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Abstract

We deal with a modification of the Savage-Hutter model describing granular material with multivalued friction force. The article examines equivalence between three types of solutions (entropy, measure-valued and kinetic). The multivalued structure of the system requires a modification of known definitions. It follows also that the obtained results are slightly different than for the classical system.

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1 Introduction

Physical phenomena are rigorously described with the mathematical language of differential equations. This language is natural for continuous processes. However, in real life applications we often encounter discontinuous phenomena. To find appropriate mathematical description of such laws it is necessary to consider discontinuous functions. We are interested in phenomena that are modeled by hyperbolic conservation laws with discontinuous terms.

The problem is to develop the proper mathematical theory to deal with discontinuous terms in PDE. One of the possible ways is to treat a given function as a monotonic graph and then extend it to a continuous graph. In that way we obtain a continuous mapping, which is not a function, but a multifunction. For example in [5] the authors consider scalar hyperbolic laws with discontinuous flux term and prove the existence and uniqueness of entropy weak solution. On the other hand in [11] such result for scalar first order hyperbolic equations with monotone discontinuous right hand side has been shown.

We consider a modification of the Savage - Hutter model of granular flow:

$$\begin{cases} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hv) = 0, \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \frac{1}{2}\beta h^2) \in h\bar{g}, \end{cases} \tag{1}$$

where (h, v) are functions depending on x, t and $\bar{g}$ is a given multifunction. The issue of definition of solutions to problem (1) is nontrivial due to the presence of a multifunction. In [9] a notation of weak entropy solutions was introduced and their existence was shown using viscous approximation method. The aim of this paper is to introduce new definitions (kinetic and measure-valued) and show relations between different types of solutions.

The first new type are solutions of kinetic problem. The theory of kinetic formulation for hyperbolic conservation laws was created at the end of twentieth century and summarized in [16]. We use those formalism to create the definition of solutions to kinetic problem for the granular flow. We generalize the definition of kinetic formulation for Euler model of gas motion as it is similar to the model that we use, but possess a zero right-hand side. Then Theorem 4.2 shows that the weak entropy solutions and solutions to the kinetic formulation are equivalent.

The last type are measure-valued solutions. Here we adapt a 2D definition from [10], where the existence was shown. Then we take the proper measures and obtain weak entropy solutions. In the classical theory to obtain weak entropy solutions from measure-valued formulation it is necessary to take the family of Dirac measures. In our situation the problem appears in the point when $u_2 = 0$. We require that the particular Young measure has to be Dirac delta in the point where right-hand side of second equation is a singular valued. In other points the formula for that measure is more complicated. We prove in Theorem 5.1 that if the family of Young measures is a.e. Dirac measures than the measure-valued solutions are also weak entropy solutions.

The main results of the paper are Theorems 4.2 and 5.1 where we show interactions between different definitions of solutions.

The paper is organized as follows. In section 2 we introduce the model of granular flow and the notation that we use. In section 3 we present definitions of three types of solutions. Section 4 proves that weak entropy solutions and solutions to the kinetic problem are equivalent (Theorem 4.2). In section 5, Theorem 5.1 shows how to obtain weak entropy solutions from measure-valued problem.

2 The model of granular flow

We would like to model avalanche motion using conservation laws [1, 9, 11, 17]. We treat snow as a granulate. We are looking for the height of snow layer $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and the velocity of snow: $v\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We assume that h, v depends on time and that the granulate is moving downhill. For the sake of simplicity we assume that the snow layer and the hill are infinitely long and invariant to the translation along vector field orthogonal to the direction of the gravity force. That is why it is possible to reduce the two-dimensional model to one-dimensional situation. It leads us to the model from [17], where using hyperbolic conservation laws, the avalanche motion is described with following system:

$$\begin{cases} \frac{\partial}{\partial t} h + \frac{\partial}{\partial x}(hv) = 0, \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \frac{1}{2}\beta h^2) = hg, \end{cases} \tag{2}$$

where β is the coefficient depending on the angle between the curve of the hill and horizontal direction. The other simplification is the assumption that β is greater than zero and that it is constant. It means that the hill has the constant slope. An easy observation is that the variables

(h,v) can be scaled in such a way that the coefficient β vanishes. The function $g(v,x)$ is defined as:

$$g(v,x) = \sin(\gamma) - \mathrm{sgn}_{\mathrm{M0}}(v)\cos(\gamma)\tan(\delta_F(x)), \tag{3}$$

where $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ is the angle between the horizontal direction and the curve of the hill. The function $\delta_F(x)$ describes the friction between the base and the granulate. It is equal to the critical angle of slope where the granulate starts to move which depends on the type of the base. That is why it change with x despite the fact that γ is constant. We also assume that $\delta_F(x) \in \mathbb{C}^1(\mathbb{R})$, and function $\mathrm{sgn}_{\mathrm{M0}}$ is given by:

$$\mathrm{sgn}_{\mathrm{M0}}(v) = \begin{cases} 0 & \text{for} \quad v = 0, \\ \frac{v}{|v|} & \text{for} \quad v \neq 0. \end{cases}$$

The main problem is the discontinuity of the function g in $v = 0$. That is why in [9] instead of the function g the multifunction was considered:

$$\bar{g}(v,x) = \sin(\gamma(x)) - \mathrm{sgn}_{\mathrm{M}}(v)\cos(\gamma(x))\tan(\delta_F(x)),$$

where

$$\mathrm{sgn}_{\mathrm{M}}(v) = \begin{cases} [-1,1] & \text{for} \quad v = 0, \\ \frac{v}{|v|} & \text{for} \quad v \neq 0. \end{cases}$$

Moreover we assume that the sign of $\bar{g}$ is opposite to the sign of velocity v.

After introducing new variables: $u = (u_1, u_2) = (h, hv)$ we obtain:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \bar{G}(u,x), \tag{4}$$

where

$$F(u) = \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + \frac{1}{2}u_1^2 \end{pmatrix}, \quad \text{and} \quad \bar{G}(u,x) = \begin{pmatrix} 0 \\ u_1\bar{g}(\frac{u_2}{u_1}, x) \end{pmatrix}.$$

In our analysis we will use both types of variables: (h,v) and (u_1,u_2). Note that the transformation from (u_1,u_2) into (h,v) is not well defined when $u_1 = 0$. This point is called vacuum.

3 Definitions of solutions

We deal with different types of solutions of equations (4). They are: the weak entropy solutions, the solutions to the kinetic problem and the measure-valued solutions. The definitions below are modifications of standard ones due to occurrence of the multifunction in (4).

3.1 The weak entropy solutions

The first type of solutions are weak solutions that fulfil additional inequality for all entropy-flux pairs. In the theory of scalar conservation laws the additional entropy inequality is required to obtain uniqueness of weak solutions. In our situation, due to the appearance of a multifunction, it is not clear if we get uniqueness even after considering weak entropy solutions. However, there is a hope that after assuming additional time regularity we obtain uniqueness. For example in [12], with this assumption, the authors proved the uniqueness of weak entropy solutions in the case of a modification of the Savage - Hutter model with nonconstant density.

Now let us recall the definition of entropy-flux pair.

Definition 3.1 *Let $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ be scalar functions in $\mathbb{C}^1(\Omega)$ such that:*

$$\nabla_{(u_1,u_2)}\eta(u_1,u_2) \cdot \nabla_{(u_1,u_2)}F(u_1,u_2) = \nabla_{(u_1,u_2)}q(u_1,u_2).$$

Then functions η, q are called entropy-flux pair. Moreover η is called the convex entropy if function η is convex.

Note that we define entropy-flux pair in the variables (u_1, u_2) which are connected with the height and the momentum of granulat. The problem is when $u_1 = 0$, where the vacuum appears. That is why instead of full family of entropies we use the family of the weak entropies (see [16]). In our case we require that for $u_1 \to 0$ the entropy is going to zero.

Definition 3.2 *The pair (u, G) such that: $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$, $G(t,x) \in \mathbb{L}^\infty_{loc}([0,T) \times \mathbb{R}; \mathbb{R}^2)$, and $G(t,x) \in \bar{G}(u(t,x),x)$ for almost all $(t,x) \in [0,T) \times \mathbb{R}$, is called the weak entropy solution of equations (4) with initial data $u^0 \in \mathbb{C}^0(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ if and only if u is a weak solution, it means:*

$$\int_{\mathbb{R}\times[0,T)} (u(t,x) \cdot \frac{\partial}{\partial t}\psi(t,x) + F(u(t,x)) \cdot \frac{\partial}{\partial x}\psi(t,x) + G(t,x) \cdot \psi(t,x))dtdx = \int_{\mathbb{R}} u^0(x) \cdot \psi(0,x)dx$$

for all test functions $\psi \in \mathbb{C}^1_0([0,T) \times \mathbb{R}; \mathbb{R}^2)$, and the entropy inequality holds:

$$\int_{\mathbb{R}\times[0,T)} (\eta(u(t,x))\frac{\partial}{\partial t}\Phi(t,x) + q(u(x,t))\frac{\partial}{\partial x}\Phi(t,x) + \nabla_u\eta(u(t,x)) \cdot G(t,x)\Phi(t,x))dtdx$$

$$\geq \int_{\mathbb{R}} \eta(u^0(x)\Phi(0,x))dx$$

for all nonnegative test functions: $\Phi \in \mathbb{C}^1_0([0,T) \times \mathbb{R}; \mathbb{R}^2)$ and for all convex entropy-flux pairs. We also require that $\nabla_u\eta(u) \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}^2)$, $\eta(\cdot,\cdot)$ is $\mathbb{C}^2$ function for $(u_1,u_2) \in (0,\infty) \times \mathbb{R}$. We extend the entropy by putting $\eta(0,\cdot) = 0$.

3.2 Kinetic formulation

The next type of solutions is related to the kinetic problem.

Definition 3.3 *The triple (h, v, G) such that: $(h,v) \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$, $G(t,x) \in \mathbb{L}^\infty_{loc}([0,T) \times \mathbb{R}; \mathbb{R}^2)$, and $G(t,x) \in \bar{G}(t,x)$ for almost all $(t,x) \in [0,T) \times \mathbb{R}$, is the kinetic solution of equations (4) with initial data $(h^0, v^0) \in \mathbb{C}^0(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ if and only if the following conditions hold:*

- *(h,v) has a finite energy, it is: for all $t \geq 0$, $h(t)$ is nonnegative and $h, \eta_E \in \mathbb{L}^\infty(\mathbb{R}^+; \mathbb{L}^1(\mathbb{R}))$, where η_E is the entropy given by: $\eta_E = \frac{1}{2}(h^2 + hv^2)$;*
- *there exists a bounded, non positive measure $m(t,x,\xi)$ such that for (h,v) the following equation:*

$$\frac{\partial}{\partial t}\chi(h, v-\xi) + \frac{1}{2}\frac{\partial}{\partial x}(\xi + v)\chi(h, v-\xi) - \nabla_{(h,hv)}\chi(h, v-\xi) \cdot G(t,x) = \frac{\partial^2}{\partial \xi^2}m(t,x,\xi) \quad (5)$$

  holds in $\mathbb{D}'(\mathbb{R}_+, \mathbb{R}^2)$. The function χ is given by:
  $\chi(h,v) = \alpha(4h - v^2)_+^{\frac{1}{2}}$, for $\alpha = \frac{1}{4}(\int_{\mathbb{R}}(1-\omega^2)_+^{\frac{1}{2}}d\omega)^{-1}$;

- *the measure m is bounded by:*

$$-\int_0^T \int_{\mathbb{R}\times\mathbb{R}} m(t,x,\xi)d\xi dxdt \leq \int_{\mathbb{R}} \eta_E(h^0,v^0)dx.$$

3.3 The measure-valued solutions

The last case concerns the measure-valued solutions. Usually they are introduced as the limit to viscous approximation. In our case the problem appears in the point where right-hand side is not continuous (this point we treat as multivalued). That is why after using Young Measure Theorem to viscous approximation we obtain only the existence of limit, but we cannot characterise it. However it is possible to treat the multifunction as the monotone graph and transform it to a continuous function. Then it is possible to make the limit passage and obtain the existence of solution (and characterisation of it) but in the language of this transformation. That point of view was used in [10] in the case of two dimensional situation. We adapt their definition to one-dimensional situation.

Definition 3.4 *The family of probabilistic measures: $\mu_{t,x} \in \mathbb{P}([0,T)\times\mathbb{R}, \mathbb{R}_+\times\mathbb{R})$ is called to be the measure-valued solution of equations (4) with initial data $u^0 \in \mathbb{C}^0(\mathbb{R};\mathbb{R}_+\times\mathbb{R})$ if following equations holds*

$$\int_{\mathbb{R}\times[0,T)} (\bar{h}\frac{\partial}{\partial t}\psi_1 + \bar{m}\frac{\partial}{\partial x}\psi_1)dtdx = \int_{\mathbb{R}} u_1^0\psi_1(0)dx,$$

$$\int_{[0,T)\times\mathbb{R}} \bar{m}\frac{\partial}{\partial t}\psi_2 + (\bar{e}+\frac{1}{2}\bar{p})\frac{\partial}{\partial x}\psi_2 + \tilde{G}\psi_2 dxdt = \int_{\mathbb{R}} u_2^0\psi_2(0)dx,$$

for all test functions: $\psi \in \mathbb{C}_0^1([0,T)\times\mathbb{R};\mathbb{R}^2)$.
Moreover the entropy inequality:

$$\int_{\mathbb{R}\times[0,T)} (\bar{\eta}\frac{\partial}{\partial t}\Phi(t,x) + \bar{q}\frac{\partial}{\partial x}\Phi(t,x) + \bar{k}\Phi(t,x))dtdx \geq \int_{\mathbb{R}} \eta(u^0(x)\Phi(0,x))dx,$$

holds for all nonnegative test functions: $\Phi \in \mathbb{C}_0^1([0,T)\times\mathbb{R};\mathbb{R}^2)$ and for all convex entropy-flux pairs.
Where:

$$\begin{aligned}
\bar{h} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \lambda_1 d\mu_{t,x}(\lambda),\\
\bar{p} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \lambda_1^2 d\mu_{t,x}(\lambda),\\
\bar{m} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2)d\mu_{t,x}(\lambda),\\
\bar{e} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} ((-\bar{g}+Id)^{-1}(x,\lambda_2))^2 d\mu_{t,x}(\lambda),\\
\tilde{G} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \lambda_1\bar{g}\circ(-\bar{g}+Id)^{-1}(x,\lambda_2)d\mu_{t,x}(\lambda),\\
\bar{\eta} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \eta(\lambda_1,\sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2))d\mu_{t,x}(\lambda),\\
\bar{q} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} q(\lambda_1,\sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2))d\mu_{t,x}(\lambda),\\
\bar{k} &= \textstyle\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial}{\partial z_2}\eta(z_1,z_2)|_{(\lambda_1,\sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2))}\lambda_1\bar{g}\circ(-\bar{g}+Id)^{-1}(x,\lambda_2)d\mu_{t,x}(\lambda),
\end{aligned}$$

for almost all $(t,x)\in[0,T)\times\mathbb{R}$;

4 The weak entropy solutions and kinetic formulation

Firstly we characterize the family of weak entropies. As the Theorem 4.1 is similar to the Perthame's result in [16] for the Euler equations describing gas dynamics, we omit that proof.

After changing variables from (u_1, u_2) into (h, v), the entropy obeys the wave equation $\frac{\partial^2}{\partial h^2}\eta - \frac{1}{h}\frac{\partial^2}{\partial v^2}\eta = 0$. Moreover the corresponding flux term is given by: $\frac{\partial}{\partial h}q = v\frac{\partial}{\partial h}\eta + \frac{\partial}{\partial v}\eta$, or equivalently by: $\frac{\partial}{\partial v}q = h\frac{\partial}{\partial h}\eta + v\frac{\partial}{\partial v}\eta$.

Definition 4.1 *The entropy that fulfil:*

$$\begin{cases} \frac{\partial^2}{\partial h^2}\eta - \frac{1}{h}\frac{\partial^2}{\partial v^2}\eta = 0, & h \geq 0, v \in \mathbb{R}, \\ \eta(h = 0, v) = 0, \\ \frac{\partial}{\partial h}\eta(h = 0, v) = b(v), \end{cases} \tag{6}$$

is called as a weak entropy.

The Theorem 4.1 like in [16] characterizes the family of weak entropies.

Theorem 4.1 *Let us consider problem (6). Then:*

- *the fundamental solution is:* $\chi(h, v) = \alpha(4h - v^2)_+^{\frac{1}{2}}$, *for* $\alpha = \frac{1}{4}(\int_{\mathbb{R}}(1 - \omega^2)_+^{\frac{1}{2}} d\omega)^{-1}$;
- *the solution of (6) is given by:*

$$\eta(h, v) = \int_{\mathbb{R}} b(\xi)\chi(h, v - \xi)d\xi; \tag{7}$$

- *the corresponding flux term is:*

$$q(h, v) = \frac{1}{2}\int_{\mathbb{R}} (\xi + v)b(\xi)\chi(h, v - \xi)d\xi; \tag{8}$$

- η *is convex (in variables* (u_1, u_2)*) if and only if* b *is convex.*

Remark 4.1 *After choosing* $b(v) = \frac{v^2}{2}$ *we obtain entropy of energy:*

$$\eta_E = \frac{1}{2}(hv^2 + h^2),$$

and corresponding flux term:

$$q_E = v(\frac{1}{2}hv^2 + h^2).$$

We show that weak entropy solutions and solutions of kinetic problem are equivalent. The proof is based on the proof for kinetic problem to the Euler model of gas motion (see [16]).

Theorem 4.2 *The pair* $(h, v) \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ *is a solution of the kinetic problem (see Def. 3.2) if and only if the pair* $(u_1, u_2) = (h, hv)$*, possess the finite energy and is a weak entropy solution (see Def. 3.3).*

Proof.
$\Leftarrow$
We assume that the pair $(u_1, u_2) = (h, hv)$, of finite energy is a weak entropy solution. Let m be a distribution defined as:

$$\frac{\partial^2}{\partial \xi^2}m(t, x, \xi) = \frac{\partial}{\partial t}\chi(h, v - \xi) + \frac{1}{2}\frac{\partial}{\partial x}(\xi + v)\chi(h, v - \xi) - \nabla_{(h,hv)}\chi(h, v - \xi) \cdot G(t, x),$$

in $\mathbb{D}'(\mathbb{R}_+,\mathbb{R}^2)$. We use test functions in the shape $b(\xi)\psi(t,x)$. That is why we can test this equation by convex function $b(\xi)$. We obtain:

$$\int_{\mathbb{R}} b(\xi)\frac{\partial}{\partial t}\chi(h,v-\xi)d\xi + \int_{\mathbb{R}} \frac{1}{2}\frac{\partial}{\partial x}(\xi+v)b(\xi)\chi(h,v-\xi)d\xi - \int_{\mathbb{R}} \nabla_{(h,hv)}(b(\xi)\chi(h,v-\xi))\cdot G(t,x)d\xi$$

$$= \int_{\mathbb{R}} b(\xi)\frac{\partial^2}{\partial\xi^2}m(t,x,\xi)d\xi.$$

And we integrate it by parts:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} b(\xi)\chi(h,v-\xi)d\xi + \frac{\partial}{\partial x}\int_{\mathbb{R}} \frac{1}{2}(\xi+v)b(\xi)\chi(h,v-\xi)d\xi - \nabla_{(h,hv)}\int_{\mathbb{R}} b(\xi)\chi(h,v-\xi)d\xi\cdot G(t,x)$$

$$= \int_{\mathbb{R}} \frac{\partial^2}{\partial\xi^2}b(\xi)m(t,x,\xi)d\xi.$$

The entropy-flux pair is given by equations (7) and (8). That is why we obtain:

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q - \nabla_{(h,hv)}\eta\cdot G = \int_{\mathbb{R}} \frac{\partial^2}{\partial\xi^2}b(\xi)m(t,x,\xi)d\xi.$$

After changing variables: $(u_1,u_2)=(h,hv)$, we have:

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q - \nabla_{(h,hv)}\eta\cdot G = \int_{\mathbb{R}} \frac{\partial^2}{\partial\xi^2}b(\xi)m(t,x,\xi)d\xi.$$

From Theorem 4.1 we know that η is a convex entropy in variables (u_1,u_2) iff b is a convex function. As u is a weak entropy solution then the entropy inequality holds for all convex entropy-flux pairs, what implies the following inequality:

$$\int_{\mathbb{R}} \frac{\partial^2}{\partial\xi^2}b(\xi)m(t,x,\xi)d\xi = \frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q - \nabla_u\eta\cdot G \le 0.$$

The fact that $\frac{\partial^2}{\partial\xi^2}b(\xi)\ge 0$, implies that m is a nonpositive measure.
It remains to show that m is bounded.
We choose the special function $b(\xi)=\frac{1}{2}\xi^2$, which lead us to the entropy of energy, it is $\eta_E = \frac{1}{2}(h^2+hv^2)$ (see Remark 4.1). We obtain:

$$\frac{\partial}{\partial t}\eta_E + \frac{\partial}{\partial x}q_E - \nabla_{(h,hv)}\eta_E\cdot G = \int_{\mathbb{R}} \frac{\partial^2}{\partial\xi^2}b(\xi)m(t,x,\xi)d\xi.$$

We test the equation by $\psi(t,x)=\phi(t)\omega_R(x)$, where $\omega_R(x)=\omega(\frac{x}{R})$ and $\omega\in\mathbb{D}(\mathbb{R})$ has support in $(-2,2)$, and in $(-1,1)$ is equal to one. We obtain:

$$-\int_0^T\int_{\mathbb{R}\times\mathbb{R}} m(t,x,\xi)\psi(t)\omega_R(x)d\xi dxdt$$

$$= \int_0^T\int_{\mathbb{R}} \nabla_{(h,hv)}\eta_E\cdot G\psi(t)\omega_R(x)dxdt - \int_0^T\int_{\mathbb{R}} \frac{\partial}{\partial t}\eta_E\psi(t)\omega_R(x)dxdt - \int_0^T\int_{\mathbb{R}} \frac{\partial}{\partial x}q_E\psi(t)\omega_R(x)dxdt$$

$$= \int_0^T\int_{\mathbb{R}} \nabla_{(h,hv)}\eta_E\cdot G\psi(t)\omega_R(x)dxdt - \int_0^T\int_{\mathbb{R}} \frac{\partial}{\partial t}\eta_E\psi(t)\omega_R(x)dxdt + \int_0^T\int_{\mathbb{R}} q_E\psi(t)\frac{\partial}{\partial x}\omega_R(x)dxdt$$

$$= \int_0^T\int_{\mathbb{R}} \nabla_{(h,hv)}\eta_E\cdot G\psi(t)\omega_R(x)dxdt - \int_0^T\int_{\mathbb{R}} \frac{\partial}{\partial t}\eta_E\psi(t)\omega_R(x)dxdt + \frac{1}{R}\int_0^T\int_{\mathbb{R}} q_E\psi(t)\frac{\partial}{\partial x}\omega(x)dxdt.$$

Remembering that (h, hv) has finite energy, it means that for all $t \geq 0$ holds: $h(t) \geq 0$ and $h, \eta_E \in \mathbb{L}^\infty(\mathbb{R}^+; \mathbb{L}^1(\mathbb{R}))$. What is more, $\nabla_{(h,hv)}\eta_E \cdot G = hv\bar{g}(v,x) \leq 0$, and q is bounded. Going with R to infinity we obtain:

$$-\int_0^T \int_{\mathbb{R}\times\mathbb{R}} m(t,x,\xi)\psi(t)d\xi dxdt \leq -\int_0^T \int_{\mathbb{R}} \frac{\partial}{\partial t}\eta_E \psi(t)dxdt,$$

and:

$$-\int_0^T \int_{\mathbb{R}\times\mathbb{R}} m(t,x,\xi)d\xi dxdt \leq \int_{\mathbb{R}} \eta_E(h^0,v^0)dx - \int_{\mathbb{R}} \eta_E(h(T,x),v(T,x))dx \leq \int_{\mathbb{R}} \eta_E(h^0,v^0)dx.$$

It shows that measure m is bounded.
$\Rightarrow$
Now we assume that the pair (h, v) is a solution of kinetic problem.
Again we test the equation:

$$\frac{\partial^2}{\partial \xi^2} m(t,x,\xi) = \frac{\partial}{\partial t}\chi(h, v-\xi) + \frac{1}{2}\frac{\partial}{\partial x}(\xi + v)\chi(h, v-\xi) - \nabla_{(h,hv)}\chi(h, v-\xi)\cdot G(t,x),$$

by a test function $b(\xi)\psi(t,x)$, where b is a convex function, and we obtain:

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q - \nabla_u \eta \cdot G = \int_{\mathbb{R}} \frac{\partial^2}{\partial \xi^2} b(\xi) m(t,x,\xi)d\xi.$$

From Theorem 4.1 we know that (η, q) is a convex entropy-flux pair. Using the fact that m is a nonpositive measure we obtain the entropy inequality:

$$0 \geq \frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q - \nabla_u \eta \cdot G,$$

that holds for every entropy such that $\nabla_u \eta(u) \in \mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R}^2)$,
$\eta(\cdot,\cdot)$ is in $\mathbb{C}^2$ for $(u_1,u_2) \in (0,\infty)\times\mathbb{R}$ and $\eta(0,\cdot) = 0$.
Taking $\eta = u_1$ and flux term $q = u_2$ we obtain following inequality:

$$\int_{\mathbb{R}\times[0,T)} (u_1(t,x)\frac{\partial}{\partial t}\Phi(t,x) + u_2(x,t)\frac{\partial}{\partial x}\Phi(t,x))dtdx \geq \int_{\mathbb{R}} \eta(u^0(x)\Phi(0,x))dx.$$

On the other hand, taking $\eta = -u_1$ we obtain the opposite inequality. Summing up, we obtained the first equation of the weak formulation.
To receive the other equation we choose $\eta = u_2$ and the corresponding flux term $q = \frac{1}{2}u_1^2 + \frac{u_2^2}{u_1}$, we obtain:

$$\int_{\mathbb{R}\times[0,T)} (u_2\frac{\partial}{\partial t}\Phi + (\frac{1}{2}u_1^2 + \frac{u_2^2}{u_1})\frac{\partial}{\partial x}\Phi + G_2\Phi)dtdx \geq \int_{\mathbb{R}} \eta(u^0(x)\Phi(0,x))dx.$$

The opposite inequality we obtain after choosing $\eta = -u_2$.
We proved that weak entropy solutions are equivalent to the solutions of kinetic formulation.
□

5 Measure-valued and weak entropy solutions

In this section we aim at proving that an entropy solution of (4) can be defined by a measure-valued solution. In the standard case the issue reduces to show that considered family of measures is indeed given by the Dirac atoms. However the discontinuity of the function g, describing the force in the original system (2), causes that representation via atom masses is no longer valid. In fact, pointwise representation holds for regular region of g, and in points of discontinuity we have to use language of transformation $(-\bar{g}(\cdot) + Id)^{-1}$ (see Def. 3.4).
Firstly, we look closer at this transformation then we define the family of measures and using it we show how to obtain weak entropy solutions.

For the sake of simplicity we assume that our hill is flat of uniform base. Under this assumption multifunction $\bar{g}$ is given by: $\bar{g} = -\mathrm{sgn}_{\mathrm{M}}(\sqrt{h}v)$. Let us observe the multifunction: $(-\bar{g}(\cdot) + Id)(\sqrt{h}v) = \mathrm{sgn}_{\mathrm{M}}(\sqrt{h}v) + \sqrt{h}v$, and the inverse mapping which is a function given by:

$$(-\bar{g}(\cdot) + Id)^{-1}(y) = \begin{cases} y - \mathrm{sgn}_{\mathrm{M}}(y) & |y| \geq 1, \\ 0 & y \in (-1, 1). \end{cases}$$

Moreover functions $(-\bar{g}(\cdot) + Id)^{-1}$ and:

$$\bar{g} \circ (-\bar{g} + Id)^{-1}(y) = \begin{cases} -\mathrm{sgn}_{\mathrm{M}}(y) & |y| > 1, \\ -y & y \in [-1, 1], \end{cases} \tag{9}$$

are continuous. To show this, it is enough to notice that $Id = -\bar{g} \circ (-\bar{g} + Id)^{-1} + Id \circ (-\bar{g} + Id)^{-1}$.

Now we define the measure μ that leads us from measure-valued solutions to weak entropy ones. We construct it by using the Dirac delta and the transformation $(-\bar{g}(\cdot) + Id)^{-1}$. We obtain the measure that is the Dirac delta in the points of continuity of right-hand side it is in the points corresponding to $\sqrt{h}v \neq 0$.

Let us take measure $\bar{\pi}_{t,x}$ equal to the Dirac delta: $\delta_{(\sqrt{h}v)(t,x)}$, it is

$$\bar{\pi}_{t,x}(y) = \begin{cases} 1 & y = (\sqrt{h}v)(t,x), \\ 0 & y \neq (\sqrt{h}v)(t,x). \end{cases}$$

Using it we define a probabilistic measure $\pi_{t,x}$. We require that for all Borel set A, π is given by the relation: $\pi_{t,x}(A) = \bar{\pi}_{t,x}((-\bar{g}(\cdot) + Id)^{-1}(A))$. Let us take $A = \{y\}$ for $|y| > 1$, then

$$\pi_{t,x}(\{y\}) = \bar{\pi}_{t,x}((-\bar{g}(\cdot) + Id)^{-1}(\{y\})) = \bar{\pi}_{t,x}(y - \mathrm{sgn}_{\mathrm{M}}(y)) = \begin{cases} 1 & y - \mathrm{sgn}_{\mathrm{M}}(y) = (\sqrt{h}v)(t,x) \\ 0 & y - \mathrm{sgn}_{\mathrm{M}}(y) \neq (\sqrt{h}v)(t,x) \end{cases}$$

and we obtain that the measure $\pi_{t,x}$ is equal to $\delta_{(-\bar{g}(\cdot)+Id)(\sqrt{h}v)}$ in the points where $\sqrt{h}v \neq 0$.

Now we are ready to define measure $\mu_{t,x} = \delta_{h(t,x)} \otimes \pi_{t,x}$ that we use in measure-valued formulation to obtain weak entropy solutions.

Theorem 5.1 *If the measure μ in measure-valued problem (see Def. 3.4) is defined as $\mu_{t,x} = \delta_{h(t,x)} \otimes \pi_{t,x}$, then solutions to that problem are also weak entropy solutions (see Def. 3.2).*

Proof.
We use the measure μ in measure-valued formulation. It means that we have to use it in each component of

$$\int_{\mathbb{R}\times[0,T)}(\bar{h}\frac{\partial}{\partial t}\psi_1+\bar{m}\frac{\partial}{\partial x}\psi_1)dtdx=\int_{\mathbb{R}}u_1^0\psi_1(0)dx,$$

$$\int_{[0,T)\times\mathbb{R}}\bar{m}\frac{\partial}{\partial t}\psi_2+(\bar{e}+\frac{1}{2}\bar{p})\frac{\partial}{\partial x}\psi_2+\tilde{G}\psi_2dxdt=\int_{\mathbb{R}}u_2^0\psi_2(0)dx.$$

Firstly we deal with:

$$\bar{h}=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1 d\mu_{t,x}(\lambda)=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1 d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda)=h(t,x)$$

and

$$\bar{p}=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1^2 d\mu_{t,x}(\lambda)=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1^2 d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda)=h^2(t,x).$$

The next part is:

$$\bar{m}=\int_{\mathbb{R}_+\times\mathbb{R}}\sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2)d\mu_{t,x}(\lambda)=\int_{\mathbb{R}_+\times\mathbb{R}}\sqrt{\lambda_1}(-\bar{g}+Id)^{-1}(x,\lambda_2)d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda)$$

$$=\int_{\mathbb{R}_+\times\mathbb{R}}\sqrt{\lambda_1}\lambda_2 d(\delta_{h(t,x)}\otimes\bar{\pi}_{t,x})(\lambda)=\int_{\mathbb{R}}\int_{\mathbb{R}_+}\sqrt{\lambda_1}\lambda_2 d\delta_{h(t,x)}d\delta_{(\sqrt{h}v)(t,x)}=hv$$

and

$$\bar{e}=\int_{\mathbb{R}_+\times\mathbb{R}}((-\bar{g}+Id)^{-1}(x,\lambda_2))^2 d\mu_{t,x}(\lambda)=\int_{\mathbb{R}_+\times\mathbb{R}}((-\bar{g}+Id)^{-1}(x,\lambda_2))^2 d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda)$$

$$=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_2^2 d(\delta_{h(t,x)}\otimes\bar{\pi}_{t,x})(\lambda)=hv^2.$$

The last part is $\tilde{G}$:

$$\tilde{G}=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1(\bar{g}\circ(-\bar{g}+Id)^{-1})(\lambda_2)d\mu_{t,x}(\lambda)$$

$$=\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1(\bar{g}\circ(-\bar{g}+Id)^{-1})(\lambda_2)d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda).$$

Using the formula (9) we obtain:

$$\tilde{G}=\begin{cases}\int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1(-\mathrm{sgn_M}(\lambda_2))d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda) & |\lambda_2|>1,\\ \int_{\mathbb{R}_+\times\mathbb{R}}\lambda_1(-\lambda_2)d(\delta_{h(t,x)}\otimes\pi_{t,x})(\lambda) & \lambda_2\in[-1,1].\end{cases}$$

In the case when $|\lambda_2|>1$, we use the properties of measure and we obtain: $\tilde{G}=-h\mathrm{sgn_M}(\sqrt{h}v)$ for $\sqrt{h}v\neq 0$. On the other hand, when $|\lambda_2|\leq 1$, we have that $\sqrt{h}v=0$. We obtain the following formula: $-h\mathrm{sgn_M}(\sqrt{h}v)$.
In all cases we use the fact that measure π is a probabilistic measure and we use Fubini theorem. Looking back to the measure-valued formulation, we obtain:

$$\int_{\mathbb{R}\times[0,T)}\bar{h}\frac{\partial}{\partial t}\psi_1+\bar{m}\frac{\partial}{\partial x}\psi_1 dtdx=\int_{\mathbb{R}\times[0,T)}h\frac{\partial}{\partial t}\psi_1+hv\frac{\partial}{\partial x}\psi_1 dtdx=\int_{\mathbb{R}}u_1^0\psi_1(0)dx,$$

$$\int_{[0,T)\times\mathbb{R}} hv\frac{\partial}{\partial t}\psi_2 + (hv^2+\frac{1}{2}h^2)\frac{\partial}{\partial x}\psi_2 + h\tilde{g}\psi_2 dxdt = \int_{\mathbb{R}} u_2^0\psi_2(0)dx.$$

After going into variables $(u_1,u_2)=(h,hv)$ we receive the weak solution for $\tilde{g}\in\bar{g}$.
It remains to show that entropy inequality holds. To prove that we use inequality

$$\int_{\mathbb{R}\times[0,T)} (\bar{\eta}\frac{\partial}{\partial t}\Phi(t,x) + \bar{q}\frac{\partial}{\partial x}\Phi(t,x) + \bar{k}\Phi(t,x))dtdx \geq \int_{\mathbb{R}} \eta(u^0(x)\Phi(0,x))dx$$

and we deal with: $\bar{\eta}$, $\bar{q}$, $\bar{k}$ in the similar way as above. After using the measure $\mu_{t,x}=\delta_{h(t,x)}\otimes\pi_{t,x}$, and changing the variables into (u_1,u_2), we obtain inequality what gives us weak entropy solutions.
□

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