Existence result to the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces

Aneta Wróblewska - Kamińska

Preprint no. 2012 - 024

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

FNP
Foundation
for Polish Science

EUROPEAN UNION
EUROPEAN REGIONAL
DEVELOPMENT FUND

Ph.D. Programme: Mathematical Methods in Natural Sciences (MMNS)
e-mail: mmns@mimuw.edu.pl
http://mmns.mimuw.edu.pl

Existence result to the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces

Aneta Wróblewska-Kamińska *

Abstract

We prove the existence of weak solutions to the problem of the motion of one or several rigid bodies immersed in a non-Newtonian fluid of rheology more general then power-law. The nonlinear viscous term in the equation is described with help of a general convex function defining Orlicz spaces. The main ingredient of the proof is convergence of nonlinear term achieved with help of the pressure localisation method.

Keywords: Non-Newtonian fluids, several rigid bodies, Orlicz spaces.

1 Introduction

We want to investigate the mathematical properties of motion of one or several non-homogenous rigid bodies immersed in homogeneous incompressible viscous fluid which occupies bounded domain $\Omega \subset \mathbb{R}^3$. In particular we are interested in fluids which viscosity increases dramatically with increasing shear rate or applied stress, i.e. we want to consider shear thickening fluids, which can behaves like a solid when it encounters mechanical stress or shear. STF moves like a liquid until an object strikes or agitates it forcefully. Then, it hardens in a few milliseconds. This is the opposite of a shear-thinning fluid, like paint, which becomes thinner when it is agitated or shaken. The fluid is a colloid, consists of solid particles dispersed in a liquid (e.g. silica particles suspended in polyethylene glycol). The particles repel each other slightly, so they float easily throughout the liquid without clumping together or settling to the bottom. But the energy of a sudden impact overwhelms the repulsive forces between the particles – they stick together, forming masses called hydroclusters. When the energy from the impact dissipates, the particles begin to repel one another again. The hydroclusters fall apart, and the apparently solid substance reverts to a liquid.

Possible application for fluids with changeable viscosity appears in military armour. The so-called STF-fabric produced by simple impregnation process of e.g. Kevlar make it applicable to any high-performance fabric. The resulting material is

*University of Warsaw, Institute of Applied Mathematics and Mechanics, ul. Banacha 2, Warszawa, Poland. A.W. is a Ph.D. student in the International Ph.D. Projects Programme of Foundation for Polish Science within the Innovative Economy Operational Programme 2007-2013 (Ph.D. Programme: Mathematical Methods in Natural Sciences). (a.wroblewska@mimuw.edu.pl)

thin and flexible, and provides protection against the risk of needle, knife or bullet contact that face police officers and medical personnel [8, 17].

Motivated by this significant shear thickening phenomenon we want to concentrate on fluids with rheology more general then of a power-law type, see Málek et al. [28, Chapter 1.], which are given by standard polynomial growth conditions for stress tensor, i.e.

$$\begin{aligned} |\mathbf{S}(\mathbf{D}\boldsymbol{u})| &\le c(1+|\mathbf{D}\boldsymbol{u}|)^{p-1} \\ \mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} &\ge c|\mathbf{D}\boldsymbol{u}|^p, \end{aligned} \tag{1}$$

where the viscous stress tensor $\mathbf{S}$ depends on the symmetric part of the velocity gradient $\mathbf{D}\boldsymbol{u}$,

$$\mathbf{D}\boldsymbol{u} = (\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u})/2,$$

of the gradient of the velocity field $\boldsymbol{u}$. We want to investigate the processes where growth is faster than polynomial. Therefore we formulate the growth conditions of the stress tensor using quite general convex function M called the $\mathcal{N}-$function (for definition see Section 2). Now we are able to describe the effect of very rapidly shear thickening fluids. For more references and existence result for fluid flow see [23, 24, 38, 39]. We assume that viscous stress tensor $\mathbf{S}$ depends on the symmetric part of the gradient of the velocity field $\boldsymbol{u}$ in the following way: $\mathbf{S} : \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}^{3\times3}_{\rm sym}$ satisfies ($\mathbb{R}^{3\times3}_{\rm sym}$ stands for the space of 3×3 symmetric matrices):

$$\mathbf{S}(\mathbf{0}) = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}(\mathbf{D}\boldsymbol{u}) \quad \text{is continuous,} \tag{2}$$

$$\Big(\mathbf{S}(\boldsymbol{\xi}) - \mathbf{S}(\boldsymbol{\eta})\Big) : \Big(\boldsymbol{\xi} - \boldsymbol{\eta}\Big) \ge 0 \text{ for all } \boldsymbol{\xi} \neq \boldsymbol{\eta},\ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{3\times3}_{\rm sym} \tag{3}$$

and there exist a positive constant c, an N–function M and M^* denotes the complementary function to M (the definitions appear in Section 2) such that for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\rm sym}$ and a.a. $x \in \Omega$ it holds

$$\mathbf{S}(\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c_c\{M(|\boldsymbol{\xi}|) + M^*(|\mathbf{S}(\boldsymbol{\xi})|)\}. \tag{4}$$

Additionally we assume that the N–function M satisfy additional growth condition

$$c_1|\cdot|^p \le M(\cdot) \le c_2 \exp^{\frac{1}{\beta+1}}(|\cdot|) \quad \text{for } p \ge 4,\ \beta > 0 \tag{5}$$

where $c_1,\ c_2$ are some positive constants, the complementary function

$$M^* \text{ satisfies } \Delta_2 - \text{condition} \tag{6}$$

and

$$M(|\cdot|^{\frac{1}{4}}) \quad \text{is convex.} \tag{7}$$

Our assumptions can capture shear dependent viscosity function which includes power-low and Carreau-type models which are quite popular among rheologist, chemical engineering, colloidal mechanics (see [27] for more references). Nevertheless we want to investigate also more general constitutive relations like non-polynomial growth $\mathbf{S} \approx |\mathbf{D}\boldsymbol{u}|^p \ln(1+|\mathbf{D}\boldsymbol{u}|)$ ($M(\tau) = \tau^p \log(\tau+1)$ grows faster then τ^p but slowlier then $\tau^{p+\varepsilon}$ for all $\varepsilon > 0$).

We can observe that the case of stress tensors having convex potentials (additionally vanishing at 0 and symmetric w.r.t. the origin) significantly simplifies verifying the coercivity condition (4). For finding $\mathcal{N}-$functions M and M^* we take an advantage of the following relation

$$M(|\boldsymbol{\xi}|) + M^*(|\nabla M(|\boldsymbol{\xi}|)|) = \boldsymbol{\xi} \cdot \nabla M(|\boldsymbol{\xi}|) \tag{8}$$

holding for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}$, cf. [30]. This corresponds to the case when the Young inequality for N–functions becomes the equality. Then obviously the choice $M(|\boldsymbol{\xi}|) = \int_0^{|\boldsymbol{\xi}|} \mu(\alpha)\, \mathbf{d}\alpha$ and if $\mathbf{S}(\mathbf{D}\boldsymbol{u}) = 2\mu(|\mathbf{D}\boldsymbol{u}|)\mathbf{D}\boldsymbol{u}$ provides that condition (4) is satisfied with a constant $c = 1$. For such chosen M we only need to verify if the $\mathcal{N}-$function–conditions, i.e, behaviour in/near zero and near infinity, are satisfied. The monotonicity (3) of $\mathbf{S}$ follows from the convexity of the potential.

The appropriate spaces to capture such formulated problem are Orlicz spaces. For definitions and preliminaries of $\mathcal{N}-$functions and Orlicz spaces see Section 2.

The motion of the body during and before the contacts with boundary of the domain Ω was studied by Starovoitov. In particular, the author gives us sufficient conditions which imply the impossibility of the collision with rigid object, see [34, Theorem 3.2.], i.e.:

c1 the domain $\Omega \subset \mathbb{R}^3$ as well as the rigid bodies in its interior have boundaries of class $C^{1,1}$;

c2 the $p - th$ power of the velocity gradient is integrable, with $p \geq 4$.

Therefore any contact of rigid body with the boundary of domain or with other one or several rigid bodies does not occur. We need just to assume that it was not present in initial time and consider certain class of non-Newtonian fluids, where the contact can be eliminated by the phenomenon of shear-thickening.

In the following paper we want to investigate the motion of several rigid bodies in non-Newtonian incompressible fluid. To construct the solution we use penalization method developed by Hoffmann and Starovoitov [25], and San Martin et al. [31]. Which is based on idea of approximating rigid objects of the system by the fluid of very high viscosity becoming singular in limiting consideration. To avoid some technical difficulties we assume that fluid density is constant in the approximate "fluid" part (what differs our considerations form those in [31], where 2-D case of a newtonian fluid is investigated).

There are two main difficulties we have to face in the proof of the existence result, more precisely in sequential stability of the approximate solutions:
1. strong compactness of the approximate velocities on time space cylinder in L^2 space;
2. passing to the limit in nonlinear term - i.e. in viscous stress tensor by mean of monotonicity method.

To solve the first problem, similarly as San Martin et al. in [31], we use the Aubin-Lions argument applied to a suitable projection of the velocity field onto the "space of rigid velocities". It is worth to notice that no-collision result by Starovoitov [34] significantly simplifies our analysis. Namely we are ensured that bodies do not penetrate each other and the boundary. We will notice that the distance between the bodies and boundary is always kept and any sharp cones appear in the fluid part.

The principal difficulty here is caused by the fact that we consider the problem in Orlicz space setting and we do not assume that the $\triangle_2$–condition is satisfied. From this reason we lose many facilitating properties.

An interesting obstacle here is the lack of the classical integration by parts formula, cf. [16, Section 4.1]. To extend it for the case of generalized Orlicz spaces we would essentially need that $C^\infty-$functions are dense in $L_M(Q)$ and tensor structure of the Orlicz space. The first one only holds if M satisfies Δ_2–condition. In general also $L_M(Q) \neq L_M(0,T;L_M(\Omega))$, unless very strong assumption are satisfied. We recall the proposition from [7].

Proposition 1.1 *Let I be the time interval and $\Omega \subset \mathbb{R}^d$, $M = M(|\boldsymbol{\xi}|)$ an $\mathcal{N}-$function, and $L_M(I\times\Omega), L_M(I;L_M(\Omega))$ the Orlicz spaces on $I\times\Omega$ and the vector valued Orlicz space on I respectively. Then*

$$L_M(I\times\Omega) = L_M(I;L_M(\Omega)),$$

if and only if there exist constants k_0, k_1 such that

$$k_0 M^{-1}(s)M^{-1}(r) \le M^{-1}(sr) \le k_1 M^{-1}(s)M^{-1}(r) \tag{9}$$

for every $s \ge 1/|I|$ and $r \ge 1/|\Omega|$.

One can conclude that (9) means that M must be equivalent to some power p, $1 \le p \le \infty$ and surely (9) would provide $L_M(Q)$ to be separable and reflexive.

The latter problem, inherent to the theory of non-Newtonian fluid, we have to identify the nonlinear term on 'fluid' part of time space cylinder. Therefore the problem is more delicate as the monotonicity argument must be localised to the "fluid" part of the system. We take the idea of Wolf [37], localise the pressure and represent it as a sum of some regular and harmonic part. Following Feireisl et al. [12] we construct the pressure with help of Riesz transform which gives result more suitable for non-standard growth conditions and such an approach can be easily adapted to more general constitutive relations for $\mathbf{S}$. The main difference from any previous works in this direction is, due to nonstandard growth conditions, that we are in Orlicz settings. Beside difficulties mentioned above, the Riesz transform in general can be not well defined on Orlicz space to itself. If M and M^* do dot satisfy Δ_2-condition it can happen that it is continuous from one Orlicz space to another one but to larger one. Therefore pressure localisation method seems to be more difficult.

We want to emphasise that we achieve the existence result for the problem of motion of rigid bodies in non-Newtonian fluids with non-polynomial growth conditions. Which allow as to consider situation of non-power-law fluids, where constitutive relation can be more general then (1) considered in [12].

Our main result, formulated below in Theorem 4.1, concerns the existence of weak solutions of the associated evolutionary system, where, in accordance with [34], collisions of two or more rigid objects do not appear in a finite time unless they were present initially, what considerably simplified analysis of the problem.

The paper is organised as follows: in Section 2 we introduce basic notion and definitions of Orlicz spaces. Some preliminary considerations, weak formulation, basic notation of investigated problem are summarised in Section 3. The main result is formulated in Section 4 as the Theorem 4.1. In Section 5 the reader can find some properties of Orlicz spaces, which we use in later considerations. The following sections contains the proof of the existence result. In Section 6 the approximate problem is introduce by replacing the bodies by the fluid of high viscosity. Section 7 contains the artificial viscosity limit. In Section 8, the previous arguments are recalled to provide the limit for the regularized velocity field.

2 Notation - Orlicz spaces

Definition 2.1 *A function $M : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an $\mathcal{N}-$*function *if it is a continuous real valued, non-negative, convex function, which has superlinear growth near zero and infinity, i.e., $\lim_{\tau\to 0} \frac{M(\tau)}{\tau} = 0$ and $\lim_{\tau\to\infty} \frac{M(\tau)}{\tau} = \infty$, and $M(\tau) = 0$ if and only if $\tau = 0$.*

Definition 2.2 *The* complementary function *M^* to a function M is defined by*

$$M^*(\varsigma) = \sup_{\tau\in\mathbb{R}_+} (\tau\varsigma - M(\tau))$$

for $\varsigma \in \mathbb{R}_+$.

The complementary function M^* is also an $\mathcal{N}-$function. Let Ω be a bounded domain in $\mathbb{R}^3$ with the Lebesgue measure $\mathrm{d}x$ then the *Orlicz class* $\mathcal{L}_M(\Omega)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}^d$ such that

$$\int_\Omega M(|f(x)|)\,\mathrm{d}x < \infty,$$

where $d = 1, 3$ or 3×3. The *Orlicz space* $L_M(\Omega)$ is defined as the set of all measurable functions $f : \Omega \to \mathbb{R}^d$ which satisfy

$$\int_\Omega M(\lambda|f(x)|)\,\mathrm{d}x \to 0 \quad \text{as } \lambda \to 0.$$

The Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|f\|_M = \inf\left\{\lambda > 0 : \quad \int_\Omega M\left(\frac{|f(x)|}{\lambda}\right)\,\mathrm{d}x \le 1\right\}.$$

Let us denote by $E_M(\Omega)$ the closure of all measurable, bounded functions on Ω in $L_M(\Omega)$. Then the space E_M is separable and the space $L_{M^*}(\Omega)$ is the dual space of $E_M(\Omega)$. It is easy to see that $E_M \subseteq \mathcal{L}_M \subseteq L_M$.

We say that M satisfies the Δ_2-*condition* if there exists a constant c_M such that $M(2\tau) \le c_M M(\tau)$ for all $\tau \in \mathbb{R}_+$. It is well known that L_M is separable and reflexive if and only if M satisfies Δ_2–condition, see [19].

The sequence $\{f^j\}_{j=1}^\infty$ converges modularly to f in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\left(\frac{|f^j - f|}{\lambda}\right)\,\mathrm{d}x \to 0 \quad \text{as } j \to \infty.$$

We will write $f^j \xrightarrow{M} f$ for the modular convergence in $L_M(\Omega)$.

In the paper Orlicz spaces are considered also on the time-space cylinder $Q = (0, T) \times \Omega$ with the Lebesgue measure $\mathrm{d}x\mathrm{d}t$. More information about properties of Orlicz spaces can be found e.g. in [19].

3 Preliminaries, weak formulation

In our investigation by $\mathcal{D}(\Omega)$ we mean the set of C^∞-functions with compact support contained in Ω. Moreover, by $L^p, W^{1,p}$ we mean the standard Lebesgue and Sobolev spaces respectively. By p' we mean the conjugate exponent to p, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

If X is a Banach space, then the symbols $L^p(0,T;X)$ stands for standard Bochner spaces.

We will use $C_{\text{weak}}([0,T];L^2(\Omega))$ to denote the space of functions $\mathbf{u} \in L^\infty(0,T;L^2(\Omega))$ which satisfy $(\mathbf{u}(t),\varphi) \in C([0,T])$ for all $\varphi \in L^2(\Omega)$. By (a,b) we mean $\int_\Omega a(x) \cdot b(x)\,\mathrm{d}x$.

Namely, we state the following problem: let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$, occupied by an incompressible fluid containing rigid bodies. Each rigid body in the considered system is identified with the connected subset of Euclidian space $\mathbb{R}^3$. The initial position of the rigid bodies is given through a family of domains

$$S_i \subset \mathbb{R}^3, \ i = 1,\dots,n,$$

which are diffeomorphic to the unit ball in $\mathbb{R}^3$. To avoid additional difficulties the boundaries of all rigid bodies are supposed to be sufficiently regular, namely there exists $\delta_0 > 0$ such that for any $x \in \partial S_i$, there are two closed balls B^{int}, B^{ext} of the radius δ_0 such that

$$x \in B^{\text{int}} \cap B^{\text{ext}}, \quad B^{\text{int}} \subset \overline{S}_i, \quad B^{\text{ext}} \subset \mathbb{R}^3 \setminus S_i \tag{10}$$

The same assumption concerns considered physical space $\Omega \subset \mathbb{R}^3$, occupied by the fluid and containing all rigid bodies. Specifically, it is supposed be a bounded domain such that for any $x \in \partial\Omega$, there are two closed balls B^{int}, B^{ext} of the radius δ_0 such that

$$x \in B^{\text{int}} \cap B^{\text{ext}}, \quad B^{\text{int}} \subset \overline{\Omega}, \quad B^{\text{ext}} \subset \mathbb{R}^3 \setminus \Omega. \tag{11}$$

The motion of the rigid body S_i is represented by the associated mapping $\boldsymbol{\eta}_i$

$$\boldsymbol{\eta}_i = \boldsymbol{\eta}_i(t,x), \ t \in [0,T), \ x \in \mathbb{R}^3, \ \boldsymbol{\eta}_i(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an isometry for all } t \in [0,T)$$
$$\text{and } \boldsymbol{\eta}_i(0,x) = x \text{ for all } x \in \mathbb{R}^3, \quad i = 1,\dots,n.$$

Therefore, the position of the body S_i at a time $t \in [0,T)$ is given the following through formula

$$S_i(t) = \boldsymbol{\eta}_i(t,S_i), \ i = 1,\dots,n. \tag{12}$$

In the above terms we introduce domains Q^f and Q^s respectively as a fluid and a rigid part of the time-space cylinder in the following way:

$$Q^s := \bigcup_{i=1,\dots,n} \left\{(t,x) \ : \ t \in [0,T], \ x \in \overline{S_i}(t)\right\} \quad Q^f := Q \setminus Q^s.$$

In the present work the concept of weak solutions is based on the Eulerian reference system and on a class of test functions which depend on the position of the rigid bodies. This idea was introduced by Judakov [26] (see also Desjardins and Esteban [4, 5], Galdi [20, 21], Hoffmann and Starovoitov [25], San Martin et al. [31] , Serre [32]). Let us denote the velocity field of the system by $\boldsymbol{u} : Q \to \mathbb{R}^3$ and introduce decomposition for a fluid and a rigid velocity as follows

$$\boldsymbol{u}^f = \boldsymbol{u} \text{ on } Q^f \quad \text{ and } \quad \boldsymbol{u}^S = \boldsymbol{u} \text{ on } Q^S$$

In our consideration we assume non-slip boundary conditions for the velocity on all surfaces and the velocity of the fluid on the boundary of each rigid body S_i ($i = 1,\dots,n$) is supposed to coincide with the velocity of rigid object. Namely

$$\boldsymbol{u}^f(t,x) = 0 \text{ on } \partial\Omega \text{ and } \boldsymbol{u}^f(t,x) = \boldsymbol{u}^S(t,x) \text{ on } \partial S(t) \text{ for all } t \in [0,T].$$

To be more precise, if we consider the mass density $\varrho = \varrho(t,x)$ and the velocity field $\boldsymbol{u} = \boldsymbol{u}(t,x)$ at a time $t \in (0,T)$ and the spatial position $x \in \Omega$, then those functions satisfy the following integral identities

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla_x \varphi \Big) \,\mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \varphi \,\mathrm{d}x \tag{13}$$

for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$, and

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \mathbf{D}\boldsymbol{\varphi} - \mathbf{S} : \mathbf{D}\boldsymbol{\varphi} \Big) \,\mathrm{d}x\mathrm{d}t \\ &= - \int_0^T \int_\Omega \varrho \nabla_x F \cdot \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_0 \boldsymbol{u}_0 \cdot \boldsymbol{\varphi} \,\mathrm{d}x \end{aligned} \tag{14}$$

for any test function $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; \mathbb{R}^3)$,

$$\boldsymbol{\varphi}(t, \cdot) \in [\mathcal{R}M](t), \tag{15}$$

which is associated with position of rigid bodies, i.e.

$$\begin{aligned} [\mathcal{R}M](t) = \big\{ \boldsymbol{\phi} \in C^1_c(\Omega; \mathbb{R}^3) \ &: \mathrm{div}_x \boldsymbol{\phi} = 0 \text{ in } \Omega, \\ &\mathbf{D}\boldsymbol{\phi} \text{ has compact support on } \Omega \setminus \cup_{i=1}^n \overline{S}_i(t) \big\}. \end{aligned} \tag{16}$$

Where the symbol $\mathbf{S}$ denotes the viscous stress tensor determined through (2 - 7), $\nabla_x F$ is a given potential driving force, and ϱ_0, $\boldsymbol{u}_0$ stand for the initial distribution of the density and the velocity, respectively.

In order to close the system we have to specify the relation between the velocity $\boldsymbol{u}$ and the motion of solids given by isometries $\boldsymbol{\eta}_i$. This can be formulated as follows. As the mappings $\boldsymbol{\eta}_i(t, \cdot)$ are isometries on $\mathbb{R}^3$, they can be written in the form

$$\boldsymbol{\eta}_i(t,x) = x_i(t) + \mathbf{O}_i(t) x,$$

where $\mathbf{O}_i(t) \in SO(3)$ (i.e. it is a matrix satisfying $\mathbf{O}_i^T \mathbf{O}_i = \mathbf{Id}$). The position $x_i(t)$ denotes the position of the center of mass of S_i at a time t and

$$x_i(t) = \frac{1}{m_i} \int_{\overline{S}_i(t)} \varrho_{S_i}(t,x) x \,\mathrm{d}x,$$

where

$$m_i = \int_{\overline{S}_i(t)} \varrho_{S_i}(t,x) \,\mathrm{d}x$$

is the total mass of ith rigid body of a mass density ϱ_{S_i}. We say that velocity field $\boldsymbol{u}$ is compatible with the family of motions $\{\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_n\}$ if

$$\boldsymbol{u}(t,x) = \boldsymbol{u}^{S_i}(t,x) = \boldsymbol{U}_i(t) + \mathbf{Q}_i(t)(x - x_i(t)) \text{ for a.a. } x \in \overline{S}_i(t),\ i = 1, \ldots, n \tag{17}$$

for a.a. $t \in [0,T)$, where $\boldsymbol{u}^{S_i}$ is solid velocity, $\boldsymbol{U}_i(t)$ denotes the translation velocity and $\mathbf{Q}$ - the angular velocity of the body s.t.

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = \boldsymbol{U}_i(t),\ \Big(\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{O}_i(t) \Big) \mathbf{O}_i^T(t) = \mathbf{Q}_i(t) \text{ a.a. on } (0,T). \tag{18}$$

4 Main result

Let us formulate now our main existence result.

Theorem 4.1 *Let Ω be a bounded domain in $\mathbb{R}^3$ and following assumption be satisfied:*

- *The initial position of the rigid bodies is given through a family of open sets*

$$S_i \subset \Omega \subset \mathbb{R}^3, \ S_i \text{ diffeomorphic to the unit ball for } i = 1, \dots, n,$$

  where both ∂S_i, $i = 1, \dots, n$, and $\partial\Omega$ belong to the regularity class specified by (10), (11).

-
$$\operatorname{dist}[\overline{S}_i, \overline{S}_j] > 0 \text{ for } i \neq j, \ \operatorname{dist}[\overline{S}_i, \mathbb{R}^3 \setminus \Omega] > 0 \text{ for any } i, j = 1, \dots, n.$$

- *The viscous stress tensor $\mathbf{S}$ satisfy hypotheses (2 - 4).*

- *The $\mathcal{N}$–function M satisfies conditions (5 - 7) with $p \geq 4$ and the complementary function M^* to M satisfies Δ_2-condition.*

- *The given forces $F \in W^{1,\infty}(\Omega)$.*

- *The initial distribution of the density is given as*

$$\varrho_0 = \begin{cases} \varrho_f = \text{const} > 0 \ \text{in } \Omega \setminus \cup_{i=1}^n \overline{S}_i, \\ \\ \varrho_{S_i} \ \text{on } S_i, \ \text{where } \varrho_{S_i} \in L^\infty(\Omega), \ \operatorname{ess\,inf}_{S_i} \varrho_{S_i} > 0, \ i = 1, \dots, n, \end{cases}$$

  while an initial velocity field $\mathbf{u}_0$ satisfies

$$\boldsymbol{u}_0 \in L^2(\Omega; \mathbb{R}^3), \ \operatorname{div}_x \boldsymbol{u}_0 = 0 \text{ in } \mathcal{D}'(\Omega), \ \mathbf{D}\boldsymbol{u}_0 = 0 \text{ in } \mathcal{D}'(S_i; \mathbb{R}^{3\times3}) \text{ for } i = 1, \dots, n.$$

Then there exist a density function ϱ,

$$\varrho \in C([0,T]; L^1(\Omega)), \ 0 < \operatorname*{ess\,inf}_{\Omega} \varrho(t,\cdot) \leq \operatorname*{ess\,sup}_{\Omega} \varrho(t,\cdot) < \infty \text{ for all } t \in [0,T],$$

a family of isometries $\{\boldsymbol{\eta}_i(t,\cdot)\}_{i=1}^n$, $\boldsymbol{\eta}_i(0,\cdot) = \mathbf{Id}$, and a velocity field $\boldsymbol{u}$,

$$\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)) \cap L^p(0,T; W_0^{1,p}(\Omega; \mathbb{R}^3)), \quad \mathbf{D}\boldsymbol{u} \in L_M(Q; \mathbb{R}^{3\times3}),$$

compatible with $\{\boldsymbol{\eta}_i\}_{i=1}^n$ in the sense specified in (17), (18), such that ϱ, $\boldsymbol{u}$ satisfy the integral identity (13) for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$, and the integral identity (14) for any $\boldsymbol{\varphi}$ satisfying (15), (16).

The aim of this paper is to prove the Theorem 4.1.

5 Properties of Orlicz spaces

In this section we collect some properties of Orlicz spaces used within the proof of Theorem 4.1.

Lemma 5.1 *Let Ω be a bounded domain and M be an $\mathcal{N}-$function s.t. $M((|\cdot|)^{1/p})$ is convex. Let $f(t,x) \in L_M((0,T)\times\Omega)$, i.e. $\|f\|_{M,\Omega} < \infty$, then $f \in L_M(0,T;L^p(\Omega))$, i.e.*

$$\|f\|_{L_M(0,T;L^p(\Omega))} := \inf\left\{\lambda > 0 : \quad \int_0^T M\left(\frac{\|f(t,\cdot)\|_{L^p(\Omega)}}{\lambda}\right) \mathrm{d}t \le 1\right\} < \infty$$

Proof: If $f \in L_M((0,T)\times\Omega)$, then there exists $0 \le \lambda < \infty$ such that

$$\int_0^T \int_\Omega M\left(\frac{|\Omega|^{\frac{1}{p}}|f(t,x)|}{\lambda}\right) \mathrm{d}x\mathrm{d}t \le 1.$$

Employing the Jensen inequality, using the non-negativity, the convexity of M and $M((|\cdot|)^{1/p})$, and that $M(0) = 0$ we infer the following

$$\begin{aligned}
&\int_0^T M\left(\frac{1}{\lambda}\left(\int_\Omega |f(t,x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right) \mathrm{d}t = \int_0^T M\left(\left(\frac{|\Omega|}{\lambda^p|\Omega|}\int_\Omega |f(t,x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}\right) \mathrm{d}t \\
&\le \frac{1}{|\Omega|}\int_0^T \int_\Omega M\left(\left(\frac{|\Omega|}{\lambda^p}|f(t,x)|^p\right)^{\frac{1}{p}}\right) \mathrm{d}x\mathrm{d}t = \frac{1}{|\Omega|}\int_0^T \int_\Omega M\left(\left(\frac{|\Omega|}{\lambda^p}|f(t,x)|^p\right)^{\frac{1}{p}}\right) \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{|\Omega|}\int_0^T \int_\Omega M\left(\frac{|\Omega|^{\frac{1}{p}}|f(t,x)|}{\lambda}\right) \mathrm{d}x\mathrm{d}t < 1.
\end{aligned} \tag{19}$$

Since $M(|\cdot|^{1/p})$ is convex and $f \in L_M((0,T)\times\Omega)$, we notice that $f \in L^p((0,T)\times\Omega)$, hence $f \in L^p(0,T;L^p(\Omega))$. Consequently $t \to f(t,x)$ is measurable and hence we obtain bochner measurability. Therefore we obtain the statement and $f \in L_M(0,T;L^p(\Omega))$. $\square$

Now we want to introduce the Riesz transform in an Orlicz space, which will be used later as a tool in the local pressure method.

Let $\beta, \gamma \in (0,\infty)$ and $\tau \in [0,\infty)$. Let us denote by $L_{\tau \log^\beta}(\Omega)$ the Orlicz space associated with the $\mathcal{N}-$function $M(\tau) = \tau(\log(\tau+1))^\beta$ and by $L_{e(\gamma)}(\Omega)$ the Orlicz space associated with the $\mathcal{N}-$function of the form $M(\tau) = \exp(\tau^\gamma)$ for sufficiently large τ. Note it holds that $L_{\tau\log^\beta}(\Omega) = E_{\tau\log^\beta}(\Omega)$ and

$$\big(E_{e(\gamma)}(\Omega)\big)^* = L_{\tau\log^{1/\gamma}}(\Omega) \quad \text{and} \quad \big(L_{\tau\log^\beta}(\Omega)\big)^* = L_{e(1/\beta)}(\Omega),$$

see [19].

Let $\mathcal{R}_{i,j}$ stands for "double" Riesz transform of an integrable function g on $\mathbb{R}^3$, which can be given by a Fourier transform $\mathcal{F}$ as

$$\mathcal{R}_{i,j}[g] = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\right)\mathcal{F}[g] = \nabla_{x_i}\nabla_{x_j}\Delta^{-1}g\,, \quad i,\, j = 1,2,3, \tag{20}$$

where

$$\Delta^{-1}g(x) = \mathcal{F}^{-1}\left(\frac{-1}{|\xi|^2}\right)\mathcal{F}[g] = \int_{\mathbb{R}^3}\frac{g(y)}{|x-y|}\mathrm{d}y.$$

Lemma 5.2 *Let Ω be a bounded domain, let $b:\mathbb{R}^3\to\mathbb{R}$ be a multiplier, α be a multi-index such that $|\alpha|\le 2$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \le C < \infty\,.$$

Then for any $\beta>0$ there exists a constant $c(\beta)$ such that for all $g\in L_{\tau\log^{\beta+1}}(\Omega)$

$$\|(\mathcal{F}^{-1}b\mathcal{F})[g]\|_{\tau\log^\beta} \le c(\beta)\|g\|_{\tau\log^{\beta+1}} \tag{21}$$

where g is prolonged to be 0 on $\mathbb{R}^3\setminus\Omega$.

We recall here the proof given by Erban in [9].
Proof: The standard Mikhlin multiplier theorem (see e.g. [2, Chapter 6]) provides that the $\mathcal{F}^{-1}b\mathcal{F}$ is bounded as a mapping

$$\mathcal{F}^{-1}b\mathcal{F}: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3) \quad\text{and}\quad \mathcal{F}^{-1}b\mathcal{F}: L^1(\mathbb{R}^3)\to L^{1,\infty}(\mathbb{R}^3),$$

where $L^{1,\infty}$ stands for a Lorenz space[1]. Employing the result from [22, Theorem B.2] (see also [3]) we conclude that there exists a constant $c(\beta)$ such that (21) is satisfied. □

Corollary 5.1 *Let Ω be a bounded domain, then for any $\beta>0$ and $g\in L_{\tau\log^{\beta+1}}(\Omega)$*

$$\|\mathcal{R}_{i,j}[g]\|_{\tau\log^\beta} \le c(\beta)\|g\|_{\tau\log^{\beta+1}}. \tag{22}$$

Proofs of properties stated below the reader can find in [23, 24]. Here $d\in\{1,3,3\times 3\}$.

Lemma 5.3 *Let $\boldsymbol{z}^j: Q\to\mathbb{R}^d$ be a measurable sequence. Then $\boldsymbol{z}^j\xrightarrow{M}\boldsymbol{z}$ in $L_M(Q;\mathbb{R}^d)$ modularly if and only if $\boldsymbol{z}^j\to\boldsymbol{z}$ in measure and there exists some $\lambda>0$ such that the sequence $\{M(|\lambda\boldsymbol{z}^j|)\}$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty}\left(\sup_{j\in\mathbb{N}}\int_{\{(t,x):|M(|\lambda\boldsymbol{z}^j|)|\ge R\}} M(|\lambda\boldsymbol{z}^j|)\,\mathrm{d}x\mathrm{d}t\right) = 0.$$

Lemma 5.4 *Let M be an $\mathcal{N}-$function and for all $j\in\mathbb{N}$ let $\int_Q M(|\in z^j|)\ \mathrm{d}x\,\mathrm{d}t\le c$, where $\boldsymbol{z}^j: Q\to\mathbb{R}^d$. Then the sequence $\{\boldsymbol{z}^j\}_{j=1}^\infty$ is uniformly integrable in $L^1(Q;\mathbb{R}^d)$.*

Proposition 5.1 *Let M be an $\mathcal{N}$–function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\psi}^j: Q\to\mathbb{R}^d$ and $\boldsymbol{\phi}^j: Q\to\mathbb{R}^d$ are uniformly bounded in $L_M(Q;\mathbb{R}^d)$ and $L_{M^*}(Q;\mathbb{R}^d)$ respectively. Moreover $\boldsymbol{\psi}^j\xrightarrow{M}\boldsymbol{\psi}$ modularly in $L_M(Q;\mathbb{R}^d)$ and $\boldsymbol{\phi}^j\xrightarrow{M^*}\boldsymbol{\phi}$ modularly in $L_{M^*}(Q;\mathbb{R}^d)$. Then $\boldsymbol{\psi}^j\cdot\boldsymbol{\phi}^j\to\boldsymbol{\psi}\cdot\boldsymbol{\phi}$ strongly in $L^1(Q)$.*

[1] i.e. $g\in L^{1,\infty}$ iff $\sup_\sigma \sigma m(\sigma,g)<\infty$, where $m(\sigma,g)=|\{x:\ |g(x)|>\sigma\}|$

Proposition 5.2 *Let σ_h be a standard mollifier, i.e., $\sigma \in C^\infty(\mathbb{R})$, σ has a compact support and $\int_{\mathbb{R}} \sigma(\tau) d\tau = 1, \sigma(t) = \sigma(-t)$. We define $\sigma_h(t) = j\sigma(t/h)$. Moreover let $*$ denote a convolution in the variable t. Then for any function $\boldsymbol{\psi} : Q \to \mathbb{R}^d$ such that $\boldsymbol{\psi} \in L^1(Q; \mathbb{R}^d)$ it holds*

$$(\sigma^j * \boldsymbol{\psi})(t,x) \to \boldsymbol{\psi}(t,x) \quad \textit{in measure.}$$

Moreover for given an $\mathcal{N}-$function M and a function $\boldsymbol{\psi} : Q \to \mathbb{R}^d$ such that $\boldsymbol{\psi} \in \mathcal{L}_M(Q; \mathbb{R}^d)$. Then the sequence $\{M(x, \varrho^j * \boldsymbol{\psi})\}$ is uniformly integrable.

Remark: Let M be any $\mathcal{N}-$function. If $f \in L_M(\mathbb{R}^3)$, then $\|f|_B\|_{L_M(B)} \leq \|f\|_{L_M(\mathbb{R}^3)}$

$$\begin{aligned}
&\|f|_B\|_{L_M(B)} \\
&= \inf\left\{\lambda > 0 \,:\, \int_B M\left(\frac{f \mathbb{1}_B}{\lambda}\right) \mathrm{d}x \leq 1\right\} = \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^3} M\left(\frac{f}{\lambda}\right) \mathbb{1}_B \,\mathrm{d}x \leq 1\right\} \\
&\leq \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^3} M\left(\frac{f}{\lambda}\right) \mathrm{d}x \leq 1\right\} = \|f\|_{L_M(\mathbb{R}^3)},
\end{aligned}$$

where the inequality is provided by non-negativity of M.

6 Approximate problem

The first steep of the proof is is to approximate the rigid objects by a fluid of a very high viscosity. For this reason we introduce a penalization problem.

As Ω is bounded, we can assume that $\Omega \subset [-L, L]^3$ for a certain $L > 0$ and consider system (23 - 26) on the spatial torus

$$\mathcal{T} = [(-L, L)|_{\{-L,L\}}]^3.$$

Then all quantities are assumed to be spatially periodic with the period $2L$, in particular we extend initial velocity field u_0 by 0 outside of Ω and density by ϱ_f - constant density of the fluid. We also extend the outer force in such way that $F \in W^{1,\infty}(\mathcal{T})$.

The construction of weak solutions is based on a two-level approximation scheme that consists in solving the system of equations:

$$\partial_t \varrho + \operatorname{div}_x(\varrho[\boldsymbol{u}]_\delta) = 0, \tag{23}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes [\boldsymbol{u}]_\delta) + \nabla_x P = \operatorname{div}_x([\mu_\varepsilon]_\delta \mathbf{S}) + \varrho \nabla_x F - \chi_\varepsilon \boldsymbol{u}, \tag{24}$$

$$\partial_t \mu_\varepsilon + \operatorname{div}_x(\mu_\varepsilon [\boldsymbol{u}]_\delta) = 0, \tag{25}$$

$$\operatorname{div}_x \boldsymbol{u} = 0, \tag{26}$$

where P is scalar function denoting the pressure. Moreover we regularise the vector field in (23) and (23) with standard regularizing kernel. Namely for δ such that $\delta < \delta_0$ (δ_0 is as in Section 3)

$$[\boldsymbol{u}]_\delta = \omega_\delta * \boldsymbol{u}$$

stands for a spatial convolution with

$$\omega_\delta(x) = \frac{1}{\delta^3} \omega\left(\frac{|x|}{\delta}\right), \tag{27}$$

where $\omega \in C^\infty(\mathcal{T})$, $\operatorname{supp}\omega \subset \mathrm{B}(0,1)$,

$$\omega(x) > 0 \text{ for } x \in B(0,1), \quad \omega(x) = \omega(-x), \quad \int_{B(0,1)} \omega(x)\,\mathrm{d}x = 1.$$

The system (23 - 26) is supplemented with the initial conditions

$$0 < \varrho(0,\cdot) = \varrho_{0,\delta} = \varrho_f + \sum_{i=1}^{n} \varrho_{S_i,\delta}, \tag{28}$$

where

$$\varrho_{0,\delta} \to \varrho_0 \text{ strongly in } L^1(\mathcal{T}) \quad \text{ as } \delta \to 0$$

and

$$\varrho_{S_i} \in \mathcal{D}(S_i),\ \varrho_{S_i,\delta}(x) = 0 \text{ whenever } \operatorname{dist}[x, \partial S_i] < \delta < \delta_0, \text{ for } i = 1, \dots, n. \tag{29}$$

Similarly, we prescribe $\varepsilon-$dependent artificial "viscosity" $\mu : (0,T) \times \mathcal{T} \to \mathbb{R}$ with initial data given by

$$\mu(0,\cdot) = \mu_{0,\varepsilon} = 1 + \frac{1}{\varepsilon}\sum_{i=1}^{n} \mu_{S_i}, \tag{30}$$

where

$$\begin{gathered} \mu_{S_i} \in \mathcal{D}(S_i),\ \mu_{S_i}(x) = 0 \text{ whenever } \operatorname{dist}[x, \partial S_i] < \delta, \\ \mu_{S_i}(x) > 0 \text{ for } x \in S_i,\ \operatorname{dist}[x, \partial S_i] > \delta \quad \text{for } i = 1, \dots, n. \end{gathered} \tag{31}$$

The "viscosity" μ can be identified as the penalization introduced by Hoffmann and Starovoitov [25] and San Martin et al. [31], where the rigid bodies are replaced by the fluid of high viscosity becoming singular for $\varepsilon \to 0$.

Furthermore, we penalize also the region out of the set Ω and we take

$$\chi_\varepsilon = \frac{1}{\varepsilon}\chi,\ \chi \in \mathcal{D}(\mathcal{T}),\ \chi > 0 \text{ on } \mathcal{T} \setminus \Omega,\ \chi = 0 \text{ in } \overline{\Omega}. \tag{32}$$

The parameters ε and δ are small positive numbers. In above formulation, the extra parameter $\delta_0 > \delta > 0$ has been introduced to keep the density constant in the approximate fluid region in order to construct the local pressure.

For fixed $\varepsilon > 0$ and $\delta > 0$, we report the following existence result that can be proved by means of monotonicity argument for nonreflexive spaces (for the existence result without regularization of velocity field see [39] and for partial results in the Sobolev space setting see Frehse et al. [13, 15] and in the Orlicz space setting Gwiazda et al. [23, 24] and Wróblewsa [38]):

Proposition 6.1 *Suppose that $p \geq 4$. Let the initial distribution of ϱ, μ be given through (28 - 31), with fixed $\varepsilon > 0$, $\delta_0 > \delta > 0$. Moreover, assume that*

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ x \in \mathcal{T},\ \boldsymbol{u}_0 \in L^2(\mathcal{T};\mathbb{R}^3),\ \operatorname{div}_x \boldsymbol{u}_0 = 0 \text{ in } \mathcal{D}'(\mathcal{T};\mathbb{R}^3), \tag{33}$$

and $\chi_\varepsilon, F \in C^\infty(\mathcal{T})$, where χ_ε is determined by (32).

Then problem (23 - 26), supplemented with the initial data (28 - 31), possesses a (weak) solution ϱ, μ, $\boldsymbol{u}$ belonging to the class

$$\varrho, \mu \in W^{1,\infty}([0,T]; C^1(\mathcal{T})),$$

$$\boldsymbol{u} \in L^\infty(0,T; L^2(\mathcal{T};\mathbb{R}^3)) \cap L^p(0,T; W^{1,p}(\mathcal{T};\mathbb{R}^3)), \quad \mathbf{D}\boldsymbol{u} \in L_M(Q;\mathbb{R}^{3\times 3}).$$

In addition, the solution satisfies the energy inequality

$$\int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(z)\,\mathrm{d}x + \int_s^z \int_{\mathcal{T}} [\mu_\varepsilon]_\delta \mathbf{S} : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \int_s^z \int_{\mathcal{T}} \chi_\varepsilon |\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \\ \leq \int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(s)\,\mathrm{d}x + \int_s^z \int_{\mathcal{T}} \varrho\nabla_x F \cdot \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \tag{34}$$

for a.a. $0 \leq s < z \leq T$ *including* $s = 0$.

The weak formulation of the equation (24) is represented by the integral identity

$$\int_0^T \int_{\mathcal{T}} \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho(\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \tag{35}$$

$$= \int_0^T \int_{\mathcal{T}} [\mu_\varepsilon]_\delta \mathbf{S} : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathcal{T}} \varrho\nabla_x F\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathcal{T}} \chi_\varepsilon \boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \\ - \int_{\mathcal{T}} \varrho_{0,\delta}\boldsymbol{u}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x$$

which is satisfied for any test function $\boldsymbol{\varphi} \in \mathcal{D}([0,T)\times\mathcal{T};\mathbb{R}^3)$, $\mathrm{div}_x\boldsymbol{\varphi} = 0$.

Using continuity equation (23) and (4), the Young and the Sobolev inequality, the condition (5) we easily deduce also that following inequality is satisfied

$$\int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(z)\,\mathrm{d}x + \int_0^z \int_{\mathcal{T}} c_c[\mu_\varepsilon]_\delta M^*(\mathbf{S})\,\mathrm{d}x\mathrm{d}t + \int_0^z \int_{\mathcal{T}} c_c\left([\mu_\varepsilon]_\delta - \frac{1}{2}\right) M(\mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \\ + \int_0^z \int_{\mathcal{T}} \chi_\varepsilon|\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \leq C(F,\boldsymbol{u}_0,\varrho_0) \tag{36}$$

for a.a. $0 \leq z \leq T$.

Let us notice that due to method of characteristic applied to (25), the artificial "viscosity" $\mu_\varepsilon \geq 1$ on $[0,T]\times\mathcal{T}$. Hence the l.h.s. of (36) is nonnegative.

7 Artificial viscosity limit

7.1 Notation

For a family $\{S_i\}_{i=1}^n$ of precompact subsets of Ω, we denote

$$\mathrm{d}[\cup_{i=1}^n S_i] = \inf\left\{ \inf_{i,j=1,\dots,n,\ i\neq j} \mathrm{dist}[S_i,S_j], \inf_{i=1,\dots,n} \mathrm{dist}[S_i,\partial\Omega] \right\}. \tag{37}$$

We define a signed distance to the boundary of a subset S of Ω by

$$\mathbf{db}_S(x) = \mathrm{dist}[x,\mathbb{R}^3\setminus S] - \mathrm{dist}[x,S].$$

We say that a sequence of sets S_n converges to S in the sense of boundaries and denote it by $S_n \overset{b}{\to} S$, if

$$\mathbf{db}_{S_n}(x) \to \mathbf{db}_S(x) \text{ uniformly for } x \text{ belonging to compact subsets of } \mathbb{R}^3. \tag{38}$$

In similar way as San Martin et al. [31] and Feireisl et al. [12], we introduce $[S]_\delta$ called the $\delta-$kernel and $]S[_\delta$ - the $\delta-$neighbourhood of the set S, i.e.:

$$[S]_\delta = \mathbf{db}_S^{-1}((\delta,\infty)), \quad]S[_\delta = \mathbf{db}_S^{-1}((-\delta,\infty)). \tag{39}$$

Moreover, we define for $k \leq 0$

$$V^{k,p} = \text{closure}_{W^{k,p}(\Omega;\mathbb{R}^3)} \left\{ \boldsymbol{v} \in \mathcal{D}(\Omega;\mathbb{R}^3) \ : \ \text{div}_x \boldsymbol{v} = 0 \right\},$$

and

$$\mathcal{K}^{k,p}(S) = \left\{ \boldsymbol{v} \in V^{k,p} \ : \ \mathbf{D}\boldsymbol{v} = 0 \text{ in } \mathcal{D}'(S;\mathbb{R}^3) \right\}, \text{ where } S \text{ is an open subset of } \Omega.$$

For $p = 2$ and the Hilbert space $V^{k,2}$ the symbol $\mathcal{P}^k(S)$ denotes the orthogonal projection of $V^{k,2}$ onto the closed subspace $\mathcal{K}^{k,2}(S)$.

7.2 Uniform estimates and the continuity equation

Let us denote $\{\varrho_\varepsilon, \mu_\varepsilon, \boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ the family of approximate solutions associated with the problem (23 - 31). For the brevity of the notation we omit the dependence of this sequence on δ. The existence of such family of solutions is assured by Proposition 6.1. In first step we fix $\delta > 0$ and identify the limit for $\varepsilon \to 0$. The limit for δ will be shown in Section 8.

At first we show briefly how behaves the continuity equation (23) as $\varepsilon \to 0$. As we noticed already, the method of characteristics applied to (25) gives us that $\mu_\varepsilon \geq 1$. Hence following the estimates (36) we infer that

$$\int_{(0,T)\times\mathcal{T}} M\left(\mathbf{D}\boldsymbol{u}_\varepsilon\right) \mathrm{d}x\mathrm{d}t \leq c \tag{40}$$

and with assumption (5) this gives

$$\int_{(0,T)\times\mathcal{T}} |\mathbf{D}\boldsymbol{u}_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t \leq c. \tag{41}$$

Let us notice that the estimate (36) provides that

$$\int_0^z \int_{\mathcal{T}\setminus\Omega} |\boldsymbol{u}| \,\mathrm{d}x \leq c.$$

Without loosing of generality we can assume that $|\mathcal{T} \setminus \Omega| > 0$, therefore employing general version of the Korn inequality (see [14, Theorem 10.16]) we obtain

$$\|\boldsymbol{u}_\varepsilon\|_{L^p(0,T;W^{1,p}(\mathcal{T}))} \leq c$$

By the Alaoglu-Banach theorem we infer that for subsequence

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \text{ weakly in } L^p(0,T;W^{1,p}(\mathcal{T};\mathbb{R}^3)) \tag{42}$$

and additionally $\text{div}_x \boldsymbol{u} = 0$ a.e. on $(0,T)\times\mathcal{T}$. Next, regularized sequence $\{[\boldsymbol{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ satisfies

$$[\boldsymbol{u}_\varepsilon]_\delta \overset{*}{\rightharpoonup} [\boldsymbol{u}]_\delta \text{ weakly-(*) in } L^p(0,T;W^{1,\infty}(\mathcal{T};\mathbb{R}^3)) \text{ and } \text{div}_x[\boldsymbol{u}]_\delta = 0. \tag{43}$$

Furthermore, employing (32) together with (34), we infer

$$\boldsymbol{u} = 0 \text{ a.a. in the set } (0,T) \times (\mathcal{T} \setminus \Omega). \tag{44}$$

Since Ω is regular (see (11)), we get in the sense of traces

$$\boldsymbol{u}|_{\partial\Omega} = 0$$

and therefore

$$\boldsymbol{u} \in L^p(0,T;V^{1,p}),$$

we mean here $\boldsymbol{u}|_{(0,T)\times\Omega}$.

Let us recall now the stability result for solutions to the transport equation obtained in [11, Proposition 5.1]:

Proposition 7.1 *Let $\boldsymbol{v}_n = \boldsymbol{v}_n(t,x)$ be a sequence of vector fields such that*

$$\{\boldsymbol{v}_n\}_{n=1}^{\infty} \textit{ is bounded in } L^2(0,T;W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)).$$

Let $\boldsymbol{\eta}_n(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ be the solution operator corresponding to the family of characteristic curves generated by $\boldsymbol{v}_n$, i.e.

$$\frac{\partial}{\partial t}\boldsymbol{\eta}_n(t,x) = \boldsymbol{v}_n(t,\boldsymbol{\eta}_n(t,x)),\ \ \boldsymbol{\eta}_n(0,x) = x \textit{ for every } x \in \mathbb{R}^3.$$

Then passing to subsequences, as the case may be,

$$\boldsymbol{v}_n \overset{*}{\rightharpoonup} \boldsymbol{v} \textit{ weakly-(*) in } L^2(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))$$

and

$$\boldsymbol{\eta}_n(t,\cdot) \to \boldsymbol{\eta}(t,\cdot) \textit{ in } C_{\mathrm{loc}}(\mathbb{R}^3) \textit{ uniformly for } t \in [0,T]$$

as $n \to \infty$, where $\boldsymbol{\eta}$ is the unique solution of

$$\frac{\partial}{\partial t}\boldsymbol{\eta}(t,x) = \boldsymbol{v}(t,\boldsymbol{\eta}(t,x)),\ \ \boldsymbol{\eta}(0,x) = x,\ \ x \in \mathbb{R}^3.$$

In addition, let $S_n \subset \mathbb{R}^3$ be the sequence of sets s.t. $S_n \xrightarrow{b} S$ and let us define $\boldsymbol{\eta}_n(t,S_n) \equiv S_n(t)$. Then

$$S_n(t) \xrightarrow{b} S(t) \tag{45}$$

with $S(t) \equiv \boldsymbol{\eta}(t,S)$, meaning $\mathbf{db}_{S_n(t)} \to \mathbf{db}_{S(t)}$ in $C_{\mathrm{loc}}(\mathbb{R}^3)$ uniformly with respect to $t \in [0,T]$.

Now let us notice that $\{\varrho_\varepsilon\}_{\varepsilon>0}$ solve the transport equation (23) with regular transport coefitients, then we can use Proposition 7.1 and (43) conclude that

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T] \times \mathcal{T}). \tag{46}$$

Moreover due to method of characteristic for all $t \in [0,T]$

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta}, \tag{47}$$

and

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta}. \tag{48}$$

Employing once more the inequality (34) we obtain

$$\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\mathcal{T};\mathbb{R}^3)) \text{ as } \varepsilon \to 0. \tag{49}$$

Using a strong-weak argument together with (43), (46) we obtain, that the limit density ϱ satisfies the equation of continuity in a weak sense

$$\partial_t \varrho + \mathrm{div}_x(\varrho[\boldsymbol{u}]_\delta) = 0 \text{ in } (0,T)\times\mathcal{T} \tag{50}$$

provided ϱ has been extended to by ϱ_f outside of Ω. Once more, according to Proposition 7.1 and assumption (29) we notice that the density is constant in the approximation of the fluid region, i.e.

$$\varrho = \varrho_f \text{ on the set } \Big((0,T)\times\Omega\Big) \setminus \cup_{t\in[0,T]} \cup_{i=1}^n \boldsymbol{\eta}(t,[S_i]_\delta), \tag{51}$$

where $[S_i]_\delta$ is the δ-kernel (see Section 7.1) and $\boldsymbol{\eta}$ is a solution of

$$\partial_t \boldsymbol{\eta}(t,x) = [\boldsymbol{u}]_\delta(t,\boldsymbol{\eta}(t,x)),\ \boldsymbol{\eta}(0,x) = x. \tag{52}$$

7.3 Position of the rigid bodies

Next we identify the position of rigid bodies.

Let us remark as in [31] that if $\boldsymbol{u}$ is a rigid velocity field in set S, then $[\boldsymbol{u}]_\delta = \boldsymbol{u}$ for all x in S for which $\mathbf{db}_S(x) > \delta$.

The replacement of $\boldsymbol{u}_\varepsilon$ by $[\boldsymbol{u}_\varepsilon]_\delta$ in (25) allows to obtain better results on characteristics of transport equations. Moreover, we are able to obtain a rigid motion as $\varepsilon \to 0$, without passing to the limit w.r.t. δ due to above remark.

Here we follow [12] and just for convenience of the reader we recall briefly some of the steps.

Step 1: First let us recall that $]\cdot[_\delta$, $[\cdot]_\omega$ denote respectively $\delta-$neighbourhood and $\omega-$kernel defined in (39). We notice that the kernels $[S_i]_\omega$ and their images $\boldsymbol{\eta}(t,[S_i]_\omega)$ are non-empty connected open sets since $0<\delta<\omega<\delta_0/2$ (δ_0 has been introduced in (10)).

Directly from the hypothesis (30) and (31) we infer

$$\mu_\varepsilon(0,x) \to \infty \text{ as } \varepsilon \to 0 \text{ uniformly for } x \in [S_i]_{\omega'},\ i=1,\dots,n,\ \omega > \omega' > \delta. \tag{53}$$

Since $\boldsymbol{\eta}_\varepsilon$ is determined as the unique solution (due to regularity of $[\boldsymbol{u}_\varepsilon]_\delta$) of the problem

$$\partial_t \boldsymbol{\eta}_\varepsilon(t,x) = [\boldsymbol{u}_\varepsilon]_\delta(t,\boldsymbol{\eta}_\varepsilon(t,x)),\ \boldsymbol{\eta}_\varepsilon(0,x) = x, \tag{54}$$

(53) provides that

$$\mu_\varepsilon(t,x) \to \infty \text{ uniformly for } t\in[0,T],\ x \in \boldsymbol{\eta}_\varepsilon(t,[S_i]_{\omega'}),\ i=1,\dots,n, \tag{55}$$

According to (45) in Proposition 7.1

$$\overline{\boldsymbol{\eta}(t,[S_i]_\omega)} \subset \boldsymbol{\eta}_\varepsilon(t,[S_i]_{\omega'}) \text{ for sufficiently small } \varepsilon > 0 \text{ and for } \delta_0/2 > \omega > \omega' > \delta. \tag{56}$$

Hence from (55) we deduce

$$\mu_\varepsilon(t,x) \to \infty \text{ as } \varepsilon \to 0 \text{ uniformly for } t\in[0,T],\ x\in\boldsymbol{\eta}(t,[S_i]_\omega),\ \text{for } i=1,\dots,n. \tag{57}$$

Therefore we infer that

$$[\mu_\varepsilon]_\delta \to \infty$$

uniformly on compact subsets of

$$\{t \in [0,T],\ x \in]\boldsymbol{\eta}(t,[S_i]_\omega)[_\delta\},\ i = 1,\dots,n.$$

Consequently, we deduce from the estimate (36) that

$$\mathbf{D}\boldsymbol{u}_\varepsilon \to \mathbf{D}\boldsymbol{u} = 0 \text{ a.a. on the set } \cup_{t\in[0,T]} \cup_{i=1}^n]\boldsymbol{\eta}(t,[S_i]_\omega)[_\delta \text{ for any } \omega > \delta, \tag{58}$$

where $\boldsymbol{\eta}$ is determined by (52).

Step 2: Using now (58) we deduce that the limit velocity $\boldsymbol{u}$ coincides with a rigid velocity field $\boldsymbol{u}^{S_i}$ on the $\delta-$neighbourhood of each of the sets $\boldsymbol{\eta}(t,[S_i]_\omega)$, where $\omega > \delta$, $i = 1,\dots,n$. Since the rigid velocity fields coincide with their regularization, namely $[\boldsymbol{u}^{S_i}]_\delta = \boldsymbol{u}^{S_i}$, we conclude that

$$\boldsymbol{u}(t,x) = \boldsymbol{u}^{S_i}(t,x) = [\boldsymbol{u}]_\delta(t,x) \text{ for } t \in [0,T],\ x \in \boldsymbol{\eta}(t,[S_i]_\delta),\ i = 1,\dots,n. \tag{59}$$

Accordingly, by (52), (59) we infer existence a family of isometries $\boldsymbol{\eta}_i(t,\cdot)$, $t \in [0,T]$, $i = 1,\dots,n$, $\boldsymbol{\eta}(0,\cdot) = \boldsymbol{Id}$, such that

$$\boldsymbol{\eta}_i(t,[S_i]_\delta) = \boldsymbol{\eta}(t,[S_i]_\delta) \text{ for all } t \in [0,T],\ i = 1,\dots,n. \tag{60}$$

Moreover by (58) the mappings $\{\boldsymbol{\eta}_i\}_{i=1}^n$ are compatible with the velocity field $\boldsymbol{u}$ and with the rigid bodies $\{S_i\}_{i=1}^n$ in the sense stated in (17), (18). In particular, hypothesis (51), (59) and the assumption $\varrho_f = \text{const}$ provide that (50) reduces to

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0 \text{ in } (0,T) \times \mathcal{T}. \tag{61}$$

Step 3: Now we concentrate on momentum equation. Since $\boldsymbol{\eta}_i$ for $i = 1,\dots,n$ are isometries, (60) implies

$$]\boldsymbol{\eta}_i(t,[S_i]_\delta)[_\delta = \boldsymbol{\eta}_i(t,S_i),\ i = 1,\dots,n.$$

Hence $[\mu_\varepsilon]_\delta$ converges uniformly locally to 1 in the complementary $\cup_{i=1}^n \overline{S_i(t)}$ for any $t \in [0,T]$. According to estimates (36) and properties of regularization we notice that

$$\boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \rightharpoonup \overline{\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{62}$$

Together with (46) and by a weak-strong argument we obtain

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \rightharpoonup \varrho(\overline{\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta}) \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{63}$$

Employing again the estimate (36) and recalling that $\mu_\varepsilon \le 1$ in $[0,T) \times \mathcal{T}$ we infer

$$\mathbf{S}_\varepsilon \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \text{ weakly–(*) in } L_{M^*}(Q;\mathbb{R}^{3\times3}) \text{ for any } t \in [0,T], \tag{64}$$

due to properties of $\mathcal{N}-$function M (convexity and superlinear growth) the Dunford-Peties lemma provides

$$\mathbf{S}_\varepsilon \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^1([0,T] \times \mathcal{T};\mathbb{R}^{3\times3}) \tag{65}$$

Moreover by (46), (48) and (49) we infer

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon \rightharpoonup \varrho \boldsymbol{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\mathcal{T};\mathbb{R}^3)). \tag{66}$$

Finally letting $\varepsilon \to 0$ in the momentum equation (35) we deduce that

$$\begin{aligned}\int_0^T \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \overline{\varrho(\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta)} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \varrho \nabla_x F \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_{0,\delta} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \, \mathrm{d}x\end{aligned} \tag{67}$$

for any test function $\boldsymbol{\varphi} \in C^1([0,T) \times \overline{\Omega})$, $\boldsymbol{\varphi}(t,\cdot) \in [\mathcal{R}M](t)$, where $[\mathcal{R}M](t)$ is defined by (16) with

$$S_i(t) = \boldsymbol{\eta}_i(t, S_i), \ i = 1, \ldots, n.$$

7.4 Convergence of the velocities

Our next goal is to identify the weak limit in (62), namely we want to show that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } L^2(0,T; L^2(\Omega; \mathbb{R}^3)). \tag{68}$$

Let us notice that due (42) and the Sobolev imbedding theorem we obtain desired convergence in space but there is still possibility for oscillations of the velocity fields $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ in time.

As it was already pointed out, according to the result obtained by Starovoitov [34, Theorem 3.1], the collisions of two rigid objects do not appear. It is provided by the fact we consider the fluid which is incompressible and the velocity gradients are assumed bounded in the Lebesgue space L^p, with $p \geq 4$. Originally in [34] this statement was proven only for one body in a bounded domain, but it is easy to observe that this result can be extended to the case of several bodies (what is also mentioned in [34]). Hence we infer

$$\mathrm{d}[\cup_{i=1}^n S_i(t)] = d(t) > 0 \text{ uniformly for } t \in [0,T], \tag{69}$$

(where d is defined in Section 7.1). Setting $S_i^\varepsilon(t) = \boldsymbol{\eta}_\varepsilon(t, S_i)$ (see (54)) and according to Proposition 7.1 we have

$$\mathrm{d}[\cup_{i=1}^n S_i^\varepsilon(t)] = d_\varepsilon \to d \text{ in } C[0,T]. \tag{70}$$

Since the contacts of rigid bodies or bodies with boundary do not occur to proof compactness of the sequence $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$, we can use the same method as in [31, 12].

Since

$$S_i^\varepsilon(t) \xrightarrow{b} S_i(t) \text{ uniformly with respect to } t \in [0,T], \ i = 1, \ldots, n,$$

we obtain, for any fixed $\sigma > 0$, and all $\varepsilon < \varepsilon_0(\sigma)$ small enough

$$S_i(t) \subset]S_i^\varepsilon(t)[_\sigma, \quad S_i^\varepsilon(t) \subset]S_i(t)[_\sigma, \text{ for all } t \in [0,T], \ i = 1, \ldots, n. \tag{71}$$

Let us now recall the following result of Fereisl et al. [12], where also the proof can be found.

Lemma 7.1 *Given a family of smooth open sets $\{S_i\}_{i=1}^n \subset \Omega$, $0 < k < 1/2$, there exists a function $h : (0, \sigma_0) \to \mathbb{R}^+$ s.t. $h(\sigma) \to 0$ when $\sigma \to 0$ and for arbitrary $\boldsymbol{v} \in V^{1,p}$:*

$$\left\| \boldsymbol{v} - \mathcal{P}^k \Big(\cup_{i=1}^n]S_i[_\sigma \Big) \boldsymbol{v} \right\|_{W^{1,k}(\Omega;\mathbb{R}^3)} \leq c \Big(\| \ \mathbf{D}(\boldsymbol{v}) \ \|_{L^2(\cup_{i=1}^n S_i; \mathbb{R}^{3\times3})} + h(\sigma) \|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \Big) \tag{72}$$

with constant $0 < c < \infty$. Moreover, h and c are independent of the position of S_i inside Ω as long as $\mathrm{d}[\cup_{i=1}^n S_i] > 2\sigma_0$.

The ptojection $\mathcal{P}^k$ is defined in Section 7.1.

Next using the local-in-time Aubin-Lions argument we show the following

Lemma 7.2 *For all $\sigma > 0$ sufficiently small, and $0 < k < 1/2$, we have*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \varrho \boldsymbol{u} \cdot \mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}]\,\mathrm{d}x\mathrm{d}t.$$

The idea of the proof follows [12, 31].

Proof: Let us fix $\sigma > 0$. According to (71) there exists $\varepsilon_0(\sigma)$ such that for all $\varepsilon < \varepsilon_0$ holds

$$\cup_{i=1}^n S_i(t) \subset \cup_{i=1}^n]S_i^\varepsilon(t)[_{\sigma/4},\ \cup_{i=1}^n S_i^\varepsilon(t) \subset \cup_{i=1}^n]S(t)[_{\sigma/4} \text{ for all } t \in [0,T].$$

If we set instead of the sequence $\{\boldsymbol{v}_n\}_n$ the sequence $\{[\boldsymbol{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ with $\boldsymbol{u}_\varepsilon \in L^\infty(0,T;L^2)$ we notice that $\boldsymbol{\eta}_\varepsilon$ is Lipshitz continous. Hence we can infer from the Proposition 7.1 that there exists $\tau > 0$ (dependent on σ) and subdivision of the time interval $0 < \tau < 2\tau < ... < J\tau = T$ such that for arbitrary $t \in I_j := [j\tau,(j+1)\tau]$ we have:

$$\cup_{i=1}^n S_i(t) \subset \cup_{i=1}^n]S_i(j\tau)[_{\sigma/4},\ \cup_{i=1}^n S_i(j\tau) \subset \cup_{i=1}^n]S_i(t)[_{\sigma/4}. \tag{73}$$

To be more precise, if we take Lip as a Lipshitz constant of the function $t \to \boldsymbol{\eta}(t,x)$, then there exist $\tau < \sigma/(2Lip)$ which satisfies (73).

Our goal now is to infer from the momentum equation (35) that

$$\mathcal{P}^0\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \text{ is precompact in } L^2(I_j;\Big[\mathcal{K}^{k,2}\Big(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\Big)\cap V^{s,2}\Big]^*) \tag{74}$$

for any $k < 1$ and $s > \frac{5}{2}$ (then $W^{s-1,2} \subset L^\infty$).

First, let us fix one of the intervals I_m, $j = 1,\dots,J$ and in momentum equation let us take as a test function $\boldsymbol{\xi}$, which is equal to zero if $t \notin I_m$ and such that

$$\boldsymbol{\xi} \in \mathcal{K}^{1,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\cap V^{s,2} \text{ for all } t \in I_j.$$

Using estimates (36) we deduce from momentum equation that

$$\left|\int_{I_m}\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \partial_t \boldsymbol{\xi}\,\mathrm{d}x\mathrm{d}t\right| \le C\|\boldsymbol{\xi}\|_{L^\infty(I_j;W^{1,2}\cap V^{s,2})} \quad \text{for all } \varepsilon > \varepsilon_0$$

According to above relation we infer that

$$\left\{\partial_t \mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\right\}_{\varepsilon>0} \text{ is bounded in } L^1(I_j;\left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\cap V^{s,2}\right]^*).$$

Moreover, since $\varrho_\varepsilon \boldsymbol{u}_\varepsilon$ is bounded also in $L^2(I_m \times \mathcal{T})$, then the sequence

$$\{\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\}_\varepsilon \text{ is bounded in } L^2(I_j;\mathcal{K}^{0,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)).$$

Since the inclusion

$$\mathcal{K}^{0,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right) \subset \left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^* \text{ is compact for } 0 < k < 1,$$

the Aubin-Lions argument provides that the sequence

$$\{\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\}_{\varepsilon>0} \text{ is precompact in } L^2(I_j;\left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^*).$$

Furthermore by (66) we have that

$$\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \to \mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho \mathbf{u}] \tag{75}$$

strongly in $L^2(I_j; \left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^*)$ for $0<k<1$.

The relation (73) provides

$$\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right) = \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right) \tag{76}$$

for all $t \in I_j$ and $0<k<1$. Since $\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)$ is self-adjoint in $L^2(\Omega)$ and by

$$\begin{aligned}
&\int_{I_j}\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{I_j}\int_\Omega \mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\cdot \mathcal{P}^k((\cup_{i=1}^n]S_i(t)[_\sigma)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{I_j}\left(\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon], \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\,\mathrm{d}t
\end{aligned} \tag{77}$$

Then by (75) we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{I_j}\left(\varrho_\varepsilon \boldsymbol{u}_\varepsilon, \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\,\mathrm{d}t \\
&\qquad = \int_{I_j}\left(\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon], \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\,\mathrm{d}t \\
&\qquad = \int_{I_j}\left(\varrho \mathbf{u}, \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\mathbf{u}]\right)_{L^2(\Omega)}\,\mathrm{d}t
\end{aligned} \tag{78}$$

Summing up the relation as above from $j=1$ to $j=J$ we obtain desired conclusion of Lemma 7.2

Combining Lemmas 7.1 and Lemma 7.2 we deduce

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon |\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \varrho|\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \tag{79}$$

what can be shown exactly step by step as in [12, Section 5.2] or [29, Section 6.1]. Therefore we achieve the conclusion (68).

Indeed, for a fixed $k\in(0,1/2)$ and for sufficiently small $\varepsilon>0$, $\sigma>0$ we set

$$\int_0^T\int_\Omega\left(\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2 - \varrho|\boldsymbol{u}|^2\right)\mathrm{d}x\mathrm{d}t = I_1^\varepsilon(\sigma) + I_2(\sigma) - I_3^\varepsilon(\sigma),$$

where

$$I_1^\varepsilon(\sigma) = \int_0^T\int_\Omega\left(\varrho_\varepsilon \boldsymbol{u}_\varepsilon\cdot\mathcal{P}^k\left(\cup_{i=1}^n]S(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon] - \varrho\boldsymbol{u}\cdot\mathcal{P}^k\left(\cup_{i=1}^n]S(t)[_\sigma\right)[\boldsymbol{u}]\right)\,\mathrm{d}x\mathrm{d}t$$

$$I_2(\sigma) = \int_0^T\int_\Omega \varrho\boldsymbol{u}\cdot\left(\mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}] - \boldsymbol{u}\right)\,\mathrm{d}x\mathrm{d}t,$$

and

$$I_3^\varepsilon(\sigma) = \int_0^T\int_\Omega \varrho_\varepsilon\boldsymbol{u}_\varepsilon\cdot\left(\mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon\right)\,\mathrm{d}x\mathrm{d}t.$$

Next let us notice that Lemma 7.2 provides

$$\lim_{\varepsilon\to 0} I_1^\varepsilon(\sigma) = 0 \quad \text{for all } \sigma \text{ sufficiently small.}$$

As $\boldsymbol{u} \in L^2(0,T;\mathcal{K}^{k,p}(\cup_{i=1}^n S_i(t)))$ by Lemma 7.1 we infer

$$\int_0^T \int_\Omega |(\mathcal{P}^k\Big(\cup_{i=1}^n S_i(t)\Big)[\boldsymbol{u}] - \boldsymbol{u})|^2 \,\mathrm{d}x\mathrm{d}t \leq h(\sigma)^2 \int_0^T \int_\Omega \|\boldsymbol{u}\|^2_{W^{1,p}(\Omega;\mathbb{R}^3)} \,\mathrm{d}x\mathrm{d}t;$$

provided that there is no contacts of two bodies or of s body and the boundary of the set Ω. Therefore we obtain

$$I_2(\sigma) \to 0 \quad \text{as } \sigma \to 0$$

Recalling that $\{\varrho_\varepsilon \boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, we have

$$I_3^\varepsilon(\sigma) \leq c\int_0^T \|\mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t.$$

Since $]S_i(t)[_\sigma \subset]S_i^\varepsilon(t)[_{2\sigma}$ we obtain also

$$I_3^\varepsilon(\sigma) \leq c\int_0^T \|(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i^\varepsilon(t)[_{2\sigma}\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t.$$

for $\varepsilon > 0$ sufficiently small. Applying again Lemma 7.1, with $\boldsymbol{u}_\varepsilon(t,\cdot)$ for arbitrary $t \in [0,T]$ and $\varepsilon > 0$ sufficiently small we have that

$$\int_0^T \|(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i^\varepsilon(t)[_{2\sigma}\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t \leq c\Big(\sum_{i=1}^n \int_0^T \int_{S^i} |\mathbf{D}\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t\Big) + cTh(2\sigma).$$

The first term on the right hand side converges to zero as $\mathbf{D}\boldsymbol{u}_\varepsilon \to 0$ on $\bigcup_{i=1}^n S_i(t)$ and a.a. $t \in [0,T]$, and $\{\mathbf{D}\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is uniformly integrable in $L^2(Q)$ since (41) holds. Finally if we pass to the limit with $\varepsilon \to 0$ and we obtain that

$$\limsup_{\varepsilon\to\infty} I^\varepsilon(\sigma) \leq C(T)h(2\sigma) + I_2(\sigma)$$

whenever σ is small enough. Letting $\sigma \to 0$ we achieve the relation (79).

7.5 Convergence in nonlinear viscous term

Our main goal now is to prove convergence in the nonlinear viscous term in the 'fluid' part of the time-space cylinder $(0,T)\times\Omega$. As $\mu = 1$ on the fluid part and boundaries $S_n \to S$, we can choose for a sufficient small epsilon small cylinders contained in the fluid part of the time space cylinder Q_f. Thus in order to obtain this result we consider the equation (35) on the set $I \times B$ such that $I \subset (0,T)$ is a time interval and a spatial ball $|B| \leq 1$ and $B \subset \Omega \setminus \cup_{i=1}^n S_i(t)$ for $t \in I$. By (51) we can assume that $\varrho = \varrho_f$ in $I \times B$. In particular, we have

$$\int_0^T \int_\Omega \varrho_f \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)) : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = 0 \tag{80}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B;\mathbb{R}^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

We cannot test the above equation with function with non-zero support on Q^s, as neither the penalizing term $\mu_\varepsilon \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$ not $\mu_\varepsilon \mathbf{D}\boldsymbol{u}_\varepsilon$ can be controlled. At this stage of our investigation, the problem must be localised in the fluid part separately from the rigid bodies. Therefore we introduce a "local" pressure

$$p = p_{\mathrm{reg}} + \partial_t p_{\mathrm{harm}}, \tag{81}$$

where p_{reg} enjoys the same regularity properties as the sum of the convective and the viscous terms in case of power-law fluids (see [12]), while p_{harm} is a harmonic function. If the tensor $\mathbf{S}$ satisfies only conditions (2)-(4) and an $\mathcal{N}-$function M does not satisfies the Δ_2-condition, then the regularity of p_{reg} can be lower then regularity of the viscous term, what in fact makes the problem different from any previous considerations in this field.

The concept of local pressure was developed by Wolf [37, Theorem 2.6]. However our construction is based on Riesz transform as in [12] and it is more suitable for application to problems with non-standard growth conditions.

We start with formulation of following lemma:

Lemma 7.3 *Let $B \subset \mathbb{R}^3$ be a bounded domain with a regular C^3 boundary and $I = (t_0, t_1)$ be a time interval. Let $\mathcal{N}-$functions $m^*(\tau) = \tau \log^{\beta+1}(\tau + 1)$ for some $\beta > 0$ and $m'(\tau) = \tau \log^{\beta}(\tau + 1)$ for $\tau \in \mathbb{R}_+$. Moreover let M^* be an $\mathcal{N}-$function such that $c_1 m^*(\tau) \leq M^*(\tau) < c_2|\tau|^2$ fore some positive constants c_1, c_2. Assume that $\boldsymbol{U} \in L^\infty(I; L^2(B; \mathbb{R}^3))$, $\mathrm{div}_x \boldsymbol{U} = 0$, and $\mathbf{T} \in L_{M^*}(I \times B; \mathbb{R}^{3\times 3})$ satisfy the integral identity*

$$\int_I \int_B \Big(\boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{T} : \nabla_x \boldsymbol{\varphi} \Big) \,\mathrm{d}x\mathrm{d}t = 0 \tag{82}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Then there exist two functions

$$p_{\mathrm{reg}} \in L^1(I; L_{m'}(B)),$$

$$p_{\mathrm{harm}}(t, \cdot) \in \mathcal{D}'(B),\ \ \Delta_x p_{\mathrm{harm}} = 0 \ \text{ in } \mathcal{D}'(I \times B),\ \ \int_B p_{\mathrm{harm}}(t, \cdot) \,\mathrm{d}x = 0$$

satisfying

$$\int_I \int_B \Big(\boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{T} : \nabla_x \boldsymbol{\varphi} \Big) \,\mathrm{d}x\mathrm{d}t = \int_I \int_B \Big(p_{\mathrm{harm}} \partial_t \mathrm{div}_x \boldsymbol{\varphi} + p_{\mathrm{reg}} \mathrm{div}_x \boldsymbol{\varphi} \Big) \,\mathrm{d}x\mathrm{d}t \tag{83}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$. Additionally,

$$\|p_{\mathrm{reg}}\|_{L_1(I; L_{m'}(B))} \leq c(m') \|\mathbf{T}\|_{L_{M^*}(I \times B; \mathbb{R}^{3\times 3})} \tag{84}$$

and

$$p(t, \cdot)|_{B'} \in C^\infty(B'),\ \text{ where } B' \subset\subset B, \tag{85}$$

$$\|p_{\mathrm{harm}}\|_{L^\infty(I; L^1(B))} \leq c(m', I, B) \Big(\|\mathbf{T}\|_{L_{M^*}(I \times B; \mathbb{R}^3)} + \|\boldsymbol{U}\|_{L^\infty(I; L^2(B; \mathbb{R}^3))} \Big). \tag{86}$$

Proof: To begin with, the "regular" component of the pressure p_{reg} is identified as

$$p_{\mathrm{reg}}(t, \cdot) = \mathcal{R} : \mathbf{T} = \sum_{i,j=1}^{i,j=3} \mathcal{R}_{i,j}[T_{i,j}](t, \cdot) \text{ in } \mathbb{R}^3 \text{ for a.a. } t \in I,$$

where $\mathcal{R}$ denotes the "double" Riesz transform (see (20)) and $\mathbf{T} = [T_{i,j}]_{i=1,2,3;j=1,2,3}$ has been extended to be zero outside B. Using (22) we obtain that the mappings

$$\mathcal{R}_{i,j}|_B : L_{m^*}(B) \to L_{m'}(B) \quad \text{are bounded for } i,j = 1,2,3.$$

As a consequence we get (84) in following way

$$\|p_{\text{reg}}\|_{L^1(I;L_{m'}(B))} = \|\mathcal{R} : \mathbf{T}\|_{L^1(I;L_{m'}(B))} \le c_1(m')\|\mathbf{T}\|_{L^1(I;L_{M^*}(B))} \le c_2(m')\|\mathbf{T}\|_{L_{M^*}(I\times B)} \tag{87}$$

where use the fact that $L_{M^*}(I\times B;\mathbb{R}^{3\times 3}) \subseteq L^1(I;L_{M^*}(B;\mathbb{R}^{3\times 3}))$ (see in the proof of [7, Corollary 1.1.0]).

Moreover,

$$\int_B p_{\text{reg}} \Delta\psi \,\mathrm{d}x = \int_B \mathbf{T} : \nabla^2\psi \;\mathrm{d}x \text{ for any} \psi \in \mathcal{D}(B). \tag{88}$$

On the other hand, (82) provides that we can redefine $\boldsymbol{U}$ w.r.t. time on the set of zero measure such that the mappings

$$t \mapsto \int_B \boldsymbol{U}\cdot\boldsymbol{\psi}\,\mathrm{d}x \in C([s,z]) \text{ for any } \quad \boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3),\ \mathrm{div}_x\boldsymbol{\psi} = 0.$$

Particularly, we infer that the Helmholtz projection $\boldsymbol{H}[\boldsymbol{U}]$ belongs to the space $C_{\text{weak}}([s,z];L^2(B;\mathbb{R}^3))$. Therefore after taking in (82) $\boldsymbol{\phi}(t,x) = \eta(t)\boldsymbol{\psi}(x)$ such that $\eta \in \mathcal{D}(I)$, $\boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3)$, $\mathrm{div}_x\boldsymbol{\psi} = 0$ it follows that

$$\int_I \Big[\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\,\mathrm{d}x\Big]\partial_t\eta\;\mathrm{d}t - \int_I \Big[\int_B \Big(\int_{t_0}^t \mathbf{T}(s,\cdot)\,\mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\,\mathrm{d}x\Big]\partial_t\eta\,\mathrm{d}t = 0.$$

Employing Lemma 2.2.1 from [33], there exists a pressure $p = p(t,\cdot)$ such that

$$\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\,\mathrm{d}x - \int_B \Big(\int_{t_0}^t \mathbf{T}(s,\cdot)\;\mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\,\mathrm{d}x = \int_B p(t,\cdot)\mathrm{div}_x\boldsymbol{\psi}\,\mathrm{d}x \tag{89}$$

for all $t \in I$ and all $\boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3)$. Note that the term on the right-hand side measurable and integrable w.r.t time variable, since the left-hand side is measurable and integrable. Moreover for a.a. $t \in I$

$$\int_B p(t,\cdot)\,\mathrm{d}x = 0 \quad \text{and} \quad p(t,\cdot) \in \mathcal{D}'(B). \tag{90}$$

Testing (89) by $\partial_t\zeta$, $\zeta \in \mathcal{D}(I)$ and integrating over time interval I and setting $\boldsymbol{\varphi}(t,x) = \zeta(t)\boldsymbol{\psi}(x)$ we conclude that

$$\int_I\int_B \Big(\boldsymbol{U}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{T} : \nabla_x\boldsymbol{\varphi}\Big)\,\mathrm{d}x\mathrm{d}t = \int_I\int_B p\partial_t\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \tag{91}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I\times B;\mathbb{R}^3)$.

Finally, the harmonic pressure is defined as

$$p_{\text{harm}}(t,\cdot) = p(t,\cdot) + \Big(\int_{t_0}^t \Big[p_{\text{reg}}(\tau,\cdot) - \frac{1}{|B|}\int_B p_{\text{reg}}(\tau,\cdot)\,\mathrm{d}x\Big]\;\mathrm{d}\tau\Big). \tag{92}$$

Now we intend to show that $p_{\rm harm}(t,\cdot)$ is a harmonic function for any t. To this end, we take $\boldsymbol{\psi} = \nabla_x \gamma$, $\gamma \in \mathcal{D}(B)$ in (89) and compare the resulting expression with (88), (92) and use that $\mathrm{div}_x \boldsymbol{U} = 0$. If we insert (92) in (91), we infer (83).

Finally Weyl's lemma (see e.g. [35]) ensure that the function $p_{\rm harm}$ is regular locally in B, i.e. $p_{\rm harm} \in C^\infty(B')$, where $B' \subset\subset B$. Hence we obtain (85).

Moreover according to (92), (89) we show that (86) holds. Indeed, let us recall first the following result concerning the Bogovski operator in the space of bounded mean oscillations (BMO): Let $v : B \to \mathbb{R}^3$ and $f : B \to \mathbb{R}$, $f \in L^\infty(B)$ and $\int_B f = 0$ then there exists at least one solution satisfying $\mathrm{div}_x v = f$ in the sense of distributions. Furthermore

$$\|v\|_{BMO},\ \|\nabla_x v\|_{BMO} \le c\|f\|_\infty$$

and $N \cdot v|_{\partial B} = 0$ in the sense of generalized traces for some constant $C > 0$. Let us notice that $BMO(B) \subset L_{\widetilde{m}}(B)$ with $\widetilde{m}(\tau) = \exp(\tau) - 1$, $\tau \in (0,\infty)$, $L_{\widetilde{m}}(B) \subset L_M$ and $L_{\widetilde{m}}(B) \subset L_{m'^*}$, where m'^* is a complementary function to the $\mathcal{N}-$function m'. Then we use in (89) a test function the function $\boldsymbol{\psi}$ such that

$$\mathrm{div}_x \boldsymbol{\psi} = \left(\mathrm{sng}\, p - \frac{1}{|B|} \int_B \mathrm{sng}\, p \right) \in L^\infty(B).$$

Using above consideration, the Hölder inequality and (90) we obtain

$$\operatorname*{ess\,sup}_{t\in I} \|p(t,\cdot)\|_{L^1(B)} \le c(B,M) \left\{ \|\boldsymbol{U}\|_{L^\infty(I;L^2(B))} + \|\mathbf{T}\|_{L_M(I\times B)} \right\}. \tag{93}$$

Therefore (92) and (84) provide (86).

□

Remark: The assumption for the lower bound for an $\mathcal{N}-$function M^*, i.e. $m^*(\tau) = \tau \log^{\beta+1}(\tau+1) \le M^*(\tau)$ for $\tau \in \mathbb{R}_+$, $\beta > 0$, implies that we have to assume also that $M(\tau) \le c \exp(\tau^{\frac{1}{\beta+1}}) - c$ for some nonnegative constant c (see (5)).

Now we apply Lemma 7.3 with the $\mathcal{N}-$function M^*, with $\boldsymbol{U} := \varrho_f \boldsymbol{u}_\varepsilon$ and $\mathbf{T} := \varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$. Accordingly, for any $\varepsilon > 0$, there exist two scalar functions $p^\varepsilon_{\rm reg}$, $p^\varepsilon_{\rm harm}$ where

$$p^\varepsilon_{\rm reg} \in L^1(I; L_{m'}(B)),\ p^\varepsilon_{\rm harm} \in L^\infty(I; L^1(B)) \ \text{ are uniformly bounded} \tag{94}$$

and $p_{\rm harm}$ is a harmonic function w.r.t. x, i.e.

$$\Delta p^\varepsilon_{\rm harm} = 0, \quad \int_B p^\varepsilon_{\rm harm}(t,\cdot) = 0, \quad \forall\, t \in I.$$

Moreover the following is satisfied

$$\int_0^T \int_\Omega \left[(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) + p^\varepsilon_{\rm reg} \mathbf{I}) : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = 0 \tag{95}$$

for any test function $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$.

The standard estimates provide that p^ε_{harm} is uniformly bounded in $L^\infty(I; W^{2,2}_{loc}(B))$, moreover we already know that $\boldsymbol{u}_\varepsilon \in L^p(I, W^{1,p}(B))$. The equations (95) provides that

$$\|\partial_t(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})\|_{L^1(I;(W^{s,2}_0(B))^*)} < c,$$

where $s > 5/3$. Then the Lions-Aubin argument gives us that

$$\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}} \to \varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}} \text{ in } L^2(I; L^2(B'; \mathbb{R}^3)), \tag{96}$$

for arbitrary $B' \subset\subset B$.

According to (68), the velocity field $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is precompact in $L^2(0,T; L^2(\Omega; \mathbb{R}^3))$, whence we can infer that

$$\nabla_x p^\varepsilon_{\text{harm}} \to \nabla_x p_{\text{harm}} \text{ in } L^2(I; L^2(B'; \mathbb{R}^3)). \tag{97}$$

As the argument is valid for any B', our goal now is to let $\varepsilon \to 0$ in (95). First we recall that the sequence $\{\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{i\times B}\}_{\varepsilon>0}$ satisfies

$$\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \text{ weakly–(*) in } L_{M^*}(I \times B; \mathbb{R}^{3\times 3}_{\text{sym}}), \tag{98}$$

$$\text{or} \quad \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^1(I \times B; \mathbb{R}^{3\times 3}_{\text{sym}}).$$

Since $\boldsymbol{u}_\varepsilon|_{I\times B}$ and $\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{I\times B}$ are uniformly bounded in $L^p(I; W^{1,p}(B; \mathbb{R}^3))$ and $L_{M^*}(I \times B; \mathbb{R}^{3\times 3})$ respectively. Hence classical imbedding theorems provides that $\mathbf{T}^\varepsilon = (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon))|_{I\times B}$ is uniformly bounded in $L_{M^*}(I \times B; \mathbb{R}^{3\times 3})$. Therefore there exists some $\mathbf{T} \in L_{M^*}(I \times B)$ such that

$$\mathbf{T}^\varepsilon \overset{*}{\rightharpoonup} \mathbf{T} \quad \text{weakly–(*) in } L_{M^*}(I \times B).$$

Moreover, since $\mathcal{R}_{i,j}$ is a linear operator, using the properties of difference quotients we show that for any function $\phi \in W^{1,r}(B)$ possessing compact support contained in an open set B holds

$$\|\mathcal{R}_{i,j}[\phi]|_B\|_{W^{1,r}(B)} \le c\|\phi\|_{W^{1,r}(B)} \quad \text{ for any } r \in (0,\infty),$$

where on the left-hand side ϕ is prolonged by zero preserving the norm. Hence the functions $\mathcal{R}_{i,i}\nabla_{x_i}\varphi_i$, $i = 1,2,3$ are sufficiently regular in order to obtain

$$\begin{aligned} &\int_I \int_B p^\varepsilon_{\text{reg}} \mathbf{I} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = \int_I \int_B (\mathcal{R} : \mathbf{T}^\varepsilon)\mathbf{I} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\ &= \int_I \int_B \sum_{i=1}^3 T^\varepsilon_{i,i} \mathcal{R}_{i,i}[\nabla_{x_i}\varphi_i] \, \mathrm{d}x\mathrm{d}t \to \int_I \int_B \sum_{i=1}^3 T \mathcal{R}_{i,i}[\nabla_{x_i}\varphi_i] \, \mathrm{d}x\mathrm{d}t \quad \text{ as } \varepsilon \to 0 \end{aligned} \tag{99}$$

Finally by (68), (98) and (96), (99) passing with $\varepsilon \to 0$ in (95) we get

$$\int_I \int_B \Big[(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u} \otimes [\boldsymbol{u}]_\delta - \overline{\mathbf{S}}) : \nabla_x \boldsymbol{\varphi} \Big] + \sum_{i=1}^3 T \mathcal{R}_{i,i} \nabla_{x_i} \varphi_i \, \mathrm{d}x\mathrm{d}t = 0 \tag{100}$$

for any test function $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$.

Our aim is to use (95) and (100) with strong convergence (97) to characterise nonlinear viscous term using monotonicity methods for nonreflexive spaces as in [24, 38, 39].

To this end we take

$$\boldsymbol{\varphi} = \sigma_h * \sigma_h * r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}) \quad \text{ with any } r \in \mathcal{D}(B)$$

as a test function in (95). Here $*$ stands for convolution in time variable with regularising kernel σ_h (i.e. $\sigma \in C^\infty(\mathbb{R})$, $\mathrm{supp}\sigma \in B_1(0)$, $\sigma(-t) = \sigma(t)$, $\int_{\mathbb{R}} \sigma(t) \, \mathrm{d}t = 1$,

$\sigma_h(t) = \frac{1}{h}\sigma(\frac{1}{h})$). Since $\boldsymbol{u}_\varepsilon|_B \in L^2(I; W^{1,2}(B))$ and $p^\varepsilon_{\rm harm} \in L^\infty(I; W^{1,2}_{\rm loc}(B))$ we infer that

$$\sigma_h * \sigma * r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}) \in C(I; L^2(B;\mathbb{R}^3)) \quad \text{for any } r \in \mathcal{D}(B).$$

Then we obtain that

$$\begin{aligned}\int_I\int_B \sigma_h * \big(\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) - \varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - p^\varepsilon_{\rm reg}\mathbf{I}\big) : \sigma_h * (\nabla_x\,(r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})))\,{\rm d}x{\rm d}t \\ = \frac{1}{2}\int_B r|\sigma_h * (\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})|^2\,{\rm d}x\,\Big|_{t=t_0}^{t=t_1} \quad \text{for any } s,\ z \in I.\end{aligned} \tag{101}$$

Let us pass to the limit with $h \to 0$ and $\varepsilon \to 0$ and start with the right-hand side of (101). The relation (96) provides that

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{h\to0}\int_B r|\sigma_h * (\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})|^2\,{\rm d}x\,\Big|_{t=t_0}^{t=t_1} &= \lim_{\varepsilon\to0}\int_B r|(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})|^2\,{\rm d}x\,\Big|_{t=t_0}^{t=t_1} \\ &= \frac{1}{2}\int_B r|\varrho_f\boldsymbol{u} + \nabla_x p_{\rm harm}|^2\,{\rm d}x\,\Big|_{t=t_0}^{t=t_1}.\end{aligned}$$

for any Lebesgue point in $[0,T]$. As $p_{\rm harm}$ is a harmonic function on B, standard elliptic estimates provide that

$$r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}) \in L^p(I; W^{1,p}(B;\mathbb{R}^3)) \cap L^\infty(I; L^2(B;\mathbb{R}^3)),$$

while

$$\varrho_f\boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \in L^{p'}(I; L^{p'}(B;\mathbb{R}^3)).$$

Employing (96) we obtain

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{h\to0}\int_I\int_B (\sigma_h * (\varrho_f\boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta)) : (\sigma_h * (\nabla_x\,(r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}))))\,{\rm d}x{\rm d}t \\ = \lim_{\varepsilon\to0}\int_I\int_B (\varrho_f\boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta) : (\nabla_x\,(r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})))\,{\rm d}x{\rm d}t \\ = \int_I\int_B (\varrho_f\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta) : (\nabla_x\,(r(\varrho_f\boldsymbol{u} + \nabla_x p_{\rm harm})))\,{\rm d}x{\rm d}t.\end{aligned} \tag{102}$$

Now we concentrate on the third term on the left-hand side of (101). Since $p^\varepsilon_{\rm harm}$ is harmonic and ${\rm div}_x\boldsymbol{u}_\varepsilon = 0$, we obtain

$$\begin{aligned}\int_I\int_B \sigma_h * \big(p^\varepsilon_{\rm reg}\mathbf{I}\big) : \nabla_x\sigma_h * (r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}))\,{\rm d}x{\rm d}t \\ = \int_I\int_B \sigma_h * p^\varepsilon_{\rm reg}{\rm div}_x\sigma_h * (r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}))\,{\rm d}x{\rm d}t \\ = \int_I\int_B \sigma_h * p^\varepsilon_{\rm reg}\nabla_x r \cdot \sigma_h * (\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})\,{\rm d}x{\rm d}t.\end{aligned} \tag{103}$$

Let us consider first the first term on the right hand side of (103). Employing the estimate (40) and assumption (7) be the Lemma Lemma 5.1 we infer

$$\|\mathbf{D}\boldsymbol{u}_\varepsilon\|_{L_M(0,T;L^4(\mathcal{T}))} < \infty.$$

The generalized version of Korn inequality [14, Theorem 10.16] gives us

$$\|\boldsymbol{u}_\varepsilon\|_{L_M(0,T;W^{1,4}(\mathcal{T}))} < \infty.$$

Since $4 > \dim(B) = 3$,

$$\|\boldsymbol{u}_\varepsilon\|_{L_M(0,T;C(\mathcal{T}))} < \infty.$$

Hence $\boldsymbol{u}_\varepsilon|_{I\times B} \in L_M(I;W^{1,4}(B;\mathbb{R}^3)) \subset L_M(I;C(B;\mathbb{R}^3)) \subset L_M(I\times B;\mathbb{R}^3)$. Using the definition of $p^\varepsilon_{\rm reg}$ and the property $(\mathcal{R}_{i,j})^* = (\mathcal{R}_{j,i})$ we get

$$\begin{aligned}
&\int_I\int_B (\sigma_h * p^\varepsilon_{\rm reg})\nabla_x r\cdot(\sigma_h * (\varrho_f \boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t\\
&\quad = \int_I\int_B \{\sigma_h * \sum_{i,j=1}^3 \mathcal{R}_{i,j}[T_{i,j}](t,x)\}\{\nabla_x r\cdot(\sigma_h * (\varrho_f\boldsymbol{u}_\varepsilon))\}\,\mathrm{d}x\mathrm{d}t\\
&\quad = \int_I\int_B \sum_{i,j=1}^3 \{\sigma_h * T^\varepsilon_{i,j}(t,x)\}\{\sigma_h * \mathcal{R}_{j,i}\left[\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)\right]\}\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Since $\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon) \in L_M(I;W^{1,4}(B))$ and $r\in\mathcal{D}(B)$, in particular $\operatorname{supp}\nabla_x r \subset\subset B$, using the properties of difference quotients, see e.g [10], we deduce that

$$\|\mathcal{R}_{j,i}[\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)]|_B\|_{W^{1,4}(B)} \le \|\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)\|_{W^{1,4}(B)} \le \|\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)\|_{W^{1,4}(\Omega)}$$

for all $t\in I$, if we extend $\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)$ by zero on the whole space $\mathbb{R}^3$ preserving the norm. Consequently we have

$$\mathcal{R}_{i,j}[\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)]|_B \in L_M(I;W^{1,4}(B)).$$

Let us denote and notice that

$$\mathbf{b}^\varepsilon = [b^\varepsilon_{i,j}]_{i=1,2,3,\,j=1,2,3} = \left[T^\varepsilon_{i,j}\right]_{i=1,2,3,\,j=1,2,3} \tag{104}$$

are uniformly bounded in $L_{M^*}(I\times B;\mathbb{R}^{3\times3})$ hence

$$\mathbf{b}^\varepsilon \rightharpoonup \mathbf{b} \text{ weakly in } L^1(I\times B;\mathbb{R}^{3\times3}). \tag{105}$$

Moreover let us denote

$$\mathbf{w}^\varepsilon = \left[w^\varepsilon_{j,i}\right]_{i=1,2,3,\,j=1,2,3} = [\mathcal{R}_{j,i}[\nabla_x r\cdot(\varrho_f\boldsymbol{u}_\varepsilon)]]_{i=1,2,3,\,j=1,2,3} \tag{106}$$

which is uniformly bounded in $L_M(I;W^{1,4}(B;\mathbb{R}^{3\times3}))$.

Now let converge with $h\to0$. Since $b_\varepsilon\in L_{M^*}(I\times B)$ and $w_\varepsilon\in L_M(I;C(B))\subset L_M(I\times B)$ then there exists $\lambda_b,\ \lambda_w\in(0,\infty)$ such that $b/\lambda_b\in\mathcal{L}_{M^*}(I\times B)$ and $w/\lambda_w\in\mathcal{L}_M(I\times B)$. Due to Proposition 5.2 we obtain that

$$\begin{aligned}
\sigma_h * \mathbf{b}^\varepsilon &\to \mathbf{b}^\varepsilon \quad \text{in measure},\\
\sigma_h * \mathbf{w}^\varepsilon &\to \mathbf{w}^\varepsilon \quad \text{in measure}.
\end{aligned}$$

and $\{M^*(\sigma_h * \mathbf{b}^\varepsilon/\lambda_b)\}_{h>0}$, $\{M(\sigma_h * \mathbf{w}^\varepsilon/\lambda_w)\}_{h>0}$ are uniformly integrable. Therefore by Lemma 5.3 we obtain

$$\begin{aligned}
\sigma_h * \mathbf{b}^\varepsilon &\xrightarrow{M^*} \mathbf{b}^\varepsilon \quad \text{modularly in } L_{M^*}(I\times B) \text{ as } h\to0,\\
\sigma_h * \mathbf{w}^\varepsilon &\xrightarrow{M} \mathbf{w}^\varepsilon \quad \text{modularly in } L_M(I\times B) \text{ as } h\to0.
\end{aligned}$$

Consequently by Proposition 5.1 we get

$$\lim_{h\to 0}\int_I\int_B(\sigma_h*\mathbf{b}^\varepsilon):(\sigma_h*\mathbf{w}^\varepsilon)\,\mathrm{d}x\mathrm{d}t=\int_I\int_B\mathbf{b}^\varepsilon:\mathbf{w}^\varepsilon\,\mathrm{d}x\mathrm{d}t \tag{107}$$

Using following interpolation

$$\|\mathbf{w}^\varepsilon\|_{W^{\alpha,r}(B)}\le c\|\mathbf{w}^\varepsilon\|^{1-\lambda}_{W^{1,4}(B)}\|w^\varepsilon\|^{\lambda}_{L^2(B)} \tag{108}$$

with $\alpha=1-\lambda$ and $\frac{1}{r}=\frac{\lambda}{2}+\frac{1-\lambda}{4}$ (see [36] Section 2.3.1, 2.4.1, 4.3.1, 4.3.2), we can find such λ ($\lambda\in(0,\frac{1}{7})$ for space dimension 3) that $W^{\alpha,r}(B)$ is continuously embedded in $L^\infty(B)$ (see [1]). Therefore for any fixed $K>0$

$$\begin{aligned}\int_I\int_B M(K(\mathbf{w}^\varepsilon-\mathbf{w}))\,\mathrm{d}x\mathrm{d}t&\le|B|\int_I M(K\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{L^\infty(B)})\,\mathrm{d}t\\&\le|B|\int_I M\left(cK\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right)\,\mathrm{d}t=I_1.\end{aligned} \tag{109}$$

As $\mathbf{w}^\varepsilon\to w$ strongly in $L^2(I\times B)$ (as $\varrho_f=\text{const}$ in $I\times B$)

$$\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{L^2(B)}\to 0\quad\text{ in measure on } I.$$

As (**??**) holds, $\mathbf{w}^\varepsilon-\mathbf{w}\in L^1(I;W^{1,4}(B))$ and consequently

$$\left|\{t\in I\,:\,\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}>\alpha\}\right|\le\frac{c}{\alpha}$$

for some c independent of ε. Then

$$cK\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{\lambda}_{L^2(B)}\to 0\quad\text{ in measure on } I.$$

Continuity of M gives that

$$M\left\{cK\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right\}\to 0\quad\text{ in measure on } I. \tag{110}$$

Next we show uniform integrability of

$$\left\{M\left(cK\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right)\right\}_{\varepsilon>0}$$

in $L^1(I)$. Let us denote $R=cK\|\mathbf{w}^\varepsilon-\mathbf{w}\|_{L^\infty(I;L^2(B))}$ and let us notice that for any subset $E\subset I$ an $\lambda\in(0,1)$

$$\begin{aligned}&\int_E M\left(R\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right)\,\mathrm{d}t\\&\quad=\int_{\{t\in E\,:\,\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}\le R^{2/\lambda}\}}M\left(R\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right)\,\mathrm{d}t\\&\quad+\int_{\{t\in E\,:\,\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}>R^{2/\lambda}\}}M\left(R\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right)\,\mathrm{d}t\\&\quad\le|E|M\left(R^{(1+\frac{2}{\lambda}(1-\lambda))}\right)+\int_E M\left(\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\frac{\lambda}{2}}_{W^{1,4}(B)}\right)\,\mathrm{d}t\end{aligned} \tag{111}$$

Let us notice that the first term on the right-hand side depends linearly on the measure of set E. It remains to show that the sequence $\left\{M\left(\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\frac{\lambda}{2}}_{W^{1,4}(B)}\right)\right\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I)$. Indeed, as M is an $\mathcal{N}-$function and for $\lambda \in (0,1)$ the following assertion holds by de l'Hôpital's rule

$$\frac{M(\tau)}{M(\tau^{1-\frac{\lambda}{2}})} \to \infty \quad \text{as } \tau \to \infty.$$

Consequently $M\left(\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|_{W^{1,4}(B)}^{1-\frac{\lambda}{2}}\right)$ is uniformly integrable in $L^1(I)$. Summarising we obtain that

$$M\left(cK\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|_{W^{1,4}(B)}^{1-\lambda}\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|_{L^2(B)}^{\lambda}\right) \text{ is uniformly integrable in } L^1(I). \tag{112}$$

By (110) and (112) the Vitaly lemma provides that the right-hand side of (109) converge to 0. Consequently

$$K\mathbf{w}^\varepsilon \to K\mathbf{w} \quad \text{modularly in } L_M(I\times B). \tag{113}$$

According to Lemma 5.3 $\{M(K\mathbf{w}^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I\times B)$ and passing to subsequence if necessary

$$\mathbf{w}^\varepsilon \to \mathbf{w} \quad \text{a.e. in } I\times B. \tag{114}$$

Our next step is to show uniform integrability of $\{\mathbf{b}^\varepsilon\mathbf{w}^\varepsilon\}_{\varepsilon>0}$ in $L^1(I\times B)$. By the Fenchel-Young inequality and convexity of M^* for $K>1$ it follows that

$$|\int_I\int_B \mathbf{b}^\varepsilon\mathbf{w}^\varepsilon \,\mathrm{d}x\mathrm{d}t| = \int_I\int_B \frac{1}{K}\mathbf{b}^\varepsilon K\mathbf{w}^\varepsilon \,\mathrm{d}x\mathrm{d}t \le \int_I\int_B \frac{1}{K}M^*(\mathbf{b}^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_B M(\frac{1}{K}\mathbf{w}^\varepsilon)\,\mathrm{d}x\mathrm{d}t$$

As K is arbitrary and $\{M(K\mathbf{w}^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I\times B)$ we obtain the assertion that $\{\mathbf{b}^\varepsilon\mathbf{w}^\varepsilon\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I\times B)$. Moreover as (105) and (114) hold we infer that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_I\int_B \sum_{i,j=1}^{3}\left(T_{i,j}^\varepsilon(t,x)\right)\left(\mathcal{R}_{j,i}\left[\nabla_x r\cdot(\boldsymbol{u}_\varepsilon)\right]\right)\,\mathrm{d}x\mathrm{d}t \\ = \int_I\int_B \sum_{i,j=1}^{3}\left(T_{i,j}(t,x)\right)\left(\mathcal{R}_{j,i}\left[\nabla_x r\cdot(\boldsymbol{u})\right]\right)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then the second term on the right hand side of (103) can be treated in similar way, since $p^\varepsilon_{\mathrm{harm}}$ is harmonic function and $r\in B$. Finally we infer

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{h\to\infty}\int_I\int_B \sigma_h * \left(p^\varepsilon_{\mathrm{reg}}\mathbf{I}\right) : \sigma_h * \nabla_x\left(r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\mathrm{harm}})\right)\,\mathrm{d}x\mathrm{d}t \\ = \int_I\int_B \sum_{i,j=1}^{3} T_{i,j}(t,x)\mathcal{R}_{i,j}[\nabla_x r\cdot(\varrho_f\boldsymbol{u} + \nabla_x p_{\mathrm{harm}})]\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

It remains to show how behave the viscous term in the limit with $h\to 0$ and $\varepsilon\to 0$, i.e.

$$\begin{aligned}\int_I\int_B \sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \nabla_x\sigma_h * (r\varrho_f\boldsymbol{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ = \int_I\int_B \sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \sigma_h * (\nabla_x r\varrho_f\boldsymbol{u}_\varepsilon + r(\varrho_f\nabla_x\boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{115}$$

As $\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{I\times B} \in L_{M^*}(I\times B;\mathbb{R}^{3\times 3})$ and $\boldsymbol{u}_\varepsilon|_{I\times B} \in L_M(I;W^{1,4}(B;\mathbb{R}^3)) \subseteq L_M(I\times B;\mathbb{R}^3)$, we proceed in similar way as in (107) passing to the limi with $h\to 0$. Then we proceed exactly as with $\mathbf{b}^\varepsilon$ and $\mathbf{w}^\varepsilon$ in order to converge with $\varepsilon to 0$. Therefore we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{h\to 0}\int_I\int_B(\sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)) : \sigma_h * (\nabla_x r\varrho_f\boldsymbol{u}_\varepsilon + r(\varrho_f\nabla_x\boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t\\
&\qquad = \lim_{\varepsilon\to 0}\int_I\int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : (\nabla_x r\varrho_f\boldsymbol{u}_\varepsilon + r(\varrho_f\nabla_x\boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t \qquad (116)\\
&\qquad = \int_I\int_B \overline{\mathbf{S}} : \nabla r\varrho_f\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \lim_{\varepsilon\to 0}\int_I\int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) r(\varrho_f\nabla_x\boldsymbol{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Summarising (116) and previous consideration, passing to the limit first with $h\to\varepsilon$ in (101) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_I\int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : (r(\varrho_f\nabla_x\boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_B r^2|\varrho_f\boldsymbol{u} + \nabla_x p_{\mathrm{harm}}|^2\,\mathrm{d}x\,\Big|_{t=t_0}^{t=t_1}\\
&\quad - \int_I\int_B \overline{\mathbf{S}} : \nabla r\varrho_f\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. + \int_I\int_B ((\varrho_f\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta)) : (\nabla_x(r(\varrho_f\boldsymbol{u} + \nabla_x p_{\mathrm{harm}})))\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_I\int_B \sum_{i,j=1}^3 T_{i,j}(t,x)\mathcal{R}_{j,i}[\nabla_x r\cdot(\boldsymbol{u}_\varepsilon)]\,\mathrm{d}x\mathrm{d}t
\end{aligned} \qquad (117)$$

Using

$$\sigma_h * \sigma_h * r(x)(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\mathrm{harm}})$$

as a test function in the limit equation (100), after passing with $\varepsilon\to 0$ we get

$$\begin{aligned}
&\int_I\int_B\left(\overline{\mathbf{S}} - \varrho_f\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta\right) : \nabla_x(r(\varrho_f\boldsymbol{u}_\varepsilon + \nabla_x p_{\mathrm{harm}}))\\
&\qquad - \sum_{i,j=1}^3 T_{i,j}\mathcal{R}_{j,i}[\nabla_x r\cdot(\varrho_f\boldsymbol{u} + \nabla_x p_{\mathrm{harm}})]\,\mathrm{d}x\mathrm{d}t \qquad (118)\\
&\qquad = \frac{1}{2}\int_B r|(\varrho_f\boldsymbol{u} + \nabla_x p_{\mathrm{harm}})|^2\,\mathrm{d}x\,\Big|_{t=t_0}^{t=t_1} \quad \text{for any } s,\ z\in I.
\end{aligned}$$

Finally we conclude

$$\limsup_{\varepsilon\to 0}\int_I\int_B r\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \nabla_x\boldsymbol{u}_\varepsilon\,\mathrm{d}x\mathrm{d}t \le \int_I\int_B r\overline{\mathbf{S}} : \nabla_x\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$

and by monotonicity argument for nonreflexive spaces used in [24, 38, 39] we finally obtain

$$\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \to \mathbf{S}(\mathbf{D}\boldsymbol{u}) \text{ a.e. in } I\times B. \qquad (119)$$

7.6 Conclusion

Considerations given in two preceding sections provides, that the (67) reduces to

$$\int_0^T\int_\Omega \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho(\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \qquad (120)$$

$$= \int_0^T \int_\Omega \mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathcal{T}} \varrho \nabla_x F \cdot \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t - \int_{\mathcal{T}} \varrho_{0,\delta} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \,\mathrm{d}x$$

for any test function $\boldsymbol{\varphi} \in C^1([0,T) \times \overline{\Omega})$, $\boldsymbol{\varphi}(t,\cdot) \in [\mathcal{R}M](t)$, with

$$[\mathcal{R}M](t) = \{\boldsymbol{\phi} \in C_c^1(\Omega;\mathbb{R}^3) \ : \mathrm{div}_x \boldsymbol{\phi} = 0 \text{ in } \Omega, \\ \mathbf{D}\boldsymbol{\phi} \text{ has compact support on } \Omega \setminus \cup_{i=1}^n \overline{S}_i(t)\},$$

where

$$S_i(t) = \eta_i(t, S_i), \ i = 1, \dots, n.$$

Furthermore, the limit solution satisfies the energy inequality

$$\int_\Omega \frac{1}{2}\varrho|\boldsymbol{u}|^2(\tau) \,\mathrm{d}x + \int_s^\tau \int_\Omega \mathbf{S} : \mathbf{D}\boldsymbol{u} \,\mathrm{d}x\mathrm{d}t \le \int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(s) \,\mathrm{d}x + \int_s^\tau \int_\Omega \varrho \nabla_x F \cdot \boldsymbol{u} \,\mathrm{d}x\mathrm{d}t \quad (121)$$

for any τ and a.a. $s \in (0,T)$ including $s = 0$.

8 The limit passage for $\delta \to 0$

In the last section we pass to the limit with $\delta \to 0$ in the system of equations (61), (120) and in the corresponding family of isometries $\{\boldsymbol{\eta}_i\}_{i=1}^n$ describing the motion of rigid bodies. Hence we denote the associated sequences of solutions by $\{\varrho_\delta, \boldsymbol{u}_\delta, \{\eta_i^\delta\}_{i=1}^n\}_{\delta>0}$.

Observe now that the initial data $\varrho_{S_i,\delta}$ in (28) can be taken in such a way that

$$\|\varrho_{S_i,\delta}\|_{L^\infty(\Omega)} \le c, \ \varrho_f + \varrho_{S_i,\delta} \to \varrho_{S_i} \ \text{ as } \delta \to 0 \text{ in } L^1(\Omega), \ i = 1, \dots, n,$$

where $\{\varrho_{S_i}\}_{i=1}^n$ are the initial distributions of the mass on the rigid bodies in Theorem 4.1. Then the theory for transport equations developed by DiPerna and Lions [6] provides that

$$\varrho_\delta \to \varrho \quad \text{ strongly in } C([0,T]; L^1(\Omega)) \text{ as } \delta \to 0$$

According to energy inequality (121), we obtain that for subsequence if necessary

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \text{ weakly in } L^p(0,T; W^{1,p}(\Omega;\mathbb{R}^3))$$

where $\boldsymbol{u}_\delta$ and the limit velocity $\boldsymbol{u}$ as well are divergence-free. Hence the continuity equation (61) reduces to a transport equation

$$\partial_t \varrho + \boldsymbol{u} \cdot \nabla_x \varrho = 0.$$

Following step by step the arguments given in previous sections we proof the rest of the convergence process. the compactness of the velocity and convergence in convective term can be done as in [39]. The convergence in nonlinear viscous term is completed by the same arguments as in Section 7.5.

References

[1] R. A. Adams and J. J. A. Fournier. *Sobolev spaces.* Academic Press, 2003.

[2] J. Bergh and J. Löfström. *Interpolation spaces: an introduction.* Springer–Verlag, Berlin, Heilderberg, New York, 1976.

[3] A. Cianchi. Strong and weak type inequalities for some classical operators in Orlicz spaces. *J. London Math. Soc.* **60**(2):187–202, 1997.

[4] B. Desjardins and M.J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.*, **146**:59–71, 1999.

[5] B. Desjardins and M.J. Esteban. On weak solutions for fluid-rigid structure interaction: Compressible and incompressible models. *Commun. Partial Differential Equations*, **25**:1399–1413, 2000.

[6] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[7] T. Donaldson. Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems. *J. Differential Equations.* 16:201-256, 1974.

[8] R.G. Egres Jr, Y.S. Lee, J.E. Kirkwood, K.M. Kirkwood, E.D. Wetzl, N.J. Wagner. "Liquid armor": Protective fabrics utilising shear thickening fluids. *Proceedings of the 4th International Conference of Safety and Protective Fabrics*, Pittsburg, PA, October, 26-27, 2004.

[9] R. Erban. On the existence of solutions to the Navier -Stokes equations of two -dimensional compressible flow. *Mathematical Methods in the Applied Sciences.* **26**:489-517, 2003.

[10] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[11] E. Feireisl. On the motion of rigid bodies in a viscous compressible fluid. *Arch. Rational Mech. Anal.*, **167**:281–308, 2003.

[12] E. Feireisl, M. Hillairet and Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian fluid. *Nonlinearity.* **21**:1349-1366, 2008.

[13] J. Frehse, J. Málek, M. Růžička. Large data existence results for unsteady flows of inhomogeneus heat-conducting incompressible fluids. *Communication in Partial Differential Equations.* 35(10):1891–1919, 2010.

[14] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids.* Birkhauser, Basel, 2009.

[15] J. Frehse and M. Růžička. Non-homogenous generalized Newtonian fluids. *Mathematische Zeitschrift.* 260:355-375, 2008.

[16] H. Gajewski, K. Gröger and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen.* Akademie-Verlag, Berlin, 1974.

[17] J.M. Houghton, B.A. Schiffman, D.P. Kalman, E.D. Wetzel, N.J. Wagner. Hypodermic Needle Puncture of Shear Thicening Fluid (STF)-Treated Fabrics. *Proceedings of SAMPE.* Baltimore MD, 2007.

[18] F. John and L. Nirenberg. On functions of bounded mean oscillation. *Comm. Pure and Applied Math.* **14**:415–426, 1961.

[19] A. Kufner, O. John and S. Fučik. *Function spaces.* Noordhoff International Publishing. Prague: Publishing House of the Czechoslovak Academy of Sciences, 1977.

[20] G. P. Galdi. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Arch. Rat. Mech. Anal.*, **148**:53–88, 1999.

[21] G. P. Galdi. On the motion of a rigid body in a viscous fluid: A mathematical analysis with applications. *Handbook of Mathematical Fluid Dynamics, Vol. I, Elsevier Sci, Amsterdam*, 2002.

[22] J. Gustavsson and J. Peetre. Interpolation of Orlicz spaces. *Studia Mathematica.* 60:33–59, 1977.

[23] P. Gwiazda and A. Świerczewska–Gwiazda. On non-Newtonian fluids with the property of rapid thickening under different stimulus. *Math. Models Methods Appl. Sci.*, 7(18):1073-1092, 2008.

[24] P. Gwiazda, A. Świerczewska–Gwiazda and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Mathematical Methods in the Applied Sciences*, 33(2):125-137, 2010.

[25] K.-H. Hoffmann and V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two dimensional case. *Adv. Math. Sci. Appl.*, **9**:633–648, 1999.

[26] N.V. Judakov. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. (Russian) *Dinamika Splošn. Sredy Vyp.* , **18**: , 249–253, 1974.

[27] J. Málek, J. Nečas, M. Rokyta and M. Růžička. Existence and regularity of solutions and the stability of the test state for fluids with shear dependent viscosity. *Math. Models and Methods in Applied Sciences.* 5:789–812, 1995.

[28] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDE's.* Chapman and Hall, London, 1996.

[29] Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian and heat-conducting fluid. *Ann Univ Ferrara.* **55**:325-352, 2009.

[30] R. T. Rockaffellar. *Convex analysis.* Princton University Press. 1970.

[31] J.A. San Martin, V. Starovoitov, and M. Tucsnak. Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.*, **161**:93–112, 2002.

[32] D. Serre. Chute libre d'un solide dans un fluid visquex incompressible. Existence. *Jap. J. Appl. Math.*, **4**:99–110, 1987.

[33] H. Sohr. *The Navier-Stokes equations: An elementary functional analytic approach.* Birkhäuser Verlag, Basel, 2001.

[34] V.N. Starovoitov. Behavior of a rigid body in an incompressible viscous fluid near boundary. *In International Series of Numerical Mathematics*, **147**:313–327, 2003.

[35] D.W. Stroock Weyl's Lemma, one of many. London Mathematical Society Lecture Note Series. No. 354. pp. 164-173. Cambridge University Press. 2008.

[36] H. Triebel. *Interpolation theory, function spaces, differential operators.* VEB Deutscher Verlag der Wissenschaften. Berlin, 1978.

[37] J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Dynamics*, **8**:1–35, 2006.

[38] A. Wróblewska. Steady flow of non-Newtonian fluids - monotonicity methods in generalized Orlicz spaces. *Nonlinear Analysis.* 72:4136-4147, 2010.

[39] A. Wróblewska. Existence results for unsteady flows of nonhomogeneous non-Newtonian incompressible fluids - monotonicity methods in generalized Orlicz spaces. Preprint 2011 - 015 http://mmns.mimuw.edu.pl/preprints.html.