_4$ and $c_4^*$

$$\begin{aligned} c_1 s^q - c_2 &\le \psi(s) \le c_3 s^r + c_4, \\ c_1^* s^{r'} - c_2^* &\le \psi^*(s) \le c_3^* s^{q'} + c_4^*, \end{aligned} \tag{1.34}$$

see [52, Chapter II, Cor. 5]. An opposite implication may not hold, the counterexample may be found also in [52, p. 27]. Note that condition $(1.34)_2$ follows from the definition of $\psi^*$ and $(1.34)_1$. We introduce the Orlicz spaces $L^\psi(\Omega)$, $L^\psi(Q)$ in Subsect. 2.1.

At this point, we can give the assumptions characterizing the subclass of implicitly constituted fluids (1.27) we shall study. Introducing an identification

$$(\mathbf{D},\mathbf{S}) \in \mathcal{A}(t,x) \quad \Longleftrightarrow \quad \mathbf{G}(t,x,\mathbf{D},\mathbf{S}) = \mathbf{0}\,, \tag{1.35}$$

we put the following assumptions on $\mathcal{A}$ (or $\mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$):

**(A1)** *$\mathcal{A}$ comes through the origin.* $(\mathbf{0},\mathbf{0}) \in \mathcal{A}(t,x)$.

**(A2)** *$\mathcal{A}$ is a monotone graph.*

$$(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) \ge 0 \qquad \text{for all } (\mathbf{D}_1,\mathbf{S}_1),(\mathbf{D}_2,\mathbf{S}_2) \in \mathcal{A}(t,x)\,.$$

**(A3)** *$\mathcal{A}$ is a maximal monotone graph.* Let $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}^{d\times d}_{\rm sym}$.

$$\text{If } (\bar{\mathbf{S}}-\mathbf{S})\cdot(\bar{\mathbf{D}}-\mathbf{D}) \geq 0 \quad \text{for all } (\bar{\mathbf{D}},\bar{\mathbf{S}}) \in \mathcal{A}(t,x) \text{ then } (\mathbf{D},\mathbf{S}) \in \mathcal{A}(t,x).$$

**(A4)** *$\mathcal{A}$ is a $\psi$ graph.* There are non-negative $m \in L^1(Q)$, $c_* > 0$ and $N$-function $\psi$ such that

$$\mathbf{S}\cdot\mathbf{D} \geq -m(t,x) + c_*(\psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}|)) \qquad \text{for all } (\mathbf{D},\mathbf{S}) \in \mathcal{A}(t,x)\,.$$

**(A5)** *The existence of a measurable selection.* **Either** there is $\mathbf{S}^* : Q \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ such that $(\boldsymbol{\xi}, \mathbf{S}^*(t,x,\boldsymbol{\xi})) \in \mathcal{A}(t,x)$ for all $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbf{S}^*$ is measurable, **or** there is $\mathbf{D}^* : Q \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ such that $(\mathbf{D}^*(t,x,\boldsymbol{\xi}),\boldsymbol{\xi}) \in \mathcal{A}(t,x)$ for all $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbf{D}^*$ is measurable.

We comment on **(A5)** and sufficient conditions that guarantee its validity in Remark 1.1 below. In the proof of the main theorem we use only the selection $\mathbf{S}^*$. At the point where we introduce an approximative scheme, we however briefly outline how to proceed in the case that only selection $\mathbf{D}^*$ is available.

Finally, we are ready to define weak solution to *Problem* $\mathcal{P}_{(t,x)}$ and establish the main theorem. Recall that the triplet $(p,\boldsymbol{v},\mathbf{S})$ is a solution of *Problem* $\mathcal{P}_{(t,x)}$ if $(p,\boldsymbol{v},\mathbf{S})$ satisfies (1.1) (1.7), (1.8) and (1.27). For simplicity, we set $\varrho = 1$.

Definition 1.1. *Assume that*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\rm div}, \quad \boldsymbol{b} \in L^{q'}(0,T;W^{-1,q'}_{\boldsymbol{n}}) \text{ and } \gamma_* \geq 0\,, \tag{1.36}$$

*We say that $(p,\boldsymbol{v},\mathbf{S})$ is weak solution to Problem $\mathcal{P}_{(t,x)}$ if*

$$p \in L^1(Q), \tag{1.37}$$

$$\boldsymbol{v} \in C_{weak}(0,T;L^2_{\boldsymbol{n},\rm div}) \cap L^q(0,T;W^{1,q}_{\mathbf{n},\rm div}) \text{ with } \mathbf{D}(\boldsymbol{v}) \in L^\psi(Q), \tag{1.38}$$

$$\mathbf{S} \in L^{\psi^*}(Q), \tag{1.39}$$

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2 = 0, \tag{1.40}$$

$$\begin{aligned}&\langle \boldsymbol{v}_{,t},\boldsymbol{w}\rangle + (\mathbf{S},\mathbf{D}(\boldsymbol{w})) - (\boldsymbol{v}\otimes\boldsymbol{v},\mathbf{D}(\boldsymbol{w})) + \gamma_*(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega} = \langle \boldsymbol{b},\boldsymbol{w}\rangle, +(p,\operatorname{div}\boldsymbol{w})\,,\\ &\qquad \text{for all } \boldsymbol{w} \in W^{1,1}_{\boldsymbol{n}} \text{ such that } \mathbf{D}(\boldsymbol{w}) \in L^\infty(\Omega)^{d\times d} \text{ and a.a. } t \in (0,T),\end{aligned} \tag{1.41}$$

$$(\mathbf{D}(\boldsymbol{v}(t,x)),\mathbf{S}(t,x)) \in \mathcal{A}(t,x) \text{ for a.a. } (t,x) \in Q. \tag{1.42}$$

Theorem 1.1. *Let $\mathcal{A}$ satisfy the assumptions* **(A1)–(A5)** *with $\psi$ satisfying $\Delta_2$- and $\nabla_2$-conditions and fulfilling*

$$c_1 s^q - c_2 \leq \psi(s) \leq c_3 s^r + c_4 \quad \text{ with } q > \frac{2d}{d+2} \text{ and arbitrary } r \in [q,\infty). \tag{1.43}$$

*Then for any $\Omega \in \mathcal{C}^{1,1}$ and $T \in (0,\infty)$ and for arbitrary $\boldsymbol{v}_0$, $\boldsymbol{b}$ and $\gamma_*$ satisfying* (1.36) *there exists weak solution to Problem $\mathcal{P}_{(t,x)}$ in the sense of Definition 1.1.*

The proof of this theorem is presented in Section 3. Several comments concerning the novel features of this result, methods incorporated in its proof and the relevance to previous studies are in order.

In the analysis of *Problem* $\mathcal{P}_{(t,x)}$ we distinguish two different cases, subcritical and supercritical[7], depending whether $\boldsymbol{v}$ is an admissible test function in (1.41) or

[7] The borderline case is included among subcritical case.

not[8]. If $\boldsymbol{v}$ is an admissible test function, the energy equality takes place. Recall that the energy equality together with the Minty's method represents a powerful tool in identifying the limit in nonlinear terms such as (1.42). The method presented here is however focused on the supercritical case. Since, in such a case, $\boldsymbol{v}$ cannot be taken as a test function in (1.41) (and the energy equality is not available), we introduce an Lipschitz approximation of $\boldsymbol{v}$ (or more precisely to $\boldsymbol{v}^n - \boldsymbol{v}$) and follow the goal to verify the assumptions of a convergence lemma established below (see Lemma 2.4) that helps us to identify the limit in (1.42) in a straightforward manner.

Regarding the construction of Lipschitz approximations to functions depending both on $t$ and $x$ for which the spatial derivatives are integrable and time derivative belongs to a dual to a suitable Bochner space (as it is typical for evolutionary (nonlinear) partial differential equations), we follow the approach developed by Kinunnen and Lewis [33] and essentially extended by Diening et al. [19] doing however several steps differently. First, our version of the Lipschitz approximation lemma is stated in Orlicz-Sobolev spaces. Also, its proof is not based on strong continuity of maximal function (used in [33] and [19]), which allows us for example to avoid the requirements on the $\Delta_2$-condition for a dual function (that we however need in other parts of the paper). Finally, we also aim to formulate the statement of the lemma as the list of properties of Lipschitz approximations to the Bochner functions taking values in Sobolev or Orlicz-Sobolev spaces and thus obtain an evolutionary variant of lemma establishing the properties of Lipschitz truncations to a sequence of Sobolev functions, see [18].

The restriction (1.43) on the parameter $q$ is due to required compact embedding into $L^2(0,T;L^2(\Omega)^d)$ used in the identification of the limit in the quadratic term. If we consider steady Stokes like systems we can relax the assumption on $q$ and require that $q \geq 1$. For the evolutionary Stokes like system (with $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = \boldsymbol{0}$) and for steady flows of considered fluids ($\boldsymbol{v}_{,t} = \boldsymbol{0}$) we need (1.43).

Since the framework of implicitly constituted fluids characterized by **(A1)-(A5)** is more general than the setting considered in previous studies, the result established in Theorem 1.1 provides large-data existence theory to a broader class of models in comparison with early studies (we refer to the survey paper [42] and the recent studies [15] and [19] for detailed summaries on long-time and large-data analysis of power-law type models). In particular, it follows from Theorem 1.1 and Lemma 1.1 that (for large-data) there is weak solution to Bingham or Herschel-Bulkley fluids (1.28) (or (1.14) with $\nu(s) \sim s^{r-2}$) if $r > \frac{6}{5}$ in three spatial dimensions - the result that is not covered by any of the previous studies[9]. The class of fluids, to which the result is applicable, is however much larger, as indicated in Subsect. 1.1.

The result stated in Theorem 1.1 can be viewed as a continuation of our previous studies [27] and [14] where similar stationary problems (that covers fluids with discontinuous or implicit constitutive equations) were studied. Even for such steady flows, Theorem 1.1 extends the results established in [14]. This is due to the Orlicz space setting and the fact that we do not require any kind of strict monotone property here - merely the assumption **(A2)** is sufficient to establish our result. References relevant to the analysis of steady flows of fluids of power-law type are listed in the length in

[8]It does not mean that $\mathbf{D}(\boldsymbol{v})$ should be bounded as required from $\mathbf{D}(\boldsymbol{w})$ in (1.41); $\boldsymbol{v}$ is admissible if all terms in the weak formulation (1.41) are for $\boldsymbol{w} = \boldsymbol{v}$ meaningful.

[9]We refer to [20, 24, 35, 41, 56, 57] for analysis of steady and unsteady flows of incompressible fluids of Bingham or Herschel-Bulkley type and to [27, 29] for analysis of flows of fluids with discontinuous power-law-like rheology - the results concern mostly the case $q > 3d/(d+2)$.

[14] or [18].

The last two comments concern the role of boundary condition and the approximative problems incorporated in our analysis. We consider the Navier slip boundary conditions (1.7) for several reasons. First of all, we are able to construct the pressure $p$ as an integrable function (while $p$ in [19] and other studies analyzing time-dependent three-dimensional flows of an incompressible non-Newtonian fluid subject to the no-slip boundary condition is merely a distribution, with respect to the time variable). Navier's slip boundary condition (1.7) thus helps us to avoid the splitting of the pressure (performed in [19]) into the regular part and the distribution, which brings additional technical difficulties that we did not want to mix up with the other tools developed here. Of course, it is also worth of observing that the analysis can be developed for boundary conditions different from (1.6). Even more, Navier's slip can be physically more appropriate kind of boundary conditions for specific applications than no-slip condition (1.6). Recall that we can approximate no-slip boundary condition by taking $\gamma_*$ large in (1.7). Theorem 1.1 does not cover flows exhibiting no-slip on the boundary. It is however possible to establish large-data existence of weak solution to *Problem* $\mathcal{P}_{(t,x)}$ with (1.6) instead of (1.7) by combining the approach developed in this study and the decomposition of the pressure developed in [62] and [19]. It is necessary to recognize that in order to obtain $p \in L^1(Q)$ we require $\mathcal{C}^{1,1}$-regularity of the boundary (such a smoothness is not needed in [19]); it is exploited in obtaining the second derivative estimates of solution to the auxiliary Neumann problem for the Laplace operator in Orlicz space setting, see Lemma B.1. We state the result for $\Omega \in \mathcal{C}^{1,1}$ - it is very likely it holds for some Lipschitz domains.

We use the following three level approximation cascade. First, we consider the selection $\mathsf{S}^*$ being a function of $\mathsf{D}$ that appears in **(A5)** and regularize $\mathsf{S}^*$ by taking its convolution with a standard regularizing kernel; thus we obtain a problem for $(p, \boldsymbol{v})$. We add the term $\frac{1}{n}|\boldsymbol{v}|^{s-2}\boldsymbol{v}$ for $s$ so large that it shifts the problem from supercritical case to subcritical case. Finally, we take finite-dimensional Galerkin approximation for such a system. In the limit process, it reveals to be more convenient to let first the regularizing parameter tend to zero, then to go from finite-dimensional approximation with maximal monotone graph to a continuous problem, and finally to investigate the limit when the penalty term $\frac{1}{n}|\boldsymbol{v}|^{s-2}\boldsymbol{v}$ vanishes.

In the final remark of this section we discuss conditions that imply the existence of measurable selection required by **(A5)**.

REMARK 1.1. *Let $\mathcal{L}(Q)$ denote the $\sigma$-algebra of Lebesgue measurable subsets of $Q$ and $\mathcal{B}(\mathbb{R}^{d\times d}_{\rm sym})$ the $\sigma$-algebra of all Borel subsets of $\mathbb{R}^{d\times d}_{\rm sym}$. The measurability of $\mathsf{S}^*$ in* **(A5)** *is meant with respect to the $\sigma$-algebra generated by $\mathcal{L}(Q)\otimes\mathcal{B}(\mathbb{R}^{d\times d}_{\rm sym})$. The existence of a measurable selection is a consequence of the measurability of the graph $\mathcal{A}(t,x)$, which in particular means that the following two conditions are satisfied (see [17] and [3, Chapter 8]):*

*(i) for all $\mathsf{D} \in \mathbb{R}^{d\times d}_{\rm sym}$ the set $\{\mathsf{S} \in \mathbb{R}^{d\times d}_{\rm sym} : (\mathsf{D},\mathsf{S}) \in \mathcal{A}(t,x)\}$ is closed,*

*(ii) for any closed $C \subset \mathbb{R}^{d\times d}_{\rm sym}$, the set*

$$\{(t,x,\mathsf{D}) \in Q \times \mathbb{R}^{d\times d}_{\rm sym} : \text{ there exists } \mathsf{S} \in C \text{ s.t. } (\mathsf{D},\mathsf{S}) \in \mathcal{A}(t,x)\}$$

*is measurable with respect to the $\sigma$-algebra $\mathcal{L}(Q)\otimes\mathcal{B}(\mathbb{R}^{d\times d}_{\rm sym})$.*

*The measurability of the graph is a standard assumption in most of considerations on abstract multi-valued elliptic and parabolic problems. Introducing the assumption* **(A5)** *weakens the above conditions, however provides a better readability for readers* *not familiar with abstract measure theory of multi-valued mappings. The analogous comments concern the measurability of* $\mathbf{D}^*$.

## 2. Tools.

**2.1. Orlicz spaces.** In this subsection we recall several facts about $N$-functions and the Orlicz spaces corresponding to them. We recall that $\psi : \mathbb{R} \to \mathbb{R}_+$ is an $N$-function if $\psi$ is an even continuous convex function satisfying (1.30). A function $\psi^*$ defined as

$$\psi^*(s) := \sup_{\ell \in \mathbb{R}} (s\,\ell - \psi(\ell)) \tag{2.1}$$

is called a complementary (conjugate, dual) function to $\psi$. It follows from its definition that $\psi^*$ is also an $N$-function and $(\psi^*)^* = \psi$.

For any open bounded set $Q \subset \mathbb{R}^{d+1}$, we define the Orlicz space $L^\psi(Q)$ as a set of all measurable functions $u : Q \to \mathbb{R}$ that satisfy

$$\lim_{\lambda \to 0} \int_Q \psi(\lambda u)\; dx\; dt = 0.$$

This space equipped with the norm

$$\|u\|_{L^\psi} = \|u\|_\psi := \inf \left\{ \lambda > 0;\; \int_Q \psi(\lambda^{-1} u)\; dx\; dt \le 1 \right\}$$

is a Banach space. By $W^{k,\psi}(Q)$ we mean the Orlicz–Sobolev space, namely the space of functions that have all distributional derivatives of the order not larger than $k$ in $L^\psi(Q)$. We say that a sequence of functions $\{s^n\}_{n\in\mathbb{N}}$ converges modularly to $s$ in $L^\psi(Q)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \int_Q \psi(\frac{1}{\lambda}(s^n - s))\; dx\; dt = 0$.

If we assume that $\psi$ satisfies $\Delta_2$-condition then $L^\psi(Q)$ is separable and moreover

$$(L^\psi(Q))^* = L^{\psi^*}(Q). \tag{2.2}$$

Next, we formulate Young and Hölder inequalities for $N$-functions and Orlicz spaces (see e.g. [52]).

LEMMA 2.1. *Let $\psi$ be an $N$-function. Then the following (Young) inequality holds*

$$|ab| \le \psi(a) + \psi^*(b) \qquad \text{for all } a, b \in \mathbb{R}. \tag{2.3}$$

*Assume that $u \in L^\psi(Q)$ and $v \in L^{\psi^*}(Q)$ then the following (Hölder) inequality holds*

$$\int_Q uv\; dxdt \le 2\|u\|_\psi \|v\|_{\psi^*}. \tag{2.4}$$

**2.2. Maximal monotone graphs.** This subsection is devoted to several important properties of a maximal monotone graph.

LEMMA 2.2 (Properties of $\mathbf{S}^*$). *Let $\mathcal{A}(t,x)$ be maximal monotone $\psi$-graph satisfying* **(A1)**–**(A5)** *with measurable selection $\mathbf{S}^* : Q \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$. Then $\mathbf{S}^*$ satisfies the following conditions:*

*(a1)* $\operatorname{Dom} \mathbf{S}^*(t,x,\cdot) = \mathbb{R}^{d\times d}_{\rm sym}$ *a.e. in $Q$;*

*(a2)* $\mathbf{S}^*$ *is monotone, i.e. for every* $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2 \in \mathbb{R}^{d\times d}_{\rm sym}$ *and a.a.* $(t,x) \in Q$

$$(\mathbf{S}^*(t,x,\boldsymbol{\xi}_1) - \mathbf{S}^*(t,x,\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq 0; \tag{2.5}$$

*(a3) There are non-negative* $m \in L^1(Q)$, $c_* > 0$ *and* $N$*-function* $\psi$ *such that for all* $\mathbf{D} \in \mathbb{R}^{d\times d}_{\rm sym}$ *the function* $\mathbf{S}^*$ *satisfies*

$$\mathbf{S}^* \cdot \mathbf{D} \geq -m(t,x) + c_*(\psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}^*|)) \tag{2.6}$$

*Moreover, let* $U$ *be a dense set in* $\mathbb{R}^{d\times d}_{\rm sym}$ *and* $(\mathbf{B}, \mathbf{S}^*(t,x,\mathbf{B})) \in \mathcal{A}(t,x)$ *for a.a.* $(t,x) \in Q$ *and for all* $\mathbf{B} \in U$*. Let also* $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}^{d\times d}_{\rm sym}$*. Then the following conditions are equivalent:*

$$\begin{aligned} &(i) \quad (\mathbf{S} - \mathbf{S}^*(t,x,\mathbf{B})) \cdot (\mathbf{D} - \mathbf{B}) \geq 0 \quad \textit{for all} \quad (\mathbf{B}, \mathbf{S}^*(t,x,\mathbf{B})) \in \mathcal{A}(t,x)\,, \\ &(ii) \quad (\mathbf{D},\mathbf{S}) \in \mathcal{A}(t,x). \end{aligned} \tag{2.7}$$

*Proof.* The proof of $(a1) - (a3)$ follows the same lines as for the standard $L^q-$setting, see e.g Chiadò Piat et al. [17]. Indeed, if $\mathbf{S}^*$ is a selection of the graph and $\mathcal{A}(t,x) \subset \mathbb{R}^{d\times d}_{\rm sym}$ for a.a $(t,x) \in Q$, then $(a1)$ holds. Moreover, since $(\mathbf{D}, \mathbf{S}^*(\mathbf{D})) \in \mathcal{A}(t,x)$, then by **(A2)** and **(A3)** also $(a2) - (a3)$ hold. To prove the second part of the lemma observe that an arbitrary monotone graph can be extended to the maximal monotone graph. In particular, for a given $(t,x) \in Q$, the set $\{(\mathbf{B}, \mathbf{S}^*(t,x,\mathbf{B})) \in \mathcal{A}(t,x);\, \mathbf{B} \in U$ where $U$ is a dense set in $\mathbb{R}^{d\times d}_{\rm sym}\} \cup \{\mathbf{D}(t,x), \mathbf{S}(t,x)\}$ can be extended to the monotone graph $\tilde{\mathcal{A}}(t,x)$. If $\mathbf{B} \in U$, which is dense in $\mathbb{R}^{d\times d}_{\rm sym}$, then due to [2, Cor. 1.5], recalled in Corollary 2.3, it holds $\mathcal{A}(t,x) = \tilde{\mathcal{A}}(t,x)$. □

Corollary 2.3. *Let $A$ and $\tilde{A}$ be given maximal monotone functions and an open convex set $U$ so that $A(\boldsymbol{\zeta}) \cap \tilde{A}(\boldsymbol{\zeta}) \neq \emptyset$ for every $\boldsymbol{\zeta}$ from a dense subset of $U$. Then $A(\boldsymbol{\zeta}) = \tilde{A}(\boldsymbol{\zeta})$ for every $\boldsymbol{\zeta}$ from $U$.*

Next, we formulate Convergence lemma that serves as a simple criterion to prove that $\mathbf{D}$ and $\mathbf{S}$, limits of weakly converging sequences $\mathbf{D}^n$ and $\mathbf{S}^n$ in $L^\psi$ and $L^{\psi*}$ respectively, fulfills the implicit constitutive relation (1.35) or equivalently (1.42).

Lemma 2.4. *Let $\mathcal{A}(t,x)$ be maximal monotone $\psi$-graph satisfying* **(A1)**–**(A5)** *and assume that there are sequences $\{\mathbf{S}^n\}_{n=1}^\infty$ and $\{\mathbf{D}^n\}_{n=1}^\infty$ such that for some $Q' \subset Q$ there hold*

$$(\mathbf{D}^n(t,x), \mathbf{S}^n(t,x)) \in \mathcal{A}(t,x) \qquad \textit{for a.a. } (t,x) \in Q', \tag{2.8}$$

$$\mathbf{D}^n \rightharpoonup \mathbf{D} \qquad \textit{weakly in } L^\psi(Q'), \tag{2.9}$$

$$\mathbf{S}^n \rightharpoonup \mathbf{S} \qquad \textit{weakly in } L^{\psi^*}(Q'), \tag{2.10}$$

$$\limsup_{n\to\infty} \int_{Q'} \mathbf{S}^n \cdot \mathbf{D}^n \, dx\, dt \leq \int_{Q'} \mathbf{S} \cdot \mathbf{D} \, dx\, dt. \tag{2.11}$$

*Then for almost all* $(t,x) \in Q'$ *we have*

$$(\mathbf{D}(t,x), \mathbf{S}(t,x)) \in \mathcal{A}(t,x). \tag{2.12}$$

*Proof.* For the proof of (2.12) we first observe that (2.8)–(2.11) imply that

$$\limsup_{n\to\infty} \int_{Q'} (\mathbf{S}^n - \mathbf{S}^*(t,x,\mathbf{D})) \cdot (\mathbf{D}^n - \mathbf{D}) \, dx\, dt \leq 0. \tag{2.13}$$

Since the graph is monotone, (2.13) is equivalent to

$$\limsup_{n\to\infty}\int_{Q'} |(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}))\cdot(\mathbf{D}^n-\mathbf{D})|\ dx\ dt=0. \tag{2.14}$$

Therefore, $(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}))\cdot(\mathbf{D}^n-\mathbf{D})$ converges strongly in $L^1(Q')$ and consequently weakly, namely, we have for all nonnegative $\varphi\in L^\infty(Q')$

$$\lim_{n\to\infty}\int_{Q'}(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}))\cdot(\mathbf{D}^n-\mathbf{D})\varphi\ dx\ dt=0. \tag{2.15}$$

From (2.15) it can be deduced that

$$\lim_{n\to\infty}\int_{Q'}\mathbf{S}^n\cdot\mathbf{D}^n\varphi\ dx\ dt=\lim_{n\to\infty}\int_{Q'}\mathbf{S}^n\cdot\mathbf{D}\varphi\ dx\ dt=\int_{Q'}\mathbf{S}\cdot\mathbf{D}\varphi\ dx\ dt. \tag{2.16}$$

Consequently, since the graph is monotone, we observe that for an arbitrary fix matrix $\mathbf{B}\in\mathbb{R}^{d\times d}_{sym}$ and all nonnegative $\varphi\in L^\infty(Q')$

$$0\le\lim_{n\to\infty}\int_{Q'}(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}^n-\mathbf{B})\varphi\ dx\ dt=\int_{Q'}(\mathbf{S}-\mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}-\mathbf{B})\varphi\ dx\ dt.$$

But since $\varphi$ is arbitrary, we get that for all $\mathbf{B}$ and a.a. $(t,x)\in Q'$

$$(\mathbf{S}-\mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}-\mathbf{B})\ge 0. \tag{2.17}$$

Since $\mathcal{A}(t,x)$ is a maximal graph and $\mathbf{B}$ is arbitrary, we conclude from (2.7) that $(\mathbf{D}(t,x),\mathbf{S}(t,x))$ is in the graph $\mathcal{A}(t,x)$ for a.a. $(t,x)\in Q'$. ∎

**2.3. Lipschitz approximation of Bochner functions taking values in the Orlicz-Sobolev spaces.** This final subsection deals with a very powerful tool that plays an important tool in the existence proof. It concerns Lipschitz approximations of Bochner functions that take values in Sobolev or more generally in Orlicz-Sobolev spaces. It carries on the study by Kinunnen and Lewis [33] who however do not control uniformly the measure of the set where the Lipschitz truncations differ from the original functions. In fact, the result presented generalizes similar approximation procedure developed by Diening, Růžička and Wolf [19] who considered the standard Sobolev space setting and used strong continuity of Hardy-Littlewood maximal functions. We present a new version of the Lipschitz approximation lemma stated for time-dependent functions taking values in the Orlicz-Sobolev spaces. In order to avoid (at least in this lemma) the assumption on the $\Delta_2$-condition for dual function we dot not use the continuity of the maximal function in the $L^p$ spaces. Finally, inspired by the approach developed for time-independent problems, where the Lipschitz approximations of Sobolev functions are introduced and studied (see [18] and the references therein), we formulate the lemma, as closely as we could, as a statement about the properties of Lipschitz truncations of Bochner functions that take values in the Sobolev or more generally Orlicz-Sobolev spaces.

LEMMA 2.5. *Let $\Omega\subset\mathbb{R}^d$ be an open bounded set and $T>0$ be the length of the time interval. Assume that $\psi$ is an $N$-function satisfying* $(1.34)_1$ *with $q,r\in(1,\infty)$ and $\psi^*$ be its conjugate automatically fullfiling* $(1.34)_2$. *For any functions $\mathbf{H},\bar{\mathbf{H}}$ and arbitrary sequences $\{\boldsymbol{u}^n\}_{n=1}^\infty$ and $\{\mathbf{H}^n\}_{n=1}^\infty$ we set*

$$a^n:=|\mathbf{H}^n|+|\mathbf{H}|+|\bar{\mathbf{H}}|\quad\text{and}\quad b^n:=|\mathbf{D}(\boldsymbol{u}^n)|$$

*such that for certain* $C^* > 1$

$$\begin{aligned} &\int_Q \psi^*(a^n) + \psi(b^n)\, dx\, dt + \sup_{t\in(0,T)} \|\boldsymbol{u}^n(t)\|_2^2 \le C^*, \\ &\boldsymbol{u}^n \to \boldsymbol{0} \qquad a.e.\ in\ Q := (0,T)\times\Omega. \end{aligned} \tag{2.18}$$

*In addition, let* $\{\mathsf{G}^n\}_{n=1}^\infty$ *and* $\{\boldsymbol{f}^n\}_{n=1}^\infty$ *be such that* $\mathsf{G}^n$ *is symmetric and*

$$\mathsf{G}^n \to \boldsymbol{0} \qquad \text{strongly in } L^1(Q)^{d\times d}, \tag{2.19}$$

$$\boldsymbol{f}^n \to \boldsymbol{0} \qquad \text{strongly in } L^1(Q)^{d}, \tag{2.20}$$

*and that the following identity holds in* $\mathcal{D}'(Q)^d$

$$\boldsymbol{u}^n_{,t} + \operatorname{div}(\mathsf{H}^n - \mathsf{H} + \mathsf{G}^n) = \boldsymbol{f}^n. \tag{2.21}$$

*Then there exists* $\beta > 0$ *such that for arbitrary* $Q_h \subset\subset Q$ *and for arbitrary* $\lambda^* \in (\lambda_{min}, \infty)$ *with* $\lambda_{min}$ *such that* $\psi(\lambda_{min}) = \lambda_{min}$ *and arbitrary* $k \in \mathbb{N}$ *there exists a sequence of* $\{\lambda_k^n\}_{n=1}^\infty$ *and the sequence of open sets* $\{E_k^n\}_{n=1}^\infty$, $E_k^n \subset Q$ *and a sequence* $\{\boldsymbol{u}^{n,k}\}_{n=1}^\infty$ *bounded in* $L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ *such that for any* $1 \le s < \infty$

$$\lambda_k^n \in [\lambda^*, (c_3 + c_4/\lambda_{min}^r)^{\frac{r^k-1}{r-1}} (\lambda^*)^{r^k}], \quad \forall n \in \mathbb{N}, \tag{2.22}$$

$$\boldsymbol{u}^{n,k} \to \boldsymbol{0} \qquad \text{strongly in } L^s(Q_h)^d, \tag{2.23}$$

$$\|\mathsf{D}(\boldsymbol{u}^{n,k})\|_{L^\infty(Q_h)} \le C(h,\Omega)\lambda_k^n, \tag{2.24}$$

$$\boldsymbol{u}^{n,k} = \boldsymbol{u}^n \qquad \text{in } Q_h \setminus E_k^n, \tag{2.25}$$

$$\limsup_{n\to\infty} |Q_h \cap E_k^n| \le C(h,\Omega)\frac{C^*}{\psi(\lambda^*)}. \tag{2.26}$$

*Moreover, for all* $g \in \mathcal{D}(Q_h)$ *the following estimates hold*

$$\limsup_{n\to\infty} \int_{Q_h\cap E_k^n} \left(|\mathsf{H}^n| + |\mathsf{H}| + |\bar{\mathsf{H}}|\right) |\mathsf{D}(\boldsymbol{u}^{n,k})|\, dx\, dt \le C(h,C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)} + \frac{1}{k^\beta}\right), \tag{2.27}$$

$$-\liminf_{n\to\infty} \int_0^T \langle \boldsymbol{u}^n_{,t}, \boldsymbol{u}^{n,k} g\rangle\, dt \le C(g,h,C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)} + \frac{1}{k}\right)^\beta. \tag{2.28}$$

*Proof.* We recall the definition of the modified parabolic metric $d_\alpha$ on $\mathbb{R}^{d+1}$ and corresponding balls that are given in Appendix. For $X, Y \in \mathbb{R}^{d+1}$ where $X := (t,x)$, $Y := (s,y)$, and for $R > 0$, $\alpha > 0$, $A \subset \mathbb{R}^{d+1}$ we define

$$\begin{aligned} d_\alpha(X,Y) &:= \max\left(|x-y|, \tfrac{|t-s|^{1/2}}{\alpha^{1/2}}\right), \\ Q_R^\alpha(X) &:= \left\{Y \in \mathbb{R}^{d+1}; d_\alpha(X,Y) < R\right\}, \\ \operatorname{diam}_\alpha A &:= \sup_{X,Y\in A} d_\alpha(X,Y). \end{aligned}$$

For $0 \le g \in L^1(0,\infty;L^1(\mathbb{R}^d))$ we introduce the parabolic maximal functions $\mathcal{M}(g)$ and $\mathcal{M}^\alpha(g)$ through

$$\begin{aligned} \mathcal{M}(g)(t,x) &:= \sup_{0<\rho<\infty} \fint_{(t-\rho,t+\rho)} \left(\sup_{0<R<\infty} \fint_{B_R(x)} g(s,y)\, dy\right) ds, \\ \mathcal{M}^\alpha(g)(t,x) &:= \sup_{Q_R^\alpha(t,x)} \fint_{Q_R^\alpha(t,x)} g(s,y)\, dy\, ds. \end{aligned}$$

Next for arbitrary open $E \subset Q$ we consider the Whitney covering $\{Q^{\alpha}_{R_i}(X_i), \zeta_i\}_{i\in\mathbb{N}}$ of the set $E$ given in Lemma A.1 and we introduce a truncation operator $\mathcal{L}^{\alpha}_{E}$ by (A.8) as

$$\mathcal{L}^{\alpha}_{E} u(t,x) := \begin{cases} u(t,x) & \text{if } (t,x) \in Q \setminus E, \\ \displaystyle\sum_{i=1}^{\infty} \overline{u}_{Q^{\alpha}_{R_i}} \zeta_i(t,x) & \text{if } (t,x) \in E, \end{cases} \tag{2.29}$$

where

$$\overline{u}_{Q^{\alpha}_{R_i}} := \fint_{Q^{\alpha}_{R_i}} u \, dx \, dt.$$

We will use the operator $\mathcal{L}^{\alpha}$ to construct $\boldsymbol{u}^{n,k}$. For this purpose we need to choose a proper set $E$ where we modify the original sequence $\boldsymbol{u}^{n}$. We proceed in the following way. For given $\lambda^* \in (\lambda_{min}, \infty)$ with $\lambda_{min}$ such that $\psi(\lambda_{min}) = \lambda_{min}$ and $k \in \mathbb{N}$ fixed, we introduce $\mu_i$ for $i = 1, \dots, k$ by the following recurrent formula

$$\mu_i := \psi(\mu_{i-1}) \text{ with } \mu_0 := \lambda^*. \tag{2.30}$$

Note that from strict monotonicity and strict convexity of $\psi$ and the definition of $\lambda_{min}$ it follows that $\mu_i < \mu_{i+1}$ and[10] $\lambda^* \le \mu_i \le (c_3 + c_4/\lambda^r_{min})^{\frac{r^k-1}{r-1}} (\lambda^*)^{r^k}$ for all $i = 0, \dots, k-1$, where $c_3$ and $c_4$ are constants that appears in (1.43). Next, using the assumption (2.18) we see that

$$\sum_{i=0}^{k-1} \int_{\{\psi(\mu_i) < \psi^*(\mathcal{M}(a^n)) + \psi(\mathcal{M}(b^n)) \le \psi(\mu_{i+1})\}} \psi^*(\mathcal{M}(a^n)) + \psi(\mathcal{M}(b^n)) \, dx \le C^*.$$

Hence, there surely exists $j_0 \in [0, \dots, k-1]$ such that

$$k \int_{\{\psi(\mu_{j_0}) < \psi^*(\mathcal{M}(a^n)) + \psi(\mathcal{M}(b^n)) \le \psi(\mu_{j_0+1})\}} \psi^*(\mathcal{M}(a^n)) + \psi(\mathcal{M}(b^n)) \, dx \le C^*. \tag{2.31}$$

Having such $j_0$, we finally define

$$\lambda^n_k := \mu_{j_0}, \tag{2.32}$$
$$H^n_k := \{(t,x) \in Q;\ \psi^*(\mathcal{M}(a^n)) + \psi(\mathcal{M}(b^n)) > \psi(\lambda^n_k)\}, \tag{2.33}$$

Thus (2.22) holds and (2.18), (A.2) leads to the estimate

$$|Q_h \cap H^n_k| \le \frac{CC^*}{\psi(\lambda^*)}. \tag{2.34}$$

[10] Using (1.43) we observe that for $s > \lambda^* > \lambda_{min}$:

$$\psi(s) \le c_3 s^r + c_4 = s^r (c_3 + c_4/s^r) \le s^r (c_3 + c_4/\lambda^r_{min}) =: c_* s^r.$$

Consequently,

$$\mu_i = \psi(\mu_{i-1}) \le c_* \mu^r_{i-1} \le c_*(\psi(\mu_{i-2}))^r \le c_*(c_* \mu^r_{i-2})^r \le \dots \le c_*^{1+r+\dots+r^{i-1}} (\lambda^*)^{r^i} = c_*^{\frac{r^i-1}{r-1}} (\lambda^*)^{r^i}.$$

We also define the sets

$$G_n := \{(t,x) \in Q; \mathcal{M}^{\alpha_k^n}(|\mathbf{G}^n|) > 1\}, \tag{2.35}$$
$$F_n := \{(t,x) \in Q; \mathcal{M}^{\alpha_k^n}(|\boldsymbol{f}^n|) > 1\}, \tag{2.36}$$

and $\alpha_k^n$ as

$$\alpha_k^n := \frac{\lambda_k^n}{(\psi^*)^{-1}(\psi(\lambda_k^n))}. \tag{2.37}$$

We observe that (2.19), (2.20) and (A.3) imply that

$$\limsup_{n\to\infty} |G_n \cup F_n| = 0. \tag{2.38}$$

In order to be able to apply Lemma A.3 we need to have control over full gradient. For this purpose we define

$$\tilde{H}^n := \{(t,x) \in Q;\ \mathcal{M}(|\nabla \boldsymbol{u}^n|) > n\}. \tag{2.39}$$

If follows from (2.18) and (1.43) and standard Korn's inequality that

$$\int_Q |\nabla \boldsymbol{u}^n|^q \, dx \, dt \le CC^* \qquad \text{with } q > \frac{2d}{d+2}\,, \tag{2.40}$$

which implies then

$$|\tilde{H}^n| \le \frac{CC^*}{n^q}. \tag{2.41}$$

Finally, we define an open set $E_k^n$ as

$$E_k^n := G_n \cup F_n \cup H_k^n \cup \tilde{H}^n. \tag{2.42}$$

With this setting, we finally define $\boldsymbol{u}^{n,k}$ as

$$\boldsymbol{u}^{n,k} := \mathcal{L}_{E_k^n}^{\alpha_k^n} \boldsymbol{u}^n, \tag{2.43}$$

and we shall investigate its properties.

First, we notice that boundedness of $\{\boldsymbol{u}^n\}$ in $L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}})$ (see $(2.18)_1$) and $L^q(0,T;W^{1,q}(\Omega)^d)$ with $q > 2d/(d+2)$ (as stated in (2.40)) implies, by a standard interpolation, that $\{\boldsymbol{u}^n\}$ is uniformly bounded in $L^{2+\eta}(0,T;L^{2+\eta}(\Omega)^d)$ with some $\eta > 0$. By Vitali's theorem, this together with the almost everywhere convergence $(2.18)_2$ leads to the observation that

$$\boldsymbol{u}^n \to \boldsymbol{0} \qquad \text{strongly in } L^2(Q) \qquad (n \to \infty).$$

Thus, referring to (A.9) and (2.43) we conclude that

$$\boldsymbol{u}^{n,k} \to \boldsymbol{0} \qquad \text{strongly in } L^2(Q) \qquad (n \to \infty). \tag{2.44}$$

Using Lemma A.3 we get $\boldsymbol{u}^{n,k} \in L^\infty(0,T;W^{1,\infty}_{loc}(\Omega)^d)$, but not uniformly w.r.t. $n$ and $k$.

Next, we show the uniform estimate (2.24) a.e. in $Q_h$. It is evident from the definition (2.43) that

$$\mathsf{D}(\boldsymbol{u}^{n,k}(t,x)) = \mathsf{D}(\boldsymbol{u}^n(t,x)) \quad \text{in} \quad Q_h \setminus E_k^n \subset Q_h \setminus H_k^n$$

and thus for a.a. $(t,x) \in Q_h \setminus E_k^n$ we have

$$\psi(|\mathsf{D}(\boldsymbol{u}^n(t,x))|) \le \psi(\mathcal{M}(b^n)) \le \psi(b^n) \le \psi(\lambda_k^n),$$

which implies

$$\|\mathsf{D}(\boldsymbol{u}^{n,k})\|_{L^\infty(Q_h \setminus E_k^n)} \le \lambda_k^n. \tag{2.45}$$

It remains to show (2.24) in $E_k^n$. Let $X \in E_k^n$ be arbitrary. Then $X \in Q_{R_i}(X_i)$ for some $i$ and we have

$$\begin{aligned}
|\mathsf{D}(\boldsymbol{u}^{n,k}(X))| &= \left|\mathsf{D}\left(\sum_j \zeta_j(X)\overline{\boldsymbol{u}^n}_{Q_{R_j}(X_j)}\right)\right| \\
&\overset{(A.4)_7}{=} \left|\mathsf{D}\left(\sum_j \zeta_j(X)(\overline{\boldsymbol{u}^n}_{Q_{R_j}(X_j)} - \overline{\boldsymbol{u}^n}_{Q_{4R_i}(X_i)})\right)\right| \\
&\overset{(A.4),(A.5)}{\le} CR_i^{-1} \sum_{j \in A_i} |\overline{\boldsymbol{u}^n}_{Q_{R_j}(X_j)} - \overline{\boldsymbol{u}^n}_{Q_{4R_i}(X_i)}| \\
&\overset{(A.4),(A.5)}{\le} CR_i^{-1} \fint_{Q_{4R_i}(X_i)} \left|\boldsymbol{u}^n - \fint_{Q_{4R_i}(X_i)} \boldsymbol{u}^n\right| \, dx\, dt \\
&\overset{(A.7)}{\le} C \fint_{Q_{4R_i}(X_i)} (|\mathsf{D}(\boldsymbol{u}^n)| + \alpha_k^n(|\mathsf{G}^n| + |\mathsf{H}^n| + |\mathsf{H}|) + \alpha_k^n R_i |\boldsymbol{f}^n|) \, dx\, dt \\
&\overset{(A.4)_2}{\le} C \fint_{Q_{16R_i}(X_{E_k^n})} (|\mathsf{D}(\boldsymbol{u}^n)| + \alpha_k^n(|\mathsf{G}^n| + |\mathsf{H}^n| + |\mathsf{H}|) + \alpha_k^n R_i |\boldsymbol{f}^n|) \, dx\, dt,
\end{aligned}$$

where $X_{E_k^n}$ is some point in $Q_h \setminus E_k^n$. Thus, using (2.35) and (2.36) we get

$$\begin{aligned}
|\mathsf{D}(\boldsymbol{u}^{n,k}(X))| &\le C \fint_{Q_{16R_i}(X_{E_k^n})} |\mathsf{D}(\boldsymbol{u}^n)| + \alpha_k^n(|\mathsf{H}^n| + |\mathsf{H}|) \, dx\, dt + C\alpha_k^n \\
&\le C \max\left\{\alpha_k^n, \fint_{Q_{16R_i}(X_{E_k^n})} |\mathsf{D}(\boldsymbol{u}^n)| \, dx\, dt, \fint_{Q_{16R_i}(X_{E_k^n})} \alpha_k^n |\mathsf{H}^n| \, dx\, dt,\right. \\
&\qquad \left. \fint_{Q_{16R_i}(X_{E_k^n})} \alpha_k^n |\mathsf{H}| \, dx\, dt\right\}.
\end{aligned} \tag{2.46}$$

If the maximum is achieved by the second term then

$$\frac{|\mathsf{D}(\boldsymbol{u}^{n,k}(X))|}{C} \le \fint_{Q_{16R_i}(X_{E_k^n})} |\mathsf{D}(\boldsymbol{u}^n)| \, dx\, dt \tag{2.47}$$

and we have

$$\psi\left(\frac{|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|}{C}\right)\le \psi\left(\fint_{Q_{16R_i}(X_{E_k^n})}|\mathbf{D}(\boldsymbol{u}^n)|\,dx\,dt\right)$$
$$\overset{(2.18)}{\le}\psi\left(\fint_{Q_{16R_i}(X_{E_k^n})}b^n\,dx\,dt\right)\le\psi(\lambda_k^n).$$

This implies

$$|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|\le C\lambda_k^n. \tag{2.48}$$

If maximum is achieved by the third term we have

$$\frac{|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|}{C\alpha_k^n}\le\fint_{Q_{16R_i}(X_{E_k^n})}|\mathbf{H}^n|\,dx\,dt. \tag{2.49}$$

Applying $\psi^*$

$$\psi^*\left(\frac{|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|}{C\alpha_k^n}\right)\le \psi^*\left(\fint_{Q_{16R_i}(X_{E_k^n})}|\mathbf{H}^n|\,dx\,dt\right)$$
$$\overset{(2.18)}{\le}\psi^*\left(\fint_{Q_{16R_i}(X_{E_k^n})}a^n\,dx\,dt\right)\le\psi(\lambda_k^n)$$

hence

$$|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|\le C(\psi^*)^{-1}(\psi(\lambda_k^n))\alpha_k^n=C\lambda_k^n. \tag{2.50}$$

It follows from the definition of $\alpha_k^n$ that the same holds if the extremum is achieved by the last term.

Consequently, using (2.37) and observing that $\psi^*(1)<\lambda_{min}=\psi(\lambda_{min})<\psi(\lambda_k^n)$, which implies that $\alpha_k^n=\frac{\lambda_k^n}{(\psi^*)^{-1}(\psi(\lambda_k^n))}<\lambda_k^n$ we get

$$|\mathbf{D}(\boldsymbol{u}^{n,k}(X))|\le C(\lambda_k^n+\alpha_k^n)\le 2C\lambda_k^n, \tag{2.51}$$

that implies (2.24).

Next, to show (2.27) we split $H_k^n$ as follows $H_k^n=H_k^{n,1}+H_k^{n,2}$, where

$$H_k^{n,1}:=\{(t,x)\in H_k^n;\ \psi(\lambda_k^n)<\psi^*(\mathcal{M}(a^n))+\psi(\mathcal{M}(b^n))\le\psi(\psi(\lambda_k^n))\} \tag{2.52}$$
$$H_k^{n,2}:=\{(t,x)\in H_k^n;\ \psi(\psi(\lambda_k^n))<\psi^*(\mathcal{M}(a^n))+\psi(\mathcal{M}(b^n))\} \tag{2.53}$$

and compute

$$\int_{Q_h\cap E_k^n}\left(|\mathbf{H}^n|+|\mathbf{H}|+|\bar{\mathbf{H}}|\right)|\mathbf{D}(\boldsymbol{u}^{n,k})|\,dxdt=\int_{Q_h\cap H_k^n}\dots+\int_{(Q_h\setminus H_k^n)\cap(F_n\cup G_n)}\dots$$
$$=\int_{Q_h\cap H_k^{n,1}}\dots+\int_{Q_h\cap H_k^{n,2}}\dots+\int_{(Q_h\setminus H_k^n)\cap(F_n\cup G_n)}\dots=:I_1^n+I_2^n+I_3^n.$$

First by using (2.18) and (2.24) we estimate $I_3^n$ with help of the Hölder inequality as

$$I_3^n\le C(h)\left(\|\mathbf{H}^n\|_{L^{\psi^*}}+\|\mathbf{H}\|_{L^{\psi^*}}+\|\bar{\mathbf{H}}\|_{L^{\psi^*}}\right)\|\lambda_k^n\chi_{F^n\cup G^n}\|_{L^\psi}\le CC^*\|\lambda_k^n\chi_{F^n\cup G^n}\|_{L^\psi}. \tag{2.54}$$

Next, denoting for a while $N := \|\lambda_k^n \chi_{F^n\cup G^n}\|_{L^\psi}$ we can use a definition of the norm in an Orlicz space to observe that

$$1 = \int_{F^n\cup G^n} \psi(\lambda_k^n/N)\, dx\, dt \implies N = \frac{\lambda_k^n}{\psi^{-1}(|F^n\cup G^n|^{-1})} \overset{(2.22)}{\leq} \frac{C(\lambda^*)^{r^k}}{\psi^{-1}(|F^n\cup G^n|^{-1})}$$

Finally, substituting this estimate into (2.54) and using (2.38) we conclude that

$$\limsup_{n\to\infty} I_3^n = 0.$$

Next, the term $I_2^n$ is estimated similarly. First, using the Hölder inequality, (2.18), (2.24) and the similar estimates for $N$ as above we find that

$$I_2^n \leq C\|\lambda_k^n \chi_{H_k^{n,2}}\|_{L^\psi} \leq \frac{C\lambda_k^n}{\psi^{-1}(|H_k^{n,2}|^{-1})}. \tag{2.55}$$

Consequently, applying (A.2) and (2.18), and using the concavity of $\psi^{-1}$ and the convexity of $\psi$ we find that[11]

$$I_2^n \leq \frac{C\lambda_k^n}{\psi^{-1}\left(\frac{\psi(\psi(\lambda_k^n))}{C^*}\right)} \leq \frac{CC^*\lambda_k^n}{\psi(\lambda_k^n)} \leq \frac{CC^*\lambda^*}{\psi(\lambda^*)}. \tag{2.56}$$

Thus, to finish the proof of (2.27) it remains to estimate $I_1^n$. Hence, using the Young inequality and (2.31) we get that

$$\begin{aligned} I_1^n &\leq C(h)\sqrt{k}\int_{Q_h\cap H_k^{n,1}} (|\mathbf{H}^n| + |\mathbf{H}| + |\bar{\mathbf{H}}|)\frac{\lambda_k^n}{\sqrt{k}}\, dx\, dt \\ &\overset{(2.31)}{\leq} \frac{C(h,C^*)}{\sqrt{k}} + \sqrt{k}\int_{Q_h\cap H_k^{n,2}} \psi\left(\frac{\lambda_k^n}{\sqrt{k}}\right)\, dx\, dt \\ &\overset{(A.2)}{\leq} \frac{C(h,C^*)}{\sqrt{k}} + \frac{C(h,C^*)\sqrt{k}\psi\left(\frac{\lambda_k^n}{\sqrt{k}}\right)}{\psi(\lambda_k^n)}. \end{aligned} \tag{2.57}$$

Next, using $\nabla_2$-condition for $\psi$ we observe that

$$\psi(s/2^m) \leq \frac{\psi(s)}{2^{m(1+\beta)}}.$$

Therefore setting $m := \frac{1}{2}\ln_2 k$ and substituting it into (2.57) we observe that

$$I_1^n \leq \frac{C(h,C^*)}{\sqrt{k}} + \frac{C(h,C^*)}{2^{\frac{1}{2}(1+\beta)\ln_2 k}} \leq C(h,C^*)\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k^{(1+\beta)}}}\right) \tag{2.58}$$

for some $\beta > 0$.

[11] The last inequality is a consequence of the fact that $\psi(\lambda)/\lambda$ is nondecreasing, which follows from the convexity of $\psi$. Indeed, we have

$$\psi(\lambda_1) = \psi\left((1-\tfrac{\lambda_1}{\lambda_2})0 + \tfrac{\lambda_1}{\lambda_2}\lambda_2\right) \leq \left(1-\tfrac{\lambda_1}{\lambda_2}\right)\psi(0) + \tfrac{\lambda_1}{\lambda_2}\psi(\lambda_2) = \tfrac{\lambda_1}{\lambda_2}\psi(\lambda_2).$$

Thus, it remains to prove (2.28). First using (A.12) and (2.44) we have

$$\begin{aligned}\limsup_{n\to\infty} -\int_0^T \langle \boldsymbol{u}^n_{,t}, \boldsymbol{u}^{n,k} g\rangle \, dt &= \limsup_{n\to\infty} \int_Q \boldsymbol{u}^{n,k}_{,t} \cdot (\boldsymbol{u}^n - \boldsymbol{u}^{n,k}) g \, dx \, dt \\ &\le \limsup_{n\to\infty} C(g) \int_{Q_h\cap E^n_k} |\boldsymbol{u}^{n,k}_{,t}||\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt.\end{aligned}$$

Next, for arbitrary $X \in E^n_k$ we can find $i$ such that $X \in Q_{R_i}(X_i)$. Then, similarly as above we have

$$\begin{aligned}R_i \alpha^n_k |\boldsymbol{u}^{n,k}_{,t}(X)| &\le C R_i^{-1} \sum_{j\in A_i} |\overline{\boldsymbol{u}^n}_{Q_{R_j}(X_j)} - \overline{\boldsymbol{u}^n}_{Q_{R_i}(X_i)}| \\ &\le C \fint_{Q_{4R_i}(X_i)} |\mathbf{D}(\boldsymbol{u}^n)| + \alpha^n_k(|\mathbf{G}^n| + |\mathbf{H}^n|) + \alpha^n_k R_i |\boldsymbol{f}^n| \, dx \, dt =: Y^n_i.\end{aligned}$$

Similarly,

$$\begin{aligned}\fint_{Q_{R_i}(X_i)} |\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt &\le C \fint_{Q_{4R_i}(X_i)} \left| \boldsymbol{u}^n - \fint_{Q_{4R_i}(X_i)} \boldsymbol{u}^n \, dx \, dt \right| \, dx \, dt \\ &\le C R_i Y^n_i.\end{aligned}$$

Consequently, we get

$$\int_{Q_h\cap E^n_k} |\boldsymbol{u}^{n,k}_{,t}||\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt \le C(\alpha^n_k)^{-1} \sum_i |Q_h \cap Q_{R_i}(X_i)| (Y^n_i)^2. \tag{2.59}$$

First, using the similar procedure as in the estimate $|\mathbf{D}(\boldsymbol{u}^{n,k})|$ we get that

$$Y^n_i \le C\lambda^n_k.$$

Therefore (2.59) can be estimated as

$$\int_{Q_h\cap E^n_k} |\boldsymbol{u}^{n,k}_{,t}||\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt \le C(\alpha^n_k)^{-1} \lambda^n_k \sum_i |Q_h \cap Q_{R_i}(X_i)| Y^n_i. \tag{2.60}$$

In addition, using the properties of the Whitney covering (A.4) and the definition of $Y^n_i$ we get that

$$\begin{aligned}&\int_{Q_h\cap E^n_k} |\boldsymbol{u}^{n,k}_{,t}||\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt \\ &\qquad \le C(\alpha^n_k)^{-1}\lambda^n_k \int_{Q_h\cap E^n_k} |\mathbf{D}(\boldsymbol{u}^n)| + \alpha^n_k(|\mathbf{G}^n| + |\mathbf{H}^n|) + \alpha^n_k |\boldsymbol{f}^n| \, dx \, dt. \end{aligned}\tag{2.61}$$

Consequently, using (2.19), (2.20) and (2.38) we have

$$\begin{aligned}&\limsup_{n\to\infty} \int_{Q_h\cap E^n_k} |\boldsymbol{u}^{n,k}_{,t}||\boldsymbol{u}^n - \boldsymbol{u}^{n,k}| \, dx \, dt \\ &\qquad \le C \limsup_{n\to\infty} (\alpha^n_k)^{-1}\lambda^n_k \int_{Q_h\cap H^n_k} |\mathbf{D}(\boldsymbol{u}^n)| + \alpha^n_k |\mathbf{H}^n| \, dx \, dt.\end{aligned}\tag{2.62}$$

Finally, we again split the remaining integral onto two parts to observe

$$(\alpha_k^n)^{-1}\lambda_k^n \int_{Q_h\cap H_k^n} |\mathbf{D}(\boldsymbol{u}^n)| + \alpha_k^n|\mathbf{H}^n| \,dx\,dt = (\alpha_k^n)^{-1}\lambda_k^n \int_{H_k^{n,1}} \dots + (\alpha_k^n)^{-1}\lambda_k^n \int_{H_k^{n,2}} \dots =: A_1^n + A_2^n.$$

Next, we proceed similarly as in the proof of (2.27). First using the Hölder inequality we can estimate the second term as

$$A_2^n \le C\lambda_k^n \left( \|\mathbf{H}^n\|_{L^{\psi^*}} \|\chi_{H_k^{n,2}}\|_{L^\psi} + (\alpha_k^n)^{-1}\|\mathbf{D}(\boldsymbol{u}^n)\|_{L^\psi}\|\chi_{H_k^{n,2}}\|_{L^{\psi^*}} \right).$$

Then by using (2.18) and the similar procedure as above and (2.37) we get

$$A_2^n \le C(h,C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)} + \frac{(\psi^*)^{-1}(\psi(\lambda_k^n))}{(\psi^*)^{-1}(\psi(\psi(\lambda_k^n)))}\right) \le C(h,C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)}\right)^\beta$$

for some $\beta > 0$. To estimate $A_1^n$ we use the Young inequality, (2.37) and (2.31) to get (see the similar procedure above)

$$\begin{aligned} A_1^n &\le \frac{C(h,C^*)}{\sqrt{k}} + \sqrt{k}|\{\psi(\lambda_k^n) < \mathcal{M}(a^n)\}| \left(\psi^*(\lambda_k^n(\alpha_k^n)^{-1}/\sqrt{k}) + \psi(\lambda_k^n/\sqrt{k})\right) \\ &\le C(h,C^*)\left(\frac{1}{\sqrt{k}} + \frac{\sqrt{k}\psi^*((\psi^*)^{-1}(\psi(\lambda_k^n))/\sqrt{k})}{\psi(\lambda_k^n)} + \frac{\sqrt{k}\psi(\lambda_k^n/\sqrt{k})}{\psi(\lambda_k^n)}\right) \\ &\le \frac{C(h,C^*)}{k^\beta}, \end{aligned} \tag{2.63}$$

where in the last inequality we used $\nabla_2$-condition. Thus, (2.28) follows. □

**3. Proof of Theorem 1.1.** In order to prove the existence of solutions we introduce a three-level approximation scheme based on the standard regularization of the selection $\mathbf{S}^*$ (that comes from **(A5)**), adding the penalty term that makes the problem subcritical[12] and then projecting such a problem to finite-dimensional Galerkin approximations. In the proof, starting from the Galerkin system for the penalized problem with regularized selection, we first let the regularization parameter tend to zero, then we take the limit from finite-dimensional approximations (with maximal monotone graph) to a continuous problem, and finally we investigate the limit when the penalty term vanishes.

**3.1.** $(\eta,\ell,n)$**–approximation.** Let us assume first that by **(A5)** there is a measurable selection $\mathbf{S}^*$ from the graph $\mathcal{A}$ having the properties collected in Lemma 2.2. We approximate $\mathbf{S}^*$ by smooth functions. For this reason, let $\rho \in C_0^\infty(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be a mollification kernel, i.e., a radially symmetric function with support in a unit ball $B(0,1) \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \rho \, d\boldsymbol{\xi} = 1$. For $\eta > 0$ we set $\rho^\eta(\boldsymbol{\xi}) = \frac{1}{\eta^{d^2}}\rho\left(\frac{\boldsymbol{\xi}}{\eta}\right)$ and define

$$\mathbf{S}^\eta(t,x,\boldsymbol{\xi}) = (\mathbf{S}^* * \rho^\eta)(t,x,\boldsymbol{\xi}) = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \mathbf{S}^*(t,x,\boldsymbol{\zeta})\rho^\eta(\boldsymbol{\xi}-\boldsymbol{\zeta})d\boldsymbol{\zeta}. \tag{3.1}$$

[12]In our understanding, it means that the velocity field is an admissible test function in the weak formulation of the balance of linear momentum.

Note that this definition can be used only in the case that the selection $\mathbf{S}^*$ is available. If this is not the case then according to **(A5)** we know that there is a measurable selection $\mathbf{D}^*$ and we can define $\mathbf{S}^\eta$ as

$$\mathbf{S}^\eta := (\mathbf{D}^* * \rho^\eta + \eta \mathbf{I})^{-1},$$

where an additional term $\eta\mathbf{I}$ guarantees that the mapping $\boldsymbol{\zeta} \mapsto (\mathbf{D}^* * \rho^\eta)(t, x, \boldsymbol{\zeta}) + \eta\boldsymbol{\zeta}$ is invertible. For clarity, we proceed with $\mathbf{S}^\eta$ defined in (3.1). One easily observes, using the convexity of $\psi$ and $\psi^*$ and the Jensen inequality, that the approximation $\mathbf{S}^\eta$ satisfies a condition analogous to (2.6).

Next, the penalty term $\frac{1}{n}|\boldsymbol{v}|^{2q'-2}\boldsymbol{v}$ is added to the equations in order to move the problem from supercritical case to subcritical case, and finally the Galerkin scheme is applied. The first limit, $\eta \to 0$, is easy since we work in finite-dimensional spaces and appropriate sequences converge strongly. In the next step, using the fact that the graph is monotone, we let $\ell \to \infty$ in the Galerkin system and apply Lemma 2.4. The main difficulty here consists in showing that assumption (2.11) of Lemma 2.4 is satisfied. On this level of approximation, for each $n \in \mathbb{N}$, the sufficient regularity of solutions (velocity) is due to the presence of the penalty term.

The final limit, $n \to \infty$, essentially uses the results of Subsection 2.3. Again Lemma 2.4 is used to verify that the limits $\mathbf{D}$ and $\mathbf{S}$ form a couple belonging to the graph $\mathcal{A}$. We shall observe that by means of the Lipschitz approximation method, which represents a key tool in the proof, we are able to verify the assumption (2.11).

Let $\{\boldsymbol{w}_i\}_{i=1}^\infty$ be an orthogonal basis of $\mathcal{V}_{\mathrm{div}}$ that is orthonormal in $L^2_{\boldsymbol{n},\mathrm{div}}$. Note that since $\mathcal{V}_{\mathrm{div}} \hookrightarrow L^2_{\boldsymbol{n},\mathrm{div}}$ compactly and densely, such a basis surely exists and can be constructed as eigenfunctions of the following problem

$$\sum_{k=1}^{d+2} (\nabla^k \boldsymbol{w}_i, \nabla^k \boldsymbol{v}) = \lambda_i (\boldsymbol{w}_i, \boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}_{\mathrm{div}}.$$

If $P^\ell$ denotes the orthogonal projection of $L^2(\Omega)^d$ on the span$\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_\ell\}$, it follows directly from the construction of the basis that

$$\|P^\ell \boldsymbol{v}\|_{\mathcal{V}_{\mathrm{div}}} \le C \|\boldsymbol{v}\|_{\mathcal{V}_{\mathrm{div}}} \qquad \text{for all } \ell \in \mathbb{N} \text{ and all } \boldsymbol{v} \in \mathcal{V}_{\mathrm{div}}. \tag{3.2}$$

Next, for an arbitrary fixed $\eta > 0$ and arbitrary fixed $\ell, n \in \mathbb{N}$ we introduce the following $(\eta, \ell, n)$-approximative problem: to find a vector-valued function $\boldsymbol{v}^\eta := \boldsymbol{v}^{\eta,\ell,n}$ such that $\boldsymbol{v}^\eta(t,x) := \sum_{i=1}^\ell c_i^{\eta,\ell}(t)\boldsymbol{w}_i(x)$, where the coefficients $c_i^{\eta,\ell}$ solve the following system of $\ell$ ordinary differential equations

$$\begin{aligned}(\boldsymbol{v}^\eta_{,t}, \boldsymbol{w}_i) + \frac{1}{n}\Big(|\boldsymbol{v}^\eta|^{2q'-2}\boldsymbol{v}^\eta, \boldsymbol{w}_i\Big) + (\mathbf{S}^\eta(\cdot, \mathbf{D}(\boldsymbol{v}^\eta)), \mathbf{D}(\boldsymbol{w}_i)) - (\boldsymbol{v}^\eta \otimes \boldsymbol{v}^\eta, \mathbf{D}(\boldsymbol{w}_i))& \\ + \gamma_*(\boldsymbol{v}^\eta, \boldsymbol{w}_i)_{\partial\Omega} = \langle \boldsymbol{b}, \boldsymbol{w}_i\rangle, \quad i = 1, \dots, \ell,& \\ \boldsymbol{v}^\eta(0) = P^\ell \boldsymbol{v}_0.&\end{aligned} \tag{3.3}$$

Using the standard Carathéodory theory it is not difficult to obtain a solution to (3.3) defined on a possibly short time interval $[0, T^*)$. This solution can be however extended to the whole time interval $[0, T]$ provided we can establish uniform estimates on $\boldsymbol{v}^\eta$ that are independent of $T^*$. We shall derive such estimates in the next subsection.

**3.2. Limit $\eta \to 0$.** Multiplying the $i$-th equation in (3.3) by $c_i^{\eta,\ell}$, summing over $i = 1, \dots, \ell$ and integrating the result over $(0,t)$, with $t \in (0,T)$, we find the identity

$$\begin{aligned} \frac{1}{2}\|\boldsymbol{v}^\eta(t)\|_2^2 + \int_{Q_t} \frac{1}{n}|\boldsymbol{v}^\eta|^{2q'} + \mathbf{S}^\eta(\cdot, \mathbf{D}(\boldsymbol{v}^\eta)) \cdot \mathbf{D}(\boldsymbol{v}^\eta)\, dx\, d\tau + \gamma_* \int_0^t \|\boldsymbol{v}^\eta\|_{2,\partial\Omega}^2 d\tau \\ = \int_0^t \langle \boldsymbol{b}, \boldsymbol{v}^\eta \rangle d\tau + \frac{1}{2}\|\boldsymbol{v}^\eta(0)\|_2^2, \end{aligned} \tag{3.4}$$

where we use notation $Q_t := (0,t) \times \Omega$. Using **(A4)**, or to be precise using Lemma 2.2, and using the Young and Korn inequalities we get

$$\begin{aligned} \sup_{t \in (0,T)} \|\boldsymbol{v}^\eta(t)\|_2^2 + \int_Q \psi(|\mathbf{D}(\boldsymbol{v}^\eta)|) + \psi^*(|\mathbf{S}^\eta(t,x,\mathbf{D}(\boldsymbol{v}^\eta))|) + \frac{1}{n}|\boldsymbol{v}^\eta|^{2q'}\, dx\, dt \\ + \gamma_* \int_0^T \|\boldsymbol{v}^\eta\|_{2,\partial\Omega}^2\, dt - \int_Q m\, dx\, dt \le C(\boldsymbol{b}, \boldsymbol{v}_0) \le C. \end{aligned} \tag{3.5}$$

Since the basis is smooth and finite dimensional, we conclude from (3.3) and (3.5) that

$$\int_0^T \left| \frac{d}{dt} c_i^{\eta,\ell}(t) \right|^{q'} dt \le C_l. \tag{3.6}$$

As a consequence of (3.5) and (3.6) we observe that $\{(c_1^{\eta,\ell}, \dots, c_\ell^{\eta,\ell})\}$ is bounded in $W^{1,q'}([0,T];\mathbb{R}^\ell)$ and by the Arzela-Ascoli theorem we can find a subsequence converging to some $(c_1^\ell, \dots, c_\ell^\ell)$ in $C^{0,\alpha}([0,T];\mathbb{R}^\ell)$ for some $\alpha \in (0,1)$. Since $\{\boldsymbol{w}_i\}_{i=1}^\ell$ is a finite fixed family of functions belonging to $W^{3,d}(\Omega)^d$ we are also able to find a subsequence that is again not relabeled such that

$$\boldsymbol{v}^\eta \to \boldsymbol{v} \qquad \text{strongly in } C^{0,\alpha}([0,T]; C^1(\bar{\Omega})), \tag{3.7}$$

$$\mathbf{S}^\eta(\cdot, \mathbf{D}(\boldsymbol{v}^\eta)) \rightharpoonup^* \mathbf{S} \qquad \text{weakly}^* \text{ in } L^\infty(Q), \tag{3.8}$$

$$\boldsymbol{v}^\eta_{,t} \rightharpoonup^* \boldsymbol{v}_{,t} \qquad \text{weakly}^* \text{ in } L^{q'}(0,T; C(\bar{\Omega})). \tag{3.9}$$

Using (3.7)–(3.9) it is quite standard to take the limit $\eta \to 0$ in (3.3) and to show that $\boldsymbol{v}^\ell := \boldsymbol{v} = \sum_{i=1}^\ell c_i^\ell \boldsymbol{w}_i$ and $\mathbf{S}^\ell := \mathbf{S}$ satisfy

$$\begin{aligned} (\boldsymbol{v}^\ell_{,t}, \boldsymbol{w}_i) + \frac{1}{n}\left(|\boldsymbol{v}^\ell|^{2q'-2}\boldsymbol{v}^\ell, \boldsymbol{w}_i\right) + (\mathbf{S}^\ell, \mathbf{D}(\boldsymbol{w}_i)) - (\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \mathbf{D}(\boldsymbol{w}_i)) \\ + \gamma_*(\boldsymbol{v}^\ell, \boldsymbol{w}_i)_{\partial\Omega} = \langle \boldsymbol{b}, \boldsymbol{w}_i \rangle, \quad i = 1, \dots, \ell, \\ \boldsymbol{v}(0) = P^\ell \boldsymbol{v}_0. \end{aligned} \tag{3.10}$$

It remains to show that $(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{S}^\ell)$ belongs to the graph $\mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$. Since $\mathbf{S}^*$ is the selection of the graph, according to Lemma 2.2, we have for all $\boldsymbol{\zeta}, \mathbf{B} \in \mathbb{R}^{d\times d}_{\text{sym}}$ and a.a. $(t,x) \in Q$

$$(\mathbf{S}^*(t,x,\boldsymbol{\zeta}) - \mathbf{S}^*(t,x,\mathbf{B})) \cdot (\boldsymbol{\zeta} - \mathbf{B}) \ge 0. \tag{3.11}$$

Adding and subtracting the term $(\mathbf{S}^*(t,x,\boldsymbol{\zeta}) - \mathbf{S}^*(t,x,\mathbf{B})) \cdot \mathbf{D}(\boldsymbol{v}^\eta)$ and then integrating the result w.r.t. the probability measure having the density $\rho^\eta(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})$, it follows from (3.11) that

$$\begin{aligned}\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} &(\mathbf{S}^*(t,x,\boldsymbol{\zeta}) - \mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}(\boldsymbol{v}^\eta) - \mathbf{B})\rho^\eta(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})d\boldsymbol{\zeta} \\ &\geq \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} (\mathbf{S}^*(t,x,\boldsymbol{\zeta}) - \mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})\rho^\eta(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})d\boldsymbol{\zeta}.\end{aligned} \tag{3.12}$$

Since the difference $(\mathbf{S}^*(t,x,\boldsymbol{\zeta}) - \mathbf{S}^*(t,x,\mathbf{B}))$ can be, for $|\boldsymbol{\zeta}| \leq \|\mathbf{D}(\boldsymbol{v}^\eta)\|_\infty + \eta$, estimated simply by a constant dependent on $\mathbf{B}$, then (3.12) can be rewritten as

$$\begin{aligned}\left(\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \mathbf{S}^*(t,x,\boldsymbol{\zeta})\rho^\eta(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})d\boldsymbol{\zeta} - \mathbf{S}^*(t,x,\mathbf{B})\right)\cdot(\mathbf{D}(\boldsymbol{v}^\eta) - \mathbf{B}) \geq \\ -C_\ell(\mathbf{B})\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} |\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta}|\rho^\eta(\mathbf{D}(\boldsymbol{v}^\eta) - \boldsymbol{\zeta})d\boldsymbol{\zeta}.\end{aligned} \tag{3.13}$$

Hence, using the strong convergence (3.7) we see that the right hand side of (3.13) tends to zero as $\eta \to 0$ and we get

$$\liminf_{\eta\to 0}\, (\mathbf{S}^\eta(t,x,\mathbf{D}\boldsymbol{v}^\eta) - \mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}(\boldsymbol{v}^\eta) - \mathbf{B}) \geq 0 \qquad \text{for a.a. } (t,x)\in Q\,, \tag{3.14}$$

which due to the strong convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$ and weak* convergence of $\mathbf{S}^\eta(x,\mathbf{D}(\boldsymbol{v}^\eta))$ yields that for all $\mathbf{B}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and for a.a. $(t,x)\in Q$

$$(\mathbf{S}^\ell - \mathbf{S}^*(t,x,\mathbf{B}))\cdot(\mathbf{D}(\boldsymbol{v}^\ell) - \mathbf{B}) \geq 0\,. \tag{3.15}$$

Thus, by Lemma 2.2, we conclude that

$$(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{S}^\ell) \in \mathcal{A}(t,x) \quad \text{ for a.a. } (t,x)\in Q\,.$$

**3.3. Limit $\ell\to\infty$.** Similarly as in preceeding subsection, multiplying the $i$-th equation in (3.10) by $c_i^\ell$ and summing the result over $i = 1,\dots,\ell$ and integrating over $(0,t)$, we get

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{v}^\ell(t)\|_2^2 + \int_{Q_t}\frac{1}{n}|\boldsymbol{v}^\ell|^{2q'} + \mathbf{S}^\ell\cdot\mathbf{D}(\boldsymbol{v}^\ell)\;dx\,d\tau + \gamma_*\int_0^t \|\boldsymbol{v}^\ell\|^2_{2,\partial\Omega}d\tau \\ = \int_0^t \langle \boldsymbol{b}, \boldsymbol{v}^\ell\rangle d\tau + \frac{1}{2}\|\boldsymbol{v}^\ell(0)\|_2^2.\end{aligned} \tag{3.16}$$

Similarly as above, this relation implies

$$\begin{aligned}\sup_{t\in(0,T)} \|\boldsymbol{v}^\ell(t)\|_2^2 + \int_Q \psi(|\mathbf{D}(\boldsymbol{v}^\ell)|) + \psi^*(|\mathbf{S}^\ell|) + \frac{1}{n}|\boldsymbol{v}^\ell|^{2q'}\;dx\,dt \\ + \gamma_*\int_0^T \|\boldsymbol{v}^\ell\|^2_{2,\partial\Omega}\;dt \leq C(\boldsymbol{b},\boldsymbol{v}_0,m) \leq C.\end{aligned} \tag{3.17}$$

As an easy consequence of (1.34), the Korn inequality and the standard interpolation, we also get that

$$\int_Q |\nabla\boldsymbol{v}^\ell|^q + |\mathbf{S}^\ell|^{r'} + |\boldsymbol{v}^\ell|^{\frac{(d+2)q}{d}}\;dx\,dt \leq C. \tag{3.18}$$

The next step concerns the uniform estimate on the time derivative of $\boldsymbol{v}^\ell$. Since now we are taking limit in infinite dimensional space, such estimate is not as trivial as in preceding subsection. First, we define

$$z := \max\left\{r, \frac{(d+2)q}{(d+2)q-2d}, 2q'\right\}. \tag{3.19}$$

In what follows, we will show that $\boldsymbol{v}^\ell_{,t}$ is bounded in $L^{z'}(0,T;\mathcal{V}^*_{\mathrm{div}})$. To establish such an uniform bound we use the fact that for any $\boldsymbol{u}\in\mathcal{V}_{\mathrm{div}}$ we have $(\boldsymbol{v}^\ell_{,t},\boldsymbol{u}) = (\boldsymbol{v}^\ell_{,t}, P^\ell(\boldsymbol{u}))$. Consequently, by using (3.10) and the continuity of $P^\ell$ (3.2), we have

$$\begin{aligned}\|\boldsymbol{v}^\ell_{,t}(t)\|_{\mathcal{V}^*_{\mathrm{div}}} &:= \sup_{\{\boldsymbol{u};\|\boldsymbol{u}\|_{\mathcal{V}_{\mathrm{div}}}=1\}} (\boldsymbol{v}^\ell_{,t}, P^\ell(\boldsymbol{u}))\\ &\le \sup_{\boldsymbol{u}} \Big| ((\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell), \nabla P^\ell(\boldsymbol{u})) + \langle \boldsymbol{b}, P^\ell(\boldsymbol{u})\rangle - (\mathbf{S}^\ell, \mathbf{D}(P^\ell(\boldsymbol{u})))\\ &\qquad -\frac{1}{n}(|\boldsymbol{v}^\ell|^{2q'-2}\boldsymbol{v}^\ell, P^\ell(\boldsymbol{u})) - \gamma_*(\boldsymbol{v}^\ell, P^\ell(\boldsymbol{u}))_{\partial\Omega}\Big|.\end{aligned} \tag{3.20}$$

In order to estimate the right hand side of (3.20) we first note that $\mathcal{V}_{\mathrm{div}}\hookrightarrow W^{1,\infty}$ and then observe that

$$\begin{aligned}\left|\int_\Omega \mathbf{S}^\ell\cdot\mathbf{D}(P^\ell(\boldsymbol{u}))\,dx\right| &\le C\|\mathbf{S}^\ell\|_{r'}\|\mathbf{D}(P^\ell(\boldsymbol{u}))\|_z \le C\|\mathbf{S}^\ell\|_{r'},\\ \left|\int_\Omega (\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell)\cdot\nabla P^\ell(\boldsymbol{u})\,dx\right| &\le \|\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell\|_{\frac{(d+2)q}{2d}}\|\nabla P^\ell(\boldsymbol{u})\|_z \le C\|\boldsymbol{v}^\ell\|^2_{\frac{(d+2)q}{d}},\\ \frac{1}{n}\left|\int_\Omega |\boldsymbol{v}^\ell|^{2q'-2}\boldsymbol{v}^\ell\cdot P^\ell(\boldsymbol{u})\,dx\right| &\le \frac{C}{n}\|\boldsymbol{v}^\ell\|^{2q'-1}_{2q'}\|P^\ell(\boldsymbol{u})\|_z \le \frac{C}{n}\|\boldsymbol{v}^\ell\|^{2q'-1}_{2q'}.\end{aligned}$$

To handle the right-hand side term we have (note that $q\le r$)

$$\left|\langle \boldsymbol{b}, P^\ell(\boldsymbol{u})\rangle\right| \le \|\boldsymbol{b}\|_{-1,q'}\|P^\ell(\boldsymbol{u})\|_{1,q} \le C\|\boldsymbol{b}\|_{-1,q'}.$$

And finally for boundary term, we have the estimate

$$\gamma_*\left|(\boldsymbol{v}^\ell, P^\ell(\boldsymbol{u}))_{\partial\Omega}\right| \le C\gamma_*\|\boldsymbol{v}^\ell\|_{2,\partial\Omega}\|P^\ell(\boldsymbol{u})\|_{2,\partial\Omega} \le \gamma_* C\|\boldsymbol{v}^\ell\|_{2,\partial\Omega}.$$

Using all these estimate in (3.20), taking then the $z'$ power, integrating the result w.r.t. $t\in(0,T)$ and using a priori estimates (3.17)–(3.18), we obtain the uniform bound

$$\|\boldsymbol{v}^\ell_{,t}\|_{L^{z'}(0,T;\mathcal{V}^*_{\mathrm{div}})} \le C. \tag{3.21}$$

Having (3.17) and (3.21), and using the Aubin-Lions lemma, we can extract a not relabeled subsequence such that

$$
\begin{aligned}
\boldsymbol{v}^\ell &\to \boldsymbol{v} && \text{strongly in } L^q(0,T;L^2(\Omega)^d), && (3.22)\\
\boldsymbol{v}^\ell &\overset{*}{\rightharpoonup} \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^d), && (3.23)\\
\mathbf{D}(\boldsymbol{v}^\ell) &\overset{*}{\rightharpoonup} \mathbf{D}(\boldsymbol{v}) && \text{weakly}^* \text{ in } L^\psi(Q), && (3.24)\\
\boldsymbol{v}^\ell &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^q(0,T;W^{1,q}_{\boldsymbol{n},\mathrm{div}}), && (3.25)\\
\mathbf{S}^\ell &\overset{*}{\rightharpoonup} \mathbf{S} && \text{weakly}^* \text{ in } L^{\psi^*}(Q), && (3.26)\\
\mathbf{S}^\ell &\rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), && (3.27)\\
\boldsymbol{v}^\ell_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^{z'}(0,T;\mathcal{V}^*_{\mathrm{div}}), && (3.28)\\
|\boldsymbol{v}^\ell|^{2q'-2}\boldsymbol{v}^\ell &\rightharpoonup |\boldsymbol{v}|^{2q'-2}\boldsymbol{v} && \text{weakly in } L^{\frac{2q'}{2q'-1}}(Q), && (3.29)\\
\boldsymbol{v}^\ell &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T;L^2(\partial\Omega)^d). && (3.30)
\end{aligned}
$$

Having all these convergence results, it is then easy to show that

$$
\begin{aligned}
&\langle \boldsymbol{v}_{,t},\boldsymbol{w}\rangle + \frac{1}{n}\Big(|\boldsymbol{v}|^{2q'-2}\boldsymbol{v},\boldsymbol{w}\Big) + (\mathbf{S},\mathbf{D}(\boldsymbol{w}_i)) - (\boldsymbol{v}\otimes\boldsymbol{v},\mathbf{D}(\boldsymbol{w}))\\
&\qquad + \gamma_*(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega} = \langle \boldsymbol{b},\boldsymbol{w}\rangle, \qquad \text{for all } \boldsymbol{w}\in\mathcal{V}_{\mathrm{div}} \text{ and a.a. } t\in(0,T),
\end{aligned} \tag{3.31}
$$

and that

$$
\lim_{t\to 0_+}\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2 = 0.
$$

Moreover, using the density of $\mathcal{V}_{\mathrm{div}}$ in any $W^{1,q}_{\boldsymbol{n},\mathrm{div}}$, we can conclude that (3.31) holds for all $\boldsymbol{w}\in Y$, where $Y := \{\boldsymbol{u}\in W^{1,z}_{\boldsymbol{n},\mathrm{div}}; \boldsymbol{u}\in L^2(\partial\Omega)^d\}$ with $z$ defined in (3.19). Note that the space $Y$ is well defined since we assume that $q > \frac{2d}{d+2}$. Moreover, we can repeat the procedure as in (3.20), and by using (3.17) and (3.18), we can deduce that

$$
\|\boldsymbol{v}_{,t}\|_{L^{z'}(0,T;Y^*)} \le C. \tag{3.32}
$$

To finish this subsection, we need to show that $(\mathbf{D}(\boldsymbol{v}),\mathbf{S})\in\mathcal{A}(t,x)$ for a.a. $(t,x)\in Q$. To do so, we set in (3.31) $\boldsymbol{w} := \varrho^\varepsilon * \varrho^\varepsilon * \boldsymbol{v}^j$ for some $j\in\mathbb{N}$ and for some standard mollifier $\varrho^\varepsilon$ depending only on time $t$. Here, $*$ denotes the standard convolution operator w.r.t. time variable, i.e., for $\varphi\in L^1(0,T;X)$ and $\varphi\equiv 0$ on $\mathbb{R}\setminus(0,T)$:

$$
(\varrho*\varphi)(t) = \int_{-\infty}^{\infty}\varrho^\varepsilon(t-\tau)\varphi(\tau)d\tau\,.
$$

Hence, if we define $\boldsymbol{v}^{\varepsilon,j} \overset{\mathrm{def}}{=} \varrho^\varepsilon * \varrho^\varepsilon * \boldsymbol{v}^j$ we get after integration over $(s_0,s)\subset(0,T)$ with $\varepsilon < \frac{1}{2}\min\{s_0, T-s\}$

$$
\begin{aligned}
&\int_{s_0}^s \langle \boldsymbol{v}_{,t},\boldsymbol{v}^{\varepsilon,j}\rangle\,dt - \int_{s_0}^s(\boldsymbol{v}\otimes\boldsymbol{v},\mathbf{D}(\boldsymbol{v}^{\varepsilon,j}))\,dt + \int_{s_0}^s(\mathbf{S},\mathbf{D}(\boldsymbol{v}^{\varepsilon,j}))\,dt\\
&\qquad + \gamma_*\int_{s_0}^s(\boldsymbol{v},\boldsymbol{v}^{\varepsilon,j})_{\partial\Omega}\,dt + \frac{1}{n}\int_{s_0}^s(|\boldsymbol{v}|^{2q'-2}\boldsymbol{v},\boldsymbol{v}^{\varepsilon,j})\,dt = \int_{s_0}^s\langle \boldsymbol{b},\boldsymbol{v}^{\varepsilon,j}\rangle\,dt.
\end{aligned} \tag{3.33}
$$

The sequence of functions $\{\boldsymbol{v}^{\varepsilon,j}\}$ is weakly convergent to $\boldsymbol{v}^\varepsilon$ in $L^q(0,T;W^{1,q}_{\mathbf{n},\mathrm{div}}(\Omega))$ as $j\to\infty$ and since the space $L^\psi(Q)$ is reflexive then also $\nabla\boldsymbol{v}^{\varepsilon,j}$ is weakly convergent in $L^\psi(Q)$. Moreover, we also have that $\boldsymbol{v}^{\varepsilon,j}$ converges weakly to $\boldsymbol{v}^\varepsilon$ in $L^{2q'}(Q)$. Consequently, taking the limit in (3.33) $j \to \infty$ we find that

$$\begin{aligned} \lim_{j\to\infty} \int_{s_0}^s \langle \boldsymbol{v}_{,t}, \boldsymbol{v}^{\varepsilon,j}\rangle\, dt - \int_{s_0}^s (\boldsymbol{v}\otimes\boldsymbol{v}, \mathbf{D}(\boldsymbol{v}^\varepsilon))\, dt + \int_{s_0}^s (\mathbf{S}, \mathbf{D}(\boldsymbol{v}^\varepsilon))\, dt \\ + \gamma_* \int_{s_0}^s (\boldsymbol{v}, \boldsymbol{v}^\varepsilon)_{\partial\Omega}\, dt + \frac{1}{n}\int_{s_0}^s (|\boldsymbol{v}|^{2q'-2}\boldsymbol{v}, \boldsymbol{v}^\varepsilon)\, dt = \int_{s_0}^s \langle \boldsymbol{b}, \boldsymbol{v}^\varepsilon\rangle\, dt. \end{aligned} \tag{3.34}$$

Then, we can observe that for a.a. $s_0$, $s$ such that $0 < s_0 < s < T$ it follows

$$\begin{aligned} \lim_{j\to\infty} \int_{s_0}^s \langle \boldsymbol{v}_{,t}, \boldsymbol{v}^{\varepsilon,j}\rangle\, dt &= \lim_{j\to\infty} \int_{s_0}^s \langle \boldsymbol{v}_{,t}, (\varrho^\varepsilon * \varrho^\varepsilon * \boldsymbol{v}^j)\rangle\, dt \\ &= \lim_{j\to\infty} \int_{s_0}^s \langle (\varrho^\varepsilon * \boldsymbol{v}_{,t}), (\varrho^\varepsilon * \boldsymbol{v}^j)\rangle\, dt = \int_{s_0}^s ((\varrho^\varepsilon * \boldsymbol{v})_{,t}, (\varrho^\varepsilon * \boldsymbol{v}))\, dt \\ &= \int_{s_0}^s \frac{1}{2}\frac{d}{dt}\|\varrho^\varepsilon * \boldsymbol{v}\|_2^2\, dt = \frac{1}{2}\|\varrho^\varepsilon * \boldsymbol{v}(s)\|_2^2 - \frac{1}{2}\|\varrho^\varepsilon * \boldsymbol{v}(s_0)\|_2^2. \end{aligned}$$

Next, we take the limit $\varepsilon \to 0$ and obtain for almost all $s_0$, $s$, namely for all Lebesgue points of the function $\boldsymbol{v}(t)$, that

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty} \int_{s_0}^s \langle \boldsymbol{v}_{,t}, \boldsymbol{v}^{\varepsilon,j}\rangle\, dt = \frac{1}{2}\|\boldsymbol{v}(s)\|_2^2 - \frac{1}{2}\|\boldsymbol{v}(s_0)\|_2^2. \tag{3.35}$$

Next, we focus on taking the limit $\varepsilon \to 0$ in the remaining terms in (3.34). First note that due to a priori estimates (3.22)–(3.29) the limiting procedure in the second, the fourth, the fifth and the sixth terms is quite standard. Also note that since $\operatorname{div} \boldsymbol{v} = 0$ we have that $(\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{v}) = 0$. It remains to discuss the convergence result for the third term in (3.34). First, it is easy to observe that

$$\int_{s_0}^s (\mathbf{S}, (\varrho^\varepsilon * \varrho^\varepsilon * \mathbf{D}(\boldsymbol{v})))\, dt = \int_{s_0}^s ((\varrho^\varepsilon * \mathbf{S}), (\varrho^\varepsilon * \mathbf{D}(\boldsymbol{v})))\, dt$$

Both of the sequences $\{\varrho^\varepsilon * \mathbf{S}\}$ and $\{\varrho^\varepsilon * \mathbf{D}(v)\}$ converge in measure in $Q$ due to [28, Prop. 2.3]. Moreover, since $\psi$ and $\psi^*$ are convex nonnegative functions, then the weak lower semicontinuity and estimate (3.5) imply that the integral

$$\int_0^T \int_\Omega \psi(|\mathbf{D}(v)|) + \psi^*(|\mathbf{S}|)\, dx\, dt$$

is finite. By [28, Prop. 2.4], the sequences $\{\varrho^\varepsilon * \bar{\mathbf{S}}\}$ and $\{\varrho^\varepsilon * \mathbf{D}\boldsymbol{v}\}$ are uniformly bounded and according to [28, Lem. 2.1] we have

$$\begin{aligned} \varrho^\varepsilon * \mathbf{D}(\boldsymbol{v}) &\to \mathbf{D}(\boldsymbol{v}) && \text{modularly in } L^\psi(Q_{s_0,s}), \\ \varrho^\varepsilon * \mathbf{S} &\to \mathbf{S} && \text{modularly in } L^{\psi^*}(Q_{s_0,s}), \end{aligned}$$

where $Q_{s_0,s} := (s_0, s)\times\Omega$. Applying [28, Prop. 2.2] allows us to conclude that

$$\lim_{\varepsilon\to 0} \int_{s_0}^s ((\varrho^\varepsilon * \mathbf{S}), (\varrho^\varepsilon * \mathbf{D}(\boldsymbol{v})))\, dt = \int_{s_0}^s (\mathbf{S}, \mathbf{D}(\boldsymbol{v}))\, dt. \tag{3.36}$$

Consequently, we can take the limit $\varepsilon \to 0$ in (3.34) and obtain that

$$\frac{1}{2}\|\boldsymbol{v}(s)\|_2^2 + \int_{s_0}^s (\mathbf{S}, \mathbf{D}(\boldsymbol{v})) + \frac{1}{n}\|\boldsymbol{v}\|_{2q'}^{2q'} + \gamma_*\|\boldsymbol{v}\|_{2,\partial\Omega}^2\, dt = \int_{s_0}^s \langle \boldsymbol{b}, \boldsymbol{v}\rangle\, dt + \frac{1}{2}\|\boldsymbol{v}(s_0)\|_2^2 \tag{3.37}$$

is valid for almost all $0 < s_0 < s < T$. Since we already know that the initial condition is attained in $L^2(\Omega)^d$ we can set in (3.37) $\lim_{s_0 \to 0+}$ (here the limit is taken over all possible $s_0$) and we can conclude that

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \int_0^t (\mathbf{S}, \mathbf{D}(\boldsymbol{v})) + \frac{1}{n}\|\boldsymbol{v}\|_{2q'}^{2q'} + \gamma_* \|\boldsymbol{v}\|_{2,\partial\Omega}^2 \, d\tau = \int_0^t \langle \boldsymbol{b}, \boldsymbol{v} \rangle \, d\tau + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 . \quad (3.38)$$

On the other hand, letting $\ell \to \infty$ in (3.16) and using weak lower semicontinuity of norms it is easy to deduce with help of (3.22)–(3.29) that

$$\begin{aligned} \limsup_{\ell\to\infty} \int_0^t (\mathbf{S}^\ell, \mathbf{D}(\boldsymbol{v}^\ell)) \, d\tau \leq - \int_0^t \frac{1}{n}\|\boldsymbol{v}\|_{2q'}^{2q'} - \gamma_* \|\boldsymbol{v}\|_{2,\partial\Omega}^2 \; + \langle \boldsymbol{b}, \boldsymbol{v} \rangle \, d\tau \\ + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 - \frac{1}{2}\|\boldsymbol{v}(t)\|_2^2. \end{aligned} \quad (3.39)$$

Consequently, comparing (3.38) and (3.39) we get for almost all $t \in (0,T)$ that

$$\limsup_{\ell\to\infty} \int_{Q_t} \mathbf{S}^\ell \cdot \mathbf{D}(\boldsymbol{v}^\ell) \, dx \, d\tau \leq \int_{Q_t} \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}) \, dx \, d\tau. \quad (3.40)$$

Thus, by virtue of Lemma 2.4 we observe that $(\mathbf{D}(\boldsymbol{v}), \mathbf{S}) \in \mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$.

**3.4. Limit $n \to \infty$.** In this subsection, $(\boldsymbol{v}^n, \mathbf{S}^n)$ denotes the couple satisfying (3.31). From weak lower semicontinuity of norms, convexity of $\psi$ and $\psi^*$ and from (3.17), (3.18) and (3.32) we observe that

$$\begin{aligned} \sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_2^2 + \int_Q \psi(|\mathbf{D}(\boldsymbol{v}^n)|) + \psi^*(|\mathbf{S}^n|) + \frac{1}{n}|\boldsymbol{v}^n|^{2q'} \, dx \, dt \\ + \int_0^T \|\nabla \boldsymbol{v}^n\|_q^q + \|\mathbf{S}^n\|_{r'}^{r'} + \|\boldsymbol{v}^n\|_{\frac{(d+2)q}{d}}^{\frac{(d+2)q}{d}} + \gamma_* \|\boldsymbol{v}^n\|_{2,\partial\Omega}^2 \, dt \\ + \|\boldsymbol{v}^n_{,t}\|_{L^{z'}(0,T;Y^*)} \leq C. \end{aligned} \quad (3.41)$$

Furthermore, we introduce the pressure: for a.a. $t \in (0,T)$ we define $\{p_1^n\}$ and $\{p_2^n\}$ through

$$\begin{aligned} p_1^n &:= \mathcal{L}^1(\mathbf{S}^n), \\ p_2^n &:= -\mathcal{L}^1(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) + \gamma_* \mathcal{L}^2(\boldsymbol{v}^n) + \frac{1}{n}\mathcal{L}^3(|\boldsymbol{v}^n|^{2q'-2}\boldsymbol{v}^n) - \mathcal{L}^4(\boldsymbol{b}), \end{aligned}$$

where the operators $\mathcal{L}^i$ are defined in Lemma C.1. Note that it is exactly the same as solving, for all $\varphi \in W^{2,\infty}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, the following problems (for almost all time in $(0,T)$)

$$(p_1^n, \triangle\varphi) = (\mathbf{S}^n, \nabla^2\varphi), \qquad \int_\Omega p_1^n dx = 0\,, \quad (3.42)$$

$$\begin{aligned} (p_2^n, \triangle\varphi) = -(\boldsymbol{v}^n \otimes \boldsymbol{v}^n, \nabla^2\varphi) + \gamma_*(\boldsymbol{v}^n, \nabla\varphi)_{\partial\Omega} + \frac{1}{n}(|\boldsymbol{v}^n|^{2q'-2}\boldsymbol{v}^n, \nabla\varphi) \\ - \langle \boldsymbol{b}, \nabla\varphi \rangle\,, \qquad \int_\Omega p_2^n dx = 0\,. \end{aligned} \quad (3.43)$$

Note that we include into the right hand side of (3.43) the terms that are compact.

Setting $p^n := p_1^n + p_2^n$, we observe (applying the result of Lemma C.1) that $p^n \in L^{1+\varepsilon}(0,T;L^{1+\varepsilon}(\Omega))$ with an $\varepsilon > 0$ and $\int_\Omega p^n \, dx = 0$ for a.a. $t \in (0,T)$. In addition, for fixed $\boldsymbol{v}^n$ and $\mathbf{S}^n$, the pressure $p^n$ constructed by the above scheme is unique and satisfies (this can be deduced by using the Helmholtz decomposition)

$$\begin{aligned}\langle \boldsymbol{v}^n_{,t}, \boldsymbol{w}\rangle + (\mathbf{S}^n, \mathbf{D}(\boldsymbol{w})) - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n, \mathbf{D}(\boldsymbol{w})) + \gamma_*(\boldsymbol{v}^n, \boldsymbol{w})_{\partial\Omega} + \tfrac{1}{n}(|\boldsymbol{v}^n|^{2q'-2}\boldsymbol{v}^n, \boldsymbol{w}) \\ = (p^n, \operatorname{div} \boldsymbol{w}) + \langle \boldsymbol{b}, \boldsymbol{w}\rangle \qquad \text{for all } \boldsymbol{w} \in W^{1,\infty}_{\boldsymbol{n}} \text{ and a.a. } t \in (0,T).\end{aligned} \tag{3.44}$$

Next, we use (3.41) and with the help of Lemma C.1 we establish uniform estimates for the pressures. First, since $\psi$ satisfies $\Delta_2$- and $\nabla_2$-conditions, we can use (C.9) to get for almost all $t \in (0,T)$ that

$$\int_\Omega \psi^*(p_1^n) \, dx \le C \left( 1 + \int_\Omega \psi^*(|\mathbf{S}^n|) \, dx \right).$$

Consequently, integrating the result w.r.t. time and using (3.41), we deduce that

$$\int_Q \psi^*(p_1^n) \, dx \, dt \le C. \tag{3.45}$$

To estimate $p_2^n$, we refer to Lemma C.1 with $z$ defined in (3.19): thus for a.a. $t \in (0,T)$ (see [15], where such an estimate is derived directly) we have

$$\|p_2^n\|_{z'} \le C \left( \|\boldsymbol{v}^n\|^2_{2z'} + \gamma_* \|\boldsymbol{v}^n\|_{\frac{z'}{d'},\partial\Omega} + \frac{1}{n} \| |\boldsymbol{v}^n|^{2q'-1} \|_{\max(\frac{2q'}{2q'-1}, \frac{dz'}{d+z'})} + \|\boldsymbol{b}\|_{-1,z'} \right).$$

Due to the definition of $z$, we can use a continuous embedding to conclude that

$$\|p_2^n\|_{z'} \le C \left( \|\boldsymbol{v}^n\|^2_{\frac{q(d+2)}{d}} + \gamma_* \|\boldsymbol{v}^n\|_{2,\partial\Omega} + \|\boldsymbol{b}\|_{-1,q'} + \frac{1}{n} \|\boldsymbol{v}^n\|^{2q'-1}_{2q'} \right).$$

Hence, applying the $z'$-power, using the definition of $z$, integrating with respect to time and using (3.41) we get that

$$\int_Q |p_2^n|^{z'} \, dx \, dt \le C. \tag{3.46}$$

Finally, using all estimates above and (3.44) we can get that

$$\int_0^T \|\boldsymbol{v}^n_{,t}\|^{z'}_{(W^{1,z}_{\boldsymbol{n}})^*} \, dt \le C. \tag{3.47}$$

As a consequence of the uniform estimates (3.41), (3.45), (3.46), (3.47) and the Aubin-

Lions lemma, we can find not relabeled subsequences such that

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;W^{\alpha,q}(\Omega)^d),\ \text{for all } \alpha \in [0,1), \tag{3.48}$$

$$\boldsymbol{v}^n \overset{*}{\rightharpoonup} \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \tag{3.49}$$

$$\mathbf{D}(\boldsymbol{v}^n) \overset{*}{\rightharpoonup} \mathbf{D}(\boldsymbol{v}) \qquad \text{weakly}^* \text{ in } L^\psi(Q), \tag{3.50}$$

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^q(0,T;W^{1,q}_{\boldsymbol{n},\mathrm{div}}), \tag{3.51}$$

$$\mathbf{S}^n \overset{*}{\rightharpoonup} \mathbf{S} \qquad \text{weakly} * \text{ in } L^{\psi^*}(Q), \tag{3.52}$$

$$\mathbf{S}^n \rightharpoonup \mathbf{S} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), \tag{3.53}$$

$$\boldsymbol{v}^n_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{z'}(0,T;(W^{1,z}_{\boldsymbol{n}})^*), \tag{3.54}$$

$$\frac{1}{n}|\boldsymbol{v}^n|^{2q'-2}\boldsymbol{v}^n \to 0 \qquad \text{strongly in } L^1(Q), \tag{3.55}$$

$$p_1^n \rightharpoonup p_1 \qquad \text{weakly in } L^1(0,T;L^1(\Omega)) \cap L^{\psi^*}(Q), \tag{3.56}$$

$$p_2^n \rightharpoonup p_2 \qquad \text{weakly in } L^{z'}(0,T;L^{z'}(\Omega)), \tag{3.57}$$

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)^d). \tag{3.58}$$

Next, using the trace theorem (Lemma D.1), (3.48) and (3.58), we can deduce that

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^s(0,T;L^s(\partial\Omega)^d) \ \text{ for all } s \in [1,2). \tag{3.59}$$

Moreover, using the construction of the pressure and continuity of the operators $\mathcal{L}^i$ (see also [15], p. 700 for details) we can deduce from (3.41), (3.43), (3.48), (3.55), (3.57) and (3.59) that

$$p_2^n \to p_2 \qquad \text{strongly in } L^s(0,T;L^s(\Omega)) \ \text{ for all } s \in [1,z'). \tag{3.60}$$

Having all these convergence results, it is then easy to show that

$$\begin{gathered}\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle + (\mathbf{S}, \mathbf{D}(\boldsymbol{w})) - (\boldsymbol{v}\otimes\boldsymbol{v}, \mathbf{D}(\boldsymbol{w})) + \gamma_*(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega} = \langle \boldsymbol{b}, \boldsymbol{w}\rangle, +(p, \operatorname{div}\boldsymbol{w}) \\ \text{for all } \boldsymbol{w} \in W^{1,1}_{\boldsymbol{n}} \text{ such that } \mathbf{D}(\boldsymbol{w}) \in L^\infty(\Omega)^{d\times d} \text{ and a.a. } t \in (0,T).\end{gathered} \tag{3.61}$$

Thus, to finish the proof, it remains to show that $(\mathbf{D}(\boldsymbol{v}), \mathbf{S}) \in \mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$. To do that we apply Lemma 2.5. Indeed, we define

$$\begin{aligned}\boldsymbol{u}^n &:= \boldsymbol{v}^n - \boldsymbol{v},\\ \mathbf{H}^n &:= -\mathbf{S}^n + p_1^n\mathbf{I},\\ \mathbf{H} &:= -\mathbf{S} + p_1\mathbf{I},\\ \bar{\mathbf{H}} &:= \mathbf{S}^*(t,x,\mathbf{D}(\boldsymbol{v})),\\ \boldsymbol{f}^n &:= -\frac{1}{n}|\boldsymbol{v}^n|^{2q'-2}\boldsymbol{v}^n,\\ \mathbf{G}^n &:= \boldsymbol{v}^n\otimes\boldsymbol{v}^n - \boldsymbol{v}\otimes\boldsymbol{v} + (p_2^n - p_2)\mathbf{I}\,.\end{aligned}$$

Hence, using (3.44), (3.61), (3.48)–(3.60), we see that all assumptions of Lemma 2.5 are satisfied. Then for some nonnegative $\varphi \in \mathcal{D}(\Omega)$ and $\eta \in \mathcal{D}(0,T)$ we define $Q_2$ as the set where $\eta(t)\varphi(x) \equiv 1$ and $Q_1 := \operatorname{supp}\eta\varphi$. Then for such given set $Q_1$ and given $\lambda^*$ and $k$ we can find a sequence $\{\boldsymbol{u}^{n,k}\}_{n=1}^\infty$ from Lemma 2.5 such that

$$\begin{aligned}\mathbf{D}(\boldsymbol{u}^{n,k}) &\overset{*}{\rightharpoonup} \mathbf{0} \text{ weakly}^* \text{ in } L^\infty(Q_1)^{d\times d},\\ \boldsymbol{u}^{n,k} &\to \mathbf{0} \text{ strongly in } L^s(Q_1)^d,\ \text{for all } s < \infty\end{aligned} \tag{3.62}$$

Next, we set in (3.44) $\boldsymbol{w} := \boldsymbol{u}^{n,k}\varphi\eta$ and integrate the result w.r.t. time $(0,T)$. Moreover, having (3.62), we see that we can add and subtract the limiting identity (3.61) to deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_0^T \langle \boldsymbol{u}^n_{,t}, \boldsymbol{u}^{n,k}\varphi\rangle\eta - (\mathbf{H}^n, \mathbf{D}(\boldsymbol{u}^{n,k})\varphi)\eta \; dt \\ = \lim_{n\to\infty}\int_Q \mathbf{G}^n\cdot\mathbf{D}(\boldsymbol{u}^{n,k}\varphi)\eta + \mathbf{H}^n\cdot(\boldsymbol{u}^{n,k}\otimes\nabla\varphi)\eta + \boldsymbol{f}^n\cdot\boldsymbol{u}^{n,k}\varphi\eta \; dx\, dt.\end{aligned} \tag{3.63}$$

Due to the strong convergence of $\mathbf{G}^n$, $\boldsymbol{u}^{n,k}$ and $\boldsymbol{f}^n$ we observe that

$$\lim_{n\to\infty}\int_0^T \langle \boldsymbol{u}^n_{,t}, \boldsymbol{u}^{n,k}\varphi\rangle\eta - (\mathbf{H}^n, \mathbf{D}(\boldsymbol{u}^{n,k})\varphi)\eta \; dt = 0. \tag{3.64}$$

Consequently, using (2.28) we find that

$$\limsup_{n\to\infty} -\int_{Q\setminus E^n_k} \mathbf{H}^n\cdot\mathbf{D}(\boldsymbol{u}^{n,k})\varphi\eta \; dt \le C(\varphi,\eta)\left(\frac{\lambda^*}{\psi(\lambda^*)}+\frac{1}{k}\right)^{\beta}. \tag{3.65}$$

Therefore, using the definition of $Q\setminus E^n_k$ and $\mathbf{H}^n$ we get that (note that the pressure term vanishes since $\operatorname{tr}\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v})=0$ a.e. in $Q$.)

$$\limsup_{n\to\infty}\int_{Q\setminus E^n_k}(\mathbf{S}^n-\mathbf{S})\cdot\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v})\varphi\eta \; dt \le C(\varphi,\eta)\left(\frac{\lambda^*}{\psi(\lambda^*)}+\frac{1}{k}\right)^{\beta}.$$

Using (3.50) and again (2.28) we can also deduce from this relation that

$$\limsup_{n\to\infty}\int_{Q\setminus E^n_k}(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}(\boldsymbol{v}))\cdot\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v})\varphi\eta \; dt \le C(\varphi,\eta)\left(\frac{\lambda^*}{\psi(\lambda^*)}+\frac{1}{k}\right)^{\beta}.$$

Since the graph $\mathcal{A}$ is monotone, we observe that the previous estimate, the uniform bound (3.41), the estimate (2.26) and the Hölder inequality imply that

$$\begin{aligned}\limsup_{n\to\infty}\int_Q |(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v})\varphi\eta|^{\frac12}\; dx\, dt \le \limsup_{n\to\infty}\int_{Q\setminus E^n_k}\ldots + \int_{E^n_k}\ldots \\ \le C(\varphi,\eta)\left(\frac{\lambda^*}{\psi(\lambda^*)}+\frac{1}{k}\right)^{\frac{\beta}{2}} + \limsup_{n\to\infty}\sqrt{|E^n_k|} \le C(\varphi,\eta)\left(\frac{\lambda^*}{\psi(\lambda^*)}+\frac{1}{k}\right)^{\frac{\beta}{2}} + \frac{C(\varphi,\eta)}{\sqrt{\psi(\lambda^*)}}.\end{aligned}$$

Consequently, letting $\lambda^*\to\infty$ we obtain that

$$\limsup_{n\to\infty}\int_{Q_2}|(\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v})|^{\frac12}\; dx\, dt = 0.$$

Since $Q_2$ was chosen arbitrarily we can deduce that at least for subsequence

$$g^n := (\mathbf{S}^n-\mathbf{S}^*(t,x,\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^n-\boldsymbol{v}) \to 0 \text{ a.e. in } Q. \tag{3.66}$$

Next, we apply a biting Lemma (see [4, p. 655]) to conclude that there is a $g\in L^1(\Omega)$, a subsequence of $\{g^n\}$ (that we do not relabel) and non-increasing sequence of sets $E_j\subset Q$ such that $\lim_{j\to\infty}|E_j|=0$ so that for arbitrary $j$ we have

$$g^n \rightharpoonup g \text{ weakly in } L^1(Q\setminus E_j).$$

The last statement is equivalent to the condition (see [21, Theorem 1.3, Chapter 8]):

$$\text{For all } \eta > 0 \text{ there is } \delta > 0 : \text{if } F \subset Q \setminus E_j \text{ and } |F| < \delta \text{ then } \sup_n \int_F g^n dx \le \eta\,. \tag{3.67}$$

Referring to Vitali's theorem, we deduce from (3.66) and (3.67) that

$$g^n \to 0 \text{ strongly in } L^1(Q \setminus E_j).$$

Consequently, using (3.50) we can finally deduce that

$$\limsup_{n\to\infty} \int_{Q\setminus E_j} \mathbf{S}^n \cdot \mathbf{D}(\boldsymbol{v}^n)\, dx\, dt = \int_{Q\setminus E_j} \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v})\, dx\, dt$$

Therefore applying Lemma 2.2 we get that $(\mathbf{D}(\boldsymbol{v}(t,x)), \mathbf{S}(t,x)) \in \mathcal{A}(t,x)$ for a.a. $(t,x) \in Q \setminus E_j$. Since the measure of $E_j$ tends to zero, we immediately observe that $(\mathbf{D}(\boldsymbol{v}(t,x)), \mathbf{S}(t,x)) \in \mathcal{A}(t,x)$ for a.a. $(t,x) \in Q$, which completes the proof.

**Appendix. Parabolic Lipschitz approximation of Sobolev functions.** In this section we recall the key tool used in the proof of the main theorem. It is a generalization of the result established in [19] within the framework of the standard Lebesgue spaces to the framework using Orlicz spaces.

We start with the definition of the modified parabolic metric $d_\alpha$ on $\mathbb{R}^{d+1}$ and corresponding balls. For $X, Y \in \mathbb{R}^{d+1}$ where $X := (t,x)$, $Y := (s,y)$, and for $R > 0$, $\alpha > 0$, $A \subset \mathbb{R}^{d+1}$ we define

$$\begin{aligned} d_\alpha(X,Y) &:= \max\left(|x-y|, \tfrac{|t-s|^{1/2}}{\alpha^{1/2}}\right), \\ Q_R^\alpha(X) &:= \left\{Y \in \mathbb{R}^{d+1}; d_\alpha(X,Y) < R\right\}, \\ \operatorname{diam}_\alpha A &:= \sup_{X,Y\in A} d_\alpha(X,Y). \end{aligned}$$

For $0 \le g \in L^\psi(Q)$ we introduce the parabolic maximal functions $\mathcal{M}(g)$ and $\mathcal{M}^\alpha(g)$ through

$$\begin{aligned} \mathcal{M}(g)(t,x) &:= \sup_{0<\rho<\infty} \fint_{(t-\rho,t+\rho)} \left( \sup_{0<R<\infty} \fint_{B_R(x)} g(s,y)\, dy \right) ds, \\ \mathcal{M}^\alpha(g)(t,x) &:= \sup_{Q_R^\alpha(t,x)} \fint_{Q_R^\alpha(t,x)} g(s,y)\, dy\, ds. \end{aligned}$$

Note that $\mathcal{M}$ and $\mathcal{M}^\alpha$ share the following property

$$\mathcal{M}^\alpha(g) \le \mathcal{M}(g) \quad \text{ in } \mathbb{R}^{d+1} \tag{A.1}$$

and we have the estimate

$$\int_Q \psi(\mathcal{M}(g))\, dx\, dt \le C \int_Q \psi(g)\, dx\, dt, \tag{A.2}$$

provided that $\psi$ satisfies $\nabla_2$- and $\Delta_2$-conditions. We refer to [34, Theorem 2.1.1., page 33]. It however also holds (see [13]) that

$$|\{(t,x) \in Q; \mathcal{M}^\alpha(g)(t,x) > \Lambda\}| \le C(Q)\Lambda^{-1} \int_Q g\, dx\, dt. \tag{A.3}$$

LEMMA A.1 (Covering Lemma). *Let $E \subset \mathbb{R}^{d+1}$ be an open bounded set. Then there exists a countable family of cubes $\{Q^\alpha_{R_i}(X_i)\}_{i\in\mathbb{N}}$ and a family of smooth functions $\{\zeta_i\}_{i\in\mathbb{N}}$ such that*

$$
\begin{aligned}
&\bigcup_{i=1}^{\infty} Q^\alpha_{R_i/2} = \bigcup_{i=1}^{\infty} Q^\alpha_{R_i} = E \\
&4R_i \le d_\alpha(X_i, \partial E) \le 8R_i, \quad \forall i \in \mathbb{N}, \ \text{with } 0 < R_i < 1 \\
&R_j > 2R_i \Rightarrow Q^\alpha_{R_i}(X_i) \cap Q^\alpha_{R_j}(X_j) = \emptyset \\
&Q^\alpha_{R_i/4}(X_i) \cap Q^\alpha_{R_j/4}(X_j) = \emptyset \quad \forall i,j \in \mathbb{N}, i \ne j \\
&\zeta_i \in \mathcal{C}^\infty_0(Q^\alpha_{2R_i/3}(X_i)), \quad \forall i \in \mathbb{N} \\
&\alpha R_i^2 |\partial_t \zeta_i| + R_i |\nabla \zeta_i| \le C(d) \ \text{in } \mathbb{R}^{d+1} \quad \forall i \in \mathbb{N} \\
&\sum_{i=1}^{\infty} \zeta_i(X) = 1, \quad \forall X \in E.
\end{aligned}
\tag{A.4}
$$

*Moreover, defining $A_i := \{j \in \mathbb{N} : \ Q^\alpha_{\frac{2R_i}{3}}(X_i) \cap Q^\alpha_{\frac{2R_j}{3}}(X_j) \ne \emptyset\}$, we have*

$$
\begin{aligned}
&\operatorname{card}(A_i) \le C(d), \ \forall i \in \mathbb{N} \\
&Q^\alpha_{R_j}(X_j) \subset Q^\alpha_{4R_i}(X_i) \subset E, \quad \forall j \in A_i \ .
\end{aligned}
\tag{A.5}
$$

*Proof.* The proof can be found in [19], note that it suffices to combine all information from Lemma 3.1 in [19], Lemma C.1 in [19] together with the estimates (3.4)–(3.7) in [19]. ◻

We also introduce the notation for mean value over an arbitrary set $A$ for an integrable function $u$:

$$
\overline{u}_A := \fint_A u \, dx \, dt.
$$

LEMMA A.2 (Poincaré inequality, [13]). *Let $u, f \in L^1(Q^\alpha_R)$ and $\nabla u, \boldsymbol{q} \in \mathbf{L}^1(Q^\alpha_R)$ satisfying*

$$
-\int_{Q^\alpha_R} u\phi_{,t} = \int_{Q^\alpha_R} \boldsymbol{q} \cdot \nabla \phi + \int_{Q^\alpha_R} f\phi \quad \forall \phi \in \mathcal{C}^\infty_0(Q^\alpha_R). \tag{A.6}
$$

*Then*

$$
\int_{Q^\alpha_R} |u - \overline{u}_{Q^\alpha_R}| \le CR \left( \int_{Q^\alpha_R} |\nabla u| + \alpha |\boldsymbol{q}| + \alpha R |f| \right). \tag{A.7}
$$

Finally, let $E \subset Q$ be an open set and $u \in L^1(Q)$. Let $\{Q^\alpha_{R_i}\}$ be the covering of $E$ from Lemma A.1 and $\{\zeta_i\}$ be the corresponding partition of unity. Then we introduce the following truncation operator $\mathcal{L}^\alpha_E$ such that

$$
\mathcal{L}^\alpha_E u(t,x) := \begin{cases} u(t,x) & \text{if } (t,x) \in Q \setminus E, \\ \displaystyle\sum_{i=1}^{\infty} \overline{u}_{Q^\alpha_{R_i}} \zeta_i(t,x) & \text{if } (t,x) \in E. \end{cases} \tag{A.8}
$$

It is easy to observe (see Lemma 3.11 in [19]) that for all $1 \le a < \infty$

$$\int_Q |\mathcal{L}_E^\alpha u|^a \, dx \, dt \le c(a) \int_Q |u|^a \, dx \, dt. \tag{A.9}$$

The last lemma of this subsection concerns the most important behavior of the operator $\mathcal{L}_E^\alpha$.

Lemma A.3. *Let $\Omega$ be an open bounded set in $\mathbb{R}^d$. Assume that $u \in L^\infty(0,T;L^2(\Omega))$, $\nabla u \in L^s(Q)$ and $\boldsymbol{q} \in L^s(Q)$ for some $s > 1$ are such that*

$$u_{,t} = \operatorname{div} \boldsymbol{q}$$

*in sense of distribution. Moreover, let $E \subset\subset Q$ be an open set such that*

$$\mathcal{M}^\alpha(|\nabla u|) + \alpha \mathcal{M}^\alpha(|\boldsymbol{q}|) \le C^* < \infty, \quad \text{a.e. in } Q \setminus E. \tag{A.10}$$

*Then for any $Q' \subset\subset Q$ there exists $C(\alpha, d, C^*, Q')$ such that*

$$\begin{aligned} \|\nabla \mathcal{L}_E^\alpha u\|_{L^\infty(Q')} &\le C, \\ \| \left(\mathcal{L}_E^\alpha u\right)_{,t} (\mathcal{L}_E^\alpha u - u)\|_{L^1(Q')} &\le C, \end{aligned} \tag{A.11}$$

*and for all $\phi_1 \in \mathcal{C}_0^\infty(\Omega)$ and all $\phi_2 \in \mathcal{C}_0^\infty(0,T)$ we have*

$$\begin{aligned} \int_0^T \langle u_{,t}, \mathcal{L}_E^\alpha u \phi_1 \rangle \phi_2 \, dt = &-\frac{1}{2} \int_Q (\mathcal{L}_E^\alpha u)^2 \phi_1 (\phi_2)_{,t} \, dx \, dt \\ &- \int_Q (u - \mathcal{L}_E^\alpha u) \left(\mathcal{L}_E^\alpha u\right)_{,t} \phi_1 \phi_2 \, dx \, dt \\ &- \int_Q (u - \mathcal{L}_E^\alpha u) \, \mathcal{L}_E^\alpha u \phi_1 \left(\phi_2\right)_{,t} \, dx \, dt. \end{aligned} \tag{A.12}$$

*Proof.* The proof of (A.11) can be found in [19, Theorem 3.9, page 15]. For the identity (A.12) see [13, Lemma A.4]. ◻

**Appendix. Estimates for the Neumann problem.**

Lemma B.1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set of class $\mathcal{C}^{1,1}$. Assume that $\psi$ is an $N$-function satisfying $\nabla_2$- and $\Delta_2$-conditions. Then there are $D_1$, $D_2 > 0$ depending only on $\Omega$, $C_1$ and $\beta$ such that for any $f \in L^\psi(\Omega)$ with $\int_\Omega f \, dx = 0$ there is a unique $u \in W^{2,1}(\Omega)$ solving*

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega, \\ \nabla u \cdot \boldsymbol{n} &= 0 \text{ on } \partial\Omega, \\ \int_\Omega \psi(|\nabla^2 u|) \, dx &\le D_1 \int_\Omega \psi(|f|) \, dx + D_2. \end{aligned} \tag{B.1}$$

Note that there are numerous similar results for the Dirichlet boundary data or others where $(B.1)_3$ is proved locally, see e.g. [32]. However, to the best of our knowledge, the result for the Neumnn problem that holds globally, up to the boundary, seems not be available in the mathematical literature. This is why we include the proof here.

Before proving Lemma B.1, we note that there is an alternative way how to prove the result using the Marcinkiewicz interpolation theorem, see [11] where the author does not even use the Orlicz spaces but proves an interpolation theorem for more general spaces (that however cover Orlicz spaces satisfying $\nabla_2$- and $\Delta_2$-condition). The statement is however again focused on homogeneous Dirichlet problem and cannot be directly applied to our setting.

*Proof.* We consider $\tilde{\psi}$ satisfying the "sharp" $\nabla_2$- and $\Delta_2$-conditions: there are $\beta > 0$ and $C_1 > 0$ such that for all $s \in \mathbb{R}_+$

$$\tilde{\psi}(s) \le \frac{\tilde{\psi}(2s)}{2^{1+\beta}} \le \frac{C_1\tilde{\psi}(s)}{2^{1+\beta}}. \tag{B.2}$$

We shall show below that for such $\tilde{\psi}$s we have

$$\int_\Omega \tilde{\psi}(|\nabla^2 u|)\, dx \le D_1 \int_\Omega \tilde{\psi}(|f|)\, dx. \tag{B.3}$$

If $\psi$ is a general $N$-function satisfying $\nabla_2$- and $\Delta_2$-conditions, we can find $\tilde{\psi}$ that is also an $N$-function, $\tilde{\psi}(s) = \psi(s)$ for all $s \ge 1$ and $\tilde{\psi}$ satisfies the "sharp" conditions (B.2). Setting (by $\psi'_+$ we understand the right-hand side derivative[13])

$$q := \frac{\psi'_+(1)}{\psi(1)}$$

and defining for all $s \in [0,1]$

$$\tilde{\psi}(s) := \psi(1)s^q,$$

we first notice that necessarily $q > 1$, otherwise $\psi$ is not an $N$-function. Then it is evident that $\tilde{\psi}$ is also an $N$-function. Moreover, it satisfies the sharp conditions for all $s \in (0,1)$. Hence we need to show that it also satisfies these conditions for $s \ge 1$. But the $\nabla_2$-condition is valid since $\psi$ satisfies it. On the other hand since $\psi$ satisfies $\Delta_2$-condition for all $s$ it is evident that $\tilde{\psi}$ satisfies sharp $\Delta_2$-condition for all $s \ge 1$. Hence, we see that $\tilde{\psi}$ satisfies both $\Delta_2$- and $\nabla_2$-sharp conditions. From (B.3) we easily conclude, that $(\text{B.1})_3$ holds with $D_2 = 2\psi(1)|\Omega|$.

To complete the proof we need to prove (B.3) for $\tilde{\psi}$ fulfilling (B.2). For this purpose we modify particular steps of the proof for the standard Marcinkiewicz theorem. First we assume that $\tilde{\psi}$ and $f$ are smooth and show (B.3) for such a $\tilde{\psi}$. Since the estimate will not depend on how $\tilde{\psi}$ is smooth we can then easily extend the result for all $\tilde{\psi}$ satisfying $\nabla_2$- and $\Delta_2$-conditions. Hence, for arbitrary nonnegative measurable $g$, we denote

$$\mu_g(t) := |\{x \in \Omega; g(x) > t\}|.$$

Then as a direct consequence of this definition we get that

$$\int_0^\infty \tilde{\psi}'_+(t)\mu_g(t)\, dt = \int_\Omega \tilde{\psi}(g(x))\, dx. \tag{B.4}$$

Next, using the standard $L^r$ theory for (B.1), we know that for any $r \in (1,\infty)$ there exists $C_r > 0$ such that any solution $u$ of (B.1) satisfies (for proof see [26, Chapter 2])

$$\int_\Omega |\nabla^2 u|^r\, dx \le C_r \int_\Omega |f|^r\, dx. \tag{B.5}$$

[13]Since $\psi$ is convex, locally Lipschitz and its effective domain is equal to $\mathbb{R}$, their left-hand side derivative and right-hand side derivative exist at all points.

Moreover, it is evident that $(-\triangle)^{-1}$ is a linear operator. Next, we define $f_1(t,x)$ and $f_2(t,x)$ such that

$$\begin{aligned} f_1(t,x) &:= f(x)\chi_{\{|f(x)|\le t\}} - \fint_\Omega f(x)\chi_{\{|f(x)|\le t\}}\,dx, \\ f_2(t,x) &:= f(x)\chi_{\{|f(x)|> t\}} - \fint_\Omega f(x)\chi_{\{|f(x)|> t\}}\,dx. \end{aligned} \tag{B.6}$$

Note that $f_1(t,x)+f_2(t,x)=f(x)$ for all $t$. Then for each $f_i$ we find $u_i$ as

$$u_1(t,x) := (-\triangle)^{-1} f_1(t,x), \qquad u_2(t,x) := (-\triangle)^{-1} f_2(t,x), \tag{B.7}$$

subjected to the Neumann homogeneous data. Again, since the problems are linear, we have $u_1(t,x)+u_2(t,x)=u(x)$ for all $t$. Next, from the definition it follows that for all $t$, $f_1$ is bounded. Consequently, we fix some $r$ that will be specified later, and by using (B.5) we get that

$$\int_\Omega |\nabla^2 u_1(t,x)|^r\,dx \le C_r \int_\Omega |f_1(t,x)|^r\,dx \le C \int_{\{|f(x)|\le t\}} |f(x)|^r\,dx, \tag{B.8}$$

where the second inequality follows from (B.6). Moreover, it directly follows from (B.8) that

$$\mu_{|\nabla^2 u_1|}(a) \le \frac{C\int_{\{|f(x)|\le t\}} |f(x)|^r\,dx}{a^r}. \tag{B.9}$$

Next, we fix some $z\in(1,q)$ that will again be specified later and in the same manner as above we derive

$$\mu_{|\nabla^2 u_2|}(a) \le \frac{C\int_\Omega |f_2(t,x)|^z\,dx}{a^z} \le \frac{C\int_{\{|f(x)|> t\}} |f(x)|^z\,dx}{a^z}, \tag{B.10}$$

that is valid for all $a$. Thus, combining (B.10) and (B.9) and using the fact that $u_1+u_2=u$ we get

$$\begin{aligned} \mu_{|\nabla^2 u|}(2a) &\le \mu_{|\nabla^2 u_1|}(a) + \mu_{|\nabla^2 u_2|}(a) \\ &\le \frac{C\int_{\{|f(x)|\le t\}} |f(x)|^r\,dx}{a^r} + \frac{C\int_{\{|f(x)|> t\}} |f(x)|^z\,dx}{a^z}. \end{aligned} \tag{B.11}$$

Setting $a=t/2$ (note that here one can choose $a$ differently to get an optimal constant in the final inequality) we have

$$\mu_{|\nabla^2 u|}(t) \le C\left(\frac{\int_{\{|f(x)|\le t\}} |f(x)|^r\,dx}{t^r} + \frac{\int_{\{|f(x)|> t\}} |f(x)|^z\,dx}{t^z}\right). \tag{B.12}$$

Finally, multiplying (B.12) by $\tilde{\psi}'_+(t)$ (which is nonnegative), integrating the result with respect to $t\in(0,\infty)$ and using (B.4) we conclude that

$$\begin{aligned} &\int_\Omega \tilde{\psi}(|\nabla^2 u|)\,dx \\ &\le C\int_0^\infty \left(\frac{\int_{\{|f(x)|\le t\}} |f(x)|^r\,dx}{t^r} + \frac{\int_{\{|f(x)|> t\}} |f(x)|^z\,dx}{t^z}\right)\tilde{\psi}'_+(t)\,dt \\ &=: CI_1 + CI_2. \end{aligned} \tag{B.13}$$

Next, we evaluate $I_1$ and $I_2$. Using the Fubini theorem we have

$$I_1 = \int_\Omega |f(x)|^r \int_{|f(x)|}^\infty \frac{\tilde{\psi}'_+(t)}{t^r}\,dt\,dx, \tag{B.14}$$

$$I_2 = \int_\Omega |f(x)|^z \int_0^{|f(x)|} \frac{\tilde{\psi}'_+(t)}{t^z}\,dt\,dx. \tag{B.15}$$

Consequently, assuming that for arbitrary $a > 0$ we know that

$$\int_a^\infty \frac{\tilde{\psi}'_+(t)}{t^r}\,dt \le C\frac{\tilde{\psi}(a)}{a^r}\,, \tag{B.16}$$

$$\int_0^a \frac{\tilde{\psi}'_+(t)}{t^z}\,dt \le C\frac{\tilde{\psi}(a)}{a^z}\,, \tag{B.17}$$

we get from (B.13) and (B.14)–(B.15) the estimate in (B.1). Hence, it remains to show (B.17)–(B.16). We start with (B.17). Using integration by parts, we find that

$$\int_0^a \frac{\tilde{\psi}'_+(t)}{t^z}\,dt = \frac{\tilde{\psi}(a)}{a^z} - \lim_{\tau\to 0_+}\frac{\tilde{\psi}(\tau)}{\tau^z} + z\int_0^a \frac{\tilde{\psi}(t)}{t^{z+1}}\,dt = \frac{\tilde{\psi}(a)}{a} + z\int_0^a \frac{\tilde{\psi}(t)}{t^{z+1}}\,dt,$$

where for the second equality we used the fact that $z < q$ and the definition of $\tilde{\psi}$ on $(0,1)$. Hence, to prove (B.17) it remains to estimate the last term. To show it, we first notice that from (B.2) it follows that for any $\alpha \in (0,1)$

$$\tilde{\psi}(\alpha s) \le C\alpha^{1+\beta}\tilde{\psi}(s). \tag{B.18}$$

Indeed, it is clear from (B.2) that for any $m \in \mathbb{N}$ $\tilde{\psi}(2^{-m}s) \le 2^{-(\beta+1)m}\tilde{\psi}(s)$. Hence for any $\alpha \in (0,1)$ we can find $m$ such that $\alpha \in [2^{-m-1}, 2^{-m})$. Consequently, there is $\gamma \in (0,1)$ such that

$$\alpha = (1-\gamma)2^{-m-1} + \gamma 2^{-m}.$$

Thus using the convexity of $\tilde{\psi}$, we have

$$\begin{aligned}\tilde{\psi}(\alpha s) &= \tilde{\psi}((1-\gamma)2^{-m-1}s + \gamma 2^{-m}s) \le (1-\gamma)\tilde{\psi}(2^{-m-1}s) + \gamma\tilde{\psi}(2^{-m}s)\\ &\le 2\tilde{\psi}(s)2^{-(\beta+1)m} \le 4\alpha^{\beta+1}\tilde{\psi}(s)\end{aligned}$$

and (B.18) follows. Consequently, we also get

$$\frac{\tilde{\psi}(\alpha t)}{(\alpha t)^{\beta+1}} \le \frac{C\tilde{\psi}(t)}{t^{\beta+1}} \implies \frac{\tilde{\psi}(t_1)}{t_1^{\beta+1}} \le \frac{C\tilde{\psi}(t_2)}{t_2^{\beta+1}} \quad \text{for all } t_1 \le t_2\,.$$

Hence, we finally fix $z \in (0,1)$ such that $z < 1+\beta$ and observe that

$$\int_0^a \frac{\tilde{\psi}(t)}{t^{z+1}}\,dt = \int_0^a \frac{1}{t^{z-\beta}}\frac{\tilde{\psi}(t)}{t^{1+\beta}}\,dt \le C\frac{\tilde{\psi}(a)}{a^{1+\beta}}\int_0^a \frac{1}{t^{z-\beta}}\,dt \le \frac{C\tilde{\psi}(a)}{a^z}$$

and (B.17) follows.

Next, we check (B.16). First, using the sharp $\Delta_2$-condition, it follows that for any $\alpha \in (2^k, 2^{k+1})$ there holds

$$\tilde{\psi}(\alpha s) \le 2C_1^{k+1}\tilde{\psi}(s) \le \frac{2C_1^{k+1}}{2^{kq}}\alpha^q\tilde{\psi}(s).$$

Thus, we see that for all

$$q \geq \ln_2(2C_1^2) \tag{B.19}$$

and any $\alpha \geq 1$ there holds,

$$\tilde{\psi}(\alpha s) \leq \alpha^q \tilde{\psi}(s) \implies \frac{\tilde{\psi}(t_1)}{t_1^q} \leq \frac{\tilde{\psi}(t_2)}{t_2^q} \quad \text{for all } t_1 \geq t_2\,. \tag{B.20}$$

Finally, we set $r := \ln_2(2C_1^2) + 2$ and integrate by parts to find that

$$\int_a^\infty \frac{\tilde{\psi}'_+(t)}{t^r}\,dt = \lim_{\tau\to\infty} \frac{\tilde{\psi}(\tau)}{\tau^r} - \frac{\tilde{\psi}(a)}{a^r} + r\int_a^\infty \frac{\tilde{\psi}(t)}{t^{r+1}}\,dt \leq r\int_a^\infty \frac{\tilde{\psi}(t)}{t^{r+1}}\,dt,$$

where the second inequality follows from (B.20) and from our choice of $r$. Finally, using a simple algebraic inequality and (B.20) with $q := r-1$ we have

$$\int_a^\infty \frac{\tilde{\psi}(t)}{t^{r+1}}\,dt = \int_a^\infty \frac{1}{t^2}\frac{\tilde{\psi}(t)}{t^{r-1}}\,dt \leq \frac{\tilde{\psi}(a)}{a^{r-1}}\int_a^\infty t^{-2}\,dt = \frac{\tilde{\psi}(a)}{a^r}.$$

Thus, the proof is complete. ∎

**Appendix. Reconstruction of the pressure.** In this part we introduce the operators $\mathcal{L}^i$ used in the reconstruction of the pressure given in (3.42)–(3.43).

LEMMA C.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^{1,1}$ boundary. Then there are linear operators that are, for arbitrary $q \in (1,\infty)$ and arbitrary $s \in (\frac{d}{d-1},\infty)$, bounded in the following sense:*

$$\mathcal{L}^1 : L^q(\Omega)^{d\times d} \to L^q(\Omega), \tag{C.1}$$

$$\mathcal{L}^2 : L^q(\partial\Omega)^d \to L^{d'q}(\Omega), \tag{C.2}$$

$$\mathcal{L}^3 : L^{\frac{ds}{d+s}}(\Omega)^d \to L^s(\Omega), \tag{C.3}$$

$$\mathcal{L}^4 : W^{-1,q}_{\boldsymbol{n}} \to L^q(\Omega)\,, \tag{C.4}$$

*and the following relations hold for all $\varphi \in W^{2,\infty}(\Omega)$ satisfying $\nabla\varphi\cdot\boldsymbol{n} = 0$ on $\partial\Omega$:*

$$(\mathcal{L}^1(\mathbf{B}), \triangle\varphi) = (\mathbf{B}, \nabla^2\varphi), \qquad \int_\Omega \mathcal{L}^1(\mathbf{B})\,dx = 0, \tag{C.5}$$

$$(\mathcal{L}^2(\boldsymbol{v}), \triangle\varphi) = (\boldsymbol{v}, \nabla\varphi)_{\partial\Omega}, \qquad \int_\Omega \mathcal{L}^2(\boldsymbol{v})\,dx = 0, \tag{C.6}$$

$$(\mathcal{L}^3(\boldsymbol{w}), \triangle\varphi) = (\boldsymbol{w}, \nabla\varphi), \qquad \int_\Omega \mathcal{L}^3(\boldsymbol{w})\,dx = 0, \tag{C.7}$$

$$(\mathcal{L}^4(\boldsymbol{b}), \triangle\varphi) = \langle\boldsymbol{b}, \nabla\varphi\rangle, \qquad \int_\Omega \mathcal{L}^4(\boldsymbol{b})\,dx = 0. \tag{C.8}$$

*Moreover, for arbitrary N-function $\psi$ satisfying $\triangle_2$- and $\nabla_2$-conditions, there exists $C > 0$ depending only on $\Omega$ and $\psi$ such that*

$$\int_\Omega \psi(|\mathcal{L}^1(\mathbf{B})|)\,dx \leq C\left(1 + \int_\Omega \psi(|\mathbf{B}|)\,dx\right), \tag{C.9}$$

*provided that the right hand side of (C.9) is finite.*

*Proof.* First, we prove the statement of lemma for the operator $\mathcal{L}^1$. Since $\mathcal{D}(\Omega)^{d\times d}$ is dense in $L^q(\Omega)^{d\times d}$ for all $q \in [1,\infty)$, it suffices to prove (C.1) and (C.5) only for $\mathsf{B} \in \mathcal{D}(\Omega)^{d\times d}$. For any such $\mathsf{B}$ we set

$$\begin{aligned} \triangle\mathcal{L}^1(\mathsf{B}) &= \operatorname{div}\operatorname{div}\mathsf{B} \quad && \text{in } \Omega,\\ \nabla\mathcal{L}^1(\mathsf{B})\cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega,\\ \int_\Omega \mathcal{L}^1(\mathsf{B})\, dx &= 0, \end{aligned} \tag{C.10}$$

i.e., we can formally write $\mathcal{L}^1(\mathsf{B}) := (\triangle)^{-1}\operatorname{div}\operatorname{div}\mathsf{B}$. Clearly, $\mathcal{L}^1$ is linear and continuous (as a consequence of the standard theory for the Laplace equation) as a mapping from $W^{2,q}(\Omega)^{d\times d}$ to $W^{2,q}(\Omega)$ for all $q \in (1,\infty)$. Moreover, multiplying (C.10) by arbitrary $\varphi \in W^{2,s}(\Omega)$ with $s \in (1,\infty)$ such that $\nabla\varphi\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, integrating twice by parts (note that all boundary terms vanish) we get (C.5). Next, we focus on the boundedness stated in (C.1). To show it, we find $\varphi$ such that

$$\begin{aligned} \triangle\varphi &= |\mathcal{L}^1(\mathsf{B})|^{q-2}\mathcal{L}^1(\mathsf{B}) - \fint_\Omega |\mathcal{L}^1(\mathsf{B})|^{q-2}\mathcal{L}^1(\mathsf{B})\, dx \quad && \text{in } \Omega,\\ \int_\Omega \varphi\, dx &= 0, \qquad \nabla\varphi\cdot\boldsymbol{n} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{C.11}$$

Using the $L^q$ theory for the Laplace equation, we know that there is a constant $C > 0$ depending only on $\Omega$ and $q$ such that

$$\begin{aligned} \int_\Omega |\nabla^2\varphi|^{q'} &\le C\int_\Omega \left| |\mathcal{L}^1(\mathsf{B})|^{q-2}\mathcal{L}^1(\mathsf{B}) - \fint_\Omega |\mathcal{L}^1(\mathsf{B})|^{q-2}\mathcal{L}^1(\mathsf{B})\, dx\right|^{q'} dx\\ &\le C\int_\Omega |\mathcal{L}^1(\mathsf{B})|^q\, dx. \end{aligned} \tag{C.12}$$

Note that since $\mathsf{B}$ is smooth, the integral on the right hand side is finite for any $q \in (1,\infty)$. Consequently, substituting $\varphi$ into $(C.5)_1$, using $(C.5)_2$, the Hölder inequality and the estimate (C.12), we find that

$$\int_\Omega |\mathcal{L}^1(\mathsf{B})|^q\, dx = (\mathsf{B},\nabla^2\varphi) \le \|\mathsf{B}\|_q\|\nabla^2\varphi\|_{q'} \le C\|\mathsf{B}\|_q\|\mathcal{L}^1(\mathsf{B})\|_q^{q-1} \tag{C.13}$$

and (C.1) follows. The proof for the operator $\mathcal{L}^4$ is almost the same with the only difference that we consider $\boldsymbol{b} \in \mathcal{V}$ and by the density argument we extend the validity of (C.4) on the whole $W^{-1,q}_{\boldsymbol{n}}$.

Next, the proof for $\mathcal{L}^3$ is even easier, it is enough to see that $\mathcal{L}^3$ is defined as

$$(\nabla\mathcal{L}^3(\boldsymbol{w}),\nabla\varphi) = -(\boldsymbol{w},\nabla\varphi) \text{ for all smooth } \varphi \qquad \int_\Omega \mathcal{L}^3(\boldsymbol{w})\, dx = 0. \tag{C.14}$$

Thus using the theory for the Laplace equation we see that $\mathcal{L}^3$ is linear and bounded even as an operator $L^{\frac{ds}{d+s}} \to W^{1,\frac{ds}{d+s}}(\Omega)$. Consequently, using the embedding theorem we get (C.3).

Finally, we focus on $\mathcal{L}^2$. Since $\mathcal{C}(\partial\Omega)^d$ is dense in $L^q(\partial\Omega)$ for any $q \in (1,\infty)$ we prove the result only for continuous $\boldsymbol{v}$. Then by continuity we can extend it onto the whole $L^q$. Hence, we first introduce an approximative linear operator $\mathcal{L}^2_\varepsilon$ as

$$\begin{aligned} (\nabla\mathcal{L}^2_\varepsilon(\boldsymbol{v}),\nabla\varphi) &= -\frac{1}{|\Omega_\varepsilon|}(\boldsymbol{v},\nabla\varphi)_{\Omega_\varepsilon} \qquad && \text{for all } \varphi \in W^{1,2}(\Omega),\\ \int_\Omega \mathcal{L}^2_\varepsilon(\boldsymbol{v})\, dx &= 0, \end{aligned} \tag{C.15}$$

where

$$\Omega_\varepsilon := \{x \in \Omega;\ \text{dist } (x, \partial\Omega) < \varepsilon\},$$

and $\boldsymbol{v} \in \mathcal{C}(\overline{\Omega})^d$ is an extension of $\boldsymbol{v}$ from $\partial\Omega$ onto $\Omega$. Note that such an operator is well-defined. Next, we investigate the limit $\varepsilon \to 0_+$. First, we find $\varphi$ solving

$$\begin{aligned} \triangle\varphi &= |\mathcal{L}^2_\varepsilon(\boldsymbol{v})|^{(2d)'-2}\mathcal{L}^2_\varepsilon(\boldsymbol{v}) - \fint_\Omega |\mathcal{L}^2_\varepsilon(\boldsymbol{v})|^{(2d)'-2}\mathcal{L}^2_\varepsilon(\boldsymbol{v})\, dx &&\text{in } \Omega,\\ \int_\Omega \varphi\, dx &= 0, \qquad \nabla\varphi\cdot\boldsymbol{n} = 0 &&\text{on } \partial\Omega. \end{aligned}$$

Consequently, we have

$$\|\varphi\|_{2,2d}^{2d} \le C\int_\Omega |\mathcal{L}^2_\varepsilon(\boldsymbol{v})|^{(2d)'}\, dx.$$

Thus, using this in (C.15) and integrating by parts we find (after using the Hölder inequality) that

$$\begin{aligned} \int_\Omega |\mathcal{L}^2_\varepsilon(\boldsymbol{v})|^{(2d)'} &= \frac{1}{|\Omega_\varepsilon|}(\boldsymbol{v}, \nabla\varphi)_{\Omega_\varepsilon} \le \|\boldsymbol{v}\|_\infty \|\nabla\varphi\|_\infty \le C\|\boldsymbol{v}\|_\infty\|\varphi\|_{2,2d}\\ &\le C\|\boldsymbol{v}\|_\infty \|\mathcal{L}^2_\varepsilon(\boldsymbol{v})\|_{(2d)'}^{(2d)'-1} \end{aligned}$$

and consequently

$$\|\mathcal{L}^2_\varepsilon(\boldsymbol{v})\|_{(2d)'} \le C\|\boldsymbol{v}\|_\infty. \tag{C.16}$$

Therefore, we can find a subsequence and $\mathcal{L}^2(\boldsymbol{v})$ such that for $\varepsilon \to 0_+$

$$\mathcal{L}^2_\varepsilon(\boldsymbol{v}) \rightharpoonup \mathcal{L}^2(\boldsymbol{v}) \qquad \text{weakly in } L^{(2d)'}(\Omega).$$

Moreover, since $\boldsymbol{v}$ is continuous it is easy to take limit in (C.15) and to show that (in fact it is (C.2))

$$\begin{aligned} &(\mathcal{L}^2(\boldsymbol{v}), \triangle\varphi) = (\boldsymbol{v}, \nabla\varphi)_{\partial\Omega} \ \text{ for all } \varphi \in W^{2,2d}(\Omega) \text{ such that } \nabla\varphi \in W^{1,2d}_{\boldsymbol{n}},\\ &\int_\Omega \mathcal{L}^2_\varepsilon(\boldsymbol{v})\, dx = 0. \end{aligned} \tag{C.17}$$

To show that the operator is well defined, i.e., that the weak limit is unique and does not depend on the extension of $\boldsymbol{v}$, one can argue by linearity of (C.17) and the estimates (boundedness) proved below. Thus, it remains to show that $\mathcal{L}^2$ fulfills (C.6). To this end, we define for arbitrary $k \in \mathbb{N}$

$$L_k := \min\{k, |\mathcal{L}^2(\boldsymbol{v})|\}.$$

Then, for arbitrary $q \in (1, \infty)$, we look for $\varphi$ solving

$$\begin{aligned} \triangle\varphi &= |L_k|^{d'q-1}\ \text{sign } \mathcal{L}^2(\boldsymbol{v}) - \fint_\Omega |L_k|^{d'q-1}\ \text{sign } \mathcal{L}^2(\boldsymbol{v})\, dx &&\text{in } \Omega,\\ \int_\Omega \varphi\, dx &= 0, \qquad \nabla\varphi\cdot\boldsymbol{n} = 0 &&\text{on } \partial\Omega. \end{aligned}$$

Consequently, for arbitrary $s \in (1, \infty)$, we have

$$\|\varphi\|_{2,s} \le C \||L_k|^{d'q-1}\|_s < \infty, \tag{C.18}$$

where the second inequality follows from the fact that $L_k$ is bounded. Thus, using such a $\varphi$ in (C.17) we find that (we use the Hölder inequality, the trace theorem and the estimate (C.18))

$$\begin{aligned}\int_\Omega |L_k|^{d'q-1} |\mathcal{L}^2(\boldsymbol{v})| \, dx = (\boldsymbol{v}, \nabla\varphi)_{\partial\Omega} &\le \|\boldsymbol{v}\|_{L^q(\partial\Omega)^d} \|\nabla\varphi\|_{L^{q'}(\partial\Omega)^d} \\ &\le C\|\boldsymbol{v}\|_{L^q(\partial\Omega)^d} \|\varphi\|_{2,\frac{d'q}{d'q-1}} \\ &\le C\|\boldsymbol{v}\|_{L^q(\partial\Omega)^d} \||L_k|^{d'q-1}\|_{\frac{d'q}{d'q-1}} < \infty.\end{aligned}$$

Next, since $L_k \le |\mathcal{L}^2(\boldsymbol{v})|$ the above estimate directly implies that

$$\|L_k\|_{d'q} \le C\|\boldsymbol{v}\|_{L^q(\partial\Omega)^d}.$$

Thus letting $k \to \infty$ we deduce that

$$\|\mathcal{L}^2(\boldsymbol{v})\|_{d'q} \le C\|\boldsymbol{v}\|_{L^q(\partial\Omega)^d},$$

which finishes the proof of the first part of lemma.

Finally, we focus on proving (C.9) for smooth compactly supported $\mathbf{B}$; the complete estimate (C.9) is then achieved by the density argument as $\psi$ satisfies $\nabla_2$- and $\Delta_2$-conditions. Thus, for $\mathbf{B}$ smooth, we know that $\mathcal{L}^1(\mathbf{B})$ belongs to any $L^p(\Omega)^{d\times d}$ for $p \in [1, \infty)$. Next, we insert into (C.5) $\varphi$ solving

$$\begin{aligned}\triangle\varphi &= \frac{\psi(\mathcal{L}^1(\mathbf{B}))}{\mathcal{L}^1(\mathbf{B})} - \fint_\Omega \frac{\psi(\mathcal{L}^1(\mathbf{B}))}{\mathcal{L}^1(\mathbf{B})} \, dx && \text{in } \Omega, \\ \int_\Omega \varphi \, dx &= 0, \qquad \nabla\varphi \cdot \boldsymbol{n} = 0 && \text{on } \partial\Omega.\end{aligned}$$

Doing so, and using the fact that $\int_\Omega \mathcal{L}^1(\mathbf{B}) \, dx = 0$, we find the identity

$$\int_\Omega \psi(\mathcal{L}^1(\mathbf{B})) \, dx = (\mathbf{B}, \nabla^2\varphi). \tag{C.19}$$

Our aim is to estimate the right hand side of (C.19) (that is finite). Since $\psi^*$ satisfies $\Delta_2$- and $\nabla_2$-conditions we can use Lemma B.1 to arrive at

$$\int_\Omega \psi^*(|\nabla^2\varphi|) \, dx \le D_1 \int_\Omega \psi^* \left( \left| \frac{\psi(\mathcal{L}^1(\mathbf{B}))}{\mathcal{L}^1(\mathbf{B})} - \fint_\Omega \frac{\psi(\mathcal{L}^1(\mathbf{B}))}{\mathcal{L}^1(\mathbf{B})} \, dx \right| \right) dx + D_2. \tag{C.20}$$

Next, we use $\Delta_2$-condition and the Jensen inequality to estimate the right hand side of (C.20) and to obtain

$$\int_\Omega \psi^*(|\nabla^2\varphi|) \, dx \le C \left( 1 + \int_\Omega \psi^* \left( \frac{\psi(\mathcal{L}^1(\mathbf{B}))}{|\mathcal{L}^1(\mathbf{B})|} \right) dx \right). \tag{C.21}$$

Finally, since $\psi$ is an $N$-function, we can use the estimate stated in [52, Chapter II, p. 14] and conclude that

$$\int_\Omega \psi^*(\nabla^2\varphi) \, dx \le C \left( 1 + \int_\Omega \psi(\mathcal{L}^1(\mathbf{B})) \, dx \right). \tag{C.22}$$

Thus, to estimate the right hand side of (C.19) we use the Young inequality and the convexity of $\psi^*$ and (C.22) and observe

$$\begin{aligned}(\mathbf{B}, \nabla^2\varphi) &\le \int_\Omega \psi(\varepsilon^{-1}|\mathbf{B}|)\,dx + \int_\Omega \psi^*(\varepsilon|\nabla^2\varphi|)\,dx\\ &\le \int_\Omega \psi(\varepsilon^{-1}|\mathbf{B}|)\,dx + \varepsilon\int_\Omega \psi^*(|\nabla^2\varphi|)\,dx\\ &\le \int_\Omega \psi(\varepsilon^{-1}|\mathbf{B}|)\,dx + \varepsilon C\left(1 + \int_\Omega \psi(\mathcal{L}^1(\mathbf{B}))\,dx\right).\end{aligned}$$

Finally, using this in (C.19) and choosing $\varepsilon$ such that $C\varepsilon = \frac{1}{2}$ we can move the second term on the left hand side and then by using $\Delta_2$-condition for $\psi$ (that is $\nabla_2$-condition for $\psi^*$) we directly obtain (C.9). The proof of Lemma C.1 is complete. ∎

**Appendix. Trace theorem for Sobolev-Slobodetski spaces.**

LEMMA D.1 (Trace theorem, [60]). *Let $\Omega \subset \mathbb{R}^d$ be bounded Lipschitz domain. Then there exists a continuous linear trace operator* tr *such that for all $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$*

$$\mathrm{tr} : W^{\alpha,p}(\Omega) \to W^{\alpha-\frac{1}{p},p}(\partial\Omega). \tag{D.1}$$

*Proof.* The proof of (D.1) is in fact not exactly stated in [60]. However, it can be proved by using several theorems stated there. First, in Subsection 2.2.2 (Remark 3) there is shown that $W^{\alpha,p}(\Omega) = \Lambda^\alpha_{p,p}(\Omega)$ where the first one is the Sobolev-Slobodetski space and the second one is the Besov space. Then in Subsection 2.3.5 one can find that $\Lambda^\alpha_{p,q}(\Omega) = B^\alpha_{p,q}(\Omega)$, where $B^\alpha_{p,q}$ is the Triebel space introduced in Subsection 2.3.1. Finally in Subsection 3.3.3, the trace theorem is proved in the setting tr : $B^\alpha_{p,q}(\Omega) \to B^{\alpha-\frac{1}{p}}_{p,q}(\partial\Omega)$. Combining all these facts we finally obtain (D.1). ∎

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