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ON THE ANISOTROPIC ORLICZ SPACES APPLIED IN THE PROBLEMS OF CONTINUUM MECHANICS

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Abstract. The paper concerns theory of anisotropic Orlicz spaces and its applications in continuum mechanics. Our main motivations are e.g. flow of non-Newtonian fluid and response of inelastic materials with non-standard growth conditions of the Cauchy stress tensor. The set of basic definitions and theorems with proofs is presented. We prove the existence of a weak solutions to the generalized Stokes system. Overview covering recent results in the referred topic is given.

1. **Introduction.** Our interest is dedicated to the properties of the anisotropic Orlicz spaces and their applications in continuum mechanics. By anisotropy in fluid mechanics we mean dependence on all components of the strain tensor, not only on the absolute value. The main motivation of the presented theory is to cover the response of the anisotropic fluids. Such fluids have one or more specific directions in which one can observe anisotropic behaviour. This phenomena could be caused by internal structure of the material or external stimulus.

Liquid crystals are the state of matter which has the properties of liquids, as well as the solid crystals. It means that they can flow like liquids, but its molecules may be oriented in a crystal-like way. On the other hand we consider materials which exhibit drastic changes in their rheological properties upon the application of an outer field (electric or magnetic). The electrorheological fluid consists of dielectric particles suspended in non-conducting oil. Dielectric in the outer electric field undergo polarization. Neighbouring dipolar particles are attracted to each other and are aligned with the lines of external field, thus producing the fibrous structures. The direction parallel to field lines is distinguished.

Anisotropic fluids are widely applicable in the common life. Most of the modern electronic displays are liquid crystal based. We can mention also magnetorheological shock absorber of buildings or in the automotive industry, magnetorheological damper and electroreheological clutch.

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One of the main aims of this paper is to collect basic definitions and theorems concerning theory of anisotropic Orlicz spaces. We present the complete proofs in order to create the comprehensive reference to considered topic.

We focus on the steady flow of non-Newtonian incompressible fluids described by the system

$$\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \operatorname{div}\boldsymbol{\Sigma}(x,\mathbb{D}(\boldsymbol{u})) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u}(x) = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a sufficiently smooth boundary $\partial\Omega$, $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity of a fluid, $\boldsymbol{f}$ given body force, $p : \Omega \to \mathbb{R}$ the pressure, $\boldsymbol{\Sigma} - \mathbb{I}p$ is the Cauchy stress tensor and $\mathbb{D}(\boldsymbol{u})$ symmetric part of velocity gradient $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$. We assume that $\boldsymbol{\Sigma}$ satisfies the following conditions

(S1) $\boldsymbol{\Sigma}$ is a Carathéodory function (i.e., measurable w.r.t. x and continuous w.r.t. the last variable).

(S2) There exists an N-function (def. 2.1) $M : \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_+$, an integrable function $k : \Omega \to \mathbb{R}_+$ and a constant $c > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$

$$\boldsymbol{\Sigma}(x,\boldsymbol{\xi}) : \boldsymbol{\xi} \geq c\left(M(x,\boldsymbol{\xi}) + M^*(x,\boldsymbol{\Sigma}(x,\boldsymbol{\xi}))\right) - k(x). \tag{4}$$

(S3) For all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym}$ and for a.a. $x \in \Omega$

$$(\boldsymbol{\Sigma}(x,\boldsymbol{\xi}) - \boldsymbol{\Sigma}(x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq 0. \tag{5}$$

By standard growth conditions we understand polynomial growth, see e.g. Ref. [8] , namely

$$\begin{aligned} |\boldsymbol{\Sigma}(x,\boldsymbol{\xi})| &\leq c(1+|\boldsymbol{\xi}|)^{q-1} \\ \boldsymbol{\Sigma}(x,\boldsymbol{\xi}) : \boldsymbol{\xi} &\geq c|\boldsymbol{\xi}|^q. \end{aligned} \tag{6}$$

We will show in Sec. 1.1 why the anisotropic Orlicz spaces are proper spaces to cover flow of non-Newtonian fluids with non-standard growth conditions of the Cauchy stress tensor.

Another field where the problem formulation in Orlicz spaces is appropriate is an inelastic deformation theory. The anisotropy of the material arises from different plastic properties in different directions. As an example we present a quasistatic system capturing the viscoplastic deformation behaviour of solids at small strain.

$$\operatorname{div}_x \boldsymbol{T}(x,t) = -\boldsymbol{F}(x,t) \quad \text{in } \Omega\times(0,T), \tag{7}$$

$$\boldsymbol{T}(x,t) = \mathcal{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}(x,t)) - \boldsymbol{\varepsilon}^p(x,t)) \quad \text{in } \Omega\times(0,T), \tag{8}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(x,t)) = \frac{1}{2}(\nabla_x\boldsymbol{u}(x,t) + \nabla_x^T\boldsymbol{u}(x,t)) \quad \text{in } \Omega\times(0,T), \tag{9}$$

$$\boldsymbol{\varepsilon}^p_t(x,t) = \mathcal{G}\left(P\boldsymbol{T}(x,t)\right) \quad \text{in } \Omega\times(0,T), \tag{10}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$, $\boldsymbol{u} : \Omega\times(0,T) \to \mathbb{R}^3$ is the displacement field, $\boldsymbol{T} : \Omega\times(0,T) \to \mathbb{R}^{3\times3}_{sym}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}^p : \Omega\times(0,T) \to \mathbb{R}^{3\times3}_{sym}$ is the inelastic deformation tensor. Moreover, the function $\boldsymbol{F} : \Omega\times(0,T) \to \mathbb{R}^3$ describes the external forces acting on the material, $\mathcal{D} : \mathbb{R}^{3\times3}_{sym} \to \mathbb{R}^{3\times3}_{sym}$ is the elasticity tensor which is assumed to be constant in time and space, symmetric and positive definite. Moreover, $\mathcal{G} : \mathbb{R}^{3\times3}_{sym} \to P\mathbb{R}^{3\times3}_{sym}$ is the inelastic constitutive function and the map P is defined by $P\boldsymbol{T} = \boldsymbol{T} - \frac{1}{3}\operatorname{tr}\boldsymbol{T}\cdot\boldsymbol{I}$. We consider system (7)–(10) with the following boundary condition of mixed type: the Dirichlet boundary condition on $\Gamma_1 \subset \partial\Omega$

$$\boldsymbol{u}(x,t) = \boldsymbol{g}_D(x,t) \ \text{ for } x \in \Gamma_1 \ \text{ and } t \geq 0 \tag{11}$$

and the Neumann boundary condition on $\Gamma_2 \subset \partial\Omega$

$$\boldsymbol{T}(x,t) \cdot \boldsymbol{n}(x) = \boldsymbol{g}_N(x,t) \ \text{ for } x \in \Gamma_2 \ \text{ and } t \geq 0 \tag{12}$$

where $\boldsymbol{n}(x)$ is the exterior unit normal vector to the boundary $\partial\Omega$ at the point x, Γ_1 and Γ_2 are open in $\partial\Omega$, disjoint, "smooth enough" sets satisfying $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\mathcal{H}_2(\Gamma_1) > 0$, where $\mathcal{H}_2$ denotes the 2-dimensional Hausdorff measure. Moreover, the functions $\boldsymbol{g}_D, \boldsymbol{g}_N$ are given boundary data. Finally, the initial condition for the inelastic strain tensor is in the form

$$\boldsymbol{\varepsilon}^p(x,0) = \boldsymbol{\varepsilon}^{p,0}(x) \tag{13}$$

with a given initial data $\boldsymbol{\varepsilon}^{p,0} : \Omega \to P\mathbb{R}^{3\times 3}_{sym}$.

We assume that $\mathcal{G}$ satisfies the following conditions

(G1) $\mathcal{G}$ is continuous and $\mathcal{G}(0) = 0$,

(G2) There exist positive constants c_1, c_2 and an $N-$function M such that for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{sym}$ it holds

$$\mathcal{G}(\boldsymbol{\xi}) : \boldsymbol{\xi} \geq c_1 \left(M(\boldsymbol{\xi}) + M^*(\mathcal{G}(\boldsymbol{\xi})) - c_2 \right.$$

(G3) $\mathcal{G}$ is strictly monotone

$$\forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{R}^{3\times 3}_{sym} \qquad \boldsymbol{\sigma}_1 \neq \boldsymbol{\sigma}_2 \ \Rightarrow \ (\mathcal{G}(\boldsymbol{\sigma}_1) - \mathcal{G}(\boldsymbol{\sigma}_2)) : (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) > 0.$$

1.1. **Constitutive theory.** To complete system (1)-(3) it is necessary to add the constitutive relation which describes the behaviour of the fluid namely the Cauchy stress tensor. In this section we will consider fluids with outer field dependence in particular electrorheological fluids. General form of the constitutive relation for electrorheological fluids, can be assumed as, cf. [12]

$$\begin{aligned} \boldsymbol{\Sigma} &= \alpha_1 \mathbb{I} + \alpha_2 \boldsymbol{E} \otimes \boldsymbol{E} + \alpha_3 \mathbb{D}(\boldsymbol{u}) + \alpha_4 \mathbb{D}(\boldsymbol{u})^2 \\ &\quad + \alpha_5 (\mathbb{D}(\boldsymbol{u}) \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \mathbb{D}(\boldsymbol{u}) \boldsymbol{E}) + \alpha_6 (\mathbb{D}(\boldsymbol{u})^2 \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \mathbb{D}(\boldsymbol{u})^2 \boldsymbol{E}), \end{aligned} \tag{14}$$

where $\boldsymbol{E}(x) : \Omega \to \mathbb{R}^d$ is a continuous function represents outer field dependence and $\alpha_i \quad i = 1, \ldots, 6$, are scalar functions of the invariants

$$|\boldsymbol{E}|^2, \ \operatorname{tr} \mathbb{D}(\boldsymbol{u}), \ \operatorname{tr} \mathbb{D}(\boldsymbol{u})^2, \ \operatorname{tr} \mathbb{D}(\boldsymbol{u})^3, \ \operatorname{tr} (\mathbb{D}(\boldsymbol{u}) \boldsymbol{E} \otimes \boldsymbol{E}), \ \operatorname{tr} (\mathbb{D}(\boldsymbol{u})^2 \boldsymbol{E} \otimes \boldsymbol{E}).$$

We consider simplified form of (14) by taking into account only two terms, namely

$$\boldsymbol{\Sigma} = \alpha_3 \mathbb{D}(\boldsymbol{u}) + \alpha_5 (\mathbb{D}(\boldsymbol{u}) \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \mathbb{D}(\boldsymbol{u}) \boldsymbol{E}), \tag{15}$$

where α_3 and α_5 are scalar functions of non-negative invariants $\operatorname{tr}\left(\mathbb{D}(\boldsymbol{u})^2\right)$ and $\operatorname{tr}\left(\mathbb{D}(\boldsymbol{u})^2 \boldsymbol{E} \otimes \boldsymbol{E}\right)$. Respectively

$$\alpha_3(\mathbb{D}(\boldsymbol{u})) = 2\beta \left(\operatorname{tr} \mathbb{D}(\boldsymbol{u})^2 \right)^{\beta - 1}, \tag{16}$$

$$\alpha_5(\mathbb{D}(\boldsymbol{u}), \boldsymbol{E}) = \gamma \left(\operatorname{tr} (\mathbb{D}(\boldsymbol{u})^2 \boldsymbol{E} \otimes \boldsymbol{E}) \right)^{\gamma - 1}, \tag{17}$$

where $\beta, \gamma \in \mathbb{R}$ and $\beta, \gamma > 1$.

Now we show that the above example of the constitutive relation satisfies conditions (S1)-(S3) and does not fulfil the standard growth conditions (6). Moreover stress tensor given by (15) is thermodynamically admissible.

The condition (S1) is obviously fulfilled. We consider the convex, N-function (def. 2.1) $\tilde{M}$ which takes as an argument symmetric part of velocity gradient and depends on x by electric field

$$\tilde{M}(x, \mathbb{D}(\boldsymbol{u})) = (\operatorname{tr}\mathbb{D}(\boldsymbol{u})^2)^{\beta} + (\operatorname{tr}(\mathbb{D}(\boldsymbol{u})^2\boldsymbol{E}(x)\otimes\boldsymbol{E}(x)))^{\gamma}.$$

The Cauchy stress $\boldsymbol{\Sigma}$ is a gradient of function $\tilde{M}$

$$\boldsymbol{\Sigma} = \nabla_{\mathbb{D}(\boldsymbol{u})}\left((\operatorname{tr}\mathbb{D}(\boldsymbol{u})^2)^{\beta} + (\operatorname{tr}(\mathbb{D}(\boldsymbol{u})^2\boldsymbol{E}(x)\otimes\boldsymbol{E}(x)))^{\gamma}\right), \tag{18}$$

which implies that Fenchel-Young inequality (rem. 1) becomes the equality, i.e.

$$\boldsymbol{\Sigma}(x, \mathbb{D}(\boldsymbol{u})) : \mathbb{D}(\boldsymbol{u}) = \tilde{M}(x, \mathbb{D}(\boldsymbol{u})) + \tilde{M}^*(x, \boldsymbol{\Sigma}(x, \mathbb{D}(\boldsymbol{u}))).$$

Monotonicity condition (S3) also follows form convexity of $\tilde{M}$ and (18)

$$(\boldsymbol{\Sigma}(\mathbb{D}(\boldsymbol{u})_1, \boldsymbol{E}) - \boldsymbol{\Sigma}(\mathbb{D}(\boldsymbol{u})_2, \boldsymbol{E})) : (\mathbb{D}(\boldsymbol{u})_1 - \mathbb{D}(\boldsymbol{u})_2) \geq 0 \tag{19}$$

for all $\mathbb{D}(\boldsymbol{u})_1, \mathbb{D}(\boldsymbol{u})_2 \in \mathbb{R}^{d\times d}_{sym}$ and $a.a.\, x \in \Omega$. More details concerning theory of convex functions can be found in [11].

Moreover we justify that given example is thermodynamically admissible, namely

$$\boldsymbol{\Sigma}(x, \mathbb{D}(\boldsymbol{u})) : \mathbb{D}(\boldsymbol{u}) \geq 0,$$

$$(\alpha_3(\mathbb{D}(\boldsymbol{u}))\mathbb{D}(\boldsymbol{u}) + \alpha_5(\mathbb{D}(\boldsymbol{u}), \boldsymbol{E})(\boldsymbol{E}\otimes\mathbb{D}(\boldsymbol{u})\boldsymbol{E} + \mathbb{D}(\boldsymbol{u})\boldsymbol{E}\otimes\boldsymbol{E})) : \mathbb{D}(\boldsymbol{u}) =$$
$$= \alpha_3(\mathbb{D}(\boldsymbol{u}))|\mathbb{D}(\boldsymbol{u})|^2 + 2\alpha_5(\mathbb{D}(\boldsymbol{u}), \boldsymbol{E})\operatorname{tr}(\mathbb{D}(\boldsymbol{u})^2\boldsymbol{E}\otimes\boldsymbol{E}) \geq 0.$$

2. **Generalized Orlicz spaces.**

Definition 2.1. Let Ω be a bounded open domain in $\mathbb{R}^d$, a function $M : \Omega\times\mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ is said to be an N-function if it satisfies the following conditions:

1. M is Carathéodory function such that $M(x, \boldsymbol{\xi}) = 0$ if and only if $\boldsymbol{\xi} = 0$, $M(x, \boldsymbol{\xi}) = M(x, -\boldsymbol{\xi})$ a.e. in Ω,
2. $M(x, \boldsymbol{\xi})$ is a convex function w.r.t $\boldsymbol{\xi}$,
3. $\lim_{|\boldsymbol{\xi}|\to 0}\sup_{x\in\Omega}\frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0$,
4. $\lim_{|\boldsymbol{\xi}|\to\infty}\inf_{x\in\Omega}\frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$.

Definition 2.2. We say that an N-function M satisfies Δ_2−condition (or equivalently is Δ_2−regular) if for some non-negative, integrable in Ω function h and a constant $k > 0$

$$M(x, 2\boldsymbol{\xi}) \leq kM(x, \boldsymbol{\xi}) + h(x) \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym} \text{ and a.a } x \in \Omega.$$

Definition 2.3. The complementary function M^* to a function M is defined by

$$M^*(x, \boldsymbol{\eta}) = \sup_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{sym}}(\boldsymbol{\xi} : \boldsymbol{\eta} - M(x, \boldsymbol{\xi}))$$

for $\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym}$, and a.a. $x \in \Omega$.

The complementary function M^* is also an N-function.

Remark 1. (Fenchel-Young inequality).
Let M be an N-function and M^* a complementary to M. Then the following inequality is satisfied

$$|\boldsymbol{\xi} : \boldsymbol{\eta}| \leq M(x, \boldsymbol{\xi}) + M^*(x, \boldsymbol{\eta}) \tag{20}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym}$ and a.a $x \in \Omega$.

Proof. The proof follows from definition 2.3 and is analogous to the isotropic case. □

Definition 2.4. The generalized Orlicz class $\mathcal{L}_M(\Omega)$ is a set of all measurable functions $\boldsymbol{\xi} : \Omega \to \mathbb{R}^{d\times d}_{sym}$ such that

$$\int_\Omega M(x, \boldsymbol{\xi})dx < \infty.$$

Definition 2.5. The generalized Orlicz space $L_M(\Omega)$ is defined as a set of all measurable functions $\boldsymbol{\xi} : \Omega \to \mathbb{R}^{d\times d}_{sym}$ which satisfy

$$\int_\Omega M(x, \lambda\boldsymbol{\xi})dx \to 0 \text{ as } \lambda \to 0.$$

Definition 2.6. Orlicz norm, for $\boldsymbol{\xi} \in \mathcal{L}_M(\Omega)$ we have

$$\|\boldsymbol{\xi}\|_M = \sup\left\{\int_\Omega |\boldsymbol{\eta} : \boldsymbol{\xi}|dx : \boldsymbol{\eta} \in \mathcal{L}_{M^*}(\Omega), \int_\Omega M^*(x, \boldsymbol{\eta})dx \le 1\right\}. \tag{21}$$

Definition 2.7. Luxemburg norm, for $\boldsymbol{\xi} \in \mathcal{L}_M(\Omega)$ we have

$$\|\boldsymbol{\xi}\|^L_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(x, \frac{\boldsymbol{\xi}}{\lambda}\right)dx \le 1\right\}. \tag{22}$$

Orlicz and Luxemburg norms are equivalent. The proof in less general case (i.e. $M(x, |\boldsymbol{\xi}|)$) can be found in [9].

Definition 2.8. By $E_M(\Omega)$ we mean the closure in $L_M(\Omega)$ of all measurable and bounded functions.

Definition 2.9. We say that a sequence $\{\boldsymbol{\xi}_i\}_{i=1}^\infty$ converges modularly to $\boldsymbol{\xi}$ in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\left(x, \frac{\boldsymbol{\xi}_i - \boldsymbol{\xi}}{\lambda}\right)dx \to 0. \tag{23}$$

We will use the notation $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ for the modular convergence in $L_M(\Omega)$.

3. Basic theorems.

Theorem 3.1. *The generalized Orlicz space is a Banach space with respect to the Orlicz norm (21) or the equivalent Luxemburg norm (22).*

Proof. We will prove the completeness w.r.t Orlicz norm. Let $\{\boldsymbol{\xi}_n\}_{n=1}^\infty$ be a Cauchy sequence in $L_M(\Omega)$ such that $\forall_{\varepsilon>0}\exists_{N_\varepsilon>0}$ holds

$$\sup\left\{\int_\Omega \boldsymbol{\eta} : (\boldsymbol{\xi}_m - \boldsymbol{\xi}_n)dx \ ; \ \boldsymbol{\eta} \in \mathcal{L}_{M^*}(\Omega), \int_\Omega M^*(x, \boldsymbol{\eta})dx \le 1\right\} < \varepsilon \quad \forall\, n, m > N_\varepsilon. \tag{24}$$

Let $\lambda > 0$ be such that

$$\int_\Omega M^*(x, \boldsymbol{\eta})dx \le 1 \text{ for all } \boldsymbol{\eta} \in L^\infty(\Omega; \mathbb{R}^{d\times d}), \|\boldsymbol{\eta}\|_\infty \le \lambda.$$

By putting

$$\boldsymbol{\eta} = \begin{cases} \lambda\frac{\boldsymbol{\xi}_m - \boldsymbol{\xi}_n}{|\boldsymbol{\xi}_m - \boldsymbol{\xi}_n|} & \text{if } \boldsymbol{\xi}_m \ne \boldsymbol{\xi}_n \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

to (24) we obtain

$$\int_\Omega |\boldsymbol{\xi}_m - \boldsymbol{\xi}_n| dx \leq \frac{\varepsilon}{\lambda} \text{ for all } m, n \geq N_\varepsilon.$$

Therefore $\{\boldsymbol{\xi}_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^1(\Omega)$. Hence, by Fatou's lemma

$$\int_\Omega |(\boldsymbol{\xi} - \boldsymbol{\xi}_n) : \boldsymbol{\eta}| dx = \int_\Omega \lim_{m\to\infty} |(\boldsymbol{\xi}_m - \boldsymbol{\xi}_n) : \boldsymbol{\eta}| dx \leq \liminf_{m\to\infty} \int_\Omega |(\boldsymbol{\xi}_m - \boldsymbol{\xi}_n) : \boldsymbol{\eta}| dx < \varepsilon.$$

Thus $\boldsymbol{\xi} \in L_M(\Omega)$ and $\|\boldsymbol{\xi} - \boldsymbol{\xi}_n\|_M \to 0$ with $n \to \infty$. This completes the proof. □

Lemma 3.2. *Generalized Hölder inequality*
Let M be an N-function and M^* its complementary, then

$$\left| \int_\Omega \boldsymbol{\xi} : \boldsymbol{\eta}\, dx \right| \leq 2\|\boldsymbol{\xi}\|_M \|\boldsymbol{\eta}\|_{M^*}, \tag{26}$$

where $\boldsymbol{\xi} \in L_M(\Omega)$ and $\boldsymbol{\eta} \in L_{M^*}(\Omega)$.

Proof. From (20) by putting $\boldsymbol{\xi} = \frac{\boldsymbol{\xi}(x)}{\|\boldsymbol{\xi}\|_M}$, $\quad \boldsymbol{\eta} = \frac{\boldsymbol{\eta}(x)}{\|\boldsymbol{\eta}\|_{M^*}}$ we obtain

$$\int_\Omega \left| \frac{\boldsymbol{\xi}(x)}{\|\boldsymbol{\xi}|_M} \frac{\boldsymbol{\eta}(x)}{\|\boldsymbol{\eta}\|_{M^*}} \right| dx \leq \int_\Omega M\left(x, \frac{\boldsymbol{\xi}(x)}{\|\boldsymbol{\xi}\|_M}\right) dx + \int_\Omega M^*\left(x, \frac{\boldsymbol{\eta}(x)}{\|\boldsymbol{\eta}\|_{M^*}}\right) dx \leq 2.$$

We finish the proof of (26) by multiplying above inequality by $\|\boldsymbol{\xi}\|_M \|\boldsymbol{\eta}\|_{M^*}$. □

Theorem 3.3. *The space $L_{M^*}(\Omega)$ is a dual space of $E_M(\Omega)$, i.e. $(E_M(\Omega))^* = L_{M^*}(\Omega)$.*

Before we prove Theorem 3.3, we will show the following lemma.

Lemma 3.4. *Let $\boldsymbol{\eta} \in L_{M^*}(\Omega)$. The linear functional $F_{\boldsymbol{\eta}}$ defined by*

$$F_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \int_\Omega \boldsymbol{\xi} : \boldsymbol{\eta}\, dx \tag{27}$$

belongs to the space $(E_M(\Omega))^*$ and its norm in that space fulfils

$$\|F_{\boldsymbol{\eta}}\| \leq 2\|\boldsymbol{\eta}\|_{M^*}. \tag{28}$$

Proof. It follows by Hölder inequality (26) that

$$|F_{\boldsymbol{\eta}}(\boldsymbol{\xi})| \leq 2\|\boldsymbol{\xi}\|_M \|\boldsymbol{\eta}\|_{M^*}$$

holds for all $\boldsymbol{\xi} \in L_M(\Omega)$ confirming the inequality (28). □

Proof. of the Theorem 3.3
Lemma 3.4 has already shown that any element $\boldsymbol{\eta} \in L_{M^*}(\Omega)$ defines a bounded linear functional $F_{\boldsymbol{\eta}}$ on $E_M(\Omega)$ which is given by (27). It remains to show that every bounded linear functional on $E_M(\Omega)$ is of the form $F_{\boldsymbol{\eta}}$ for any $\boldsymbol{\eta} \in L_{M^*}(\Omega)$.

Let $F \in (E_M(\Omega))^*$, we define a measure λ on the measurable subsets S of Ω

$$\lambda(S) = F(\boldsymbol{\tau}\mathbb{I}_S)$$

where $\mathbb{I}_S$ denotes the characteristic function of S, $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{sym}$, $|\tau| = 1$. Let

$$A(r) = \sup_{x\in\Omega, |\boldsymbol{\xi}|=r} M(x, \boldsymbol{\xi})$$

be an auxiliary function and $r \in [0, \infty)$. This function is needed to generalise the approach presented in [1]. Since

$$\int_\Omega M\left(x, A^{-1}\left(\frac{1}{|S|}\right)\mathbb{I}_S \boldsymbol{\tau}\right) dx \leq \int_S \sup_{x\in S} M\left(x, A^{-1}\left(\frac{1}{|S|}\right)\boldsymbol{\tau}\right) dx \leq \int_S \frac{1}{|S|} \leq 1,$$

we have

$$|\lambda(S)| = |F(\boldsymbol{\tau}\mathbb{I}_S)| \leq \|F\|\|\boldsymbol{\tau}\mathbb{I}_S\|_M \leq \frac{c\|F\|}{A^{-1}(1/|S|)}. \tag{29}$$

Since the right-hand side of (29) converges to zero when $|S|$ converges to zero, the measure λ is absolutely continuous w.r.t Lebesgue measure. By Radon-Nikodym and Riesz theorems, cf. [15], λ can be expressed in the form

$$\lambda(S) = \int_S \boldsymbol{\eta}(x)dx$$

for some $\boldsymbol{\eta}$ integrable on Ω. Therefore

$$F(\boldsymbol{\xi}) = \int_\Omega \boldsymbol{\xi} : \boldsymbol{\eta}\, dx$$

holds for measurable bounded functions $\boldsymbol{\xi}$.

If $\boldsymbol{\xi} \in E_M(\Omega)$ we can find a sequence of measurable functions $\boldsymbol{\xi}_i$ which converges a.e. to $\boldsymbol{\xi}$ and satisfies $|\boldsymbol{\xi}_i| \leq |\boldsymbol{\xi}|$ on Ω. Since $|\boldsymbol{\xi}_i : \boldsymbol{\eta}|$ converges a.e to $|\boldsymbol{\xi} : \boldsymbol{\eta}|$, Fatou's lemma yields

$$\begin{aligned}\left|\int_\Omega \boldsymbol{\xi} : \boldsymbol{\eta}\, dx\right| &\leq \int_\Omega |\boldsymbol{\xi} : \boldsymbol{\eta}| dx \leq \liminf_{i\to\infty} \int_\Omega |\boldsymbol{\xi}_i : \boldsymbol{\eta}| dx \\ &\leq \liminf_{i\to\infty} 2\|\boldsymbol{\xi}_i\|_M \|\boldsymbol{\eta}\|_{M^*} \leq 2\|\boldsymbol{\xi}\|_M\|\boldsymbol{\eta}\|_{M^*}.\end{aligned}$$

Hence the linear functional

$$F_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \int_\Omega \boldsymbol{\xi} : \boldsymbol{\eta}\, dx$$

is bounded on $E_M(\Omega)$ when $\boldsymbol{\eta} \in L_{M^*}(\Omega)$. Since $F_{\boldsymbol{\eta}}$ and F achieve the same values on the measurable, simple functions, a set which is dense in $E_M(\Omega)$, they agree on $E_M(\Omega)$ and the proof is completed. □

Theorem 3.5. *The space $E_M(\Omega)$ is separable.*

Proof. The theorem can be proved in two steps. First we approximate $u \in E_M(\Omega)$ by simple functions. Then a dominated convergence argument shows that the simple functions converge in norm to u in $E_M(\Omega)$. □

Theorem 3.6. *The space $L_M(\Omega)$ is separable if and only if M is Δ_2-regular.*

Proof. We begin the proof with the fact that if M is Δ_2-regular then $E_M(\Omega) = L_M(\Omega)$. Let $\boldsymbol{\tau} \in E_M(\Omega)$ and $\boldsymbol{\tau}_b$ be a bounded function on Ω such that $\|\boldsymbol{\tau} - \boldsymbol{\tau}_b\|_M < \frac{1}{2}$. Then from Δ_2-condition we obtain $M(x, 2\boldsymbol{\tau} - 2\boldsymbol{\tau}_b) \leq 2M(x, \boldsymbol{\tau} - \boldsymbol{\tau}_b) + h(x)$ what implies that $2\boldsymbol{\tau} - 2\boldsymbol{\tau}_b \in \mathcal{L}_M(\Omega)$. As $2\boldsymbol{\tau}_b \in \mathcal{L}_M(\Omega)$, convexity of $\mathcal{L}_M(\Omega)$ provides that $\boldsymbol{\xi} = \frac{1}{2}[2(\boldsymbol{\tau} - \boldsymbol{\tau}_b) + 2\boldsymbol{\tau}_b] \in \mathcal{L}_M(\Omega) \subset L_M(\Omega)$.

The second part of the proof can be shown by contradiction using Luzin's theorem, cf. [7]. And by definition of $E_M(\Omega)$ we obtain separability of $L_M(\Omega)$. □

Theorem 3.7. *$L_M(\Omega)$ is reflexive if and only if both M and M^* are Δ_2-regular.*

Proof. Form the proof of the theorem 3.6 we know that if M is Δ_2-regular then $E_M(\Omega) = L_M(\Omega)$. By theorem 3.3 $(E_M(\Omega))^* = L_{M^*}(\Omega)$, so we obtain reflexivity

$$((L_M(\Omega))^*)^* = ((E_M(\Omega))^*)^* = (L_{M^*}(\Omega))^* = (E_{M^*}(\Omega))^* = L_M(\Omega).$$

□

Lemma 3.8. *Let* $\boldsymbol{\xi}_i : \Omega \to \mathbb{R}^{d\times d}_{sym}$ *be a measurable sequence. Then* $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ *in* $L_M(\Omega)$ *modularly if and only if* $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$ *in measure and there exists some* $\lambda > 0$ *such that the sequence* $\{M(x, \lambda\boldsymbol{\xi}_i)\}_{i=1}^{\infty}$ *is uniformly integrable, i.e.,*

$$\lim_{R\to\infty}\left(\sup_{i\in\mathbb{N}}\int_{\{x:|M(x,\lambda\boldsymbol{\xi}_i)|\geq R\}} M(x,\lambda\boldsymbol{\xi}_i)dx\right) = 0.$$

For the proof see [3].

Proposition 1. *Let* M *be an* N*-function and* M^* *its complementary function. Suppose that the sequences* $\boldsymbol{\psi}^j : \Omega \to \mathbb{R}^{d\times d}$ *and* $\boldsymbol{\phi}^j : \Omega \to \mathbb{R}^{d\times d}$ *are uniformly bounded in* $L_M(\Omega)$ *and* $L_{M^*}(\Omega)$ *respectively. Moreover* $\boldsymbol{\psi}^j \xrightarrow{M} \boldsymbol{\psi}$ *modularly in* $L_M(\Omega)$ *and* $\boldsymbol{\phi}^j \xrightarrow{M^*} \boldsymbol{\phi}$ *modularly in* $L_{M^*}(\Omega)$*. Then* $\boldsymbol{\psi}^j : \boldsymbol{\phi}^j \to \boldsymbol{\psi} : \boldsymbol{\phi}$ *strongly in* $L^1(\Omega; \mathbb{R}^{d\times d}_{sym})$*.*

For the proof see [3].

In the statement of the next theorem, an $N-$function has the same properties as before, but it is defined on $\mathbb{R}_+$. To avoid confusion, we denote it with a small letter m.

Theorem 3.9. *Let* Ω *be a bounded domain with a Lipschitz boundary. Let* m *be an* N*-function satisfying* Δ_2*-condition and such that* m^γ *is quasiconvex for some* $\gamma \in (0,1)$*. Then, for any* $f \in L_m(\Omega;\mathbb{R})$ *such that*

$$\int_\Omega f dx = 0,$$

the problem of finding a vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ *such that*

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f \quad \text{in } \Omega \\ \boldsymbol{v} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{30}$$

has at least one solution $\boldsymbol{v} \in L_m(\Omega;\mathbb{R}^n)$ *and* $\nabla\boldsymbol{v} \in L_m(\Omega;\mathbb{R}^{n\times n})$*. Moreover, for some positive constant* c

$$\int_\Omega m(|\nabla\boldsymbol{v}|)dx \leq c\int_\Omega m(|f|)dx.$$

For the proof see e.g. [14].

4. **Generalized Stokes problem.** The present section is directed to the existence of weak solutions of the steady generalized Stokes problem. Omitting the convective term we can skip the assumption on the lower growth of the $N-$function. We shall also provide the proof without an assumption that M^* satisfies Δ_2-condition. Contrary to most of the results mentioned in Section 5 where shear thickening fluids were considered, here we direct our attention to shear thinning fluids. The framework is based on the closures of smooth functions with respect to various topologies. Because of the smoothing procedure we may consider the case of still anisotropic, but homogenous function $M : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$. We will consider simplification of system (1)-(3) namely the generalized Stokes system

$$-\operatorname{div} \boldsymbol{\Sigma}(x, \mathbb{D}(\boldsymbol{u})) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{31}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{32}$$

$$\boldsymbol{u}(x) = 0 \quad \text{on } \partial\Omega, \tag{33}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a sufficiently smooth boundary $\partial\Omega$, $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity of a fluid and $p : \Omega \to \mathbb{R}$ the pressure, $\boldsymbol{\Sigma} - p\mathbb{I}$ is the Cauchy stress tensor. We assume that $\boldsymbol{\Sigma}$ satisfies conditions (S1)-(S3). The main result of this section concerns existence of the weak solutions to the system (31)-(33).

4.1. **Definitions and notation.** The following notation is used

$$\mathcal{V} := \{\boldsymbol{u} \in C_c^\infty(\Omega);\ \operatorname{div} \boldsymbol{u} = 0\},$$
$$L^2_{\operatorname{div}}(\Omega) := \text{closure of } \mathcal{V} \text{ in } L^2\text{-norm},$$

where $C_c^\infty(\Omega)$ is the set of compactly supported smooth functions.

We define the closure of $C_c^\infty(\Omega)$ with respect to two topologies, namely

1. modular topology of $L_M(\Omega)$, which we denote by Y_0^M, namely

$$\begin{aligned} Y_0^M = &\{\boldsymbol{u}\ ;\ \operatorname{div} \boldsymbol{u} = 0,\ \mathbb{D}(\boldsymbol{u}) \in L_M(\Omega)\ \ \exists\ \{\boldsymbol{u}^j\}_{j=1}^\infty \subset \mathcal{V} : \\ &\mathbb{D}(\boldsymbol{u}^j) \xrightarrow{M} \mathbb{D}(\boldsymbol{u}) \text{ modularly in } L_M(\Omega)\} \end{aligned} \tag{34}$$

2. weak-star topology of $\mathcal{L}_M(\Omega)$, which we denote by Z_0^M, namely

$$\begin{aligned} Z_0^M = &\{\boldsymbol{u}\ ;\ \operatorname{div} \boldsymbol{u} = 0,\ \mathbb{D}(\boldsymbol{u}) \in L_M(\Omega)\ \ \exists\ \{\boldsymbol{u}^j\}_{j=1}^\infty \subset \mathcal{V} : \\ &\mathbb{D}(\boldsymbol{u}^j) \xrightharpoonup{*} \mathbb{D}(\boldsymbol{u}) \text{ weakly star in } L_M(\Omega)\}. \end{aligned} \tag{35}$$

Moreover, by $BD_M(\Omega)$ we denote the space of functions with symmetric gradient in $L_M(\Omega)$, namely

$$BD_M(\Omega) := \{\boldsymbol{u} \in L^1(\Omega;\mathbb{R}^d)\,;\, \mathbb{D}(\boldsymbol{u}) \in L_M(\Omega)\}.$$

Remark 2. The space $BD_M(\Omega)$ is a Banach space with a norm

$$\|\boldsymbol{u}\|_{BD_M(\Omega)} := \|\boldsymbol{u}\|_{L^1(\Omega)} + \|\mathbb{D}(\boldsymbol{u})\|_M$$

and it is a subspace of the space of bounded deformations $BD(\Omega)$

$$BD(\Omega) := \{\boldsymbol{u} \in L^1(\Omega;\mathbb{R}^d)\ ;\ [\mathbb{D}(\boldsymbol{u})]_{i,j} \in \mathcal{M}(\Omega), \text{ for } i,j = 1,\dots,d\},$$

here $\mathcal{M}(\Omega)$ denotes the space of bounded measures on Ω.

We also define the subspace and the subset of $BD_M(\Omega)$ as follows

$$BD_{M,0}(\Omega) := \{\boldsymbol{u} \in BD_M(\Omega) \mid \gamma_0(\boldsymbol{u}) = 0\},$$
$$\mathcal{BD}_{M,0}(\Omega) := \{\boldsymbol{u} \in BD_M(\Omega) \mid \mathbb{D}(\boldsymbol{u}) \in \mathcal{L}_M(\Omega) \text{ and } \gamma_0(\boldsymbol{u}) = 0\}.$$

Where according to [13, Theorem 1.1.] γ_0 is a unique continuous operator from $BD(\Omega)$ onto $L^1(\partial\Omega;\mathbb{R}^d)$ such that the generalized Green formula

$$2\int_\Omega \phi[\mathbb{D}(\boldsymbol{u})]_{i,j}dx = -\int_\Omega \left(u_j\frac{\partial\phi}{\partial x_i} + u_i\frac{\partial\phi}{\partial x_j}\right)dx + \int_{\partial\Omega} \phi\,(\gamma_0(u_i)n_j + \gamma_0(u_j)n_i)\,d\mathcal{H}^{d-1} \tag{36}$$

holds for every $\phi \in C^1(\overline{\Omega})$, where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$ and $\gamma_0(u_i)$ is the i-th component of $\gamma_0(\boldsymbol{u})$ and $\mathcal{H}^{d-1}$ is the $(d-1)-$Hausdorff measure.

The operator γ_0 is a generalization of the concept of trace in Sobolev spaces to the case of $BD(\Omega)$ space. Moreover, if $\boldsymbol{u} \in C(\overline{\Omega};\mathbb{R}^d)$, then $\gamma_0(\boldsymbol{u}) = \boldsymbol{u}|_{\partial\Omega}$. Notice that in case of $\boldsymbol{u} \in W_0^{1,1}(\Omega;\mathbb{R}^d)$ it coincides with the classical trace operator in Sobolev spaces.

By [13, Proposition 1.1.] there exists an extension operator from $BD(\Omega)$ to $BD(\mathbb{R}^d)$.

Let us define two auxiliary functions $\underline{m}$, $\overline{m}:\mathbb{R}_+\to\mathbb{R}_+$ as follows

$$\begin{aligned}\underline{m}(r) &:= \min_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{sym},|\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}),\\ \overline{m}(r) &:= \max_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{sym},|\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}).\end{aligned}\tag{37}$$

The $N-$functions $\underline{m}$ and $\overline{m}$ are defined on $\mathbb{R}_+$.

4.2. **Existence result.**

Theorem 4.1. *Let condition C1. or C2. be satisfied*

(C1) Ω is a bounded star-shaped domain,

(C2) Ω is a bounded non-star-shaped domain and

$$\overline{m}(r)\le c_m\left((\underline{m}(r))^{\frac{d}{d-1}}+1\right)\tag{38}$$

for all $r\in\mathbb{R}_+$, and $\underline{m}$ satisfies Δ_2-condition.

Let M be an N-function and $\boldsymbol{\Sigma}$ satisfy conditions (S1)-(S3). Then, for given $|\boldsymbol{f}|\in E_{\underline{m}^*}(\Omega)$ there exists $\boldsymbol{u}\in Z_0^M$ such that

$$\int_\Omega \boldsymbol{\Sigma}(x,\mathbb{D}(\boldsymbol{u})):\mathbb{D}(\boldsymbol{\varphi})dx=\int_\Omega \boldsymbol{f}\,\boldsymbol{\varphi}dx\tag{39}$$

for all $\boldsymbol{\varphi}\in\mathcal{V}$.

We will not present the complete proof of existence of the weak solutions to the boundary value problem (31)-(33). The following lemmas 4.3 and 4.4 provide the crucial step in the proof. The omitted parts follow the same steps as presented in [4].

Lemma 4.2. *Let m be an N-function and Ω be a bounded domain, $\bar{\Omega}\subset[-\frac14,\frac14]^d$, and $\boldsymbol{u}\in\mathcal{BD}_{M,0}(\Omega)$. Then*

$$\|m(|\boldsymbol{u}|)\|_{L^{\frac{d}{d-1}}(\Omega)}\le C_d\|m(|\mathbb{D}(\boldsymbol{u})|)\|_{L^1(\Omega)}.\tag{40}$$

Proof. The above variant of the Korn-Sobolev inequality was prooved in [5]. □

Lemma 4.3. *Let $M:\mathbb{R}^{d\times d}_{sym}\to\mathbb{R}_+$ be an N-function, Ω be a bounded star-shaped domain and Y_0^M, Z_0^M be the function spaces defined by (34) and (35). Then $Y_0^M=Z_0^M$.*

Moreover, if $\boldsymbol{\chi}\in\mathcal{L}_{M^*}(\Omega)$, $\boldsymbol{f}\in\mathcal{L}_{\underline{m}^*}(\Omega)$ and

$$-\operatorname{div}\boldsymbol{\chi}=\boldsymbol{f}\quad\text{in } \mathcal{D}'(\Omega),\tag{41}$$

then

$$\int_\Omega \boldsymbol{\chi}:\mathbb{D}(\boldsymbol{u})dx=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx.\tag{42}$$

Proof. The proof follows the similar lines as the proof of unsteady case from [5]. We focus only on the inclusion

$$Z_0^M\subset Y_0^M.\tag{43}$$

As the modular topology is stronger than weak-star, obviously we have $Y_0^M\subset Z_0^M$. For this purpose we want to extend $\boldsymbol{u}$ by zero outside of Ω to the whole $\mathbb{R}^d$ and then mollify it. To extend $\boldsymbol{u}$ we observe that $Z_0^M\subset BD_{M,0}(\Omega)$. Notice that $\boldsymbol{u}\in Z_0^M$ is an element of $BD(\Omega)$. We concentrate on showing that it vanishes on the boundary. Take the sequence $\{\boldsymbol{u}^k\}_{k=1}^\infty$ of compactly supported smooth functions with the properties prescribed in the definition of the space Z_0^M. By putting this sequence into (36) we get

$$2\int_\Omega \phi[\mathbb{D}(\boldsymbol{u}^k)]_{i,j}dx = -\int_\Omega \left(u_j^k\frac{\partial\phi}{\partial x_i} + u_i^k\frac{\partial\phi}{\partial x_j}\right)dx. \tag{44}$$

Now we pass to the weak-star limit in (44), it is possible, due to linearity of all terms. It implies that the boundary term vanishes. Let x_0 be a vantage point of Ω and $\lambda\in(0,1)$. We define auxiliary function $\boldsymbol{v}^\lambda(x)$ as follows

$$\boldsymbol{v}^\lambda(x) := \boldsymbol{v}(\lambda(x-x_0)+x_0). \tag{45}$$

Let $\varepsilon_\lambda = \frac{1}{2}\mathrm{dist}\,(\partial\Omega,\lambda\Omega)$ where $\lambda\Omega := \{y=\lambda(x-x_0)+x_0 \mid x\in\Omega\}$. Define then

$$\boldsymbol{u}^{\lambda,\varepsilon}(x) := \boldsymbol{\varrho}_\varepsilon * \boldsymbol{u}^\lambda(x) \tag{46}$$

where $\boldsymbol{\varrho}_\varepsilon = \frac{1}{\varepsilon^d}\boldsymbol{\varrho}(\frac{x}{\varepsilon})$ is a standard regularizing kernel on $\mathbb{R}^d$ (i.e. $\boldsymbol{\varrho}\in C^\infty(\mathbb{R}^d)$, $\boldsymbol{\varrho}$ has a compact support in $B(0,1)$ and $\int_{\mathbb{R}^d}\boldsymbol{\varrho}(x)dx = 1, \boldsymbol{\varrho}(x)=\boldsymbol{\varrho}(-x)$) and the convolution is done w.r.t. space variable x, $\varepsilon<\frac{\varepsilon_\lambda}{2}$. Note that $\boldsymbol{u}^{\lambda,\varepsilon}$ also vanishes on the boundary.

Now we pass to the limit with $\varepsilon\to 0$ and hence $\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}) \xrightarrow{\varepsilon\to 0} \mathbb{D}(\boldsymbol{u}^\lambda)$ in $L^1(\Omega;\mathbb{R}^{d\times d})$. The function $\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon})\in L^1(\Omega;\mathbb{R}^{d\times d})$ and $\boldsymbol{\varrho}_\varepsilon * \mathbb{D}(\boldsymbol{u}^\lambda) \xrightarrow{\varepsilon\to 0} \mathbb{D}(\boldsymbol{u}^\lambda)$ in $L^1(\Omega;\mathbb{R}^{d\times d})$ and hence $\boldsymbol{\varrho}_\varepsilon * \mathbb{D}(\boldsymbol{u}^\lambda) \xrightarrow{\varepsilon\to 0} \mathbb{D}(\boldsymbol{u}^\lambda)$ in measure on the set Ω.

By the analogous argumentation as in the proof of Lemma 4.2, we show the uniform integrability of $\{M(\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}))\}_{\varepsilon>0}$.

The convexity of M implies that for all $\theta>0$ the following inequality holds

$$\int_\Omega \left|M(\mathbb{D}(\boldsymbol{u}^\lambda)) - \frac{1}{\sqrt{\theta}}\right|_+ dx \geq \int_\Omega \left|M(\boldsymbol{\varrho}_\varepsilon * \mathbb{D}(\boldsymbol{u}^\lambda)) - \frac{1}{\sqrt{\theta}}\right|_+ dx. \tag{47}$$

Since $\beta\mathbb{D}(\boldsymbol{u}^\lambda)\in\mathcal{L}_M(\Omega)$ for some $\beta>0$, then also $\int_\Omega |M(x,\beta\mathbb{D}(\boldsymbol{u}^\lambda)) - \frac{1}{\sqrt{\theta}}|_+dx$ is finite. Hence taking supremum over $\varepsilon\in(0,\frac{\varepsilon_\lambda}{2})$ in (47) provides the uniform integrability of the sequence $\{M(\beta\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}))\}_{\varepsilon>0}$.

Modular convergence in $L_M(\Omega)$ of $\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}) \xrightarrow{\varepsilon\to 0} \mathbb{D}(\boldsymbol{u}^\lambda)$ is provided by lemma 3.8 We pass to the lmit with $\lambda\to 1$ and obtain that $\mathbb{D}(\boldsymbol{u}^\lambda) \xrightarrow{\lambda\to 1} \mathbb{D}(\boldsymbol{u})$ in $L^1(\Omega;\mathbb{R}^{d\times d})$ and $\mathbb{D}(\boldsymbol{u}^\lambda) \xrightarrow{\lambda\to 1} \mathbb{D}(\boldsymbol{u})$ modularly in $L_M(\Omega)$. Consequently $Y_0^M = Z_0^M$, which completes the first part of the proof.

It remains to prove (41). We define

$$\boldsymbol{u}^{\lambda,\varepsilon}(x) := \boldsymbol{\varrho}_\varepsilon * \boldsymbol{u}^\lambda(x) \tag{48}$$

where $\varepsilon<\frac{\varepsilon_\lambda}{2}$. We test equation (41) with sufficiently regular test function $\boldsymbol{u}^{\lambda,\varepsilon}$

$$\int_\Omega \boldsymbol{\chi} : \mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{\lambda,\varepsilon}dx. \tag{49}$$

To treat the left-hand side of (49) we follow the ideas presented in the first part of the proof. For proving the convergence of the term $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{\lambda,\varepsilon}dx$ we apply Lemma 4.2 to $\underline{m}$ and obtain

$$\left(\int_\Omega (\underline{m}(|\boldsymbol{u}^{\lambda,\varepsilon}|))^{\frac{d}{d-1}}dx\right)^{\frac{d-1}{d}} \leq C_d\int_\Omega \underline{m}(|\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon})|)dx.$$

Moreover using the Hölder inequality and the definition of $\underline{m}$ we get

$$\int_\Omega (\underline{m}(|\boldsymbol{u}^{\lambda,\varepsilon}|))dx \le C_{\Omega,d}\int_\Omega (\underline{m}(|\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon})|)dx \le C_{\Omega,d}\int_\Omega M(\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}))dx. \tag{50}$$

Hence (50) provides that modular convergences $\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon}) \xrightarrow{\varepsilon\to 0} \mathbb{D}(\boldsymbol{u}^{\lambda})$, $\mathbb{D}(\boldsymbol{u}^{\lambda}) \xrightarrow{\lambda\to 1} \mathbb{D}(\boldsymbol{u})$ in $L_M(\Omega)$ imply that $\boldsymbol{u}^{\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{u}^{\lambda}$, $\boldsymbol{u}^{\lambda} \xrightarrow{\lambda\to 1} \boldsymbol{u}$ modularly in $L_{\underline{m}}(\Omega)$. Using Proposition 1 for N-functions $\underline{m}^*$ and $\underline{m}$ we obtain

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{\lambda,\varepsilon}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx.$$

Analogously Proposition 1 for N-functions M and M^* provides the convergence

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_\Omega \boldsymbol{\chi}:\mathbb{D}(\boldsymbol{u}^{\lambda,\varepsilon})dx = \int_\Omega \boldsymbol{\chi}:\mathbb{D}(\boldsymbol{u})dx.$$

By passing to the limit with ε, λ in (49) we obtain that

$$\int_\Omega \boldsymbol{\chi}:\mathbb{D}(\boldsymbol{u})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx. \tag{51}$$

□

In the case of non-star-shaped the additional condition is assumed. We control the anisotropy in terms of the spread between auxiliary functions $\underline{m}(r)$ and $\overline{m}(r)$.

Lemma 4.4. *Let M be an N-function such that $\overline{m}(r) \le c_m((\underline{m}(r))^{\frac{d}{d-1}}+1)$ for $r \in \mathbb{R}_+$ and let $\underline{m}$ satisfy Δ_2-condition. Let Ω be a bounded domain with a sufficiently smooth boundary, Y_0^M, Z_0^M be the function spaces defined by (34) and (35). Then $Y_0^M = Z_0^M$.*

Moreover, let $\boldsymbol{\chi} \in \mathcal{L}_{M^*}(\Omega)$, $|\boldsymbol{f}| \in \mathcal{L}_{\underline{m}^*}(\Omega)$ and

$$-\operatorname{div}\boldsymbol{\chi} = \boldsymbol{f} \quad \text{in } \mathcal{D}'(\Omega). \tag{52}$$

Then

$$\int_\Omega \boldsymbol{\chi}:\mathbb{D}(\boldsymbol{u})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx \tag{53}$$

holds.

Proof. If Ω is a Lipschitz domain, then there exists a finite family of star-shaped domains $\{\Omega_i\}_{i\in J}$ such that

$$\Omega = \bigcup_{i\in J}\Omega_i$$

see e.g. [10]. We introduce the partition of unity θ_i with $0 \le \theta_i \le 1$, $\theta_i \in C_c^\infty(\Omega_i)$, $\operatorname{supp}\theta_i = \Omega_i, \sum_{i\in J}\theta_i(x) = 1$ for $x \in \Omega$. By Lemma 4.2 applied to $\underline{m}$ and function $(\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}$ and the definition of $\underline{m}$ we obtain

$$\begin{aligned}\int_\Omega (\underline{m}(|(\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}|))^{\frac{d}{d-1}}dx &\le C_d\left(\int_\Omega \underline{m}(|\mathbb{D}\left((\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}\right)|)dx\right)^{\frac{d}{d-1}}\\ &\le C_d\left(\int_\Omega M(\mathbb{D}\left((\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}\right)dx)^{\frac{d}{d-1}}\right).\end{aligned}$$

Since $\overline{m}(r) \le c_m((\underline{m}(r))^{\frac{d}{d-1}}+1)$, $\nabla\theta \in L^\infty(\Omega;\mathbb{R}^d)$ and

$$(\mathbb{D}(\boldsymbol{u}^\lambda)\theta_i^\lambda)^\varepsilon + \frac{1}{2}(\boldsymbol{u}\otimes\nabla\theta_i)^{\lambda,\varepsilon} + \frac{1}{2}(\nabla\theta_i\otimes\boldsymbol{u})^{\lambda,\varepsilon} = \mathbb{D}\left((\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}\right),$$

where $\Omega_i = \operatorname{supp}\,\theta_i$, we conclude that $\mathbb{D}\left((\boldsymbol{u}\theta_i)^{\lambda,\varepsilon}\right) \in L_M(\Omega_i)$.

We concentrate on the function $\sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\ \theta_i\}^\lambda$, where $\{\cdot\}^\lambda$ is defined by (45). To solve the problem that $\sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\ \theta_i\}^\lambda$ may be not divergence-free, we introduce the function $\boldsymbol{\varphi}^{\lambda,\varepsilon} \in L_{\underline{m}^{\frac{d}{d-1}}}(\Omega)$ which is a solution to the problem

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi}^{\lambda,\varepsilon} &= \sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\cdot\nabla\theta_i\}^\lambda \quad \text{in } \Omega,\\ \boldsymbol{\varphi}^{\lambda,\varepsilon} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{54}$$

The existence of such $\boldsymbol{\varphi}^{\lambda,\varepsilon}$ is provided by Theorem 3.9 applied to the N-function $\underline{m}^{\frac{d}{d-1}}$ which satisfies Δ_2-condition. The function $\underline{m}^\gamma$ with $\gamma = \frac{d-1}{d}$ is quasiconvex. Then we complete the proof in a similar way as in the case of star-shaped domains. Instead of the sequence defined by (46), we take

$$\boldsymbol{\psi}^{\lambda,\varepsilon}(x) := \sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\theta_i\}^\lambda - \boldsymbol{\varphi}^{\lambda,\varepsilon}(x).$$

It remains to show that $\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ with $\lambda \to 1$ and $\varepsilon \to 0$. Indeed, Theorem 3.9 implies that

$$\begin{aligned} \int_\Omega \underline{m}^{\frac{d}{d-1}}(|\mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon})|)dx &\le \int_\Omega \underline{m}^{\frac{d}{d-1}}(|\nabla\boldsymbol{\varphi}^{\lambda,\varepsilon}|)dx\\ &\le c\int_\Omega \underline{m}^{\frac{d}{d-1}}\left(\left|\sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\cdot\nabla\theta_i\}^\lambda\right|\right). \end{aligned} \tag{55}$$

Since for every $i \in J$ the sequence

$$\boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\ \cdot\nabla\theta_i\}^\lambda \xrightarrow{\underline{m}^{\frac{d}{d-1}}} \boldsymbol{u}\cdot\nabla\theta_i \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(\Omega) \tag{56}$$

as $\varepsilon \to 0$ and $\lambda \to 1$ and $\sum_{i\in J} \boldsymbol{u}\ \cdot\nabla\theta_i = 0$, we conclude that

$$\sum_{i\in J} \boldsymbol{\varrho}_\varepsilon * \{\boldsymbol{u}\cdot\nabla\theta_i\}^\lambda \xrightarrow{\underline{m}^{\frac{d}{d-1}}} 0 \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(\Omega) \tag{57}$$

as $\varepsilon \to 0$ and $\lambda \to 1$. Therefore

$$\mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon}) \xrightarrow{\underline{m}^{\frac{d}{d-1}}} 0 \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(\Omega). \tag{58}$$

Again, following the similar lines as the star-shape case, instead of the function defined by (48), we test (52) with

$$\boldsymbol{\zeta}^{\lambda,\varepsilon}(x) := \sum_{i\in J} \varrho_\varepsilon * \{\boldsymbol{u}\,\theta_i\}^\lambda - \boldsymbol{\varphi}^{\lambda,\varepsilon}(x). \tag{59}$$

Passing to the limit with $\lambda \to 1$ and $\varepsilon \to 0$ in (49) it again remains to show that all the terms related with function $\boldsymbol{\varphi}^{\lambda,\varepsilon}$ vanish in the limit, namely

$$\begin{aligned} \lim_{\varepsilon\to 0,\lambda\to 1} \int_\Omega \boldsymbol{\chi} : \mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon})\ dx = 0,\\ \lim_{\varepsilon\to 0,\lambda\to 1} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}^{\lambda,\varepsilon}\ dx = 0. \end{aligned} \tag{60}$$

Since $\mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon}) \xrightarrow{\underline{m}^{\frac{d}{d-1}}} 0$ (see (58)), $\overline{m}(r) \le c_m((\underline{m}(r))^{\frac{d}{d-1}} + 1)$, then $M(\alpha\mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon}))$ is uniformly integrable with some $\alpha > 0$. Moreover, by Lemma 3.8 the modular convergence in $L_{\underline{m}^{\frac{d}{d-1}}}(\Omega)$ to zero implies the convergence in measure to zero. Consequently by Lemma 3.8 applied to a function M we obtain that $\mathbb{D}(\boldsymbol{\varphi}^{\lambda,\varepsilon}) \to 0$ modularly in $L_M(\Omega)$ as $\varepsilon \to 0$ and $\lambda \to 1$. Therefore $(60)_1$ holds.

The convergence passage in $(60)_2$ is a simple consequence of (58). Finally we conclude that $\nabla\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ modularly in $L_{\underline{m}}(\Omega)$ and since $\boldsymbol{\varphi} = 0$ on $\partial\Omega$ we obtain $\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ modularly in $L_{\underline{m}}(\Omega;\mathbb{R}^n)$ as $\varepsilon \to 0$ and $\lambda \to 1$. To complete the proof we follow the case of star-shaped domains. □

5. **Overview of recent results.** We provide an overview of recent results concerning applications of anisotropic Orlicz-Musielak spaces.

The system (1)-(3) with assumptions of type (S1)–(S3) has been extensively studied both for steady and unsteady flows. The convective term $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u})$ imposes always some condition for the lower growth of the function M. We formulate it as a consecutive assumption.

(S4) Let an N-function M satisfy for some $c > 0$ and $q \ge \frac{3d+2}{d+2}$ the condition
$$M(x,\boldsymbol{\xi}) \ge c|\boldsymbol{\xi}|^q \tag{61}$$
and M^* satisfies Δ_2-condition.

First results in this field are related with the strict monotonicity condition in the place of monotonicity condition (S3), namely

(S3') For all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym}, \boldsymbol{\xi} \ne \boldsymbol{\eta}$ and for a.a. $x \in \Omega$
$$(\boldsymbol{\Sigma}(x,\boldsymbol{\xi}) - \boldsymbol{\Sigma}(x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0. \tag{62}$$

In this case the following facts on existence of weak solutions has been proved, cf. [4, Thm. 1.1]

Theorem 5.1. *Let f be in the form $\boldsymbol{f} = \operatorname{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \mathbb{R}^{d\times d}_{sym}$ and $\boldsymbol{F} \in L_{M^*}(\Omega)$. Moreover, let $\boldsymbol{T}$ satisfy (S1), (S2), (S3'), (S4). Then there exists a weak solution to (1)-(3), namely $\boldsymbol{v} \in L^2_{\operatorname{div}}(\Omega)$, $\mathbb{D}(\boldsymbol{v}) \in L_M(\Omega)$ and the following is satisfied for all $\boldsymbol{\varphi} \in \mathcal{V}$*
$$\int_\Omega (\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi} + S(x,\mathbb{D}(\boldsymbol{v})) : \mathbb{D}(\boldsymbol{\varphi}))\,dx = -\langle \boldsymbol{F}, \mathbb{D}(\boldsymbol{\varphi})\rangle_M. \tag{63}$$

An analogue for the unsteady case was proved in [3, Thm. 1.1]. We formulate the problem and state the existence theorem below
$$\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \operatorname{div}\boldsymbol{\Sigma}(x,\mathbb{D}(\boldsymbol{u})) + \nabla p = \boldsymbol{f} \quad \text{in } (0,T)\times\Omega, \tag{64}$$
$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega, \tag{65}$$
$$\boldsymbol{u}(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{66}$$
$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{67}$$

Theorem 5.2. *Let (S1), (S2), (S3') and (S4) be satisfied. Given $\boldsymbol{f} \in W^{-1,q'}(Q)$ and $v_0 \in L^2_{\operatorname{div}}(\Omega)$ there exists a weak solution to (64) -(67), that is to say $\boldsymbol{v} \in L^\infty(0,T;L^2_{\operatorname{div}}(\Omega)) \cap L^q(0,T;W^{1,q}_0(\Omega))$, $\mathbb{D}(\boldsymbol{v}) \in L_M(Q)$ and the following is satisfied for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V}(\Omega))$*
$$\int_Q (-\boldsymbol{v}\boldsymbol{\varphi}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi} + \boldsymbol{S}(x,\mathbb{D}(\boldsymbol{v})) : \mathbb{D}(\boldsymbol{\varphi}))\,dxdt + \int_\Omega \boldsymbol{v}_0\boldsymbol{\varphi}dx = \int_Q \boldsymbol{f}\boldsymbol{\varphi}dxdt. \tag{68}$$

In the unsteady case the Orlicz spaces are understood as the spaces defined on time-space cylinder $Q := (0,T) \times \Omega$ with $x-$dependent function M, namely by an Orlicz class we mean the set of all measurable functions $\boldsymbol{\xi} : (0,T) \times \Omega \to \mathbb{R}^{d\times d}_{sym}$ such that

$$\int_{(0,T)\times\Omega} M(x,\boldsymbol{\xi})dxdt < \infty.$$

The remaining definitions of Orlicz spaces, norms, modular convergence, etc. are formulated analogously.

The existence proof in both of the mentioned papers used the Young measures tools what required strict monotonicity for showing the reduction of the Young measure to the Dirac measure. This restriction was abandoned in [6] for unsteady and in [16] for steady case, where the authors used generalization of Minty trick for non-reflexive spaces.

An analogous formulation was considered for Stokes problem in unsteady case. We recall here from [5]

Theorem 5.3. *Let condition C1. or C2. be satisfied*

(C1) Ω is a bounded star-shaped domain,

(C2) Ω is a bounded non-star-shaped domain and

$$\overline{m}(r) \le c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1) \tag{69}$$

for all $r \in \mathbb{R}_+$, and $\underline{m}$ satisfies Δ_2-condition.

Let M be an N-function and $\boldsymbol{S}$ satisfy conditions (S1)-(S3). Then, for given $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ and $f \in E_{\underline{m}^*}(Q;\mathbb{R}^d)$ there exists $\boldsymbol{u} \in Z^M_0$ such that

$$\int_Q -\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \boldsymbol{S}(t,x,\mathbb{D}(\boldsymbol{u})) : \mathbb{D}(\boldsymbol{\varphi})dxdt = \int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}dxdt - \int_\Omega \boldsymbol{u}_0\boldsymbol{\varphi}(0)dx \tag{70}$$

for all $\boldsymbol{\varphi} \in C^\infty_c(-\infty,T;\mathcal{V})$.

In the above unsteady setting the meaning of the space Z^M_0 differs from (35), namely

$$\begin{aligned} Z^M_0 =&\{\boldsymbol{u} \in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)),\, \mathbb{D}(\boldsymbol{u}) \in L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid \\ &\exists\ \{\boldsymbol{u}^j\}_{j=1}^\infty \subset C^\infty_c((-\infty,T);\mathcal{V}) : \boldsymbol{u}^j \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \\ &\text{and } \mathbb{D}(\boldsymbol{u}^j) \overset{*}{\rightharpoonup} \mathbb{D}(\boldsymbol{u}) \text{ weakly star in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\}. \end{aligned} \tag{71}$$

The studies of the non-homogeneous case with the stress tensor depending on the density can be found in [17], where the existence of weak solutions to the following problem is proved

$$\rho_t + \mathrm{div}\,(\rho\boldsymbol{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{72}$$

$$(\rho\boldsymbol{u})_t + \mathrm{div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \mathrm{div}\,\boldsymbol{\Sigma}(t,x,\rho,\mathbb{D}(\boldsymbol{u})) + \nabla p = \rho\boldsymbol{f} \quad \text{in } (0,T)\times\Omega, \tag{73}$$

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega, \tag{74}$$

$$\boldsymbol{u}(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{75}$$

$$\rho(0,x) = \rho_0,\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{76}$$

We complete this section by recalling the result on existence of weak solutions to the quasistatic system describing viscoplastic deformation behaviour of solids at small strain, namely the system (7)–(8). For simplicity we provide the statement of the theorem in the case of homogenous boundary conditions. Moreover, let $\boldsymbol{F} = 0$, which immediately provides that the so-called save load conditions are satisfied ([2, Def. 3.2]).

Theorem 5.4. *Let (G1)-(G3) be satisfied and let* M^* *satisfy* Δ_2-*condtion. Given* $\varepsilon^{p,0} \in L^2(\Omega; P\mathbb{R}^{3\times 3}_{sym})$ *there exists a weak solution to* (7)–(8).

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