An Age-Structured Two-Sex Model in the Space of Radon Measures: Well Posedness.

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An Age-Structured Two-Sex Model in the Space of Radon Measures: Well Posedness.

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Abstract

In the following paper the well-posedness of an age-structured two-sex model in the space of Radon measures equipped with the flat metric is presented. Existence and uniqueness of measure valued solutions is proved by a regularization technique. This approach allows to obtain Lipschitz continuity of solutions with respect to time and stability estimates, which are crucial when one analyse convergence of numerical algorithms. Moreover, a brief discussion on the marriage function, which is the main source of nonlinearity, is carried out and example of the marriage function fitting into this framework is given.

1 Introduction

Age-structured two-population models have been widely studied over the last several decades. This subject is in the centre of researcher interest, because results can be directly applied to human population dynamics. Such models can be also useful in modeling sexually transmitted diseases (e.g. HIV), see [4], [9], [12], [23]. One of the first attempts of including both sexes and tracking interactions between them was made in [19]. Although age structure was left out of consideration, this paper became a starting point for many recently used models (see [3] and [14] for extensions). Significant generalization based on incorporating an age structure was formulated for the first time in [7] by Fredrickson and reintroduced in [15] by Hoppensteadt in 70's. It was further analysed under particular assumptions in several papers. In [13] symmetry with respect to males and females of all coefficients is required. In [17] it is assumed that new individuals are produced just in the first marriage. A specific form of a marriage function is postulated in [25]. Well-posedness in a general case was established in [21] (see also [16]). However, as it was shown in [20], in the long time period exponential growth of population is observed, which is not a realistic phenomenon. Due to this fact, environmental influences have been taken into consideration, e.g., by introducing dependency of birth and death rates on a state of the whole population in [24].

The Fredrickson–Hoppensteadt model, being a base of our analysis, describes the evolution of males and females and the process of heterogenous couples formation. Dynamics of males and females is given by McKendrick-type equations with boundary terms determining the influx of newborn individuals. Evolution of couples is described by a similar equation, however the source term for couples is much more complicated than birth and death processes for males and females. Indeed, marriage is a really complex process influenced by many social

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and economical factors (e.g. religion, health, education, marriage preferences and interactions between both sexes). Fortunately, most researchers agree that essential components from a mathematical modeling point of view are sex and age.

In the literature we mentioned in the introduction above, authors assume that a distribution of populations is given by a density. However, it is often necessary to describe a population state with a measure, which is not absolutely continuous with respect to the Lebesgue measure. In fact, setting some models of population dynamics in the space of measures was suggested for the first time in [26, Section III.5]. The novelty of our framework consists in setting an age-structured two-sex model in the space of nonnegative Radon measures equipped with the flat metric d, what follows the approach used in [2], [10] and [11]. The main advantage of such a setting is ability to prove Lipschitz continuity with respect to time of measure valued solutions and stability estimates. The choice of the space is also strictly connected with advantages coming from numerics. From the perspective of numerical simulations, in case of equations considered in the space of Radon measures, particle method is the natural one, as each compactly supported Radon measure μ can be approximated in the flat metric d with a sum of Dirac deltas with accuracy $\Delta x/2$ on a fixed interval of length Δx. More precisely, $d(\mu, \sum_{i=1}^{N} \delta_i) \leq N \cdot \Delta x/2$, where $\delta_i = \int_{x_i}^{x_{i+1}} \mathrm{d}\mu$ and $\Delta x = \max_{i \in \{1,\ldots,N\}} |x_{i+1} - x_i|$. Such an approximation allows to transform differential equation into a system od ODE's, what is often an advantage in numerical simulations.

This paper is organized as follows. Section 2 consists of analytical results. First, we introduce the model. Subsection 2.1 is devoted to the brief discussion on the marriage function. We shall give an example of the marriage function, which is reasonable from the biological point of view and satisfies the model assumptions. Subsection 2.2 contains results concerning the linear non-autonomous case. We analyse the equation describing the dynamics for couples separately, as it is independent on other equations. By a regularization argument and theory of dual equations we obtain existence, uniqueness and continuous dependence of solutions in the flat metric. Estimates for the linear non-autonomous case are crucial in the passage to the nonlinear case, which bases on the Banach Fixed Point Theorem This subject is held in Subsection 2.3, where we formulate the theorem for the nonlinear case. All proofs are deferred to Section 3.

2 Results

sec_results

In this paper we consider the following nonlinear PDE's system

$$
\begin{cases}
\partial_t \mu_t^m + \partial_x \mu_t^m + \xi_m(t,\mu_t^m,\mu_t^f)\mu_t^m = 0, & t\in[0,T],\ x\in\mathbb{R}_+,\\
D_\lambda \mu_t^m(0^+) = \int_{\mathbb{R}_+^2} b_m(t,\mu_t^m,\mu_t^f)(z)\,\mathrm{d}\mu_t^c(z), & \\
\mu_o^m \in \mathcal{M}_+(\mathbb{R}_+), &
\end{cases}
$$

$$
\begin{cases}
\partial_t \mu_t^f + \partial_x \mu_t^f + \xi_f(t,\mu_t^m,\mu_t^f)\mu_t^f = 0, & t\in[0,T],\ x\in\mathbb{R}_+,\\
D_\lambda \mu_t^f(0^+) = \int_{\mathbb{R}_+^2} b_f(t,\mu_t^m,\mu_t^f)(z)\,\mathrm{d}\mu_t^c(z), & \\
\mu_o^f \in \mathcal{M}_+(\mathbb{R}_+), &
\end{cases}
\tag{2.1}
$$

$$
\begin{cases}
\partial_t \mu_t^c + \partial_{z_1} \mu_t^c + \partial_{z_2} \mu_t^c + \xi_c(t,\mu_t^m,\mu_t^f,\mu_t^c)\mu_t^c = \mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c), & t\in[0,T],\ z=(z_1,z_2)\in\mathbb{R}_+^2,\\
\mu_t^c(\{0\}\times B) = \mu_t^c(B\times\{0\}) = 0, & \\
\mu_o^c \in \mathcal{M}_+(\mathbb{R}_+^2). &
\end{cases}
$$

Through this paper, $\mathcal{M}_+(\mathbb{R}_+^i)$ denotes the space of nonnegative Radon measures with bounded total variation on $\mathbb{R}_+^i = \{x\in\mathbb{R}^i : x\geq 0\}$, for $i=1,2$. $B\in\mathcal{B}(\mathbb{R}_+)$ is a Borel set, μ_t^m, μ_t^f, μ_t^c are measures on $\mathcal{M}_+(\mathbb{R}_+)$, $\mathcal{M}_+(\mathbb{R}_+)$ and $\mathcal{M}_+(\mathbb{R}_+^2)$ determining the distribution of males, females and couples at time t. Functions ξ_m, ξ_f are death rates for males and females respectively. ξ_c is a rate of disappearance of couples, which incorporates divorces and death of one of the spouses as well. Influx of the new individuals is described by nonlocal boundary conditions. $D_\lambda\mu_t^m(0^+)$ and $D_\lambda\mu_t^f(0^+)$ are Radon-Nikodym derivatives of μ_t^m and μ_t^f respectively, with respect to the one dimensional Lebesgue measure λ at the point 0 (we assume that the support of the singular part of measures μ_t^m, μ_t^f does not contain 0). Functions b_m and b_f are birth rates for males and females respectively. The source term for couples is given by the operator $\mathcal{T}$. For any Borel set $B\in\mathcal{B}(\mathbb{R}_+^2)$, value $\mathcal{T}(B)$ is a measure of the set containing couples formed between males being at age x and females being at age y, such that $(x,y)\in B$. Distribution of single males and single females is given by measures s_t^m and s_t^f respectively. Formally, s_t^m and s_t^f are measures on $\mathbb{R}_+$, such that for each Borel set $B\in\mathcal{B}(\mathbb{R}_+)$

$$
s_t^i(B) = (\mu_t^i - \sigma_t^i)(B), \quad \text{for } i=m,f, \tag{2.2}
$$

single

holds. Measures σ_t^m and σ_t^f are projections of μ_t^c on $\mathbb{R}_+$ and describe a distribution of males and females respectively, who enter a marriage at time t. More precisely, for each Borel set $B\in\mathcal{B}(\mathbb{R}_+)$

$$
\sigma_t^m(B) = \mu_t^c(B\times\mathbb{R}_+) \quad\text{and}\quad \sigma_t^f(B) = \mu_t^c(\mathbb{R}_+\times B). \tag{2.3}
$$

projection_s

We define a metric on $\mathcal{M}_+(\mathbb{R}_+^i)$ as

$$
d_i(\mu_1,\mu_2) = \sup\left\{ \int_{\mathbb{R}_+^i} \varphi \,\mathrm{d}(\mu_1-\mu_2) \colon \varphi\in\mathbf{C^1}(\mathbb{R}_+^i;\mathbb{R}) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}}\leq 1 \right\}, \tag{2.4}
$$

distance_i

where $\|\varphi\|_{\infty,\mathbf{Lip}} = \max\left\{\|f\|_{\mathbf{L}^\infty}, \mathbf{Lip}(f)\right\}$.

Remark 2.1. *For each $\lambda > 0$, it holds that*

$$d_i(\mu_1,\mu_2) = \frac{1}{\lambda}\sup\left\{\int_{\mathbb{R}_+^i}\varphi\,\mathrm{d}(\mu_1-\mu_2)\colon \varphi\in\mathbf{C^1}(\mathbb{R}_+^i;\mathbb{R}) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}}\leq\lambda\right\}. \tag{2.5}$$ distance_i_b

Note that in this paper space $\mathcal{M}_+(\mathbb{R}_+^i)$ is equipped with a metric d_i and this shall remain until said differently. Metric d_i is often called *flat* metric or *bounded Lipschitz distance*. Space $(\mathcal{M}_+(\mathbb{R}_+^i), d_i)$ is complete and separable. For the proof we refer to [11, Theorem 2.6]. This proof is related to $(\mathcal{M}_+(\mathbb{R}_+), d_1)$, however the same technique can be applied to obtain analogous result for $(\mathcal{M}_+(\mathbb{R}_+^2), d_2)$. Space $(\mathcal{M}_+(\mathbb{R}_+), d_1)$ is a subspace of dual to $\mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})$ endowed with the norm $\|f\|_{\mathbf{W^{1,\infty}}(\mathbb{R})} = \max\left\{\|f\|_{\mathbf{L^\infty}}, \|\partial_x f\|_{\mathbf{L^\infty}}\right\}$. Respectively, $(\mathcal{M}_+(\mathbb{R}_+^2), d_2)$ is a subspace of dual to $\mathbf{W^{1,\infty}}(\mathbb{R}_+^2;\mathbb{R})$ endowed with the norm $\|f\|_{\mathbf{W^{1,\infty}}(\mathbb{R}^2)} = \max\left\{\|f\|_{\mathbf{L^\infty}}, \|\partial_{x_1} f\|_{\mathbf{L^\infty}}, \|\partial_{x_2} f\|_{\mathbf{L^\infty}}\right\}$ [6]. Due to the Rademacher theorem, Lipschitz functions are differentiable almost everywhere, what implies that metric d_i is a metric dual to $\|\cdot\|_{\mathbf{W^{1,\infty}}}$ distance. For transparency purposes we shall write $\|\cdot\|_{\mathbf{W^{1,\infty}}}$ instead of $\|\cdot\|_{\mathbf{W^{1,\infty}}(\mathbb{R})}$ or $\|\cdot\|_{\mathbf{W^{1,\infty}}(\mathbb{R}^2)}$, what should not lead to misunderstandings. We define product spaces

$$\mathcal{U} = \mathcal{M}_+(\mathbb{R}_+)\times\mathcal{M}_+(\mathbb{R}_+)\times\mathcal{M}_+(\mathbb{R}_+^2) \quad\text{and}\quad \mathcal{V} = \mathcal{M}_+(\mathbb{R}_+)\times\mathcal{M}_+(\mathbb{R}_+).$$

The solution to (2.1) is defined as follows.

efinition:sol **Definition 2.2.** *A triple $\mathbf{u} = (\mu^m, \mu^f, \mu^c)$, such that $\mathbf{u} : [0,T] \to \mathcal{U}$ is a weak solution to the system (2.1) on the time interval $[0,T]$, if μ^m, μ^f, μ^c are narrowly continuous with respect to time and for all $(\varphi_m, \varphi_f, \varphi_c)$ such that $\varphi_m, \varphi_f \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+;\mathbb{R})$ and $\varphi_c \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+^2;\mathbb{R})$, the following equalities hold*

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}_+}\Big(\partial_t\varphi_i(t,x) + \partial_x\varphi_i(t,x) - \xi_i(t,\mu_t^m,\mu_t^f)\varphi_i(t,x)\Big)\,\mathrm{d}\mu_t^i(x)\,\mathrm{d}t\\
&+\int_0^T\varphi_i(t,0)\int_{\mathbb{R}_+^2} b_i(t,\mu_t^m,\mu_t^f)(z)\,\mathrm{d}\,\mu_t^c(z)\,\mathrm{d}t\\
&=\int_{\mathbb{R}_+}\varphi_i(T,x)\,\mathrm{d}\mu_T^i(x) - \int_{\mathbb{R}_+}\varphi_i(0,x)\,\mathrm{d}\mu_o^i(x) \quad \text{for } i=f,m \qquad\text{and}\\
&\int_0^T\int_{\mathbb{R}_+^2}\Big(\partial_t\varphi_c(t,z) + \partial_x\varphi_c(t,z) + \partial_y\varphi_c(t,z) - \xi_c(t,\mu_t^m,\mu_t^f,\mu_t^c)\varphi_c(t,z)\Big)\,\mathrm{d}\mu_t^c(z)\,\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{R}_+^2}\varphi_c(t,z)\,\mathrm{d}\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c)(z)\,\mathrm{d}t = \int_{\mathbb{R}_+}\varphi_c(T,z)\,\mathrm{d}\mu_T^c(z) - \int_{\mathbb{R}_+}\varphi_c(0,z)\,\mathrm{d}\mu_o^c(z).
\end{aligned}$$

Here, by *narrowly continuous functions* we refer to the narrow convergence introduced in [1, § 5.1]. See also [11, Definition 2.1, Theorem 2.6] for more details.

The assumptions on the model functions are:

$$\xi_m,\xi_f \in \mathbf{BC^{0,1}}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})) \tag{2.6}$$

$$b_m,b_f \in \mathbf{BC^{0,1}}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2;\mathbb{R})) \tag{2.7}$$

$$\xi_c \in \mathbf{BC^{0,1}}([0,T]\times\mathcal{U};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2;\mathbb{R})) \tag{2.8}$$

$$\mathcal{T} \in \mathbf{BC^{0,1}}([0,T] \times \mathcal{U};\ \mathcal{M}_+(\mathbb{R}_+^2)) \tag{2.9}$$

Here, $\mathbf{BC^{0,1}}([0,T]\times\mathcal{V};X)$ and $\mathbf{BC^{0,1}}([0,T]\times\mathcal{U};X)$ are spaces of X valued functions, bounded with respect to the $\|\cdot\|_X$ norm, continuous with respect to time and Lipschitz continuous with respect to measure variables. Boundedness with respect to the $\|\cdot\|_X$ norm for operator $\mathcal{T}$ is equivalent to boundedness with respect to the $\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}$ norm, as $\mathcal{T}$ takes values in the space of nonnegative measures. The norm $\|\cdot\|_{\mathbf{BC}}$ in the $\mathbf{BC^{0,1}}$ space is defined as

$$\|f\|_{\mathbf{BC}} = \sup_{t\in[0,T],\mathbf{v}\in Y} \left(\|f(t,\mathbf{v})\|_X + \mathbf{Lip}\,(f_t)\right), \tag{2.10}$$

BC_alfa_norm

where $Y = \mathcal{V}$ (for (2.6), (2.7)) or $Y = \mathcal{U}$ (for (2.8), (2.9)) and $\mathbf{Lip}\,(f_t)$ and is a Lipschitz constant of $f(t,\cdot)$.

2.1 Marriage function

tion_marriage

The *marriage function* is a function, which provides a number of new marriages between males and females in particular age in a certain moment. The major difficulty in modeling marriages is defining an exact form of the marriage function. Several functions have been proposed, e.g., male dominance, female dominance, minimum function, geometric mean, harmonic mean (see [13], [14], [18], [21], [22], [25] and references therein for more details and examples), but none of functions mentioned above can be rigorously derived from sociological data or a microscopic description of the marriage process. Even though we cannot point out the one marriage function which should be preferred over another, there are still some general properties accepted by most of researchers, i.e., *non-negativity, heterosexuality, homogeneity, consistency, monotonicity, competition* (see [16, Section 2.5], [17] for details). The property, which raises the most serious concerns is homogeneity. It is intuitively clear that each individual has a limited number of contacts with other individuals. However, in populations which are dense enough this fact does not influence the marriage process, what the homogeneity property is supposed to reflect. On the other hand, it is believed that homogeneity assumption does not hold at low densities, when the time needed for finding appropriate mate increases significantly. Also some rigorous derivations of a marriage function leads to non-homogeneous functions ([8]).

Remark 2.3. *It is more convenient to formulate and analyse the model* (2.1) *for the whole populations of males and females. Therefore, instead of the marriage function $\mathcal{F}$ dependent on single males and single females distributions we use operator $\mathcal{T}$, such that*

$$\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c) = \mathcal{F}(t,\mu_t^m-\sigma_t^m,\mu_t^f-\sigma_t^f), \tag{2.11}$$

T_F

where σ_t^m and σ_t^f are given by (2.3).

Following [17], in this paper we assume that the marriage function is defined as

$$\mathcal{F}(t,s_t^m,s_t^f) = \left(\frac{\theta(x,y)h(x)g(y)}{\gamma + \int_{\mathbb{R}_+} h(z)\,\mathrm{d}s_t^m(z) + \int_{\mathbb{R}_+} g(\omega)\,\mathrm{d}s_t^f(\omega)}\right)(s_t^m\otimes s_t^f), \tag{2.12}$$

marriage_f_e

where $(s_t^m\otimes s_t^f)$ is a product measure on $\mathbb{R}_+^2$. Therefore, we include homogeneous and non-homogeneous case assuming that $\gamma\in[0,1]$. Functions $h,g\in(\mathbf{L}^\infty\cap\mathbf{L^1})(\mathbb{R}_+^2;\mathbb{R}_+)$ are Lipschitz and describe preferences distributions. More precisely, h is a function describing distribution of preferred males on the marriage market. Function h is independent on y, what reflects that preferences do not depend on the age of a female. Although it is not a highly realistic assumption, we shall use it to simplify the analysis. Function g has the analogous meaning. Function $\theta(x,y)$ describes a marriage rate between a male of age x and a female of age y. In our case $\mathcal{F}$ is defined as a product measure on $\mathbb{R}_+^2$. According to our definition

$$\mathcal{F}(t, s_t^m, s_t^f)(B_m \times B_f) = 0,$$

whenever $s_t^m(B_m) = 0$ or $s_t^f(B_f) = 0$, what ensures that the single males and single females distribution is a nonnegative measure for each time t.

arriage_f_ass

Lemma 2.4. *Operator $\mathcal{T}$ defined by* (2.11) *satisfies assumption* (2.9).

ction_non_aut

2.2 The Linear Non-Autonomous Case

In this section we consider the non autonomous version of (2.1),

$$(2.13.1)\begin{cases} \partial_t \mu^m + \partial_x \mu^m + \xi_m(t,x)\mu^m = 0, & t \in [0,T],\ x \in \mathbb{R}_+, \\ D_\lambda \mu_t^m(0^+) = \int_{\mathbb{R}_+^2} b_m(t,z)\,\mathrm{d}\mu_t^c(z), & \\ \mu_o^m \in \mathcal{M}_+(\mathbb{R}_+), & \end{cases}$$

$$(2.13.2)\begin{cases} \partial_t \mu^f + \partial_x \mu^f + \xi_f(t,x)\mu^f = 0, & t \in [0,T],\ x \in \mathbb{R}_+, \\ D_\lambda \mu_t^f(0^+) = \int_{\mathbb{R}_+^2} b_f(t,z)\,\mathrm{d}\mu_t^c(z), & \\ \mu_o^f \in \mathcal{M}_+(\mathbb{R}_+), & \end{cases} \tag{2.13}$$

$$(2.13.3)\begin{cases} \partial_t \mu^c + \partial_{z_1} \mu^c + \partial_{z_2} \mu^c + \xi_c(t,z)\mu^c = \mathcal{T}(t), & t \in [0,T],\ z = (z_1,z_2) \in \mathbb{R}_+^2, \\ \mu_t^c(\{0\} \times B) = \mu_t^c(B \times \{0\}) = 0, & \\ \mu_o^c \in \mathcal{M}_+(\mathbb{R}_+^2). & \end{cases}$$

We assume that

$$\xi_m, \xi_f \in \mathbf{BC}([0,T];\ \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})) \tag{2.14}$$
$$b_m, b_f \in \mathbf{BC}([0,T];\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2;\mathbb{R})) \tag{2.15}$$
$$\xi_c \in \mathbf{BC}([0,T];\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2;\mathbb{R})) \tag{2.16}$$
$$\mathcal{T} \in \mathbf{BC}([0,T];\ \mathcal{M}_+(\mathbb{R}_+^2)). \tag{2.17}$$

The space $\mathbf{BC}\left([0,T];X\right)$ consists of continuous, X valued functions bounded with respect to the norm $\|f\|_{\mathbf{BC_t}} = \sup_{t\in[0,T]} \|f(t)\|_X$. Before we proceed, we introduce some useful notation.

- Define the auxiliary function $F^{i,\nu}(t) = \int_{\mathbb{R}_+^2} b_i(t,z)d\nu(z)$, where index ν will be often omitted for transparency purposes.
- Let $(x_1,\dots,x_n) \in X^n$. We define $\|(x_1,\dots,x_n)\|_{X^n} = \sum_{k=1}^n \|x_k\|_X$.

To deal with the non-autonomous system (2.13) we begin with the analysis of equation (2.13.3), which is independent on equations (2.13.1) and (2.13.2). A convenient way to deal with (2.13.3) relies on its dual formulation, that is,

$$\begin{cases} \partial_t\varphi(t,z) + D_z\varphi(t,z) - \xi_c(t,z)\varphi(t,z) &= 0, \qquad t\in[0,T],\ z=(z_1,z_2)\in\mathbb{R}^2_+, \\ \varphi(T,z) &= \psi(z), \end{cases} \tag{2.18}$$

where $D_z\varphi = \left(\partial_{z_1}\varphi + \partial_{z_2}\varphi\right)$ and $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$. Function $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}^2_+;\mathbb{R})$ is a solution to the dual problem to (2.13.3), if it satisfies (2.18) in the classical strong sense. In the following Lemma we present some results about the problem (2.18).

lemma3:dual **Lemma 2.5.** *For all $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ there exists the unique solution to (2.18). Moreover, the following estimates hold:*

$$\|\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty}\,\mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}, \tag{2.19}$$

$$\mathbf{Lip}\left(\varphi_{T,\psi}(t,\cdot)\right) \le \|\psi\|_{\infty,\mathbf{Lip}}\,\mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}(T-t)}, \tag{2.20}$$

$$\sup_{s\in[T-t,T]}\|\partial_s\varphi_{T,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty}\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)\mathrm{e}^{(1+\|\xi_c\|_{\mathbf{BC_t}})(T-s)}, \tag{2.21}$$

If moreover $\tilde\varphi$ solves (2.18) with terminal data ψ and parameter $\tilde\xi_c$, then

$$\|\varphi_{T,\psi}(t,\cdot)-\tilde\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le 2\,C\,\|\psi\|_{\infty,\mathbf{Lip}}\|\xi_m-\tilde\xi_m\|_{\mathbf{BC_t}}(T-t)\,\mathrm{e}^{C(T-t)}, \tag{2.22}$$

where $C = \|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}$.

Since the result is classical, we do not present the proof. The relation between (2.13.3) and (2.18) is explained by the following Lemma.

lemma3 **Lemma 2.6.** *Fix $\mu^c_o\in\mathcal{M}_+(\mathbb{R}^2_+)$ and let ξ_c, $\mathcal{T}$ satisfy assumptions (2.16), (2.17). Then:*

i) Problem (2.13.3) admits the unique solution $\mu^c\in\mathbf{Lip}([0,T],\mathcal{M}_+(\mathbb{R}^2_+))$, that is, for all $0\le t_1\le t_2\le T$,

$$d_2\left(\mu^c_{t_2},\mu^c_{t_1}\right) \le \max\left\{1,\mu^c_o(\mathbb{R}_+)\right\}\cdot K\,\mathrm{e}^{3KT}\,(t_2-t_1),$$

where $K = \left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)$.

ii) Let μ^c and ν^c be solutions to (2.13.3) with initial data μ^c_o and ν^c_o respectively. Then,

$$d_2(\mu^c_t,\nu^c_t) \le \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}t}\,d_2(\mu^c_o,\nu^c_o).$$

iii) Let $\tilde\mu^c_o\in\mathcal{M}_+(\mathbb{R}^2_+)$ and $\tilde\xi_c,\tilde{\mathcal{T}}$ satisfy assumptions (2.16), (2.17). Let $\tilde\mu^c_t$ be solution to (2.13.3) with initial data $\tilde\mu^c_o$ and coefficients $\tilde\xi_c,\tilde{\mathcal{T}}$. Then,

$$d_2\left(\mu^c_t,\tilde\mu^c_t\right) \le K_1\,t\,\|(\xi_c-\tilde\xi_c,\mathcal{T}-\tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\,\mathrm{e}^{K_2\,t},$$

where $K_1 = K_1\left(T,\mu^c_o,\ \xi_c,\ \tilde\xi_c,\mathcal{T}\right)$ and $K_2 = 2\,\|(\xi_c,\tilde\xi_c)\|_{\mathbf{BC_t}}$.

iv) Fix $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. If $\mu^c \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}^+)\right)$ solves (2.13.3), then for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_1,t_2] \times \mathbb{R}^2_+; \mathbb{R})$ we have

$$\int_{t_1}^{t_2} \left(\int_{\mathbb{R}^2_+} \left(\partial_t \varphi(t,z) + D_z \varphi(t,z) - \xi_c(t,z)\varphi(t,z)\right) \mathrm{d}\mu^c_t(z) + \int_{\mathbb{R}^2_+} \varphi(t,z)\, \mathrm{d}[\mathcal{T}(t)](z) \right) \mathrm{d}t$$
$$= \int_{\mathbb{R}^2_+} \varphi(t_2,z)\, \mathrm{d}\mu^c_{t_2}(z) - \int_{\mathbb{R}^2_+} \varphi(t_1,z)\, \mathrm{d}\mu^c_{t_1}(z). \tag{2.23}$$

v) If $\mu^c \in \mathbf{Lip}([0,T]; \mathcal{M}_+(\mathbb{R}^2_+))$ solves (2.13.3), then for any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+; \mathbb{R})$ there exists a function $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^2_+; \mathbb{R})$ solving the dual problem (2.18) *and such that*

$$\int_{\mathbb{R}^2_+} \psi(z)\, \mathrm{d}\mu^c_t(z) = \int_{\mathbb{R}^2_+} \varphi_{T,\psi}(T-t,z)\, \mathrm{d}\mu^c_o(z) + \int_0^t \int_{\mathbb{R}^2_+} \varphi_{T,\psi}(s+(T-t),z)\, \mathrm{d}[\mathcal{T}(s)](z)\, \mathrm{d}s\,. \tag{2.24}$$

pf-formula

vi) For any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+; \mathbb{R})$ let $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^2_+; \mathbb{R})$ solves the dual problem (2.18)*. Then the measure defined by* (2.24) *solves (2.13.3).*

Analysis of problems (2.13.1) and (2.13.2) is based on their dual formulations as well. These problems are analogous, therefore we restrict ourselves to performing analysis just for one of them, that is, (2.13.1). We define the dual problem to (2.13.1) as

$$\begin{cases} \partial_t \varphi(t,x) + \partial_x \varphi(t,x) - \xi_m(t,x)\varphi(t,x) &= 0, \qquad t \in [0,T],\ x \in \mathbb{R}_+, \\ \varphi(T,x) &= \psi(x). \end{cases} \tag{2.25}$$

dual1:lemma **Lemma 2.7.** *For all $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+; \mathbb{R})$ there exists the unique solution to* (2.25)*. Moreover, the following estimates hold:*

$$\|\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}(T-t)}, \tag{2.26}$$
$$\mathbf{Lip}\left(\varphi_{T,\psi}(t,\cdot)\right) \le \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{2\|\xi_m\|_{\mathbf{BC_t}}(T-t)}, \tag{2.27}$$
$$\sup_{s \in [T-t,T]} \|\partial_s \varphi_{T,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \left(1 + \|\xi_m\|_{\mathbf{BC_t}}\right) \mathrm{e}^{(1+\|\xi_m\|_{\mathbf{BC_t}})(T-s)}, \tag{2.28}$$

If moreover $\tilde{\varphi}$ solves (2.25) *with terminal data ψ and parameter $\tilde{\xi}_m$, then*

$$\|\varphi_{T,\psi}(t,\cdot) - \tilde{\varphi}_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le 2\,C\, \|\psi\|_{\infty,\mathbf{Lip}} \|\xi_m - \tilde{\xi}_m\|_{\mathbf{BC_t}} (T-t)\, \mathrm{e}^{C(T-s)}, \tag{2.29}$$

where $C = \|(\xi_m, \tilde{\xi}_m)\|_{\mathbf{BC_t}}$.

Since the result is classical, we do not present the proof. The relation between (2.13.1) and (2.25) is explained by the following Lemma.

lemma1 **Lemma 2.8.** *Fix $\mu^m_o \in \mathcal{M}_+(\mathbb{R}^2_+)$ and let ξ_m, b_m satisfy assumptions* (2.14), (2.15)*. Then:*

i) Problem (2.13.1) admits the unique solution $\mu^m \in \mathbf{Lip}([0,T], \mathcal{M}_+(\mathbb{R}_+))$, that is, for all $0 \le t_1 \le t_2 \le T$,

$$d_1\left(\mu^m_{t_2}, \mu^m_{t_1}\right)$$
$$\le \max\left\{1, \mu^m_o(\mathbb{R}_+), \mu^c_o(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right) \mathrm{e}^{5\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T}\|_{\mathbf{BC_t}}\right)T} (t_2 - t_1).$$

ii) *Let* μ^m *and* ν^m *be solutions to (2.13.3) with initial data* μ_o^m *and* ν_o^m *respectively. Then,*

$$d_1(\mu_t^m, \nu_t^m) \le e^{2\|\xi_m\|_{\mathbf{BC_t}} t} d_1(\mu_o^m, \nu_o^m).$$

iii) *Let* $\tilde{\xi}_m, \tilde{b}_m$ *satisfy assumptions* (2.14), (2.15) *and* $\tilde{\mu}_t^m$ *be a solution to (2.13.3) with coefficients* $\tilde{\xi}_m, \tilde{b}_m$*. Then,*

$$d_1\left(\mu_t^m, \tilde{\mu}_t^m\right) \le K_1\, t\, \|(\xi_m - \tilde{\xi}_m,\ \xi_c - \tilde{\xi}_c,\ b_m - \tilde{b}_m,\ \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\, e^{K_2\, t},$$

where $K_1 = K_1\left(T,\ \mu_o^m, \mu_o^c,\ \xi_m,\ \tilde{\xi}_m,\ \xi_c,\ \tilde{\xi}_c,\ \tilde{b}_m, \mathcal{T}\right)$ *and* $K_2 = K_2\left(\xi_m,\ \tilde{\xi}_m,\ \xi_c,\ \tilde{\xi}_c,\ \mathcal{T}\right)$.

iv) *Fix* $t_1, t_2 \in [0,T]$ *with* $t_1 < t_2$*. If* $\mu^m \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}^+)\right)$ *solves (2.13.3), then for any* $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_1,t_2] \times \mathbb{R}_+^2; \mathbb{R})$ *we have*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}_+^2} \left(\partial_t\varphi(t,x) + \partial_x\varphi(t,x) - \xi_c(t,x)\varphi(t,x)\right) d\mu_t^m(x)\, dt$$
$$= \int_{\mathbb{R}_+} \varphi(t_2,x)\, d\mu_{t_2}^m(x) - \int_{\mathbb{R}_+} \varphi(t_1,x)\, d\mu_{t_1}^m(x) - \int_{t_1}^{t_2} \varphi(t,0) F^{m,\,\mu^c}(t)\, dt\,. \quad (2.30)$$

v) *If* $\mu^m \in \mathbf{Lip}([0,T]; \mathcal{M}_+(\mathbb{R}_+))$ *solves (2.13.1), then for any* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+; \mathbb{R})$*, there exists a function* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}_+; \mathbb{R})$ *solving the dual problem* (2.18) *and such that*

$$\int_{\mathbb{R}_+} \psi(x)\, d\mu_t^m(x) = \int_{\mathbb{R}_+^2} \varphi_{T,\psi}(T-t,x)\, d\mu_o^m(x) + \int_0^t \varphi(s,0) F^{m,\,\mu^c}(s)\, ds\,. \quad (2.31)$$ pf-formula:r

vi) *For any* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+; \mathbb{R})$ *let* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}_+; \mathbb{R})$ *solves the dual problem* (2.25)*. Then the measure defined by* (2.31) *solves (2.13.1).*

All results from this lemma are valid for the problem (2.13.2) when one change the index m for index f.

2.3 The Nonlinear Case

ion_nonlinear

To study the fully nonlinear system we need to equip the space $\mathcal{U} = \mathcal{M}_+(\mathbb{R}_+) \times \mathcal{M}_+(\mathbb{R}_+) \times \mathcal{M}_+(\mathbb{R}_+^2)$ in a metric. For each $\mathbf{u} = (\mu_1, \mu_2, \mu_3), \mathbf{v} = (\nu_1, \nu_2, \nu_3) \in \mathcal{U}$ we define

$$\mathbf{d}(\mathbf{u}, \mathbf{v}) = d_1(\mu_1, \nu_2) + d_1(\mu_2, \nu_2) + d_2(\mu_3, \nu_3).$$

thm:Main **Theorem 2.9.** *Let* $\mathbf{u}_o = (\mu_o^m, \mu_o^f, \mu_o^c) \in \mathcal{U}$ *and* (2.6) – (2.9) *hold. Then, there exists a solution* $\mathbf{u} : [0,T] \to \mathcal{U}$ *to the full nonlinear problem* (2.1)*. Moreover,*

i) *for all* $0 \le t_1 \le t_2 \le T$ *there exist constants* K_1 *and* K_2*, such that*

$$\mathbf{d}\left(\mathbf{u}_{t_1}, \mathbf{u}_{t_2}\right) \le K_1\, e^{K_2 t_2}\, (t_2 - t_1),$$

where K_1*,* K_2 *depend on all model coefficients and additionally* K_1 *depends on the initial data.*

ii) Let $\tilde{\mathbf{u}}_o \in \mathcal{U}$ *and* $\tilde{b}_m$, $\tilde{b}_f$, $\tilde{\xi}_m$, $\tilde{\xi}_f$, $\tilde{\xi}_c$, $\tilde{\mathcal{T}}$ *satisfy assumptions* (2.6) – (2.9). *Let* $\tilde{\mathbf{u}}$ *solve* (2.1) *with initial data* $\tilde{\mathbf{u}}_o$ *and coefficients* $\tilde{b}_m, \tilde{b}_f, \tilde{\xi}_m, \tilde{\xi}_f, \tilde{\xi}_c, \tilde{\mathcal{T}}$. *Then, there exist constants* K_1, K_2 *and* K_3 *such that for all* $t \in [0,T]$

$$\mathbf{d}\left(\mathbf{u}_t, \tilde{\mathbf{u}}_t\right) \le \mathrm{e}^{K_1 t}\, \mathbf{d}\left(\mathbf{u}_o, \tilde{\mathbf{u}}_o\right) + K_2\ t\ \mathrm{e}^{K_3 t} \cdot \|(b-\tilde{b}, \xi - \tilde{\xi}, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}}.$$

where $(b-\tilde{b}) = (b_m - \tilde{b}_m, b_f - \tilde{b}_f)$, $(\xi - \tilde{\xi}) = (\xi_m - \tilde{\xi}_m, \xi_f - \tilde{\xi}_f, \xi_c - \tilde{\xi}_c)$ *and* $K_1 = K_1(\xi)$, $K_2 = K_2\left(T,\ \mathbf{u}_o\, \xi,\ \tilde{\xi},\ \tilde{b},\ \mathcal{T}\right)$, $K_3 = K_3\left(\xi,\ \tilde{\xi},\ \mathcal{T}\right)$.

3 Proofs

sec_proofs

In the proof of Lemma 2.4 we shall need the following

slicing

Lemma 3.1. *Let* $\mu, \tilde{\mu} \in \mathcal{M}_+(\mathbb{R}^2_+)$ *and* $\sigma, \tilde{\sigma} \in \mathcal{M}_+(\mathbb{R}_+)$ *be projections of measures* $\mu, \tilde{\mu}$ *on* $\mathbb{R}_+$*, respectively, defined as in* (2.3). *Then,*

$$d_1(\sigma, \tilde{\sigma}) \le d_2(\mu, \tilde{\mu}).$$

Proof of Lemma 3.1. According to the Slicing Lemma [5, Section 1.5.2], there exists a Borel set N, such that $\mu(N) = 0$ and for each $x \notin N$ there exists a Radon probability measure ν_x, such that

$$\int_{\mathbb{R}^2_+} f(x,y)\, \mathrm{d}\mu = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f(x,y)\, \mathrm{d}\nu_x(y) \right) \mathrm{d}\sigma(x),$$

for each measurable and μ–integrable function f. Define a set $\mathcal{Z} \subset \mathbf{W^{1,\infty}}(\mathbb{R}^2_+;\mathbb{R})$ as

$$\mathcal{Z} = \left\{ f \in \mathbf{W^{1,\infty}}(\mathbb{R}^2_+;\mathbb{R})\ :\ f(x,y) = g(x),\ g \in \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R}),\ \|g\|_{\infty,\mathbf{Lip}} \le 1 \right\}.$$

It is straightforward, that $\|f\|_{\infty,\mathbf{Lip}} \le 1$, for each $f \in \mathcal{Z}$. Moreover,

$$\int_{\mathbb{R}^2_+} f(x,y)\, \mathrm{d}\mu\,(x,y) = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} g(x)\, \mathrm{d}\nu_x(y) \right) \mathrm{d}\sigma(y) = \int_{\mathbb{R}_+} g(x)\, \mathrm{d}\sigma(x).$$

Analogous equality holds for $\tilde{\mu}$ and $\tilde{\sigma}$, what implies that

$$\sup_{f \in \mathcal{Z}, \|f\|_{\infty,\mathbf{Lip}} \le 1} \left\{ \int_{\mathbb{R}^2_+} f(x,y)\, \mathrm{d}(\,\mu - \tilde{\mu})(x,y) \right\} = \sup_{g \in \mathbf{W^{1,\infty}}, \|g\|_{\infty,\mathbf{Lip}} \le 1} \int_{\mathbb{R}_+} g(x)\, \mathrm{d}(\,\sigma - \tilde{\sigma})(x).$$

According to Remark 2.5, the left hand side is not greater than $d_2(\mu, \tilde{\mu})$ and the right hand side is equal to $d_1(\sigma, \tilde{\sigma})$ what ends the proof. □

Proof of Lemma 2.4. Fix $t \in [0,T]$. Let $(\mu^m_t, \mu^f_t, \mu^c_t), (\nu^m_t, \nu^f_t, \nu^c_t) \in \mathcal{U}$ and $\sigma^{m,\mu^c_t}_t$, $\sigma^{f,\mu^c_t}_t$, $\sigma^{m,\nu^c_t}_t$, $\sigma^{f,\nu^c_t}_t$ be measures defined as in (2.3), that is,

$$\sigma^{m,\mu}_t(B) = \mu(B \times \mathbb{R}_+) \quad \text{and} \quad \sigma^{f,\mu}_t(B) = \mu(\mathbb{R}_+ \times B),$$

for any $\mu \in \mathcal{M}_+(\mathbb{R}^2_+)$ and Borel set $B \in \mathcal{B}(\mathbb{R}_+)$. It is sufficient to show that $\mathcal{F}(t,\cdot,\cdot) \in \mathbf{Lip}(\mathcal{V};\ \mathcal{M}_+(\mathbb{R}^2_+))$. Indeed, if that holds, then

$$d_2\left(\mathcal{T}(t, \mu^m_t, \mu^f_t, \mu^c_t), \mathcal{T}(t, \nu^m_t, \nu^f_t, \nu^c_t)\right)$$

$$
\begin{aligned}
&= d_2\left(\mathcal{F}\left(t, \mu_t^m - \sigma_t^{m,\mu_t^c}, \mu_t^f - \sigma_t^{f,\mu_t^c}\right), \mathcal{F}\left(t, \nu_t^m - \sigma_t^{m,\nu_t^c}, \nu_t^f - \sigma_t^{f,\nu_t^c}\right)\right) \\
&\le \mathbf{Lip}(\mathcal{F}(t))\left(d_1\left(\mu_t^m - \sigma_t^{m,\mu_t^c}, \nu_t^m - \sigma_t^{m,\nu_t^c}\right) + d_1\left(\mu_t^f - \sigma_t^{f,\mu_t^c}, \nu_t^f - \sigma_t^{f,\nu_t^c}\right)\right) \\
&\le \mathbf{Lip}(\mathcal{F}(t))\left(d_1(\mu_t^m, \nu_t^m) + d_1(\sigma_t^{m,\mu_t^c}, \sigma_t^{m,\nu_t^c}) + d_1(\mu_t^f, \nu_t^f) + d_1(\sigma_t^{f,\mu_t^c}, \sigma_t^{f,\nu_t^c})\right) \\
&\le 2\ \mathbf{Lip}(\mathcal{F}(t))\left(d_1(\mu_t^m, \nu_t^m) + d_1(\mu_t^f, \nu_t^f) + d_1(\mu_t^c, \nu_t^c)\right),
\end{aligned}
$$

where the last inequality holds due to Lemma (3.1). Define $Z = \int_{\mathbb{R}_+^2} h(z)\, \mathrm{d}s_t^m(z)$, $\tilde{Z} = \int_{\mathbb{R}_+^2} h(z)\, \mathrm{d}\, \tilde{s}_t^m(z)$ and $W = \int_{\mathbb{R}_+^2} g(w)\, \mathrm{d}s_t^f(w)$. Let $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+^2; \mathbb{R}_+)$ such that $\|\varphi\|_{\infty,\mathbf{Lip}} \le 1$. Adding and subtracting the following term

$$
\int_{\mathbb{R}_+^2} \frac{\theta(x,y)h(x)g(y)}{\int_{\mathbb{R}_+} h(z)\, \mathrm{d}s_t^m(z) + \int_{\mathbb{R}_+} g(w)\, \mathrm{d}s_t^f(w)}\ \mathrm{d}(\,\tilde{s}_t^m \otimes s_t^f)(x,y)
$$

to the expression $\int_{\mathbb{R}_+^2} \varphi(x,y)\, \mathrm{d}\left(\mathcal{F}(t, s_t^m, s_t^f) - \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right)(x,y)$ yieds

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2} \varphi(x,y)\, \mathrm{d}\left(\mathcal{F}(t, s_t^m, s_t^f) - \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right)(x,y) \\
&= \int_{\mathbb{R}_+^2} \varphi(x,y)\ \frac{\theta(x,y)h(x)g(y)}{Z+W}\ \mathrm{d}(\,s_t^m \otimes s_t^f - \tilde{s}_t^m \otimes s_t^f)(x,y) \\
&+ \int_{\mathbb{R}_+^2} \varphi(x,y)\theta(x,y)h(x)g(y)\left(\frac{1}{Z+W} - \frac{1}{\tilde{Z}+W}\right)\mathrm{d}(\,\tilde{s}_t^m \otimes s_t^f)(x,y) \\
&= \int_{\mathbb{R}_+} h(x)\ \frac{\int_{\mathbb{R}_+} \varphi(x,y)\theta(x,y)g(y)\, \mathrm{d}s_t^f(y)}{Z+W}\ \mathrm{d}(\,s_t^m - \tilde{s}_t^m)(x) \\
&- \int_{\mathbb{R}_+} h(x)\ \frac{\int_{\mathbb{R}_+} h(z)\left(\int_{\mathbb{R}_+} \varphi(z,y)\theta(z,y)g(y)\, \mathrm{d}s_t^f(y)\right)\tilde{s}_t^m(z)}{(Z+W)(\tilde{Z}+W)}\ \mathrm{d}(\,s_t^m - \tilde{s}_t^m)(x) \\
&= \int_{\mathbb{R}_+} h(x)\Phi(x)\, \mathrm{d}(\,s_t^m - \tilde{s}_t^m)(x),
\end{aligned}
$$

where

$$
\Phi(x) = \frac{\int_{\mathbb{R}_+} \varphi(x,y)\theta(x,y)g(y)\, \mathrm{d}s_t^f(y)}{Z+W} - \frac{\int_{\mathbb{R}_+} h(z)\left(\int_{\mathbb{R}_+} \varphi(z,y)\theta(z,y)g(y)\, \mathrm{d}s_t^f(y)\right)\tilde{s}_t^m(z)}{(Z+W)(\tilde{Z}+W)}.
$$

One can easily check that

$$
\begin{aligned}
\|\Phi\|_{\mathbf{L}^\infty} &\le 2\|\varphi\|_{\mathbf{L}^\infty}\|\theta\|_{\mathbf{L}^\infty} \\
\|\Phi'\|_{\mathbf{L}^\infty} &\le 2\|\varphi\|_{\infty,\mathbf{Lip}}\|\theta\|_{\infty,\mathbf{Lip}}.
\end{aligned}
$$

Therefore, it holds that $h \cdot \Phi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+; \mathbb{R})$, $\|h \cdot \Phi\|_{\mathbf{L}^\infty} \le 2\|\varphi\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty}\|\theta\|_{\mathbf{L}^\infty}$ and $\|(h \cdot \Phi)'\|_{\mathbf{L}^\infty} \le 4\|\varphi\|_{\infty,\mathbf{Lip}}\ \|h\|_{\infty,\mathbf{Lip}}\ \|\theta\|_{\infty,\mathbf{Lip}}$. By Remark 2.5 we conclude that

$$
d\left(\mathcal{F}(t, s_t^m, s_t^f), \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right) \le 4\|h\|_{\infty,\mathbf{Lip}}\ \|\theta\|_{\infty,\mathbf{Lip}}
$$

Analogous arguments leads to inequality $d\left(\mathcal{F}(t,s_t^m,s_t^f),\mathcal{F}(t,s_t^m,\tilde{s}_t^f)\right)\le 4\|g\|_{\infty,\mathbf{Lip}}\,\|\theta\|_{\infty,\mathbf{Lip}}$. Therefore,

$$d\left(\mathcal{F}(t,s_t^m,s_t^f),\mathcal{F}(t,\tilde{s}_t^m,\tilde{s}_t^f)\right)\le 4\,\left(\|h\|_{\infty,\mathbf{Lip}}+\|g\|_{\infty,\mathbf{Lip}}\right)\,\|\theta\|_{\infty,\mathbf{Lip}}.$$

□

Proof of Lemma 2.6.

i) We shall prove that problem (2.13.3) admits the unique solution. The proof bases on the regularization technique. More precisely, we regularize initial data μ_o^c and coefficient $\mathcal{T}$, what leads to the standard problem, which can be solved by the method of characteristics. Then we show a convergence of the sequence of regularized solutions and prove that the limit is a solution to (2.13.3) in the sense of Definition 2.2. Let $\rho\in\mathbf{C_c^\infty}(\mathbb{R}^2;\mathbb{R}_+)$ be such that $\int_{\mathbb{R}^2}\rho(z)\,\mathrm{d}z=1$. For $\varepsilon>0$ define a family of mollifiers $\rho^\varepsilon(z)=\rho(z/\varepsilon)/\varepsilon$. We define a covolution $*$ as

$$(\nu*\rho)(z)=\int_{\mathbb{R}^2_+}\rho(z-\varepsilon-\zeta)\,\mathrm{d}\nu\,(\zeta).$$

The reason why we shifted ρ by ε to the right is that $\operatorname{supp}\left((\nu\star\rho)(z)=\int_{\mathbb{R}^2}\rho(z-\zeta)\,\mathrm{d}\nu\,(\zeta)\right)\subseteq[-\varepsilon,+\infty)$, where $\star$ is a standard convolution. We consider (2.13.3) with initial data $u_o^{c,\varepsilon}$ and coefficient $\mathcal{T}^\varepsilon$, where

$$u_o^{c,\varepsilon}=(\mu_o^c*\rho^\varepsilon)\in(\mathbf{BC}^\infty\cap\mathbf{L^1})(\mathbb{R}^2_+;\mathbb{R}_+)\quad\text{and}\quad\mathcal{T}^\varepsilon(t)=(\mathcal{T}(t)*\rho^\varepsilon)\in(\mathbf{BC}^\infty\cap\mathbf{L^1})(\mathbb{R}^2_+;\mathbb{R}_+).$$

Due to assumption on $\mathcal{T}$, it holds that $\mathcal{T}^\varepsilon\in\mathbf{BC}\left([0,T];(\mathbf{BC}^\infty\cap\mathbf{L^1})(\mathbb{R}^2;\mathbb{R}_+)\right)$. It can be shown that

$$\|\mathcal{T}^\varepsilon\|_{\mathbf{BC_t}}\le\|\mathcal{T}\|_{\mathbf{BC_t}},\quad d_2(\mu_o^c,u_o^{c,\varepsilon})\stackrel{\varepsilon\to0}{\longrightarrow}0\quad\text{and}\quad\sup_{t\in[0,t]}d_2(\mathcal{T}(t),\mathcal{T}(t)^\varepsilon)\stackrel{\varepsilon\to0}{\longrightarrow}0.\tag{3.1}$$

reg:f

The proof of the analogous statement is contained in [2, Proof of Lemma 4.1], hence we do not present it here. Consider the equation (2.13.3) in the regular case, that is,

$$\begin{aligned}\partial_t u^{c,\varepsilon}(t,z)+D_z u^{c,\varepsilon}(t,z)+\xi_c(t,z)u^{c,\varepsilon}(t,z)&=\mathcal{T}^\varepsilon(t),\quad t\in[0,T],\ z\in\mathbb{R}^2_+,\\ u^{c,\varepsilon}(0,z)&=u_o^{c,\varepsilon}(z).\end{aligned}\tag{3.2}$$

The change of variables $(t,z)\stackrel{\Phi}{\longrightarrow}(t,X(t;z))$ in (3.2), where $X(t;z)$ is a solution to

$$\frac{d}{dt}X(t;z)=1,\quad X(0;z)=z,$$

transforms the original equation into the ODE

$$\begin{aligned}\partial_t v^{c,\varepsilon}(t,y)&=-\xi_c(t,y)v^{c,\varepsilon}(t,y)+\mathcal{T}^\varepsilon(t),\quad t\in[0,T],\ y\in\mathbb{R}^2_+,\\ v^{c,\varepsilon}(0,y)&=u_o^{c,\varepsilon}(y),\end{aligned}\tag{3.3}$$

where $y=X(t;z)$. This equation is an ODE in $\mathbf{L^1}(\mathbb{R}^2_+;\mathbb{R})$ with globally Lipschitz right hand side. Therefore, existence and uniqueness of a classical solution $v^{c,\varepsilon}\in\mathbf{BC^1}([0,T];\mathbf{L^1}(\mathbb{R}^2_+;\mathbb{R}))$ of (3.3) follows from the Banach Fixed Point Theorem. Moreover, $v^{c,\varepsilon}$ is a nonnegative function on $\mathbb{R}_+$. The solution of (3.2) is obtained by taking the inverse transform Φ^{-1}, that is, $u^{c,\varepsilon}(t,\Phi^{-1}(y)) = v^{c,\varepsilon}(t,y)$. Φ is a $\mathbf{C^1}$ diffeomorphism what implies that the regularity of solutions under the inverse transform does not change. Integrating (3.2) we obtain that for every $0 \le t_1 \le t_2 \le T$ and for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}^2_+;\mathbb{R})$

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\big(\partial_t\varphi(t,z) + D_z\varphi(t,z) - \xi_c(t,z)\varphi(t,z)\big)\,u^{c,\varepsilon}(t,z)\,\mathrm{d}z\,\mathrm{d}t \qquad (3.4)\\
&= \int_{\mathbb{R}^2_+}\varphi(t_2,z)u^{c,\varepsilon}(t_2,z)\,\mathrm{d}z - \int_{\mathbb{R}^2_+}\varphi(t_1,z)u^{c,\varepsilon}(t_1,z)\,\mathrm{d}z - \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\varphi(t,z)\,\mathrm{d}[\mathcal{T}^{\varepsilon}(t)](z)\,\mathrm{d}t\,.
\end{aligned}$$

Choosing φ as a solution to the dual problem (2.18) with $T = t_2$, we obtain

$$\int_{\mathbb{R}^2_+}\psi(z)u^{c,\varepsilon}(t_2,z)\,\mathrm{d}z = \int_{\mathbb{R}^2_+}\varphi_{t_2,\psi}(t_1,z)u^{c,\varepsilon}(t_1,z)\,\mathrm{d}z + \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\varphi_{t_2,\psi}(t,z)\,\mathrm{d}[\mathcal{T}^{\varepsilon}(t)](z)\,\mathrm{d}t\,. \qquad (3.5)$$

Let u^{c,ε_m}, respectively u^{c,ε_n}, solve problem (3.2) with ε replaced by ε_m, respectively ε_n. Moreover, let v be the solution to

$$\begin{aligned}
\partial_t v(t,z) + D_z v(t,z) + \xi_c(t,z)v(t,z) &= \mathcal{T}^{\varepsilon_m}(t), \quad t\in[0,T],\ z\in\mathbb{R}^2_+,\\
v(0,z) &= u_o^{c,\varepsilon_n}(z).
\end{aligned}$$

Let $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$. Using the formula (3.5) with $t_1 = 0$ and $t_2 = t$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2_+}\psi(z)\left(u^{c,\varepsilon_n}(t,z) - v(t,z)\right)\mathrm{d}z &= \int_0^t\int_{\mathbb{R}^2_+}\varphi_{t,\psi}(s,z)\,\mathrm{d}[\mathcal{T}^{\varepsilon_n}(s) - \mathcal{T}^{\varepsilon_m}(s)](z)\,\mathrm{d}t\\
&\le \int_0^t d_2(\mathcal{T}^{\varepsilon_n}(s),\mathcal{T}^{\varepsilon_m}(s))\,\mathrm{d}s \ \le\ T\cdot\sup_{s\in[0,T]} d_2(\mathcal{T}^{\varepsilon_n}(s),\mathcal{T}^{\varepsilon_m}(s)).
\end{aligned}$$

Taking supremum over all functions ψ gives

$$d_2\left(u^{c,\varepsilon_n}(t,\cdot),v(t,\cdot)\right) \le\ T\cdot\sup_{s\in[0,T]} d_2(\mathcal{T}^{\varepsilon_n}(s),\mathcal{T}^{\varepsilon_m}(s)).$$

Due to (3.1) $d_2(u^{c,\varepsilon_n}(t,\cdot),v(t,\cdot))$ converges to 0 uniformly wih respect to time. Analogously,

$$\int_{\mathbb{R}^2_+}\psi(z)\left(u^{c,\varepsilon_m}(t,z) - v(t,z)\right)\mathrm{d}z \ = \ \int_{\mathbb{R}^2_+}\varphi_{t,\psi}(0,z)\left(u_o^{c,\varepsilon_m}(z) - u_o^{c,\varepsilon_n}(z)\right)\mathrm{d}z\,.$$

Taking supremum over all functions ψ yields

$$d_2\left(u^{c,\varepsilon_m}(t,\cdot),v(t,\cdot)\right) \ \le\ \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}t}\,d_2(u_o^{c,\varepsilon_m},u_o^{c,\varepsilon_n}) \ \le\ \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}T}\,d_2(u_o^{c,\varepsilon_m},u_o^{c,\varepsilon_n}), \qquad (3.6)$$

what holds due to estimates (2.19) and (2.20) for a dual problem. Therefore, by (3.1), $d\left(u^{\varepsilon_n}(t,\cdot),u^{\varepsilon_m}(t,\cdot)\right) \overset{n,m\to\infty}{\longrightarrow} 0$ uniformly with respect to time.
Note that $\mathbf{BC^1}\left([0,T];\mathbf{L^1}(\mathbb{R}^2;\mathbb{R}_+)\right) \subset \mathbf{BC}\left([0,T];\left(\mathcal{M}_+(\mathbb{R}^2),\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}\right)\right)$ and metric d_2 is equal to $(\mathbf{W^{1,\infty}})^*$ distance on the set $\mathcal{M}_+(\mathbb{R}^2_+)$. Completeness of $(\mathcal{M}_+(\mathbb{R}^2),d_2)$ implies that the sequence $u^{\varepsilon_n}(t,\cdot)$ converges uniformly with respect to t to the unique limit μ^c_t. Notice that $\partial_t\varphi(t,\cdot)$, $D_z\varphi(t,\cdot)$, $\xi_c(t,\cdot)\varphi(t,\cdot)$ are continuous functions bounded uniformly with respect

to t. The integral $\int_{\mathbb{R}^2_+} \varphi(t,z)\,\mathrm{d}[\mathcal{T}^\varepsilon(t)](z)$ converges to $\int_{\mathbb{R}^2_+} \varphi(t,x)\,\mathrm{d}[\mathcal{T}(t)](z)$ uniformly with respect to t. Therefore, passage to the limit with u^{ε_n} and $\mathcal{T}^{\varepsilon_n}$ yields

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2_+} \big(\partial_t\varphi(t,z) + D_z\varphi(t,z) - \xi_c^\varepsilon(t,z)\varphi(t,z)\big)\,\mathrm{d}\,\mu_t^c(z)\,\mathrm{d}t \tag{3.7}$$
$$= \int_{\mathbb{R}^2_+} \varphi(t_2,z)\,\mathrm{d}\,\mu_{t_2}^c(z)\,\mathrm{d}z - \int_{\mathbb{R}^2_+} \varphi(t_1,z)\,\mathrm{d}\mu_{t_1}^c(z)\,\mathrm{d}z - \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+} \varphi(t,z)\,\mathrm{d}[\mathcal{T}(t)](z)\,\mathrm{d}s\,,$$

what proves that μ^c is a solution to (2.13.3) in the sense of Definition 2.2 (for the proof of uniqueness we refer to the proof of claim **iv**). Similarly we prove that passage to the limit in (3.5) yields

$$\int_{\mathbb{R}^2_+} \psi(z)\,\mathrm{d}\mu_{t_2}^c(z)\,\mathrm{d}z \;=\; \int_{\mathbb{R}^2_+} \varphi_{t_2,\psi}(t_1,z)\,\mathrm{d}\mu_{t_1}^c(z)\,\mathrm{d}z + \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+} \varphi_{t_2,\psi}(t,z)\,\mathrm{d}[\mathcal{T}(t)](z)\,\mathrm{d}t\,. \tag{3.8}$$

Using (2.21) and (3.8) for $t_1 = 0$ and $t_2 = t$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} \psi(z)\,\mathrm{d}\left(\mu_t^c - \mu_o^c\right)(z) = \int_{\mathbb{R}^2_+} \big(\varphi_{t,\psi}(0,z) - \psi(z)\big)\,\mathrm{d}\mu_o^c(z) + \int_0^t \varphi_{t,\psi}(s,z)\,\mathrm{d}[\mathcal{T}(s)](z)\,\mathrm{d}s \\
&\le \sup_{s\in[0,t]} \|\partial_s\varphi_{t,\psi}(s,\cdot)\|\; t\;\mu_o^c(\mathbb{R}^2_+) + t \sup_{s\in[0,t]} \|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \cdot \sup_{s\in[0,t]} \|\mathcal{T}(s)\|_{(\mathbf{W}^{\mathbf{1},\infty})^*} \\
&\le \|\psi\|_{\infty,\mathbf{Lip}}\left(1 + \|\xi_c\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\; t\;\mu_o^c(\mathbb{R}^2_+) + t\;\|\psi\|_{\infty,\mathbf{Lip}}\,\mathrm{e}^{\|\xi_c\|t}\;\|\mathcal{T}\|_{\mathbf{BC_t}} \\
&\le \|\psi\|_{\infty,\mathbf{Lip}} \cdot \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|\xi_c\|_{\mathbf{BC_t}} + \|\mathcal{T}\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\; t.
\end{aligned}$$

Taking supremum over all functions ψ such that $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ and $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$ gives

$$d(\mu_t^c, \mu_o^c) \le \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\; t.$$

for all $t \in [0,T]$. This allows us to estimate the total mass of μ^c in time t.

$$\begin{aligned}
\mu_t^c(\mathbb{R}^2_+) &\le d(\mu_t^c, \mu_o^c) + \mu_o^c(\mathbb{R}^2_+) \\
&\le \left[\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\; t + 1\right] \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \\
&\le \left[\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right) t + 1\right] \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t} \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \\
&\le \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\,.
\end{aligned}$$

Again, by (2.21), formula (3.8) for $0 \le t_1 \le t_2 \le T$ and the inequality above we obtain the following uniform Lipschitz estimate

$$\begin{aligned}
d\left(\mu_{t_2}^c, \mu_{t_1}^c\right) &\le \max\left\{1, \mu_{t_1}^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)(t_2-t_1)}\,(t_2 - t_1) \\
&\le \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right) \mathrm{e}^{3\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)T}\,(t_2 - t_1).
\end{aligned}$$

ii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \leq 1$. By the formula (3.8),

$$\int_{\mathbb{R}_+} \psi(z)\,\mathrm{d}\left(\mu^c_t - \nu^c_t\right)(z) = \int_{\mathbb{R}_+} \varphi_{t,\psi}(0,z)\,\mathrm{d}\left(\mu^c_o - \nu^c_o\right)(z).$$

Taking supremum over all functions ψ finishes the proof due to estimates (2.19) and (2.20) for a dual problem.

iii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ be such that $\|\psi\|_{\infty,\mathbf{Lip}} \leq 1$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} \psi(z)\,\mathrm{d}(\,\mu^c_t - \tilde{\mu}^c_t)(z) = \int_{\mathbb{R}^2_+} \big(\varphi_{t,\psi}(0,z) - \tilde{\varphi}_{t,\psi}(0,z)\big)\,\mathrm{d}\mu^c_o\,(z) \\
&\qquad + \int_0^t \int_{\mathbb{R}^2_+} \varphi_{t,\psi}(0,z)\,\mathrm{d}\mathcal{T}(s)\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}^2_+} \tilde{\varphi}_{t,\psi}(0,z)\,\mathrm{d}\,\tilde{\mathcal{T}}(s) \\
&\qquad = \int_{\mathbb{R}^2_+} \big(\varphi_{t,\psi}(0,z) - \tilde{\varphi}_{t,\psi}(0,z)\big)\,\mathrm{d}\mu^c_o\,(z) + \int_0^t \int_{\mathbb{R}^2_+} \big(\varphi_{t,\psi}(0,z) - \tilde{\varphi}_{t,\psi}(0,z)\big)\,\mathrm{d}[\,\mathcal{T}(s)](z)\,\mathrm{d}s \\
&\qquad + \int_0^t \int_{\mathbb{R}^2_+} \tilde{\varphi}_{t,\psi}(0,z)\,\mathrm{d}[\,\mathcal{T}(s) - \tilde{\mathcal{T}}(s)](z)\,\mathrm{d}s \\
&\qquad \leq 2C\,\|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}}\,t\,\mathrm{e}^{Ct}\,\mu^c_o(\mathbb{R}^2_+) + 2\,CT\,\|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}}\,t\,\mathrm{e}^{Ct} \cdot \sup_{s\in[0,t]} \|\mathcal{T}(s)\|_{(\mathbf{W^{1,\infty}})^*} \\
&\qquad + t \cdot \sup_{s\in[0,t]}\ \sup \left\{ \int_{\mathbb{R}^2_+} f(x)\,\mathrm{d}(\,\mathcal{T}(s) - \tilde{\mathcal{T}}(s))(x)\ :\ \|f\|_{\infty,\mathbf{Lip}} \leq \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}s} \right\} \\
&\qquad \leq C_1\,t\,\mathrm{e}^{Ct}\left(\mu_o(\mathbb{R}^2_+) + \|\mathcal{T}\|_{\mathbf{BC_t}}\right)\|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}} + t\,\mathrm{e}^{2\|\tilde{\xi}_c\|_{\mathbf{BC_t}}t} \sup_{s\in[0,t]} d_2(\mathcal{T}(s),\tilde{\mathcal{T}}(s)) \\
&\qquad \leq C_2\,t\,\mathrm{e}^{C_3 t}\,\|(\xi_c - \tilde{\xi}_c, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}},
\end{aligned}$$

where $C = \|(\xi_c, \tilde{\xi}_c)\|_{\mathbf{BC_t}}$, $C_1 = 2C(1+T)$, $C_2 = C_1\left(\mu^c_o(\mathbb{R}^2_+) + \|\mathcal{T}\|_{\mathbf{BC_t}}\right) + 1$ and $C_3 = 2\|(\xi_c, \tilde{\xi}_c)\|_{\mathbf{BC_t}}$.

iv) Assume that $\mu^c \in \mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}^2_+)\right)$ is a solution to (2.13.3) in the sense of Definition 2.2. Fix a $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}^2_+;\mathbb{R})$, then we prove that (2.30) holds for $t \in [t_1, t_2]$. Define $\varphi^\varepsilon(t,z) = \kappa_\varepsilon(t)\,\varphi(t,z)$, where

$$\kappa_\varepsilon \in \mathbf{C^\infty_c}([t_1,t_2)\,,[0,1])\,, \quad \kappa_\varepsilon(t_1) = 1\,, \quad \lim_{\varepsilon\to 0} \kappa_\varepsilon(s) = \chi_{[t_1,t_2)}(s)$$

and

$$\lim_{\varepsilon\to 0} \kappa'_\varepsilon = \delta(t = t_1) - \delta(t = t_2) \text{ in } \mathcal{M}_+([0,T])\,.$$

Use φ^ε as a test function in the definition of weak solution.

$$\begin{aligned}
0 \ &= \ \int_0^T \left(\int_{\mathbb{R}^2_+} \partial_t\varphi^\varepsilon(t,z) + D_z\varphi^\varepsilon(t,z) - \xi_c(t,z)\varphi^\varepsilon(t,z)\,\mathrm{d}\mu^c_t\,(z) + \int_{\mathbb{R}^2_+} \varphi^\varepsilon(t,z)\,\mathrm{d}[\,\mathcal{T}(t)](z) \right)\mathrm{d}t \\
&= \ \int_0^T k'_\varepsilon(t) \int_{\mathbb{R}^2_+} \varphi(t,z)\,\mathrm{d}\mu^c_t\,(z)\,\mathrm{d}t
\end{aligned}$$

$$+ \int_0^T k_\varepsilon(t) \left(\int_{\mathbb{R}^2_+} \partial_t \varphi(t,z) + D_z \varphi(t,x) - \xi_c(t,z)\varphi(t,z)\, \mathrm{d}\mu_t^c(z) + \int_{\mathbb{R}^2_+} \varphi(t,z)\, \mathrm{d}[\mathcal{T}(t)](z) \right) \mathrm{d}t$$

Passing to the limit with ε and using Dominated Convergence Theorem finishes the proof.

v) Equality follows from **iv)** by setting $t_1 = 0$, $t_2 = t$ and $\varphi(s,x) = \varphi_{T,\psi}(s + (T - t_2), x)$.

vi) We proved that there exists a solution to (2.13.3) which also fulfils (2.24). This equation characterizes μ^c uniquely, hence each μ^c given by (2.24) is a solution to (2.13.3).

□

Proof of Lemma 2.8.

i) We shall show that problem (2.13.1) admits the unique solution. The proof is analogous to the proof of Lemma 2.6. Let $\rho \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R};\mathbb{R}_+)$ be such that $\int_\mathbb{R} \rho(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$ define a family of mollifiers $\rho^\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon$. The convolution is defined as $(\nu * \rho)(x) = \int_{\mathbb{R}_+} \rho(x - \varepsilon - \zeta)\,\mathrm{d}\nu(\zeta)$. We consider (2.13.1) with initial data $u_o^{m,\varepsilon}$, where

$$u_o^{m,\varepsilon} = (\mu_o^m * \rho^\varepsilon) \in (\mathbf{BC} \cap \mathbf{L^1})(\mathbb{R}_+;\mathbb{R}_+).$$
$$d_1(u_o^{m,\varepsilon}, \mu_o^m) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.9}$$

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Consider the equation (2.13.1) in the regular case, that is,

$$\begin{aligned} \partial_t u^{m,\varepsilon}(t,x) + \partial_x u^{m,\varepsilon}(t,x) + \xi_m(t,x) u^{m,\varepsilon}(t,x) &= 0, \quad t \in [0,T],\ x \in \mathbb{R}_+, \\ u^{m,\varepsilon}(t,0) &= F^{m,\mu^c}(t), \\ u^{m,\varepsilon}(0,x) &= u_o^{m,\varepsilon}(x). \end{aligned} \tag{3.10}$$

Due to assumption (2.15) on b_m and properties of μ_t^c from Lemma 2.6 it holds that $F^{m,\mu^c} \in (\mathbf{BC} \cap \mathbf{L^1})([0,T];\mathbb{R}_+)$. Moreover,

$$\begin{aligned} \sup_{s\in[0,t]} \left| F^{m,\mu^c}(s) \right| &\leq \|b_m\|_{\mathbf{BC_t}} \cdot \sup_{s\in[0,t]} \mu_s^c(\mathbb{R}_+) \\ &\leq \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \|b_m\|_{\mathbf{BC_t}}\, \mathrm{e}^{2\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}. \end{aligned} \tag{3.11}$$

Let $\tilde F^{m,\mu^c}(t) = \int_{\mathbb{R}^2_+} \tilde b_m(t,z)\,\mathrm{d}\,\tilde\mu_t^c(z)$. Then,

$$\begin{aligned} F^{m,\mu^c}(t) - \tilde F^{m,\mu^c}(t) &= \int_{\mathbb{R}^2_+} \left(b(t,z) - \tilde b(t,z)\right) \mathrm{d}\mu_t^c(z) + \int_{\mathbb{R}^2_+} \tilde b(t,z)\,\mathrm{d}\left(\mu_t^c - \tilde\mu_t^c\right)(z) \\ &\leq \|b_m - \tilde b_m\|_{\mathbf{BC_t}}\, \mu_t^c(\mathbb{R}^2_+) + \|\tilde b_m\|_{\mathbf{BC_t}}\, d_2\left(\mu_t^c, \tilde\mu_t^c\right). \end{aligned} \tag{3.12}$$

Existence and uniqueness of solutions to (3.10) follow from the method of characteristics. The method leads to the explicit formula on the solution $u^{m,\varepsilon}(t,x)$, that is,

$$u^{m,\varepsilon}(t,x) = \begin{cases} u_o^{m,\varepsilon}(x-t) \exp\left(\int_0^t \xi_m\left(s,\ s + (x-t)\right) \mathrm{d}s\right) & \text{for} \quad x \in [t,+\infty), \\ F^{m,\mu^c}(t-x) \exp\left(\int_{t-x}^t \xi_m\left(s,\ s + (x-t)\right) \mathrm{d}s\right) & \text{for} \quad x \in [0,t). \end{cases} \tag{3.13}$$

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for $t > 0$ and $x \in \mathbb{R}_+$. We shall prove that $u^{m,\varepsilon}(t,\cdot)$ is a Cauchy sequence in in $(\mathcal{M}_+(\mathbb{R}_+); d_1)$. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be such that $\varepsilon_n \to 0$ as $n \to +\infty$. Formula (3.13) implies that $u^{m,\varepsilon_n}(t,\cdot) \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)$, if $u_o^{m,\varepsilon_n} \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)$. Moreover, $\|u^{m,\varepsilon_n}(t,\cdot)\|_{\mathbf{L^1}}$ is uniformly bounded. It can be checked that $u^{m,\varepsilon_n} \in \mathbf{BC}([0,T];\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)) \subset \mathbf{BC}\left([0,T];\left(\mathcal{M}_+(\mathbb{R}_+), \|\cdot\|_{(\mathbf{W^{1,\infty}})^*}\right)\right)$. Now, let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R};\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \leq 1$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \psi(x)\left(u^{m,\varepsilon_n}(t,x) - u^{m,\varepsilon_k}(t,x)\right)\mathrm{d}x \\
&\qquad = \int_t^{+\infty} \psi(x)\, \mathrm{e}^{\int_0^t \xi_m\left(s,(x-t)+s\right)\mathrm{d}s}\left(u_o^{m,\varepsilon_n}(x-t) - u_o^{m,\varepsilon_k}(x-t)\right)\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}_+} \psi(x+t)\, \mathrm{e}^{\int_0^t \xi_m(s,\, x+s)\mathrm{d}s}\left(u_o^{m,\varepsilon_n}(x) - u_o^{m,\varepsilon_k}(x)\right)\mathrm{d}x \\
&\qquad \leq\ \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}T}\, d_1(u_o^{m,\varepsilon_n}, u_o^{m,\varepsilon_k}).
\end{aligned}$$

Taking supremum over all functions ψ yields

$$d_1(u^{m,\varepsilon_n}(t,\cdot), u^{m,\varepsilon_k}(t,\cdot)) \leq \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}T}\, d_1(u_o^{m,\varepsilon_n}, u_o^{m,\varepsilon_k}) \to 0.$$

Therefore, for each t there exists a limit μ_t^m, such that $d_1(u^{m,\varepsilon}(t,\cdot), \mu_t)$ converges to zero uniformly with respect to time. We need to show that μ_t^m is a solution in the sense of Definition 2.2. Integrating (3.10) we obtain that for every $0 \leq t_1 \leq t_2 \leq T$ and $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$,

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\left(\partial_t\varphi(t,x)(t,x) + \partial_x\varphi(t,x) - \xi_m(t,x)\varphi(t,x)\right)u^{m,\varepsilon}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\varphi(t,0)F^{m,\mu^c}(t)\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}_+}\varphi(t_2,x)u^{m,\varepsilon}(t_2,x)\,\mathrm{d}x - \int_{\mathbb{R}_+}\varphi(t_1,x)u^{m,\varepsilon}(t_1,x)\,\mathrm{d}x\,. \qquad (3.14)
\end{aligned}$$

Choosing φ as a solution to the dual problem (2.25) with $T = t_2$ we obtain

$$\int_{\mathbb{R}_+}\psi(x)u^{m,\varepsilon}(t_2,x)\,\mathrm{d}x = \int_{\mathbb{R}_+}\varphi_{t_2,\psi}(t_1,x)u^{m,\varepsilon}(t_1,x)\,\mathrm{d}x + \int_{t_1}^{t_2}\varphi_{t_2,\psi}(s,0)F^{m,\mu^c}(t)\,\mathrm{d}t\,. \quad (3.15)$$

Notice that $\partial_t\varphi(t,\cdot)$, $\partial_x\varphi(t,\cdot)$, $\varphi(t,\cdot)$ and $\xi_m(t,\cdot)\varphi(t,\cdot)$ are continuous functions uniformly bounded with respect to t. Therefore, passage to the limit in (3.14) for $t_1 = 0$ and $t_2 = T$ yields

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\left(\partial_t\varphi(t,x)(t,x) + \partial_x\varphi(t,x) - \xi_m(t,x)\varphi(t,x)\right)\mathrm{d}\mu_t^m(x)\,\mathrm{d}t + \int_{t_1}^{t_2}\varphi(t,0)F^{m,\mu^c}(t)\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}_+}\varphi(t_2,x)\,\mathrm{d}\mu_t^m(x) - \int_{\mathbb{R}_+}\varphi(t_1,x)\,\mathrm{d}\mu_t^m(x), \qquad (3.16)
\end{aligned}$$

what proves that μ^m is a solution to (2.13.1). Passage to the limit in (3.15) yields

$$\int_{\mathbb{R}_+}\psi(x)\,\mathrm{d}\mu_{t_2}^m(x)\,\mathrm{d}x = \int_{\mathbb{R}_+}\varphi_{t_2,\psi}(t_1,x)\,\mathrm{d}\mu_{t_1}^m(x)\,\mathrm{d}x + \int_{t_1}^{t_2}\varphi_{t_2,\psi}(s,0)F^{m,\mu^c}(t)\,\mathrm{d}t\,. \qquad (3.17)$$

Using (2.26), (2.28), (3.11) and (3.17) for $t_1 = 0$ and $t_2 = t$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+} \psi(x)\,\mathrm{d}\left(\mu_t^m - \mu_o^m\right)(x) = \int_{\mathbb{R}_+} \left(\varphi_{t,\psi}(0,x) - \psi(x)\right) \mathrm{d}\mu_o^m(x) + \int_0^t \varphi_{t,\psi}(s,0) F^{m,\mu^c}(s)\,\mathrm{d}s \\
&\le \sup_{s\in[0,t]} \|\partial_s \varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \mu_o^m(\mathbb{R}_+)\, t + \sup_{s\in[0,t]} \|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \cdot \sup_{s\in[0,t]} \left|F^{m,\mu^c}(s)\right| t \\
&\le \|\psi\|_{\infty,\mathbf{Lip}} \left(1 + \|\xi_m\|_{\mathbf{BC_t}}\right) \mathrm{e}^{\left(1+\|\xi_m\|_{\mathbf{BC_t}}\right)t} \mu_o^m(\mathbb{R}_+)\, t \\
&+ \|\psi\|_{\infty,\mathbf{Lip}}\, \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}} t} \cdot \max\left\{1, \mu_o^c(\mathbb{R}_+)\right\} \cdot \|b_m\|_{\mathbf{BC_t}}\, \mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\ t \\
&\le \|\psi\|_{\infty,\mathbf{Lip}} \cdot \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right) \mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\ t.
\end{aligned}$$

Taking supremum over all functions ψ such that $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$ and $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$ gives

$$d_1(\mu_t^m, \mu_o^m) \le \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right) \mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\ t,$$

for all $t \in [0,T]$. This allows us to estimate the total mass of μ^m in time t.

$$\begin{aligned}
\mu_t(\mathbb{R}_+) &\le \left[\max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right) \cdot \mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\ t + 1\right] \mu_o^m(\mathbb{R}_+) \\
&\le \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \cdot \left[\left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right)\ t + 1\right] \mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\
&\le \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \cdot \mathrm{e}^{3\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}.
\end{aligned}$$

Again, by (2.28), formula (3.17) for $0 \le t_1 \le t_2 \le T$ and the inequality above we obtain the following Lipschitz estimate

$$d_1(\mu_{t_2}^m, \mu_{t_1}^m) \le \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\} \cdot \left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right) \mathrm{e}^{5\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T}\|_{\mathbf{BC_t}}\right)T}\ (t_2 - t_1).$$

ii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$. By the formula (3.17)

$$\int_{\mathbb{R}_+} \psi(x)\,\mathrm{d}\,(\mu_t^c - \nu_t^c)(x) = \int_{\mathbb{R}_+} \varphi_{t,\psi}(0,x)\,\mathrm{d}\,(\mu_o^c - \nu_o^c)(x).$$

Taking supremum over all functions ψ finishes the proof due to estimates (2.26) and (2.27) for a dual problem.

iii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$. Let $\tilde{\mu}_t^c$ be a solution to (2.13.1) with boundary condition given by $\tilde{\mu}_t^c(\{0\}) = \tilde{F}^{m,\mu^c}(t) = \int_{\mathbb{R}_+^2} \tilde{b}(t,z)\,\mathrm{d}\,\tilde{\mu}_t^c(z)$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \psi(x)\,\mathrm{d}\left(\mu_t^m - \tilde{\mu}_t^m\right)(x) \\
&= \int_{\mathbb{R}_+} \left(\varphi_{t,\psi}(0,x) - \tilde{\varphi}_{t,\psi}(0,x)\right) \mathrm{d}\mu_o(x) + \int_0^t \varphi_{t,\psi}(s,x) \left(F^{m,\mu^c}(s) - \tilde{F}^{m,\mu^c}(s)\right) \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\varphi_{t,\psi}(0,\cdot)-\tilde\varphi_{t,\psi}(0,\cdot)\|_{\mathbf{L}^\infty}\mu_o^m(\mathbb{R}_+)+t\sup_{s\in[0,t]}\|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty}\cdot\sup_{s\in[0,t]}\left|F^m(s)-\tilde F^m(s)\right|\\
&\leq 2\,\|\psi\|_{\infty,\mathbf{Lip}}\|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}\|\xi_m-\tilde\xi_m\|_{\mathbf{BC_t}}\,t\,\mathrm{e}^{\|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}t}\,\mu_o^m(\mathbb{R}_+)\\
&+t\,\|\psi\|_{\infty,\mathbf{Lip}}\,\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\left(\|b_m-\tilde b_m\|_{\mathbf{BC_t}}\cdot\sup_{s\in[0,t]}\mu_s^c(\mathbb{R}_+^2)+\|\tilde b_m\|_{\mathbf{BC_t}}\cdot\sup_{s\in[0,t]}d_2(\mu_s^c,\tilde\mu_s^c)\right)\\
&\leq 2\,\|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}\|\xi_m-\tilde\xi_m\|_{\mathbf{BC_t}}\,t\,\mathrm{e}^{\|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}t}\,\mu_o^m(\mathbb{R}_+)\\
&+t\,\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\,\|b_m-\tilde b_m\|_{\mathbf{BC_t}}\cdot\max\left\{1,\mu_o^c(\mathbb{R}_+)\right\}\mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\\
&+t\,\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\,\|\tilde b_m\|_{\mathbf{BC_t}}C\,t\,\mathrm{e}^{2\,\|(\xi_c,\tilde\xi_c)\|_{\mathbf{BC_t}}t}\,\|(\xi_c-\tilde\xi_c,\mathcal{T}-\tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\\
&\leq C_1\,t\,\|(\xi_m-\tilde\xi_m,\ \xi_c-\tilde\xi_c,\ b_m-\tilde b_m,\ \mathcal{T}-\tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\,\mathrm{e}^{C_2\,t},
\end{aligned}
$$

where $C=2\,\|(\xi_c,\tilde\xi_c)\|_{\mathbf{BC_t}}(1+T)\cdot\left(\mu_o(\mathbb{R}_+^2)+\|\mathcal{T}\|_{\mathbf{BC_t}}\right)$, $C_2=2\left(1+\|(\xi_m,\tilde\xi_m,\xi_c,\tilde\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)$ and

$$
\begin{aligned}
C_1 \;=\;& 2\max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+^2)\right\}\\
&\cdot\left[1+\|(\xi_m,\tilde\xi_m,)\|_{\mathbf{BC_t}}+\|\tilde b_m\|_{\mathbf{BC_t}}\|(\xi_c,\tilde\xi_c,)\|_{\mathbf{BC_t}}(1+T)\cdot\left(\mu_o(\mathbb{R}_+^2)+\|\mathcal{T}\|_{\mathbf{BC_t}}\right)\right].
\end{aligned}
$$

Taking supremum over all functions ψ finishes the proof.

iv) The proof is analogous to the proof of Lemma 2.8, claim *iv)*.

v) The equality follows from **iv)** by setting $t_1=0$, $t_2=t$, $\varphi(s,x)=\varphi_{T,\psi}(s+(T-t_2),x)$.

vi) We proved that there exists a solution to (2.13.1) which also fulfils (3.17). This equation characterizes μ^m uniquely, hence each μ^m given by (3.17) is a solution to (2.13.1). □

Proof of Theorem 2.9. Let $\mathbf{u}_o=(\mu_o^m,\mu_o^f,\mu_o^c)\in\mathcal{U}$ be an initial measure in (2.1) and b_m, b_f, ξ_m, ξ_f, ξ_c, $\mathcal{T}$ satisfy assumptions (2.6)–(2.9). Let us introduce a complete metric space $\mathbf{BC}(I;\bar B_R(\mathbf{u}_o))$ where $I=[0,\varepsilon]$ with ε to be chosen later on and $\bar B_R(\mathbf{u}_o)=\{\mathbf{v}\in\mathcal{U}\ :\ \mathbf{d}(\mathbf{u}_o,\mathbf{v})\leq R\}$. The space $\mathbf{BC}(I;\bar B_R(\mathbf{u}_o))$ is equipped with a norm given by $\|\mathbf{u}\|_{\mathbf{BC}}=\sup_{t\in[0,T]}\left(\|\mu_1(t)\|_{(\mathbf{W^{1,\infty}})^*}+\|\mu_2(t)\|_{(\mathbf{W^{1,\infty}})^*}+\|\mu_3(t)\|_{(\mathbf{W^{1,\infty}})^*}\right)$ for any $\mathbf{u}=(\mu_1,\mu_2,\mu_3)$. This space is complete since $\bar B_R(\mathbf{u}_o)$ is a closed subset of the complete metric space $\mathcal{U}$. Note that $\|\mathbf{u}-\mathbf{v}\|_{\mathbf{BC}}=\sup_{t\in[0,T]}\left(d_1(\mu_1,\nu_1)+d_1(\mu_2,\nu_2)+d_1(\mu_3,\nu_3)\right)$ for any $\mathbf{u}=(\mu_1,\mu_2,\mu_3)$ and $\mathbf{v}=(\nu_1,\nu_2,\nu_3)$. We define an operator $\mathcal{Z}$ on $\mathbf{BC}(I;\bar B_R(\mathbf{u}_o))$ as follows

$$
\begin{aligned}
&\mathcal{Z}:\mathbf{BC}(I;\bar B_R(\mathbf{u}_o))\longrightarrow\mathbf{BC}(I;\bar B_R(\mathbf{u}_o))\\
&\mathcal{Z}(\mathbf{u})=\mathbf{v}_{(b,\,\xi,\mathcal{T})(\mathbf{u})},
\end{aligned}
$$

where $\mathbf{v}_{(b,\,\xi,\mathcal{T})(\mathbf{u})}$ is the solution to the non-autonomous system (2.13) with coefficients $b_m(\cdot,\mu^m,\mu^f)$, $b_f(\cdot,\mu^m,\mu^f)$, $\xi_m(\cdot,\mu^m,\mu^f)$, $\xi_f(\cdot,\mu^m,\mu^f)$, $\xi_c(\cdot,\mu^m,\mu^f,\mu^c)$, $\mathcal{T}(\cdot,\mu^m,\mu^f,\mu^c)$ and initial data $\mathbf{u}_o$. First we need to prove that the operator $\mathcal{Z}$ is well defined, meaning that its image is a bounded continuous function taking values in $\bar B_R(\mathbf{u}_o)$. Continuity of $\mathbf{v}_{(b,\,\xi,\mathcal{T})(\mathbf{u})}$ follows from statement *i)* in Lemma 2.8 and statement *i)* in Lemma 2.6. Moreover, for each

$t \in [0, \varepsilon]$ we have

$$\mathbf{d}(\mathcal{Z}(\mathbf{u})(t), \mathbf{u}_o) \leq 3\ C_1 C_2\ t\ \mathrm{e}^{5C_2 t} \leq R,$$

where $C_1 = \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^f(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+^2)\right\}$ and $C_2 = 1 + \|(\xi_m, \xi_f, \xi_c, b_m, b_f, \mathcal{T})\|_{\mathbf{BC_t}}$. We need to assume that $\varepsilon < 1$. Then, $3\ C_1C_2\ \varepsilon\ \mathrm{e}^{5C_2\varepsilon} \leq 3\ C_1C_2\ \varepsilon\ \mathrm{e}^{5C_2} \leq R$ or equivalently

$$\varepsilon \quad \leq \quad R\ \left(3\ \mathrm{e}^{5C_2}\, C_1 C_2\right)^{-1} =: \upsilon_1. \tag{3.18}$$

Now, we prove that $\mathcal{Z}$ is a contraction for ε small enough. We shall show that $\mathcal{Z}$ is a Lipschitz operator with Lipschitz constant smaller than 1.

$$\begin{aligned}
&\|\mathcal{Z}(\mathbf{u}) - \mathcal{Z}(\bar{\mathbf{u}})\|_{\mathbf{BC}} = \sup_{t\in[0,\varepsilon]} \mathbf{d}\left(\mathcal{Z}(\mathbf{u})(t), \mathcal{Z}(\bar{\mathbf{u}})(t)\right) = \sup_{t\in[0,\varepsilon]} \mathbf{d}\left(\mathbf{v}_{(b,\,\xi,\mathcal{T})(\mathbf{u})}(t), \mathbf{v}_{(b,\,\xi,\mathcal{T})(\bar{\mathbf{u}})}(t)\right)\\
&= \sup_{t\in[0,\varepsilon]} C_3 t\, \mathrm{e}^{C_4 t} \cdot \Big(\|\xi_m(t,\mathbf{u}) - \xi_m(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}} + \|\xi_f(t,\mathbf{u}) - \xi_f(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}} +\\
&+ \|\xi_c(t,\mathbf{u}) - \xi_c(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}} + \|b_m(t,\mathbf{u}) - b_m(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}} + \|b_f(t,\mathbf{u}) - b_f(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}} +\\
&+ \|\mathcal{T}(t,\mathbf{u}) - \mathcal{T}(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}\Big)\\
&\leq \sup_{t\in[0,\varepsilon]} C_3 t\, \mathrm{e}^{C_4 t} \cdot \Big(\mathbf{Lip}(\xi_m(t,\cdot)) + \mathbf{Lip}(\xi_f(t,\cdot)) + \mathbf{Lip}(\xi_c(t,\cdot)) + \mathbf{Lip}(b_m(t,\cdot)) +\\
&\quad + \mathbf{Lip}(b_f(t,\cdot)) + \mathbf{Lip}(\mathcal{T}(t,\cdot))\Big) \cdot \sup_{t\in[0,T]} \mathbf{d}\left(\mathbf{u}(t), \bar{\mathbf{u}}(t)\right) \leq C_3\varepsilon\, \mathrm{e}^{C_4\varepsilon} \cdot \mathbf{Lip} \cdot \|\mathbf{u}(t) - \bar{\mathbf{u}}(t)\|_{\mathbf{BC}},
\end{aligned}$$

where $C_3 = C_3\left(T,\ \mu_o^m,\ \mu_o^f,\ \mu_o^c,\ \xi_m,\ \tilde{\xi}_m,\ \xi_f,\ \tilde{\xi}_f,\ \xi_c,\ \tilde{\xi}_c,\ \tilde{b}_m,\ \tilde{b}_f,\ \mathcal{T}\right)$,
$C_4 = C_4\left(\xi_m,\ \tilde{\xi}_m,\ \xi_f,\ \tilde{\xi}_f,\ \xi_c,\ \tilde{\xi}_c,\ \mathcal{T}\right)$ and $\mathbf{Lip} = \mathbf{Lip}(\xi_m) + \mathbf{Lip}(\xi_f) + \mathbf{Lip}(\xi_c) + \mathbf{Lip}(b_m) + \mathbf{Lip}(b_f) + \mathbf{Lip}(\mathcal{T})$. These constants are finite due to assumptions (2.6)–(2.9), We need to assume that $\varepsilon < 1$. Then,

$$\|\mathcal{Z}(\mathbf{u}) - \mathcal{Z}(\bar{\mathbf{u}})\|_{\mathbf{BC}} \leq C_3\varepsilon\, \mathrm{e}^{C_4} \cdot \mathbf{Lip} \cdot \|\mathbf{u}(t) - \bar{\mathbf{u}}(t)\|_{\mathbf{BC}}.$$

Lipschitz constant of $\mathcal{Z}$ is smaller then 1, if the following inequality holds

$$\mathbf{Lip}(\mathcal{Z}) \quad = \quad C_3\varepsilon\, \mathrm{e}^{C_4} \cdot \mathbf{Lip} < 1.$$

Hence, $\varepsilon \leq \left(C_3\, \mathrm{e}^{C_4} \cdot \mathbf{Lip}\right)^{-1} =: \upsilon_2$. We proved that $\mathcal{Z}$ is a contraction on a complete metric space $\mathbf{BC}(I, \bar{B}_R(\mathbf{u}_o))$, where $\varepsilon = \min\{1, \upsilon_1, \upsilon_2\} > 0$. From the Banach Fixed Point Theorem it follows that there exists unique $\mathbf{u}^*$, such that $\mathcal{Z}(\mathbf{u}^*) = \mathbf{u}^*$. This solution can be extended on the whole $[0, T]$ interval, because υ_1 and υ_2 do not depend on time. Moreover, the sequence of solutions to the non-autonomus system defined inductively by

$$\mathbf{u}_1 = \mathcal{Z}(\mathbf{u}_o), \quad \mathbf{u}_{n+1} = \mathcal{Z}(\mathbf{u}_n)$$

converges in $\|\cdot\|_{\mathbf{BC}}$ to $\mathbf{u}^*$. Thus, passage to the limit in the integrals (3.7), (3.16) (and in the analogous integral for μ^f) proves that $\mathbf{u}^*$ is the solution to the system (2.1) in the sense of Definition 2.2. From *i)* in Lemma 2.8 and *i)* in Lemma 2.6 it follows that $\mathbf{u}^*$ is Lipschitz continuous with respect to time. Estimates in claims *i)* and *ii)* are consequences of estimates for the linear non-autonomous case (see Lemma 2.8 and Lemma 2.6).

□

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