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LOCAL PRESSURE METHODS IN ORLICZ SPACES FOR THE MOTION OF RIGID BODIES IN A NON-NEWTONIAN FLUID WITH GENERAL GROWTH CONDITIONS

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Abstract. In the present paper we provide the decomposition and local estimates for the pressure function for the non-stationary flow of incompressible non-Newtonian fluids in Orlicz spaces. We show that this method can be applied to prove the existence of weak solutions to the problem of motion of one or several rigid bodies in a non-Newtonian incompressible fluid with growth conditions given by an N-function.

1. **Introduction.**

The problem of existence and construction of a pressure function arises in the theory of non-stationary incompressible non-Newtonian fluids.

Since in mechanics of incompressible fluids mainly divergence-free test functions are considered, the pressure usually does not appear in the definition of a weak solution. However the pressure function p can be identified a posteriori in some cases [7]. It can be obtained that there exists p of the form

$$p = p_{\rm reg} + \partial_t p_{\rm harm},$$

where $p_{\rm reg} \in L^q(I; L^q(\Omega))$ and $p_{\rm harm} \in L^2(I; L^2(\Omega))$ (where I stands for the time interval and Ω is a spatial domain occupied by the fluid). But since the time derivative $\partial_t p_{\rm harm}$ is present here, we do not know if p is an integrable function on the time space cylinder, and even if it is a measurable function in $(x, t) \in Q$. Moreover when the pressure is introduced by the De Rham theorem, we still do not know what is the best function space for the pressure. In the present paper we want to develop the method of a local pressure. The concept of the local pressure is given by J. Wolf in [21] in order to obtain the existence result for non-stationary motion of a non-Newtonian fluid with shear rate dependent viscosity of a power-law type with no restriction on shape or size of the spatial domain. There local pressure estimates are based on variational methods. Here and in [6, 21] the pressure is decomposed into a measurable function $p_{\rm reg}$ and the singular part $\partial_t p_{\rm harm}$, where $p_{\rm harm}$ is harmonic w.r.t. space variable. In [21] the author provides optimal a-priori estimates for the components $p_{\rm reg}$ and $p_{\rm harm}$, which are achieved mainly by L^q-estimates for weak solutions to the Laplace equation obtained by Simader and variational estimates

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for the bipotential operator established by R. Müller (for references see [21]). Later Feireisl, Hillairet and Nečasová used different tools to derive the decomposition of the pressure and estimates for p_{reg} and p_{harm}. Their construction was based on the Riesz transform which seems to be more suitable for application to problems associated with fluids of a power-law type. This construction allowed to obtain that the regular part p_{reg} shares the same regularity properties as the nonlinear viscous part in the momentum equation for the velocity field of the fluid.

Our construction of the local pressure is also based on the Riesz transform as in [6], but we state the problem in a more general setting. Motivated by [6] we want to generalize the result obtained therein and formulate the problem of the motion of rigid bodies in a non-Newtonian fluid in Orlicz spaces, what is more suitable for applications in fluid mechanics with nonstandard rheology.

The paper is organized as follows. In Section 2 we introduce notation and give some definitions and preliminaries concerning Orlicz spaces. The main result concerning decomposition and estimates for the function which in fluid mechanics is associated with the pressure is presented in Section 3. An example of an application in the existence theory for the motion of rigid bodies in a viscous fluid with growth conditions in Orlicz spaces can be found in Section 4.

2. **Notation - Orlicz spaces.** In our investigation by $\mathcal{D}(\Omega)$ we mean the set of C^∞-functions with compact support contained in Ω. Moreover, by $L^q, W^{1,q}$ we denote the standard Lebesgue and Sobolev spaces respectively.

If X is a Banach space, then the symbol $L^q(0,T;X)$ means the standard Bochner space. We use $C_{\text{weak}}([0,T];L^2(\Omega))$ to denote functions $\mathbf{u}\in L^\infty(0,T;L^2(\Omega))$ which satisfy $(\mathbf{u}(t),\varphi)\in C([0,T])$ for all $\varphi\in L^2(\Omega)$.

Definition 2.1. A function $M:\mathbb{R}_+\to\mathbb{R}_+$ is said to be an *N-function* if it is a continuous, non-negative, convex function with a superlinear growth near zero and infinity, i.e., $\lim\limits_{\tau\to 0}\frac{M(\tau)}{\tau}=0$ and $\lim\limits_{\tau\to\infty}\frac{M(\tau)}{\tau}=\infty$, and if $M(\tau)=0$ iff $\tau=0$.

Definition 2.2. The *complementary function* M^* to a function M is defined by

$$M^*(\varsigma)=\sup_{\tau\in\mathbb{R}_+}(\tau\varsigma-M(\tau))\quad\text{for }\varsigma\in\mathbb{R}_+.$$

The complementary function M^* is also an N-function. Let Ω be a bounded domain in $\mathbb{R}^3$ with Lebesgue measure $\mathrm{d}x$. The *Orlicz class* $\mathcal{L}_M(\Omega)$ is the set of all measurable functions $f:\Omega\to\mathbb{R}^d$ such that

$$\int_\Omega M(|f(x)|)\,\mathrm{d}x<\infty$$

(d stands for 1, 3 or 3×3). The *Orlicz space* $L_M(\Omega)$ is defined as the set of all measurable functions $f:\Omega\to\mathbb{R}^d$ which satisfy

$$\int_\Omega M(\lambda|f(x)|)\,\mathrm{d}x\to 0\quad\text{as }\lambda\to 0.$$

The Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|f\|_M=\inf\left\{\lambda>0:\quad\int_\Omega M\left(\frac{|f(x)|}{\lambda}\right)\,\mathrm{d}x\le 1\right\}.$$

Let us denote by $E_M(\Omega)$ the closure of all measurable, bounded functions on Ω in $L_M(\Omega)$. Then the space E_M is separable and the space $L_{M^*}(\Omega)$ is the dual space of $E_M(\Omega)$. It is easy to see that $E_M\subseteq\mathcal{L}_M\subseteq L_M$.

We say that M satisfies the Δ_2-*condition* if there exists a constant c_M such that $M(2\tau) \leq c_M M(\tau)$ for all $\tau \in \mathbb{R}_+$. It is well known that L_M is separable and reflexive if and only if M satisfies the Δ_2-condition.

In this paper Orlicz spaces are considered also on the time-space cylinder $Q = I \times \Omega$ with the Lebesgue measure $\mathrm{d}x\mathrm{d}t$, where I stands for a time interval. Let us recall that in general $L_M(I \times \Omega) \neq L_M(I; L_M(\Omega))$, unless very strong assumptions are satisfied, see [3]. More information about Orlicz spaces can be found e.g. in [9].

Now we introduce Riesz transforms in Orlicz spaces, which will be used later as a tool providing local estimates for the pressure function.

Let $\beta, \gamma \in (0, \infty)$ and denote by $L_{\tau \log^\beta}(\Omega)$ the Orlicz space associated with the N-function $M(\tau) = \tau(\log(\tau+1))^\beta$ and by $L_{e(\gamma)}(\Omega)$ the Orlicz space associated with the N-function of the form $M(\tau) = \exp(\tau^\gamma)$ for sufficiently large τ. Let us notice that $L_{\tau \log^\beta}(\Omega) = E_{\tau \log^\beta}(\Omega)$ (see [9]) and $\left(E_{e(\gamma)}(\Omega)\right)^* = L_{\tau \log^{1/\gamma}}(\Omega)$, $\left(L_{\tau \log^\beta}(\Omega)\right)^* = L_{e(1/\beta)}(\Omega)$.

By $\mathcal{R}_{i,j}$ let us denote the "double" Riesz transform on $\mathbb{R}^3$, which can be defined by the Fourier transform $\mathcal{F}$ as

$$\mathcal{R}_{i,j}[g] = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2}\right)\mathcal{F}[g] = \nabla_{x_i}\nabla_{x_j}\Delta^{-1}g\,, \quad i, j = 1,2,3, \tag{1}$$

where $\Delta^{-1}g(x) = \mathcal{F}^{-1}\left(-1/|\xi|^2\right)\mathcal{F}[g]$.

Lemma 2.3. *Let Ω be a bounded domain, let $b : \mathbb{R}^3 \to \mathbb{R}$ be a Fourier multiplier, α be a multi-index and*

$$\sup_{\xi \in \mathbb{R}^3, \xi \neq 0} |\xi|^{|\alpha|}|D^\alpha b(\xi)| \leq C < \infty \quad \textit{for all } \alpha \textit{ such that } |\alpha| \leq 2.$$

Then for $\beta > 0$ there exists a constant $c(\beta)$ such that for all $g \in L_{y \log^{\beta+1}}(\Omega)$

$$\|(\mathcal{F}^{-1}b\mathcal{F})[g]\|_{\tau \log^\beta} \leq c(\beta)\|g\|_{\tau \log^{\beta+1}} \tag{2}$$

where g is prolonged to be 0 on $\mathbb{R}^3 \setminus \Omega$.

Proof. (We recall here the proof given by Erban in [5].) The standard Mikhlin multiplier theorem (see e.g. [1, Chapter 6]) provides that $\mathcal{F}^{-1}b\mathcal{F}$ is bounded as a mapping

$$\mathcal{F}^{-1}b\mathcal{F} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \quad \text{and} \quad \mathcal{F}^{-1}b\mathcal{F} : L^1(\mathbb{R}^3) \to L^{1,\infty}(\mathbb{R}^3),$$

where $L^{1,\infty}$ stands for a Lorenz space (i.e. $g \in L^{1,\infty}$ iff $\sup_\sigma \sigma m(\sigma, g) < \infty$, where $m(\sigma, g) = |\{x : |g(x)| > \sigma\}|$). Employing the result from [10, Theorem B.2] (see also [2]) we conclude that there exists a constant $c(\beta)$ such that (2) is satisfied. □

Corollary 1. *Let Ω be a bounded domain. Then for $\beta > 0$*

$$\|\mathcal{R}_{i,j}[g]\|_{\tau \log^\beta} \leq c(\beta)\|g\|_{\tau \log^{\beta+1}}. \tag{3}$$

3. **Decomposition of the pressure.** In the present section our aim is to derive estimates for the decomposition of the function which in fluid mechanics is associated with the pressure.

Theorem 3.1. *Let M^*, m^* and m' be N-functions s.t. $m^*(\tau) = \tau \log^{\beta+1}(\tau+1) \leq c_1 M^*(\tau) < c_2|\tau|^2$ for some constants $c_1, c_2 > 0$ and $m'(\tau) = \tau \log^\beta(\tau+1)$ for $\tau \in \mathbb{R}_+$ and $\beta \in (0, \infty)$. Let $I = [t_0, t_1]$ be a time interval and let $B \subset \mathbb{R}^3$ be a domain* *with a regular C^3 boundary. Assume that $\boldsymbol{U} \in L^\infty(I; L^2(B;\mathbb{R}^3))$, $\mathrm{div}_x \boldsymbol{U} = 0$, and $\mathsf{T} \in L_{M^*}(I\times B;\mathbb{R}^{3\times 3})$ satisfy the integral identity*

$$\int_I \int_B \Big(\boldsymbol{U}\cdot \partial_t \boldsymbol{\varphi} + \mathsf{T} : \nabla_x \boldsymbol{\varphi}\Big)\ \mathrm{d}x\mathrm{d}t = 0 \tag{4}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(I\times B;\mathbb{R}^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Then there exist two functions

$$p_{\mathrm{reg}} \in L^1(I; L_{m'}(B))$$

and

$$p_{\mathrm{harm}}(t,\cdot) \in \mathcal{D}'(B),\ \Delta_x p_{\mathrm{harm}} = 0 \ \textit{in}\ \mathcal{D}'(I\times B),\ \int_B p_{\mathrm{harm}}(t,\cdot)\ \mathrm{d}x = 0$$

satisfying

$$\int_I \int_B \Big(\boldsymbol{U}\cdot \partial_t \boldsymbol{\varphi} + \mathsf{T} : \nabla_x \boldsymbol{\varphi}\Big)\ \mathrm{d}x\mathrm{d}t = \int_I \int_B \Big(p_{\mathrm{harm}} \partial_t \mathrm{div}_x \boldsymbol{\varphi} + p_{\mathrm{reg}} \mathrm{div}_x \boldsymbol{\varphi}\Big)\ \mathrm{d}x\mathrm{d}t \tag{5}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I\times B;\mathbb{R}^3)$. Additionally,

$$\|p_{\mathrm{reg}}\|_{L_1(I;L_{m'}(B))} \le c(m')\|\mathsf{T}\|_{L_{M^*}(I\times B;\mathbb{R}^{3\times 3})} \tag{6}$$

and for a.a. $t\in I$

$$p_{\mathrm{harm}}(t,\cdot)|_{B'} \in C^\infty(B'),\ \textit{where}\ B' \subset\subset B. \tag{7}$$

Proof. To begin with, the "regular" component of the pressure p_{reg} is identified as

$$p_{\mathrm{reg}}(t,\cdot) = \mathcal{R} : \mathsf{T} = \sum_{i,j=1}^{i,j=3} \mathcal{R}_{i,j}[T_{i,j}](t,\cdot) \text{ in } \mathbb{R}^3 \text{ for a.a. } t\in I,$$

where $\mathcal{R}$ denotes the "double" Riesz transform (see (1)) and $\mathsf{T} = [T_{i,j}]_{i=1,2,3;j=1,2,3}$ has been extended to be zero outside B. Using (3) we obtain that the mappings

$$\mathcal{R}_{i,j}|_B : L_{m^*}(B) \to L_{m'}(B) \quad \text{are bounded for } i,j = 1,2,3.$$

Therefore by the imbedding $L_{M^*}(I\times B)) \subseteq L^1(I; L_{M^*}(B))$ we infer

$$\|p_{\mathrm{reg}}\|_{L^1(I;L_{m'}(B))} = \|\mathcal{R}:\mathsf{T}\|_{L^1(I;L_{m'}(B))} \le c\|\mathsf{T}\|_{L^1(I;L_{M^*}(B))} \le c\|\mathsf{T}\|_{L_{M^*}(I\times B)},$$

and obtain (6) with some constant $c(m') > 0$. Moreover,

$$\int_B p_{\mathrm{reg}} \Delta\boldsymbol{\psi}\ \mathrm{d}x = \int_B \mathsf{T} : \nabla^2 \boldsymbol{\psi}\ \mathrm{d}x \text{ for any } \boldsymbol{\psi}\in\mathcal{D}(B). \tag{8}$$

On the other hand, (4) provides that we can redefine $\boldsymbol{U}$ with respect to time on a set of measure zero such that

$$t \mapsto \int_B \boldsymbol{U}\cdot\boldsymbol{\psi}\ \mathrm{d}x \in C([t_0,t_1]) \text{ for any } \boldsymbol{\psi}\in\mathcal{D}(B;\mathbb{R}^3),\ \mathrm{div}_x\boldsymbol{\psi} = 0.$$

Particularly, we infer that the Helmholtz projection $\boldsymbol{H}[\boldsymbol{U}]$ belongs to the class $C_{\mathrm{weak}}([t_0,t_1]; L^2(B;\mathbb{R}^3))$. Therefore (4) with $\boldsymbol{\varphi}(t,x) = \eta(t)\boldsymbol{\psi}(x)$ provides

$$\int_I \Big[\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\ \mathrm{d}x\Big]\partial_t\eta\ \mathrm{d}t - \int_I \Big[\int_B \Big(\int_{t_0}^t \mathsf{T}(s,\cdot)\ \mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\ \mathrm{d}x\Big]\partial_t\eta\ \mathrm{d}t = 0$$

for any $\boldsymbol{\psi}\in\mathcal{D}(B;\mathbb{R}^3)$, $\mathrm{div}_x\boldsymbol{\psi} = 0$, $\eta\in\mathcal{D}(t_0,t_1)$. Employing Lemma 2.2.1 from [17], we see that there exists a pressure $p = p(t,\cdot)$ such that

$$\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\ \mathrm{d}x - \int_B \Big(\int_{t_0}^t \mathsf{T}(s,\cdot)\ \mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\ \mathrm{d}x = \int_B p(t,\cdot)\mathrm{div}_x\boldsymbol{\psi}\ \mathrm{d}x \tag{9}$$

for all $t \in [t_0, t_1]$ and all $\boldsymbol{\psi} \in \mathcal{D}(B; \mathbb{R}^3)$. Note that the term on the right-hand side is measurable and integrable w.r.t. time, since the left-hand side is measurable and integrable. Moreover,

$$\int_B p(t, \cdot)\, \mathrm{d}x = 0 \quad \text{and} \quad p(t, \cdot) \in \mathcal{D}'(B).$$

Testing (9) by $\partial_t \zeta$, $\zeta \in \mathcal{D}(I)$, integrating over the time interval I, and setting $\boldsymbol{\varphi}(t, x) = \zeta(t)\boldsymbol{\psi}(x)$ we conclude that for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$

$$\int_I \int_B \Big(\boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{T} : \nabla_x \boldsymbol{\varphi} \Big)\, \mathrm{d}x\mathrm{d}t = \int_I \int_B p \partial_t \mathrm{div}_x \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t. \tag{10}$$

Finally, the harmonic pressure is defined for a.a. $t \in (t_0, t_1)$ as

$$p_{\text{harm}}(t, \cdot) = p(t, \cdot) + \Big(\int_{t_0}^{t} \Big[p_{\text{reg}}(\tau, \cdot) - \frac{1}{|B|} \int_B p_{\text{reg}}(\tau, \cdot)\, \mathrm{d}x \Big]\, \mathrm{d}\tau \Big). \tag{11}$$

Now we intend to show that $p_{\text{harm}}(t, \cdot)$ is in fact a harmonic function for any t. To this end, we take $\boldsymbol{\psi} = \nabla_x \eta$, $\eta \in \mathcal{D}(B)$ in (9) and compare the resulting expression with (8). If we insert (11) in (10), we will obtain (5). Finally Weyl's lemma (see e.g. [19]) ensures that the function p_{harm} is regular locally in B, i.e. $p_{\text{harm}} \in C^\infty(B')$, where $B' \subset\subset B$. Hence we infer (7). □

4. **An application of the method.** The method of local pressure can facilitate investigation of mathematical properties of the motion of one or several rigid bodies immersed in an incompressible viscous fluid which occupies a bounded domain $\Omega \subset \mathbb{R}^3$. For the existence results see [6, 15, 22].

In particular with the help of Orlicz spaces we can investigate mathematical properties of fluids for which viscosity increases dramatically with increasing shear rate or applied stress. We want to consider shear thickening fluids, which can behave almost like a solid when they encounter mechanical stress or shear. Such fluids found their application in some of high-performance fabrics produced by a simple impregnation process. The resulting material is thin and flexible, and provides protection against the risk of needle, knife or bullet contact that face police officers and medical personnel [4, 8].

Motivated by this significant shear thickening phenomenon we want to concentrate on fluids with rheology more general than that of power-law type, see Málek et al. [14, Chapter 1.], given by standard polynomial growth conditions for the stress tensor, i.e.

$$|\mathbf{S}(\mathbf{D}\boldsymbol{u})| \le c(1 + |\mathbf{D}\boldsymbol{u}|)^{q-1} \quad \text{and} \quad \mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \ge c|\mathbf{D}\boldsymbol{u}|^q, \tag{12}$$

where $\mathbf{D}\boldsymbol{u} = (\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u})/2$ denotes the symmetric part of the velocity gradient.

We want to investigate the processes where the growth is faster than polynomial. Therefore we formulate growth conditions of the stress tensor using a quite general convex function M which is an N-function. First we assume that the viscous stress tensor $\mathbf{S}$ depends on the symmetric part of the gradient of the velocity field $\boldsymbol{u}$, i.e. $\mathbf{S} : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ ($\mathbb{R}^{3\times 3}_{\text{sym}}$ stands for the space of 3×3 symmetric matrices), and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$, $\mathbf{S} = \mathbf{S}(\mathbf{D}\boldsymbol{u})$ is continuous, and monotone, i.e.

$$\Big(\mathbf{S}(\boldsymbol{\xi}) - \mathbf{S}(\boldsymbol{\eta}) \Big) : \Big(\boldsymbol{\xi} - \boldsymbol{\eta} \Big) \ge 0 \text{ for all } \boldsymbol{\xi} \ne \boldsymbol{\eta},\ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{3\times 3}_{\text{sym}}.$$

Finally we assume that there exist a positive constant c, N–functions M and M^* (M^* denotes the complementary function to M), such that

$$\mathsf{S}(\boldsymbol{\xi}) : \boldsymbol{\xi} \geq c\{M(|\boldsymbol{\xi}|) + M^*(|\mathsf{S}(\boldsymbol{\xi})|)\} \quad \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}. \tag{13}$$

Our assumptions can capture shear dependent viscosity functions including power-law and Carreau-type models which are quite popular in rheology, chemical engineering and colloidal mechanics. Nevertheless, we want to investigate also constitutive relations more general than of type (12), e.g. $\mathsf{S} \approx |\mathsf{D}\boldsymbol{u}|^q \ln(1 + |\mathsf{D}\boldsymbol{u}|)$ (here $M(\tau) = \tau^q \log(\tau + 1)$ grows faster than τ^q but slower than $\tau^{q+\nu}$ for all $\nu > 0$).

The appropriate spaces to capture the problem with growth and coercivity conditions given by (13) are Orlicz spaces. Existence results for problems devoted to fluid mechanics stated in the Orlicz space setting can be found e.g. in [11, 12]

Investigating the existence of weak solutions to the motion of one or several rigid bodies in such a fluid we use the concept of weak solutions based on the Eulerian reference system and on a class of test functions which depend on the position of the rigid bodies, see e.g. Judakov [20], Starovoitov [13], San Martin et al. [16] and for more references see Feireisl et al. [6].

We state the following problem: let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$, occupied by an incompressible fluid containing rigid bodies. The initial position of the rigid bodies is determined through a family of sufficiently regular domains $S_i \subset \Omega \subset \mathbb{R}^3,\ i = 1, \ldots, n$. The motion η_i associated with the body S_i is a mapping

$$\eta_i = \eta_i(t, x),\ t \in [0, T),\ x \in \mathbb{R}^3,\ \eta_i(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an isomorphism,}$$
$$\eta_i(0, x) = x \text{ for all } x \in \mathbb{R}^3, \quad i = 1, \ldots, n.$$

Accordingly, the position of the body S_i at time t is given through the formula

$$S_i(t) = \eta_i(t, S_i),\ i = 1, \ldots, n.$$

In the above terms we introduce domains Q^f and Q^s respectively as a fluid and a rigid part of the time-space cylinder in the following way:

$$Q^s := \bigcup_{i=1,\ldots,n} \{(t, x)\ :\ t \in [0, T],\ x \in \overline{S_i}(t)\}, \quad Q^f := Q \setminus Q^s.$$

A weak solution is a pair $(\varrho, \boldsymbol{u})$ with mass density $\varrho = \varrho(t, x)$ and velocity field $\boldsymbol{u} = \boldsymbol{u}(t, x)$ satisfying the integral identity

$$\int_0^T \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\boldsymbol{u} \cdot \nabla_x\varphi\Big)\mathrm{d}x\mathrm{d}t = -\int_\Omega \varrho_0\varphi\mathrm{d}x \quad \text{ for any } \varphi \in C^1([0, T) \times \overline{\Omega})$$

and

$$\int_0^T \int_\Omega \Big(\varrho\boldsymbol{u} \cdot \partial_t\boldsymbol{\varphi} + \varrho\boldsymbol{u} \otimes \boldsymbol{u} : \mathsf{D}\boldsymbol{\varphi} - \mathsf{S} : \mathsf{D}\boldsymbol{\varphi}\Big)\mathrm{d}x\mathrm{d}t = -\int_\Omega \varrho_0\boldsymbol{u}_0 \cdot \boldsymbol{\varphi}\mathrm{d}x$$

for any test function $\boldsymbol{\varphi} \in C^1([0, T) \times \overline{\Omega}; \mathbb{R}^3)$, $\boldsymbol{\varphi}(t, \cdot) \in [\mathcal{R}M](t)$, where

$$[\mathcal{R}M](t) = \{\boldsymbol{\phi} \in C^1_c(\Omega; \mathbb{R}^3) \mid \mathrm{div}_x\boldsymbol{\phi} = 0 \text{ in } \Omega,$$
$$\mathsf{D}\boldsymbol{\phi} \text{ has a compact support on } \Omega \setminus \cup_{i=1}^n \overline{S}_i(t)\}.$$

In addition we assume that $\boldsymbol{u}(t, x)|_{\partial\Omega} = 0$ for $t \in [0, T]$ and the velocity of the fluid on the boundary of each rigid body S_i $(i = 1, \ldots, n)$ coincides with the velocity of the rigid object. More detailed description of the problem can be found in [6].

The construction of the weak solution is based on a two-level approximation scheme as in [6, 15] and consists in solving the system of equations:

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x(\varrho[\boldsymbol{u}]_\delta) &= 0,\\ \partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes [\boldsymbol{u}]_\delta) + \nabla_x p &= \operatorname{div}_x([\mu_\varepsilon]_\delta \mathbf{S})\\ \partial_t \mu_\varepsilon + \operatorname{div}_x(\mu_\varepsilon [\boldsymbol{u}]_\delta) &= 0,\\ \operatorname{div}_x \boldsymbol{u} &= 0,\end{aligned}$$

where p denotes the pressure. The rigid bodies are replaced in this approximation by a fluid of high viscosity μ_ε becoming singular for $\varepsilon \to 0$. In the fluid part (where is no rigid body) μ_ε stays equal to 1. For simplicity we assume that the density of the fluid is constant. The extra parameter $\delta > 0$ is introduced to improve properties of the approximation and $[\ ]_\delta$ denotes spatial convolution with the regularizing kernel.

The first step is to pass to the limit with $\varepsilon \to 0$ for fixed δ and identify the positions of the rigid bodies. The main problem, inherent to the theory of non-Newtonian fluids, is that we have to identify the nonlinear viscous term. Passing to the limit with $\varepsilon \to 0$, the problem appears to be more delicate as the monotonicity argument must be localised to the "fluid" part of the time-space cylinder. We cannot test the momentum equation by functions with non-zero support on Q^s, since neither the penalizing term $\mu_\varepsilon \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$ nor $\mu_\varepsilon \mathbf{D}\boldsymbol{u}_\varepsilon$ can be controlled. Therefore we localise the problem in the fluid part and hence need local estimates for the pressure. To this end we consider the momentum equation of the approximate problem on the time interval $I \subset [0,T]$ and the spatial domain $B \subset \Omega$ such that $I \times B$ is in the "fluid" part of the time space cylinder. We can assume that $\varrho = \varrho_f = \text{const}$ in $I \times B$. In particular, we have for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$, $\operatorname{div}_x \boldsymbol{\varphi} = 0$ that

$$\int_I \int_B \varrho_f \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = 0.$$

At this stage the problem is separated from the rigid bodies and we can apply Theorem 3.1 with the N-function M^* and the function $\boldsymbol{U} := \varrho_f \boldsymbol{u}$ and $\mathbf{T} = \varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$. Therefore for any $\varepsilon > 0$ there exist two scalar functions $p^\varepsilon_{\text{reg}}$, $p^\varepsilon_{\text{harm}}$ satisfying $p = p^\varepsilon_{\text{reg}} + \partial_t p^\varepsilon_{\text{harm}}$ and such that for any test function $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$

$$\int_I \int_B \Big[(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) + p^\varepsilon_{\text{reg}} \mathbf{I}) : \nabla_x \boldsymbol{\varphi} \Big] \, \mathrm{d}x\mathrm{d}t = 0.$$

The reader can find more details and the existence results in [6, 15] for the case of power-law fluids and in [22] for the case of the growth condition given by (13). The assumption on the lower bound for the N-function M^*, i.e. $m^*(\tau) = \tau \log^{\beta+1}(\tau + 1) \leq c_1 M^*(\tau)$ for $\tau \in \mathbb{R}_+$, $\beta > 0$ and some constant c_1, implies that we have to assume also that $M(\tau) \leq c_2 \exp(\tau^{\frac{1}{\beta+1}})$ with constant c_2.

In the above consideration and in [22] the main difference from any previous work in this direction is, due to condition (13), that we are in the Orlicz-space setting. Let us notice that the Riesz transform in general cannot be well defined as an operator from one Orlicz space to the same one. If M and M^* do not satisfy the Δ_2-condition it can turn out that it is continuous from one Orlicz space to another but a larger one. Therefore the method of local pressure seems to be more difficult.

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REFERENCES

[1] J. Bergh and J. Löfström. *Interpolation spaces: an introduction.* Springer–Verlag, Berlin, Heidelberg, New York, 1976.

[2] A. Cianchi. Strong and weak type inequalities for some classical operators in Orlicz spaces. *J. London Math. Soc.* **60**(2):187–202, 1997.

[3] T. Donaldson. Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems. *J. Differential Equations.* 16:201-256, 1974.

[4] R.G. Egres Jr, Y.S. Lee, J.E. Kirkwood, K.M. Kirkwood, E.D. Wetzl, N.J. Wagner. "Liquid armor": Protective fabrics utilising shear thickening fluids. *Proceedings of the 4th International Conference of Safety and Protective Fabrics*, Pittsburg, PA, October, 26-27, 2004.

[5] R. Erban. On the existence of solutions to the Navier -Stokes equations of two -dimensional compressible flow. *Math. Methods Appl. Sci.* **26**:489-517, 2003.

[6] E. Feireisl, M. Hillairet and Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian fluid. *Nonlinearity.* **21**:1349-1366, 2008.

[7] J. Frehse, J. Málek, M. Růžička. Large data existence results for unsteady flows of inhomogeneus heat-conducting incompressible fluids. *Comm. in Partial Differential Equations.* 35(10):1891–1919, 2010.

[8] J. Houghton, B. Schiffman, D. Kalman, E. Wetzel, N. Wagner. Hypodermic Needle Puncture of Shear Thickening Fluid (STF)-Treated Fabrics. *Proceedings of SAMPE.* Baltimore, 2007.

[9] A. Kufner, O. John and S. Fučik. *Function spaces.* Noordhoff International Publishing. Prague: Publishing House of the Czechoslovak Academy of Sciences, 1977.

[10] J. Gustavsson, J. Peetre. Interpolation of Orlicz spaces. *Studia Mathematica.* 60:33–59, 1977.

[11] P. Gwiazda, A. Świerczewska-Gwiazda. On non-Newtonian fluids with the property of rapid thickening under different stimulus. *Math. Models Methods Appl. Sci.*, 7(18):1073-1092, 2008.

[12] P. Gwiazda, A. Świerczewska-Gwiazda, A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.*, 33(2):125-137, 2010.

[13] K. H. Hoffmann, V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two dimensional case. *Adv. Math. Sci. Appl.*, **9**:633–648, 1999.

[14] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDE's.* Chapman and Hall, London, 1996.

[15] Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian and heat-conducting fluid. *Ann. Univ. Ferrara.* **55**:325-352, 2009.

[16] J.A. San Martin, V. Starovoitov, M. Tucsnak. Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, **161**:93–112, 2002.

[17] H. Sohr. *The Navier-Stokes equations: An elementary functional analytic approach.* Birkhäuser Verlag, Basel, 2001.

[18] V.N. Starovoitov. Behavior of a rigid body in an incompressible viscous fluid near boundary. *In International Series of Numerical Mathematics*, **147**:313–327, 2003.

[19] D.W. Stroock. Weyl's Lemma, one of many. London Mathematical Society Lecture Note Series. No. 354. pp. 164-173. Cambridge University Press. 2008.

[20] N.V. Judakov. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. (Russian) *Dinamika Splošn. Sredy Vyp.* , **18**:249–253, 1974.

[21] J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mech.*, **8**:1–35, 2006.

[22] A. Wróblewska. Existence result to the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth condition in Orlicz spaces. Work in progress.

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