Does the fully parabolic quasilinear 1D Keller-Segel system enjoy long-time asymptotics analogous to its parabolic-elliptic simplification?

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Does the fully parabolic quasilinear 1D Keller-Segel system enjoy long-time asymptotics analogous to its parabolic-elliptic simplification?

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Abstract

We show that the one-dimensional fully parabolic Keller-Segel system with nonlinear diffusion possesses global-in-time solutions, provided the nonlinear diffusion is equal to $\frac{1}{(1+u)^{\alpha}}$, $\alpha < 1$, independently on the volume of the initial data. We also show that in the critical case, i.e. for $\alpha = 1$, the same result holds for initial masses smaller than a prescribed constant. Additionally, we prove existence of initial data for which solution blows up in a finite time for any nonlinear diffusion integrable at infinity. Thus we generalize the known blow-up result of parabolic-elliptic case to the fully parabolic one. However, in the parabolic-elliptic case the above mentioned integrability condition on nonlinear diffusion sharply distinguishes between global existence and blow-up cases. We are unable to recover the entire global existence counterpart of this result in a fully parabolic case, as, as already mentioned, for the critical diffusion a certain smallness assumption is needed. Nevertheless, we introduce some functional inequalities which would yield a global existence result for the fully parabolic case for any nonintegrable at infinity nonlinear diffusion. When discussing them, we prove a set of similar inequalities which, hopefully, may shed some light on the desired ones.

keywords: fully parabolic Keller-segel system, global existence, finite-time blowup

1 Introduction

The Keller-Segel model was introduced as a system of four parabolic quasilinear equations in [9] to describe in a mathematical way the motion of small organisms under the chemotactic forces. One of its issues was to provide a model whose solution aggregates in a finite-time as a result of activity of chemotactic force attracting cells. In [13] Nanjundiah simplified the original model introducing the so-called minimal version of the Keller-Segel model consisiting of two parabolic equations. Basing on numerical experiments it was claimed that the simplified model is still a proper description of an aggregation phenomenon. Moreover the attention was also paid to the threshold value of initial mass guaranteeing the finite-time blowup of solution, a phenomenon interpreted as aggregation of cells. However, rigorous results that followed were showing finite-time blowups of solutions merely of further parabolic-elliptic simplifications of a model (see: [8], [10], [11]). Nagai and collaborators ([10], [11], [12]) have even found the values of initial mass yielding finite-time blowups both for radial solutions in a ball and nonradial ones in more general two-dimensional domains. In a radial case the conjecture in [2] based on numerical computations was confirmed. Let us emphasize that the models were parabolic-elliptic ones, though there is a wide agreement in the community that model with two parabolic equations is the one describing the reality better. In [7] the authors pointed one solution blowing up in a fully parabolic case. Unluckily, no exact information on a threshold value which distinguishes between finite-time blowup and infinite lifespan of solution is available. There is no mathematical proof that results from parabolic-elliptic case hold also in a fully parabolic one. Recently, in [14], there was presented an argument matching behaviour of solutions in parabolic-elliptic case with fully parabolic one. However, this result shows the stability properties of a model when passing from a fully parabolic to the parabolic-elliptic case in a very wide class of functions. Due to this obstacle one cannot say if the finite-time blowup from a parabolic-elliptic case is inherited in a fully parabolic one.

On the other hand, recently in [3] it was proved that solutions to the one-dimensional quasilinear fully parabolic case blow up provided nonlinear diffusion is weak enough. Next in [4] and [5] the exact strength of the diffusion distinguishing between finite-time blowup and global solutions in a corresponding parabolic-elliptic one-dimensional systems was identified. Our aim in the present paper is to study if the same results are available in a fully parabolic case. The confirmation would be a nice evidence that the results in a parabolic-elliptic case can be believed to hold also in a fully parabolic one.

In a present paper we are unfortunately unable to realize this program to a full extent. This is a first step of studying the connection between one-dimensional parabolic-elliptic and fully parabolic models. Actually, with respect to the class of nonlinear diffusions admitting finite-time blowup in a fully parabolic case we prove an analogous result as in [5], where the parabolic-elliptic system was studied. However our global-in-time existence result is weaker than its parabolic-elliptic counterparty. Namely, we prove that global solutions exist for the subcritical diffusions (which is already a new result in the fully parabolic case and fulfills the global existence counterpart to the finite-time blowup result in [3]) without any restriction on the size of the initial mass, but the result is not pushed far enough to cover all the nonlinearities nonintegrable at infinity. Basing on our result one cannot say what happens if one starts with an arbitrarily large initial mass in the case of the nonlinearity which could be a candidate for a critical one, namely $a(u) = \frac{1}{1+u}$. In this case we prove a global existence for initial mass smaller than a certain threshold. The question about behaviour of solutions emanating from masses larger than our threshold remains still an open problem. So is the question of qualitative behaviour of solutions to the Keller-Segel system ran by diffusions which are nonintegrable at infinity, but weaker than $\frac{1}{1+u}$. In spite of this, we believe that the method of the proof we introduce in the case of diffusion $\frac{1}{1+u}$ brings us closer to understanding the behavior of solutions in that case even for a large mass initial data. Namely, we introduce inequalities (that are stated in the last section), which so far we were not able to prove, that would yield the global existence result entirely analogous to the one in parabolic-elliptic case (see [4] and [5]). These highly nonlinear inequalities seem to be interesting on their own. In the last section we prove a set of similar inequalities suggesting that the one which we need could be critical in some sense.

Let us emphasize that studying the lifespan of solutions in a fully parabolic case requires completely different methods than those used in a parabolic-elliptic one. A change of variable introduced in [4] which reduces a parabollic-elliptic system to one equation posessing a Liapunov functional is not available in a fully parabolic case. That reformulation essentially simplifies further studies of properties of solutions to the parabolic-elliptic Keller-Segel system in one dimension.

Let us now describe our results in a more precise way.

We consider a following one dimensional Keller-Segel problem with nonlinear difussion a

$$[KS]\begin{cases} u_t = (a(u)u_x - \chi u v_x)_x \quad in\ (0,\infty)\times(0,1),\ \chi > 0,\\ \varepsilon v_t = v_{xx} + u - M \quad in\ (0,T)\times(0,1),\ \varepsilon > 0,\\ u_x = v_x = 0 \quad on\ (0,T)\times\{0;1\}\\ (u,v)(0) = (u_0, v_0). \end{cases}$$

We save the letters u, v to denote the solution of the above [KS] system and in the entirety of this paper we usually refer to solution of [KS] system simply as to u, v. We denote absolute value of a real number as $|\cdot|$, L_p norm as $|\cdot|_p$ and $W^{1,p}$ norm as $|\cdot|_{1,p}$. C, K are constants, which may change even in a single line.

This paper is divided into four main sections and organized as follows: In the remainder of this introduction we collect well-known facts on short-time existence, uniqueness and regularity of solutions as well as availability of the Liapunov functional. The next two research sections deal with dependence of time-asymptotics of solutions to [KS] on nonlinear diffusion. The second one is devoted to problem of the global-in-time existence of L^∞-bounded solutions of [KS]. In the third part, we present finite-time blowup result for supercritical (integrable) diffusions a. Finally, we conclude with some remarks on possible extensions of our results and its connections to some functional inequalities.
Let us state the standard well-posedness and classical solvability result:

Proposition 1 (Well-posedness, classical solvability, conservation of mass for [KS] system)**.** *Assume that*

$$a \in C^2(\mathbb{R}_+),\ a > 0, \quad M := \int_0^1 u_0(t,x)dx > 0,\ u_0 \geq 0, \quad \int_0^1 v_0(t,x)dx = 0, \quad u_0, v_0 \in W^{1,2}(0,1) \tag{1}$$

in [KS]. Then it admits local-in-time, unique, classical solution with maximal time of existence $T_m < \infty$

$$(u,v) \in\ C([0,T_m) \times [0,1]; \mathbb{R}^2)\ \cap\ C^{2,1}((0,T_m) \times [0,1]; \mathbb{R}^2), \tag{2}$$

which additionally satisfies

$$\int_0^1 u(t,x)dx = \int_0^1 u_0(t,x)dx = M,\ u \geq 0\ \text{ for }\ t > 0, \quad \int_0^1 v(t,x)dx = 0. \tag{3}$$

Moreover, if $T_m < \infty$ we have blowup of L^∞ norm, i.e.

$$\lim_{t \to T_m} [|u(t)|_\infty + |v(t)|_\infty] \to \infty. \tag{4}$$

The proof follows the general theory of parabolic (triangular) systems. For precise references consult [3, p.440].
Interestingly, the [KS] system posesses the Liapunov functional, which, in 1D setting, is additionally bounded from below. Let us introduce:

Definition 1. *For $b : \mathbb{R}_+ \to \mathbb{R}$ such that: $b''(s) = \frac{a(s)}{s}$, $b'(1) = 0$, $b(1) = 0$ and $b(u_0) \in L^1$, where a denotes diffusion associated with [KS], let*

$$\lambda(t) := \int_0^1 \left(b(u(t,x)) - u(t,x)v(t,x) + \frac{1}{2}|v_x(t,x)|^2 dx \right) \tag{5}$$

for (u,v) solving [KS].

Proposition 2. *Under assumptions of Proposition 1 it holds for $t \in [0,T_m)$:*

$$\frac{d}{dt}\lambda(t) \leq -\varepsilon |v_t|_2^2(t), \qquad \lambda(t) \geq -\frac{M^2}{2}. \tag{6}$$

This proposition is shown in [3, Lemmas 4, 5].

Next we have a Corollary which is a simple consequence of (5) and (6).

Corollary 1. *Under assumptions of Proposition 1 there exists $C > 0$ such that*

$$\int_0^{T_m} \int_0^1 |v_t|^2 dx dt < C. \tag{7}$$

2 Global existence

The main result of this sections reads

Theorem 1. *For (u, v) solving [KS] it holds:*

$$|u(t)|_\infty \leq C \tag{8}$$

provided one of the following assumptions is valid:

i. (subcritical case) nonlinear diffusion takes the form: $a(s) = \frac{1}{(1+s)^\alpha}$ for $\alpha \in [0, 1)$ and the initial mass $\int_0^1 u_0(x)dx = M$ is arbitrary;

ii. (critical case) nonlinear diffusion takes the form: $a(s) = \frac{1}{(1+s)}$ and the initial mass $\int_0^1 u_0(x)dx = M$ satisfies $M < \frac{2}{\sqrt{\chi}} - 1$.

For the sake of clarity, first we derive the main ingredients needed to show the above theorem and in the final part of this section we combine them in the main proof. Let us begin with quoting the following regularity result:

Proposition 3 (boundedness of highly integrable nonnegative solutions to quasilinear parabolic equation)**.** *Consider the following system*

$$\begin{cases} u_t = \operatorname{div}(D(x,t,u)\nabla u + f(x,t)) & in\ (0,T) \times (0,1) \\ \partial_\nu v = 0 \quad on\ (0,T) \times \{0,1\} \end{cases} \tag{9}$$

Assume that:

(1) $D \in C^1([0,1] \times [0,T) \times [0,\infty))$, $D > 0$ and there exist $m \in \mathbb{R}, s_0 \geq 0, \delta > 0$ such that $D(x,t,s) \geq \delta s^{m-1}$ for $s \geq s_0$;

(2) $f \in C^0(0,T; C^0(\bar{\Omega}) \cap C^1(\Omega))$, $f_x(0) = f_x(1) = 0$ for $t \in (0,T)$, $f \in L^\infty(L^{q_1})$;

(3) $u \geq 0$ solves (9) and $u \in L^\infty(L^{p_0})$;

(4) following inequalities for q_1, p_0 hold:

$$q_1 > 3 \tag{10}$$

$$p_0 \geq 1, \quad p_0 > 1 - m\frac{2q_1 - 3}{q_1 - 3}, \quad p_0 > 1 - m, \quad p_0 > \frac{1-m}{2} \tag{11}$$

then

$$|u(t)|_\infty \leq C. \tag{12}$$

Proof. This theorem is a version of [15, Lemma 4.1]. More precisely, taking $n = 1, g \equiv 0$ in [15, Lemma 4.1] one obtains exactly Proposition 3. □

Later on we need the following maximum regularity result for the one-dimensional heat equation. The formulation which we need involves a right hand-side in $L^\infty(0,T; L^1(0,1))$. This is a non-classical case and since we didn't find a proper reference, we attach a proof of it.

Proposition 4 (maximum regularity for 1D heat equation)**.** *Let $v \in C^{1,2}([0,T) \times [0,1])$ be a solution to the heat equation*

$$v_t - v_{xx} = f \text{ in } (0,T) \times (0,1) \ \ , v(0,x) = v_0(x) \in C[0,1], \tag{13}$$

where $f \in L^\infty(0,T;L^1(0,1)) \cap C([0,T) \times [0,1])$. Moreover v satisfies boundary conditions

$$v_x(0) = v_x(1) = 0. \tag{14}$$

Then

$$\sup_{t\in[0,T)} |v_x|_q < C$$

for any $q < \infty$.

The proof of this statement can be found as Lemma 4.1. in [16].

The following lemma allows us to prove virtually the entire Theorem 1 in the subcritical diffusion case.

Lemma 1. *Assume that nonlinear diffusion of [KS] $a(u) = \frac{1}{(1+u)^\alpha}$ with $\alpha \in [0,1]$. Additionally, assume that solution (u,v) of [KS] satisfies for time independent constant C:*

$$|(1+u)^{\frac{s-\alpha+1}{2}}|_p(t) \le C \tag{15}$$

where s,p are positive numbers satisfying

$$p(s-\alpha+1) > 2\alpha, \quad p \ge 1 \tag{16}$$

then

$$|(1+u)|_{s+1}(t) \le C(u_0,v_0,p,\alpha) \tag{17}$$

Proof. Without loss of generality set $\varepsilon = \chi = 1$ in [KS]. Multilplying the first equation of [KS] by $(1+u)^s$, integrating over space and performing one integration by parts we have:

$$\frac{1}{s+1}\frac{d}{dt}\int (u+1)^{s+1} = -s\int a(u)|(u+1)_x|^2(u+1)^{s-1} + s\int (u+1-1)v_x(u+1)_x(u+1)^{s-1},$$

which by assumption $a(u) = \frac{1}{(1+u)^\alpha}$ gives

$$\begin{aligned}\frac{d}{dt}\int (u+1)^{s+1} + (s+1)s\int |(u+1)_x|^2(u+1)^{s-1-\alpha} = \\ s(s+1)\int v_x(u+1)_x(u+1)^s - s(s+1)\int v_x(u+1)_x(u+1)^{s-1} = \\ s(s+1)\int [(u+1)_x(u+1)^{\frac{s-1-\alpha}{2}}](u+1)^{\frac{s+1+\alpha}{2}}v_x - s(s+1)\int [(u+1)_x(u+1)^{\frac{s-1-\alpha}{2}}](u+1)^{\frac{s-1+\alpha}{2}}v_x.\end{aligned}$$

Hence,

$$\frac{d}{dt}\int (u+1)^{s+1} + \frac{4s(s+1)}{2(s-\alpha+1)^2}\int |[(u+1)^{\frac{s-\alpha+1}{2}}]_x|^2 \le \frac{s(s+1)}{2}\int |v_x|^2[(u+1)^{s+1+\alpha} + (1+u)^{s-1+\alpha}],$$

adding to both sides of the above inequality term $\int (u+1)^{s+1}$ we arrive at:

$$\begin{aligned}\frac{d}{dt}\int (u+1)^{s+1} + \int (u+1)^{s+1} + \frac{2s(s+1)}{(s-\alpha+1)^2}\int |[(u+1)^{\frac{s-\alpha+1}{2}}]_x|^2 \le \\ \frac{s(s+1)}{2}\int |v_x|^2[(u+1)^{s+1+\alpha} + (1+u)^{s-1+\alpha}] + \int (u+1)^{s+1}.\end{aligned} \tag{18}$$

To proceed further, recall that the Gagliardo-Nirenberg interpolation inequality in 1D yields:

$$|w|_r^r \leq K|w|_{1,2}^{r\theta}|w|_p^{r(1-\theta)}, \qquad \frac{1}{r} = -\frac{\theta}{2} + \frac{1-\theta}{p}.$$

Therefore, requiring $r\theta = 2$ one obtains $r = 2p + 2$. Setting $w := (u+1)^{\frac{s-\alpha+1}{2}}$ we have:

$$\int (u+1)^{(2p+2)\frac{s-\alpha+1}{2}} \leq K|(u+1)^{\frac{s-\alpha+1}{2}}|_{1,2}^2|(u+1)^{\frac{s-\alpha+1}{2}}|_p^{2p} \tag{19}$$

In order to use this inequality, we add $\frac{2s(s+1)}{(s-\alpha+1)^2}\int(u+1)^{s-\alpha+1}$ to both sides of (18), thus:

$$\frac{d}{dt}\int(u+1)^{s+1} + \int(u+1)^{s+1} + \frac{2s(s+1)}{(s-\alpha+1)^2}|(u+1)^{\frac{s-\alpha+1}{2}}|_{1,2}^2 \leq$$
$$\frac{2s(s+1)}{(s-\alpha+1)^2}\int(u+1)^{s-\alpha+1} + \frac{s(s+1)}{2}\int|v_x|^2[(u+1)^{s+1+\alpha} + (1+u)^{s-1+\alpha}] + \int(u+1)^{s+1}$$
$$\leq \left(1 + \frac{2(s+1)}{s}\right)\int(u+1)^{s+1} + \frac{2(s+1)}{s} + \frac{s(s+1)}{2}\int|v_x|^2[1 + 2(u+1)^{s+1+\alpha}]$$

where the second inequality holds by $cx \leq c + cx^\gamma,\ \gamma \geq 1,\ c, x$ positive and $\alpha \leq 1$. In view of assumption (15) the above inequality and Young's inequality imply for any positive, fixed smallness constants η, δ:

$$\frac{d}{dt}\int(u+1)^{s+1} + \int(u+1)^{s+1} + \left(K|(u+1)^{\frac{s-\alpha+1}{2}}|_{1,2}^2|(u+1)^{\frac{s-\alpha+1}{2}}|_p^{2p}\right)\frac{2s(s+1)}{K(s-\alpha+1)^2}|(u+1)^{\frac{s-\alpha+1}{2}}|_p^{-2p} \leq$$
$$\frac{3s+2}{s}\int(u+1)^{s+1} + \delta\int(u+1)^{s+1+\alpha+\eta} + \frac{\eta}{s+1+\alpha+\eta}\left(\delta\frac{s+1+\alpha+\eta}{s+1+\alpha}\right)^{-\frac{s+1+\alpha}{\eta}}\int|v_x|^{\frac{2(s+1+\alpha+\eta)}{\eta}}$$
$$+ \frac{s(s+1)}{2}\int|v_x|^2 + \frac{2(s+1)}{s}.$$

Next, by (15) and (19)

$$\frac{d}{dt}\int(u+1)^{s+1} + \int(u+1)^{s+1} + \left(\int(u+1)^{(p+1)(s-\alpha+1)}\right)\frac{2s(s+1)}{K(s-\alpha+1)^2}C^{-2p} \leq$$
$$2\delta\int(u+1)^{s+1+\alpha+\eta} + \frac{\eta}{s+1+\alpha+\eta}\left(\delta\frac{s+1+\alpha+\eta}{s+1+\alpha}\right)^{-\frac{s+1+\alpha}{\eta}}\int|v_x|^{\frac{2(s+1+\alpha+\eta)}{\eta}} +$$
$$\frac{s(s+1)}{2}\int|v_x|^2 + \frac{2(s+1)}{s} + \frac{3s+2}{s}\left(\frac{(3s+2))s+1)}{\delta s(s+1+\alpha+\eta)}\right)^{\frac{s+1}{\alpha+\eta}}. \tag{20}$$

Taking η, δ such that

$$2\delta = \frac{2s(s+1)}{K(s-\alpha+1)^2}C^{-2p}$$

and

$$(p+1)(s-\alpha+1) = s+1+\alpha+\eta,$$

the latter is possible due to assumption (16) which is equivalent to:

$$(p+1)(s-\alpha+1) > s+1+\alpha,$$

and abandoning a precise control over constants, we finally arrive at

$$\frac{d}{dt}\int(u+1)^{s+1} + \int(u+1)^{s+1} \leq C(s,\alpha)\left(1 + |v_x|_q^q\right) \tag{21}$$

for some number $q = q(s, \alpha)$.

Thus in view of Proposition 4 we have global-in-time bounds for $\int(u+1)^{s+1}$. □

At this point we want to reason in an iterative manner, using at each step the above proposition. This procedure is implemented in a following two corollaries. First we consider the subcritical case which allows us to obtain global-in-time boundedness of arbitrarily large L_p norm of u.

Corollary 2. *Let u, v solve the [KS] system with subcritical diffusion $a(u) = (1+u)^{-\alpha}$, $\alpha \in [0,1)$, then:*

$$|u|_p(t) \le C(u_0, v_0, p, \alpha) \quad \forall_{p\in[1,\infty)} \tag{22}$$

where $C(u_0, v_0, p, \alpha)$ is time independent.

Proof. We choose s_i, p_i; $i = 1, 2, ..$ as follows:

$$s_1 = 1+\alpha, \qquad p_1 = \frac{2}{s_1 - \alpha + 1} = 1 \tag{23}$$

$$s_{i+1} = 2s_i + 1 + \alpha, \qquad p_{i+1} = \frac{2(s_i+1)}{s_{i+1} - \alpha + 1} = 1 \tag{24}$$

which implies

$$s_i = (1+\alpha)(2^i - 1), \qquad p_i(s_i - \alpha + 1) \ge 2 > 2\alpha. \tag{25}$$

Therefore such choice of s_i, p_i satisfies assumption (16) for every i. Now we recursively obtain that:

1. for s_1, p_1 from Proposition 1 we have

$$|(u+1)^{\frac{s_1-\alpha+1}{2}}|_{p_1} = \int (u+1) = M+1$$

so assumption (15) holds and by Lemma 1 we obtain $|u|_{s_1+1}(t) \le C$;

2. for s_{i+1}, p_{i+1} we get:

$$|(u+1)^{\frac{s_{i+1}-\alpha+1}{2}}|_{p_{i+1}} = \int (u+1)^{s_i+1},$$

so so assumption (15) is valid by virtue of inductive assumption and consequently via Lemma 1 one obtains:

$$|u|_{s_{i+1}+1}(t) \le C$$

Hence we lift stepwise the integrability of u to any fixed number $p < \infty$. □

Observe that for the case of critical diffusion (i. e. for $\alpha = 1$) one cannot embark on with analogous iteration, knowing merely that $|u|_1(t) \le C$. The reason is that for assumption (15) to hold, one would need to set $p_1 = \frac{2}{s_1}$, which in turn violates (16). Recalling the proof of Lemma 1 it turns out, that it is impossible to supersede (16) with non-sharp inequality, which would suffice. Nevertheless, we can obtain a following weaker result for the critical diffusion:

Corollary 3. *Let u, v solve the [KS] system with critical diffusion $a(u) = (1+u)^{-1}$, then:*

$$|u|_{1+\varepsilon}(t) \le C \;\; \textit{implies} \;\; |u|_p(t) \le C(u_0, v_0, p, \alpha) \quad \forall_{p\in[1,\infty)} \tag{26}$$

where $C(u_0, v_0, p, \alpha)$ is time independent and $\varepsilon > 0$ is arbitrarily small.

Proof. Choose s_i, p_i, $i = 1, 2, ..$ as follows:

$$s_1 = 2(1+\varepsilon), \qquad p_1 = \frac{2(1+\varepsilon)}{s_1} = 1, \tag{27}$$

$$s_{i+1} = 2s_i + 2, \qquad p_{i+1} = \frac{2(s_i+1)}{s_{i+1}} = 1. \tag{28}$$

Again we can proceed inductively using Lemma 1 to show that

$$|u|_p(t) \le C(u_0, v_0, p, \alpha)$$

for any $p < \infty$. We see that:

(i=1) $|(u+1)^{\frac{s_1}{2}}|_{p_1} = |(u+1)^{1+\varepsilon}|_1 \leq C, \quad p_1 s_1 = 2+2\varepsilon > 2 = 2\alpha$ by definitions of s_1, p_1 and assumption $|u|_{1+\varepsilon}(t) \leq C$;

(i+1) $|(u+1)^{\frac{s_{i+1}}{2}}|_{p_{i+1}} = |(u+1)^{1+s_i}|_1 \leq C, \quad p_{i+1}s_{i+1} = 2+2s_i > 2 = 2\alpha$ by definitions of s_i, p_i and recursive assumption.

□

At this stage, we posses all the needed ingredients for showing Theorem 1 in the subcritical case, because Corollary 2 allows us to obtain high integrability and Proposition 3 enables us to perform the step from high integrability to boundedness. However, in the critical diffusion case, we lack the bound on $|u|_{1+\varepsilon}(t)$. In what follows we struggle to obtain one. Firstly we derive global-in-time bounds for $|u|_{LlogL}$ under assumptions on smallness of M (see Lemma 2). Next we utilize an idea from [1] to perform the step from the bound $|u|_{LlogL}(t) \leq C$ to $|u|_{1+\varepsilon}(t) \leq C$.
In order to show Lemma 2 we need a following Proposition 5.

Proposition 5. *For a function $m \in W^{1,2}(0,1)$, it holds for every $\nu > 0$:*

$$\int e^{2m} < \frac{1+\nu}{4}\left(\int e^m\right)^2 \int |m_x|^2 + \left(1+\frac{1}{\nu}\right)\left(\int e^m\right)^2. \tag{29}$$

Proof. First, for arbitrary $m \in W^{1,1}(0,1)$ a constant in the critical Sobolev imbedding is 1, as for any $x \in (0,1)$ holds:

$$|m(x)| \leq |m(x) - m(z)| + |m(z)| \leq \int_0^1 |m_x| + |m(z)| \leq |m|_{1,1}. \tag{30}$$

because by absolute continuity of such u one can fix $z = \arg\min_{[0,1]} u$. Hence:

$$\int e^{2m} \leq \left(\sup_{x\in(0,1)} e^{\frac{m(x)}{2}}\right)^2 \int e^m \leq |e^{\frac{m}{2}}|_{1,1}^2 \int e^m = \left(\frac{1}{2}\int |e^{\frac{m}{2}} m_x| + \int e^{\frac{m}{2}}\right)^2 \int e^m$$

Next, by Hölder's inequality:

$$\int e^{2m} \;\leq\; \left[\frac{1}{2}\left(\int e^m\right)^{\frac{1}{2}}\left(\int |m_x|^2\right)^{\frac{1}{2}} + \int e^{\frac{m}{2}}\right]^2 \int e^m \;\leq\; \left(\int e^m\right)^2 \left[\frac{1+\nu}{4}\left(\int |m_x|^2\right) + \left(1+\frac{1}{\nu}\right)\right]$$

□

Corollary 4. *Let u be a solution to [KS] with $\int_0^1 u_0(x)dx = M$. For $M < \frac{2}{\sqrt{\chi}} - 1$ the following inequality holds:*

$$\chi \int_0^1 u^2 dx < \int_0^1 |(\log(1+u))_x|^2\, dx + C(M,\chi) \tag{31}$$

Proof. Due to (3), u is a positive function with fixed mass. Letting $m := \log(1+u)$, i.e. $u+1 = e^m$, in the Proposition 5 leads to:

$$\int_0^1 u^2 dx < (M+1)^2 \frac{1+\nu}{4} \int_0^1 |(\log(1+u))_x|^2\, dx + C(M,\nu)$$

which gives thesis provided $\chi(M+1)^2\frac{1+\nu}{4} < 1$ holds. In view of Proposition 5 one can take ν as small as needed, thus the assumed inequality $M < \frac{2}{\sqrt{\chi}} - 1$ equivalent to $\chi(M+1)^2\frac{1}{4} < 1$ suffices. □

Next lemma is inspired by the additional Liapunov functional for Keller-Segel system that appeared in [6].

Lemma 2. *For u, v solving the [KS] system with critical diffusion $a(u) = (1+u)^{-1}$, such that $M = \int_0^1 u_0(x)dx < \frac{2}{\sqrt{\chi}} - 1$, there is a global bound for LlogL norm of $u+1$, i.e. $\int |(u+1)log(u+1)|(t) \leq C$.*

Proof. We test first equation of [KS] system with $log(u+1)$ and the second one two times: with $\chi(u+1)$ and $-\chi log(u+1)$, obtaining:

$$\begin{aligned}
\int (u+1)_t log(u+1) &= -\int \tfrac{|(u+1)_x|^2}{(u+1)^2} + \chi \int \tfrac{u}{u+1} v_x (u+1)_x \\
&= -\int \tfrac{|(u+1)_x|^2}{(u+1)^2} + \chi \int v_x (u+1)_x - \chi \int \tfrac{1}{u+1} v_x (u+1)_x, \\
\varepsilon\chi \int v_t (u+1) &= -\chi \int v_x (u+1)_x + \chi \int u(u+1) - \chi M \int (u+1), \\
-\varepsilon\chi \int v_t log(u+1) &= +\chi \int \tfrac{1}{u+1} v_x (u+1)_x - \chi \int u log(u+1) + \chi M \int log(u+1).
\end{aligned}$$

By conservation of mass one has $\frac{d}{dt}\int u = 0$, so, adding the above equalities we arrive at:

$$\frac{d}{dt}\int (u+1)log(u+1) + \int |(log(u+1))_x|^2 =$$

$$= \chi \int u(u+1) - \chi \int u \log(u+1) - \chi M(M+1) + \chi M \int \log(u+1) - \chi\varepsilon \int v_t((u+1) - \log(u+1)),$$

Since for any $\mu > 0$ we can find such C that $\int (u+1)log(u+1) \leq \int |(u+1)log(u+1)| \leq \mu \int u^2 + C$ and $\log(u+1) \leq u$ for $u \geq 0$, Young's inequality yields:

$$\frac{d}{dt}\int (u+1)log(u+1) + \int (u+1)log(u+1) + \int |(log(u+1))_x|^2 \leq$$

$$\chi \int u^2 - \chi\varepsilon \int |v_t|(2u+1) + \mu \int u^2 + C \leq (\chi + 2\mu)\int u^2 + C|v_t|_2^2 + C. \quad (32)$$

Recall Proposition 5. Thanks to sharpness of the inequality (31) and arbitrary smallness of μ the inequality (32) folds to:

$$\frac{d}{dt}\int (u+1)log(u+1) + \int (u+1)log(u+1) \leq C|v_t|_2^2 + C(M,\chi)$$

which in turn gives for $t \in [0, T_m)$ boundedness of $\left[\int_0^1 (u(t,x)+1)log(u(t,x)+1)dx\right](t)$ in view of Corollary 1. Finally, for $w > 0$ there holds:

$$\int |wlogw| = \int wlogw(\mathbb{1}_{\{w\geq 1\}} - \mathbb{1}_{\{w \leq 1\}}) = \int wlogw - 2\int wlogw \mathbb{1}_{\{w\leq 1\}} \leq \int wlogw + \frac{2}{e},$$

which for $w = u+1$ translates into

$$\left[\int_0^1 |(u(t,x)+1)log(u(t,x)+1)|dx\right](t) \leq \left[\int_0^1 (u(t,x)+1)log(u(t,x)+1)dx\right](t) + \frac{2}{e} \leq C, \quad t \in [0, T_m)$$

□

Remark 1. *Observe that we need a smallness assumptions on both χ and M in Lemma 2 only due to restrictions of Proposition 5 and Corollary 4. However notice that the positive answer to the following open problem would suffice in order to prove Lemma 2 without any restriction neither on the size of initial mass, nor χ.*

Problem. *If there exists a continuous function $h : [0,\infty) \to \mathbb{R}$ or family of such functions h_δ, such that for any $\delta > 0$ and for any $m \in W^{1,2}(0,1)$ the following inequality holds:*

$$\int e^{2m} < \delta \int |m_x|^2 + h_\delta\left(\int e^m\right)? \quad (33)$$

We will comment further on this inequality and its connections to the problem studied in the paper in the last section.

Now we state an inequality which we need in the sequel. It is actually a one-dimensional version of the inequality in [1]. The proof is exactly the same, we present it only for reader's convenience.

Proposition 6. *For $w \in W^{1,2}(0,1)$ and any $\delta > 0$ there exists C_δ such that the following inequality holds*

$$|w|_4^4 \leq \delta |w|_{1,2}^2 |w|_{LlogL} + C_\delta |w|_1. \tag{34}$$

Proof. Define $\eta_N : \mathbb{R} \to \mathbb{R}$ as follows:

$$\eta_N(s) = \begin{cases} 0 & for\ |x| \leq N, \\ 2(|s| - N) & for\ |x| \in (N, 2N], \\ |s| & for\ |x| > 2N. \end{cases} \tag{35}$$

The Gagliardo-Nirenberg inequality gives

$$|\eta(w)|_4^4 \leq K |\eta(w)|_{1,2}^2 |\eta(w)|_1^2. \tag{36}$$

We estimate r.h.s of (36) in a following manner

$$\begin{array}{rcl} |\eta(w)|_{1,2}^2 & = & \int |\eta'(w) w_x|^2 + \int |\eta(w)|^2 \leq 4|w|_{1,2}^2, \\ |\eta(w)|_1 & \leq & \int |w| \mathbb{1}_{\{|w|>N\}} = \int |wlogw| (logw)^{-1} \mathbb{1}_{\{|w|>N\}} \leq (logN)^{-1} |w|_{LlogL}, \end{array} \tag{37}$$

where in the first equation we used $|\eta'| \leq 2$ and in the second we take $N > e$. Next we estimate the difference between w and $\eta(w)$

$$||w| - \eta(w)|_4^4 = \int ||w| - \eta(w)|^4 \mathbb{1}_{\{|w|>N\}} \leq (2N)^3 \int ||w| - \eta(w)| \mathbb{1}_{\{|w|>N\}} \leq 8N^3 |w|_1^1. \tag{38}$$

Considering (37), (38) we can write in view of (36)

$$|w|_4^4 \leq 8[||w| - \eta(w)|_4^4 + |\eta(w)|_4^4] \leq 64N^3|w|_1^1 + 8K|\eta(w)|_{1,2}^2|\eta(w)|_1^2 \leq 64N^3|w|_1^1 + 32K|w|_{1,2}^2 (logN)^{-2} |w|_{LlogL}^2,$$

which for $N = \max\left\{e; e^{\sqrt{\frac{32K}{\delta}}}\right\}$ is (34). ∎

The next lemma follows the method introduced in [12].

Lemma 3. *If u solving [KS] with critical diffusion $a(u) = (1+u)^{-1}$ admits $|u+1|_{LlogL}(t) \leq C$, then $|u+1|_3(t) \leq C$.*

Proof. Test equation for u_t in [KS] with $(u+1)^2$ to get

$$\frac{1}{3}\frac{d}{dt}\int (u+1)^3 = -2\int |(u+1)_x|^2 + 2\int (u+1)^2 u_x v_x - 2\int (u+1) u_x v_x$$

$$= -2\int |(u+1)_x|^2 - 2/3\int (u+1)^3 v_{xx} + \int (u+1)^2 v_{xx}$$

$$= -2\int |(u+1)_x|^2 - 2/3\int (u+1)^3 (v_t + M - u) + \int (u+1)^2 (v_t + M - u)$$

$$= -2\int |(u+1)_x|^2 - \frac{2M}{3}\int (u+1)^3 - 2/3\int (1+u)^3 v_t + \int (1+u)^2 v_t + 2/3\int u(u+1)^3 - \int u(1+u)^2 + M\int (u+1)^2$$

Thus for $w := u+1$

$$\frac{1}{3}\frac{d}{dt}\int w^3 + \frac{2M}{3}\int w^3 + 2\int |w_x|^2 \le 8/3\int w^3|v_t| + \int w^4 + \frac{3M^2}{4}. \tag{39}$$

Recall that in view of (34) for arbitrary small $\delta > 0$ holds

$$\int w^4 \le \delta\left[\int |w_x|^2 + \int w^2\right]\int |wlogw| + C_\delta \int w. \tag{40}$$

on the other hand by the Gagliardo-Nirenberg inequality used twice

$$|w|_6^3 \le C|w|_{1,2}^{\frac{3}{5}}|w|_3^{\frac{12}{5}} = C|w|_{1,2}^{\frac{3}{5}}|w|_3^{\frac{3}{2}}|w|_3^{\frac{9}{10}} \le C|w|_{1,2}^{\frac{3}{5}}|w|_3^{\frac{3}{2}}|w|_{1,2}^{\frac{2}{5}}|w|_1^{\frac{1}{2}} = C(M)|w|_{1,2}|w|_3^{\frac{3}{2}}$$

which means for any $\theta > 0$

$$\int w^3|v_t| \le |v_t|_2|w|_6^3 \le \theta|w_x|_2^2 + \theta|w|_2^2 + C_\theta|v_t|_2^2|w|_3^3. \tag{41}$$

In view of (40) and (41) in tandem with assumption $|w|_{LlogL}(t) \le C$, (39) implies that $f(t) := \int_0^1 w^3(x,t)dx$ satisfies

$$\frac{d}{dt}f(t) + Mf(t) \le C|v_t|_2^2 f(t) + C \tag{42}$$

Finally, utilizing Gronwall's inequality for $f(t)e^{Mt}$ and inequality (7) one arrives for time-independent C at:

$$f(t) \le e^{C-Mt}f(0) + C \quad \forall_{t\in[0,T_m)}$$

□

Proof of Theorem 1. First we argue that inequality

$$|u|_p(t) \le C(u_0, v_0, p, \alpha) \quad \forall_{p\in[1,\infty)} \tag{43}$$

holds. In the case of subcritical diffusion we have it for any initial mass by Corollary 2. In the critical diffusion case we need the assumption on smallness of χ and the initial mass M. Then we are in a position to apply Lemma 2 and obtain global in time bounds for $|u+1|_{LlogL}$, which in turn gives $|u+1|_3(t) \le C$ via Lemma 3. This allows us by Corollary 3 to obtain (43). In order to perform the final step, i.e. reach the bound $|u(t)|_\infty \le C$, we resort to Proposition 3. We make the following choices in the setting of this theorem, in line with its assumptions

(1) $D(x,t,s) := a(s)$, then by defintion of $a(s)$, $D(x,t,s) \in C^\infty([0,1]\times[0,\infty)^2)$ and it holds: $D(x,t,s) \ge (1+u)^{-1} \ge (2s)^{-1}$ for $s \ge 1$. Therefore assumption (1) of Proposition 3 holds with $\delta = \frac{1}{2}, m = 0, s_0 = 1$.

(2) $f := uv_x$. By Proposition 2 $uv_x \in C^1((0,T)\times[0,1])$ and as [KS] boundary condition is Neumann's one, $uv_x(0) = uv_x(1) = 0$. Moreover by Proposition 4 and inequality (43) one has $uv_x \in L^\infty(L^{q_1})$ for any $q_1 < \infty$; fix $q_1 = 17$.

(3) $u \ge 0$ with the above choices solves [KS] and fixing $p_0 = 2$, $u \in L^\infty(L^{p_0})$ because of (43).

(4) $17 = q_1 > 3$, $2 = p_0 > 1$.

Thereby assumptions of Proposition 3 are fulfilled and by its thesis we have $|u+1|_\infty(t) \le C$ □

Remark 2. *Theorem 1 can be generalized to hold for system considered in [3], i.e.:*

$$\begin{cases} u_t = (a(u)u_x - uv_x)_x \quad in\ (0,\infty)\times(0,1) \\ \varepsilon v_t = Dv_{xx} + u - M + \gamma v \quad in\ (0,T)\times(0,1) \\ a(u)u_x = v_x = 0 \quad on\ (0,T)\times\{0;1\} \\ (u,v)(0) = (u_0,v_0) \end{cases} \tag{44}$$

where ε, γ, D are nonnegative.

3 Finite-time blowup

In this section we analyze the opposite situation to that of the previous section, i.e. a possibility of a finite-time blowup of $|u(t)|_\infty$, u being the solution to [KS]. Our method is a slight extension of results in [3, 4]. Actually we are going to modify a method in [3] in the spirit of [4, Theorem 10]. The reason for it is that we want to include a wider class of nonlinearities in our result than covered in [3]. For a simplicity of presentation we assume $\chi = 1$.

Theorem 2. *Let $\chi = 1$ in [KS].For any diffusion $a \in C[0,\infty) \cap L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ there exist: such a small $\varepsilon > 0$ and (u_0, v_0) with initial mass $M = \int_0^1 u_0(x)dx$ large enough such that a solution u to [KS] emanating from initial data (u_0, v_0) blows up in a finite time.*

We postpone the proof of this result until several technical propositions are proven. Let us begin with some definitions

Definition 2. *Let L, U, V be*

(i) $L(t) := \frac{1}{q}\int_0^1 |U(t,z)|^q dz$, where $U(t,x) := \int_0^x u(t,z)dz$ for $x \in [0,1]$ and $q > 2$;

(ii) $V(t,x) := \int_0^x v(t,z)dz$ for $x \in (0,1)$.

For function B (specified further), M- the initial mass of [KS] and $q > 2$, let us formally introduce function $A_{B,q}(L)$ as follows

Definition 3. $A_{B,q} := (q-1)B^{\frac{2}{q}}(M)\left[\frac{M^{q+1}}{q+1}\right]^{\frac{q-2}{q}} \beta^{\frac{q-2}{q}}\left(\frac{M^{q+1}}{Lq(q+1)}\right) + ML[1+\frac{\varepsilon M^{q-1}}{4q^2}] - \frac{M^{q+1}}{q(q+1)}$ *for* $\beta(x) := \frac{B(x)}{x}$.

Next we state a following technical result

Proposition 7. *If there exists such a concave real function B, for which $\lim_{x\to\infty} \frac{B(x)}{x} = 0$ and $0 \le r\int_r^\infty a(s)ds \le B(r)$ where $a(s) : [0,\infty) \to \mathbb{R}_+$ is a diffusion in [KS], then a following differential inequality holds*

$$\frac{d}{dt}\left(L + \lambda + \frac{M^2}{2}\right)(t) \le A\left[\left(L + \lambda + \frac{M^2}{2}\right)\right], \tag{45}$$

where L complies with Definition 2 and λ denotes the Liapunov functional associated to [KS] by Proposition 2.

Proof. As we have already mentioned the proof is a modification of the method in [3] along the lines of [4, Theorem 10]. By nonnegativity of u solving [KS] we have: $L(t) = \frac{1}{q}\int_0^1 |U(t,z)|^q = \frac{1}{q}\int_0^1 U(t,z)^q$ and by differentiation

$$\frac{d}{dt}L(t) = \int_0^1 U(t,z)^{q-1}U_t(t,z)dz = \int_0^1 U(t,z)^{q-1}[a(u)u_z - uv_z](t,z)dz \tag{46}$$

The later inequality results from integration over interval $(0;z)$ of the first equation of [KS] and its boundary conditions. Denoting $A(s) := -\int_s^\infty a(z)dz$ and computing v_z by integration over interval $(0;z)$ of the second equation of [KS], we see that by (46) it holds

$$\frac{d}{dt}L(t) = \int_0^1 U^{q-1}[A(u)]_z - \int_0^1 U^{q-1}u[\varepsilon V_t - U + Mz]. \tag{47}$$

Integrating by parts we have

$$\frac{d}{dt}L(t) = -(q-1)\int_0^1 U^{q-2}uA(u) + [U^{q-1}A(u)]\Big|_0^1 + \int_0^1 U^{q-1}u[U - Mz - \varepsilon V_t] =$$
$$(q-1)\int_0^1 U^{q-2}[-uA(u)] + M^{q-1}A(u(1,t)) + \frac{1}{q+1}\int_0^1 (U^{q+1})_z - \frac{1}{q}\int_0^1 (U^q)_z[Mz + \varepsilon V_t]. \quad (48)$$

Observe that $A(u(1,t)) \leq 0$ because of the nonnegativity of a as well as the definition of A. Next $-uA(u) = u\int_u^\infty a(s)ds \leq B(u)$, integrating by parts and using definition $L(t) := \frac{1}{q}\int U^q dz$, we arrive at

$$\frac{d}{dt}L(t) \leq (q-1)\int_0^1 U^{q-2}B(u) + \frac{1}{q+1}M^{q+1} + \frac{1}{q}\int_0^1 U^q M - \frac{M}{q}M^q + \frac{\varepsilon}{q}\left[\left(\int U^q v_t\right) - M^q V_t(1)\right]$$
$$\leq (q-1)\int_0^1 U^{q-2}B(u) + ML - \frac{M^{q+1}}{q(q+1)} + \frac{\varepsilon}{q}M^{\frac{q}{2}}L^{\frac{1}{2}}|v_t|_2. \quad (49)$$

where the last summand comes from the definition of U and Hölder's inequality. Let us focus on the term $\int_0^1 U^{q-2}B(u)$. We use the estimates from [4, Theorem 10] involving Jensen's inequality for probabilistic measures. We present them for reader's convenience. The used measures are $\frac{B(u)dx}{\int B(u)}$ and dx, $\frac{U^q dx}{qL}$.

$$\int_0^1 U^{q-2}B(u)dx = \left(\int_0^1 B(u)dx\right)\int_0^1 (U^q)^{\frac{q-2}{q}}\frac{B(u)dx}{\int_0^1 B(u)dx} \leq \left(\int_0^1 B(u)dx\right)\left(\int_0^1 U^q\frac{B(u)dx}{\int_0^1 B(u)dx}\right)^{\frac{q-2}{q}} =$$
$$\left(\int_0^1 B(u)dx\right)^{\frac{2}{q}}(qL)^{\frac{q-2}{q}}\left(\int_0^1 B(u)\frac{U^q dx}{qL}\right)^{\frac{q-2}{q}} \leq (qL)^{\frac{q-2}{q}}\left[B\left(\int_0^1 u\right)\right]^{\frac{2}{q}}\left[B\left(\int_0^1 \frac{uU^q dx}{qL}\right)\right]^{\frac{q-2}{q}}. \quad (50)$$

Recalling that $\beta(x) := \frac{B(x)}{x}$ and in virtue of $B\left(\int_0^1 \frac{uU^q dx}{qL}\right) = B\left(\frac{U^{q+1}|_0^1}{(q+1)qL}\right)$, inequalities (49), (50) yield:

$$\frac{d}{dt}L(t) \leq (q-1)B^{\frac{2}{q}}(M)\left[\frac{M^{q+1}}{q+1}\right]^{\frac{q-2}{q}}\beta^{\frac{q-2}{q}}\left(\frac{M^{q+1}}{Lq(q+1)}\right) + ML - \frac{M^{q+1}}{q(q+1)} + \frac{\varepsilon}{q}M^{\frac{q}{2}}L^{\frac{1}{2}}|v_t|_2. \quad (51)$$

Adding (51) and (6) we arrive at

$$\frac{d}{dt}(L+\lambda)(t) \leq A_{B,q}(L) - \varepsilon\left[|v_t|_2 + \frac{1}{4q}M^{\frac{q}{2}}L\right]^2 \leq A_{B,q}(L).$$

Hence

$$\frac{d}{dt}\left(L + \lambda + \frac{M^2}{2}\right)(t) \leq A_{B,q}(L). \quad (52)$$

In view of the bound $\lambda(t) \geq -\frac{M^2}{2}$, see (6), (52) implies the claim, provided function $A_{B,q} : \mathbb{R}_+ \to \mathbb{R}$ is nondecreasing. Observe that it suffices to check that the term $\beta^{\frac{q-2}{q}}\left(\frac{M^{q+1}}{Lq(q+1)}\right)$ is nondecreasing. Its monotonicity corresponds to the one of $\beta(x^{-1})$, which is nondecreasing if and only if $\beta(x) = \frac{B(x)}{x}$ is nonincreasing. Assume the contrary, i.e. there are $x_0 > y_0 > 0$, such that $\frac{B(x_0)}{x_0} > \frac{B(y_0)}{y_0}$. Consequently, the graph of B over the interval $[y_0, x_0]$ must remain above the interval in the plane joining points $(x_0, B(x_0))$ and $(y_0, B(y_0))$ (denote it by I), because of concavity of B. Hence tangent to the graph of B at $(y_0, B(y_0))$ must be at least as steep as the straight line containing I, which forces the graph of B to lie below zero for some strictly positive values. This violates nonnegativity of B. □

In order to make use of Proposition 7 we need a following fact, which can be found as [5, Lemma 3.1].

Proposition 8. *Let $a \in C^1(0,\infty) \cap L^1(0,\infty)$, $a \geq 0$ and $\sup_{r\in(0,1)} r\int_r^\infty a(s)ds < \infty$, then there exists a concave function $B \in C[0,\infty)$ such that $B(r) \geq r\int_r^\infty a(s)ds$ and $\lim_{r\to\infty} \frac{B(r)}{r} = 0$.*

Remark 3. *Using the function constructed in [5, Lemma 3.1] automatically allows us to justify that $\beta(x)$ is nonincreasing. However we have already shown it for general B.*

Proof of Theorem 2. By Proposition 8 assumptions of Proposition 7 are fulfilled, hence we obtain

$$\frac{d}{dt}\left(L + \lambda + \frac{M^2}{2}\right)(t) \leq A_{B,q}\left[\left(L + \lambda + \frac{M^2}{2}\right)(t)\right]. \tag{53}$$

We aim now at showing that for some u_0, v_0 one has

$$A_{B,q}\left[\left(L + \lambda + \frac{M^2}{2}\right)(0)\right] < 0. \tag{54}$$

To this end recall that by definition (compare Proposition 2)

$$\lambda(u,v) := \int\left[b(u) - uv + \frac{1}{2}|v_x|^2\right], \tag{55}$$

where $b : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying: $b''(s) = \frac{a(s)}{s}$, $b'(1) = 0$, $b(1) = 0$, therefore one can set

$$b(x) = \begin{cases} \int_x^1 \left(\int_r^1 \frac{a(s)}{s} ds\right) dr & for\ x \leq 1, \\ \int_1^x \left(\int_1^r \frac{a(s)}{s} ds\right) dr & for\ x \geq 1, \end{cases} \tag{56}$$

which implies a following bound on b,

$$b(x) \leq |a|_\infty \left|\int_x^1 \left(\int_r^1 \frac{ds}{s}\right) dr\right| = C|1 - x + xlnx| \leq \begin{Bmatrix} C(1+x) & for\ x \leq 1 \\ C(1+x^2) & for\ x \geq 1 \end{Bmatrix} \leq C(1+x^2). \tag{57}$$

Using the inequality $|v|_\infty \leq K|v|_{1,2}$ one can merge (55) with (57) into

$$\lambda(u,v) + \frac{M^2}{2} \leq C\int u^2 + |v|_\infty \int u + \frac{1}{2}|v|_{1,2}^2 + \frac{M^2}{2} + C \leq C\int u^2 + |v|_{1,2}\int u + \frac{1}{2}|v|_{1,2}^2 + \frac{M^2}{2} + C. \tag{58}$$

Since it is satisfactory to give any examples of u_0, v_0 leading to the validity of (54), we are free to make a following choice for $M > 1$

$$\begin{aligned} u_0 &= 2M^3\left(x + \tfrac{1}{M} - 1\right)\mathbb{1}_{[1-\frac{1}{M},1]}(x), \\ v_0 &= Mx - \tfrac{M}{2}, \end{aligned} \tag{59}$$

which yields

$$\int u_0 = M, \quad \int u_0^2 = \frac{4}{3}M^3, \quad |v_0|_{1,2}^2 = \frac{13}{12}M^2, \tag{60}$$

$$L(0) := \frac{1}{q}\int_0^1 \left[\int_0^x u_0(z)dz\right]^q dx = \frac{2^q M^{q-1}}{q(2q+1)}. \tag{61}$$

In view of (61), (60), (58) one estimates

$$\frac{2^q M^{q-1}}{2q(q+1)} \leq L(0) + \lambda(u_0,v_0) + \frac{M^2}{2} \leq \frac{2^q M^{q-1}}{(q+1)q} + CM^3 + C. \tag{62}$$

Recall that for B it holds $\lim_{x\to\infty} \frac{B(x)}{x} = 0$. Let us choose M_0 such that

$$\forall_{M\geq M_0}\ B(M) \leq M; \quad B\left(\frac{M^{q+1}}{\left(L(0) + \lambda(u_0,v_0) + \frac{M^2}{2}\right)q(q+1)}\right) \leq c\frac{M^{q+1}}{\left(L(0) + \lambda(u_0,v_0) + \frac{M^2}{2}\right)q(q+1)}, \tag{63}$$

while the latter is possible for fixed $q > 2$ in view of (62). Let us now calculate, using definition of A as well as (63)

$$A_{B,q}\left[\left(L+\lambda+\frac{M^2}{2}\right)(0)\right]=$$

$$(q-1)q^{q-2}B^{\frac{2}{q}}(M)\left[L(0)+\lambda(u_0,v_0)+\frac{M^2}{2}\right]^{\frac{q-2}{q}}B^{\frac{q-2}{q}}\left(\frac{M^{q+1}}{\left(L(0)+\lambda(u_0,v_0)+\frac{M^2}{2}\right)q(q+1)}\right)$$

$$+M\left[1+\frac{\varepsilon M^{q-1}}{4q^2}\right]\left[L(0)+\lambda(u_0,v_0)+\frac{M^2}{2}\right]-\frac{M^{q+1}}{q(q+1)}\leq$$

$$c(q)M^{\frac{2}{q}}M^{\frac{(q+1)(q-2)}{q}}+cM\left[\frac{2^qM^{q-1}}{2q(q+1)}+CM^3+C\right]-\frac{M^{q+1}}{q(q+1)}\leq$$

$$C(q)[M^q+M^4+1]-\frac{M^{q+1}}{q(q+1)},$$

where we chose $\varepsilon := \frac{1}{M^{q-1}}$, which gives us the smallness condition on ε already mentioned in the Theorem 2. Therefore assuming $q > 4$ (54) holds for M large enough. Knowing this, we conclude in a following standard way that $|u|_\infty$ blows up in a finite time. First assume that $T_{\max} = \infty$, then by (54) and (53) we obtain that for finite t, $L(t)$ becomes negative, which is absurd. By Proposition 1 this implies $T_{\max} < \infty$ and a finite-time blowup. □

4 Conclusions

The present paper has been aimed as a first step in the studies of correspondence between qualitative behaviour of solutions to one-dimensional quasilinear parabolic-elliptic and fully parabolic Keller-Segel systems. As we mentioned in the introduction, this question is worth of studies, since in higher dimensions and biologically more relevant cases, rigorous results are (almost always) presented in a parabolic-elliptic case, while this is a fully parabolic one which forms an original model. We focused on the question whether in a fully parabolic case, like in the parabolic-elliptic one, there is no critical nonlinearity. It is known that such a phenomenon takes place in the latter (see [4][5]). We did not succeed in answering this question, however we managed to make a first step in studying this problem. On the one hand we fully proved global-in-time existence of solutions in subcritical cases, on the other we proved finite-time blowup of solutions when nonlinear diffusion is integrable at infinity.

Moreover, our studies of case when a nonlinear diffusion is of the form $\frac{1}{u+1}$ not only gives the global lifespan of a solution starting from initial mass and chemosensitivity small enough. It also suggests that studying nonlinear functional inequalities of the type (33), i.e.

$$\int e^{2m} < \delta \int |m_x|^2 + h_\delta\left(\int e^m\right) \tag{64}$$

is meaningful for full understanding the case of the critical nonlinear diffusion $\frac{1}{u+1}$. Having such inequality, Corollary 4 holds with no extra restrictions on initial mass or chemosensitivity, therefore by Lemma 2 we get the boundedness of $|u+1|_{LlogL}$. Next, we notice that we can apply Lemma 3 and Corollary 3.

So far we are unable to prove neither the above relaxed version nor (33). However, we state and prove a similar inequalities that seem to be a perturbation of what is required. Namely, we have:

Proposition 9. *Let $3/2 < c < 2$. For $m \in W^{1,2}(0,1)$ it holds*

$$\int e^{cm} \leq \delta \int |m_x|^2 + C_\delta\left(\int e^m\right)^{\frac{c}{2-c}} + \left(\int e^m\right)^c,$$

where $C_\delta = C\delta^{\frac{1-c}{2-c}}$.

Proof. By the Gagliardo-Nirenberg inequality

$$\int e^{cm} = |e^{\frac{m}{2}}|_{2c}^{2c} \leq |e^{\frac{m}{2}}|_{1,1}^{2\theta c} |e^{\frac{m}{2}}|_{2}^{(1-\theta)2c},$$

where $\theta = \frac{c-1}{c}$. Next, by Hölder's inequality

$$\int e^{cm} \leq \left[\frac{1}{2}\left(\int e^m\right)^{\frac{1}{2}} \left(\int |m_x|^2\right)^{\frac{1}{2}} + \int e^{\frac{m}{2}}\right]^{2(c-1)} \int e^m \leq \frac{1}{2}\left(\int e^m\right)^c \left(\int |m_x|^2\right)^{c-1} + 2^{2c-3}\left(\int e^m\right)^c$$

Now we are in a position to apply Young's inequality which completes the proof. □

One sees that the right-hand side blows up when c approaches 2. Unfortunately, it explodes in a very bad manner, while it's not only a polynomial nonlinearity in the power of $\int_0^1 e^m$ which is blowing up, since the same happens to the constant C_δ. One may argue that applying the Gagliardo-Nirenberg and Hölder's inequalities may be too generous. Another possibility is that perhaps seeking a proper form of (33) one should be looking for function f which is faster than polynomial, in the spirit of Trudinger-Moser inequality.

Let us notice that in a case of diffusion a which is still nonintegrable at infinity but decays faster than $a(u) = \frac{1}{u+1}$ the following inequality if it is available for any $\delta > 0$, $m \in W^{1,2}(0,1)$

$$\int e^{2m} \leq \delta \int \frac{a(u)}{u}|m_x|^2 + C_\delta f\left(\int e^m\right)$$

yields an estimate for $|u|_{LlogL}$.

Finally let us mention that our finite-time blowup result holds true only for large initial masses and small factors $\varepsilon > 0$ by the time derivative of v. It is an interesting open problem if one can, like in the parabolic-elliptic case, achieve this result for any size of the initial mass. Another challenging problem is if the finite-time explosion of the solution can happen for $\varepsilon > 0$ large.

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