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On the flow of chemically reacting gaseous mixture.

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Abstract: We consider the Cauchy problem for the system of equations governing flow of isothermal reactive mixture of compressible gases. Our main contribution is to prove sequential stability of weak solutions when the state equation essentially depends on the species concentration and the viscosity coefficients vanish on vacuum. Moreover, under additional assumption on the "cold" component of the pressure in the regions of small density, we prove the existence of weak solutions for arbitrary large initial data.
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1 Introduction

We investigate the system of equations describing flow of two-component compressible gaseous mixture in the whole space $\Omega = \mathbb{R}^3$ or in the periodic domain $\Omega = \mathbb{T}^3$. The species A and B undergo an isothermal, reversible chemical reaction

$$A \leftrightarrows B.$$

The dynamics of such fluid may be characterized by the total mass density $\varrho = \varrho(t,x)$ being the sum of species densities $\varrho = \varrho_A + \varrho_B$, the velocity vector field $\mathbf{u} = \mathbf{u}(t,x)$ and the species A mass fraction $Y_A = Y_A(t,x)$. The following equations express the physical laws of conservation of mass, momentum and the balance of species mass, respectively:

$$\left.\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \\ \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(2\mu\mathbf{D}(\mathbf{u})) - \nabla(\nu\operatorname{div}\mathbf{u}) + \nabla p = 0 \\ \partial_t(\varrho Y_A) + \operatorname{div}(\varrho Y_A\mathbf{u}) + \operatorname{div}(\mathcal{F}_A) = \varrho\omega_k, \quad k = 1,\dots,n \end{aligned}\right\} \text{in} \quad (0,T)\times\Omega. \tag{1}$$

Here, $\mathbf{D}(\mathbf{u})$ denotes the symmetric part of the velocity gradient $\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^T\mathbf{u}\right)$, $p = p(\varrho, Y_A, Y_B)$ is the internal pressure, $\omega = \omega(\varrho, Y_A, Y_B)$ is the species A production rate, $\mathcal{F}_A = \mathcal{F}_A(\varrho, Y_A, Y_B)$ denotes the diffusion flux of the species A and $\mu = \mu(\varrho)$, $\nu = \nu(\varrho)$ are the two Lamé viscosity coefficients satisfying

$$\mu(\varrho) > 0, \quad 2\mu(\varrho) + 3\nu(\varrho) \geq 0.$$

We remark that the model is consistent with the principle of mass conservation, thus necessarily

$$\int_\Omega \varrho(t)\ \mathrm{d}x = const.$$

for all $t \in [0,T]$. In some cases it will be more convenient to switch to another formalism, i.e. to use the notion of partial densities ϱ_A, ϱ_B instead of mass fractions Y_A, Y_B. They are related by $Y_i = \frac{\varrho_i}{\varrho}$,

for $i \in S$, where $S = \{A, B\}$.
We assume that the pressure $p = p(\varrho, Y_A, Y_B)$ obeys the following state equation

$$p(\varrho, Y_A, Y_B) = p_E(\varrho) + p_M(\varrho, Y_A, Y_B), \tag{2}$$

where $p_E(\varrho) = \varrho^\gamma$, $\gamma > 1$ is the barotropic part of the pressure also referred to as a "cold pressure", since for the heatconducting gases this is the only nonvanishing part when temperature tends to absolute 0. By p_M we denote the classical molecular pressure given, in accordance with the Boyle law, by the constitutive equation

$$p_M = \sum_{k \in S} p_k = \varrho \left(\sum_{k \in S} \frac{Y_k}{m_k} \right), \tag{3}$$

where m_k is the molar mass of k-th species (we take the perfect gas constant=1) and we assume that $m_A \neq m_B$.
The species mass flux $\mathcal{F}_A$ yields diffusion effects due to the mole fraction gradients and pressure gradients and is given in a general form

$$\mathcal{F}_k = -\sum_{l \in S} C_{kl} \mathbf{d}_l, \quad k \in S, \tag{4}$$

where $\mathbf{d}_k$– the diffusion force for the k-th species depends on the gradient of molecular pressure in the following way

$$\mathbf{d}_k = \nabla \left(\frac{p_k}{p_M} \right) + \left(\frac{p_k}{p_M} - \frac{\varrho_k}{\varrho} \right) \nabla \log p_M.$$

and C_{kl}, $k, l \in S$ are the multicomponent flux diffusion coefficients. Supposing the following form of the matrix C (see Giovangigli [13], Chapter 7):

$$C = C_0(\varrho, Y_A, Y_B) \begin{pmatrix} Y_B & -Y_A \\ -Y_B & Y_A \end{pmatrix}, \tag{5}$$

we verify, by use of (4), that

$$\mathcal{F}_A = -C_0 \mathbf{d}_A = -\frac{C_0}{p} \left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B} \right) \nabla \varrho_A - \frac{\varrho_A}{\varrho m_B} \nabla \varrho \right),$$

$$\mathcal{F}_B = -C_0 \mathbf{d}_B = -\frac{C_0}{p} \left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B} \right) \nabla \varrho_B - \frac{\varrho_B}{\varrho m_A} \nabla \varrho \right).$$

and we assume that the diffusion coefficient C_0 is proportional to the Boyle pressure $C_0 \approx p_M$ (we take $\frac{C_0}{p_M} = 1$).
An important consequence of (5) is that $\mathcal{F}_B + \mathcal{F}_A = 0$, therefore we can consider only the first mass fraction as unknown and use the relation

$$Y_A + Y_B = 1, \tag{6}$$

to evaluate the mass fraction of the remaining species.
The molar production rate ω is usually approximated by a continuously differentiable function proportional to concentration of substrates and products. We will additionally postulate existence of constants $\underline{\omega}$ and $\overline{\omega}$ such that

$$-\underline{\omega} \leq \omega(Y_A, Y_B) \leq \overline{\omega}, \qquad \text{for all } 0 \leq Y_A, Y_B \leq 1, \tag{7}$$

and we suppose

$$\omega(Y_A, Y_B) \geq 0 \quad \text{whenever} \quad Y_A = 0. \tag{8}$$

Following Mellet & Vasseur [17] we assume that the viscosity coefficients $\mu(\varrho)$, $\nu(\varrho)$ are $C^2(0, \infty)$ functions satisfying

$$\nu(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho), \tag{9}$$

known as the Bresch-Desjardins relation [1].
Furthermore, we stipulate that there exists positive constant $r \in (0,1)$ such that

$$\begin{aligned} \mu'(\varrho) \geq r, \quad \mu(0) \geq 0, \\ |\nu'(\varrho)| \leq \tfrac{1}{r}\mu'(\varrho), \\ r\mu(\varrho) \leq 2\mu(\varrho) + 3\nu(\varrho) \leq \tfrac{1}{r}\mu(\varrho). \end{aligned} \tag{10}$$

In addition, for arbitrary small $\varepsilon > 0$ and $\gamma \geq 3$ we suppose that

$$\liminf_{\varrho \to \infty} \frac{\mu(\varrho)}{\varrho^{\frac{\gamma}{3}+\varepsilon}} > 0. \tag{11}$$

The main difficulty concerning systems with viscosity coefficients vanishing when density equals 0 is lack of information about the velocity vector field. It is no longer in $L^2_{\rm loc}((0,T)\times\Omega)$ as in the case for constant viscosity coefficients. In fact, it cannot even be defined on vacuum. Although this degeneracy causes additional difficulties, it also contributes some benefits. Namely, it provides particular mathematical structure that yields global in time integrability of $\nabla\sqrt{\varrho}$. This property was observed for the first time by Bresch, Desjardins & Lin [4] for the Korteweg equations and for the 2-dimensional viscous shallow water model [1]. Later on, Mellet & Vasseur coupled these ideas with the additional estimate for the norm of $\varrho\mathbf{u}^2$ in $L^\infty(0,T;L\log L(\Omega))$ and proved the sequential stability of weak solutions to the barotropic compressible Navier-Stokes system with the viscosity coefficients given by (9-11). Concerning the stability result, it is possible to extend this approach to treat the case of selfgravitating [9] gases, however existence of regular approximate solutions in this framework is still elusive. The main difficulty is to preserve the logarithmic estimate for the velocity at the level of construction of solution. To the best of our knowledge, when no additional drag terms are present, this is still an open problem.

Nevertheless, some progress has been achieved in the case when further assumption on the zero Kelvin isothermal curve of the equation of state in the neighbourhood of small densities is enforced. This strategy was proposed in the work of Bresch and Desjardins [3] for the heat conducting fluids as a way to get close to a solid state in tension. Their condition was designed to recover the standard cold component of the pressure ϱ^γ far from vacuum and to encompass plasticity and elasticity effects of solid materials, for which low densities may lead to negative pressures. By this modification the compactness of velocity can be obtained without requiring more a priori regularity than expected from the usual energy approach. In this framework the globally well posed system can be constructed by parabolic regularization of the total and partial masses conservation equations and by adding to the momentum equation the capillarity force regularizing the density together with the hyperdiffusive term providing integrability of higher derivatives of velocity. Then, the existence of solutions follows from the fixed point argument applied to the momentum equation combined with the standard theory for the semi-linear parabolic equation of species production.

This is, in a sense, opposite with respect to systems with constant viscosity coefficients, for which the main difficulty is lack of sufficient information about density. The first rigorous existence theory in this field was performed in the seminal work of Lions [16]. He was able to show global in time weak solvability of compressible Navier-Stokes system for arbitrary large initial data and for $\gamma \geq \frac{9}{5}$. Later on, his ideas were extended by Feireisl to handle the case when the density is not square integrable [10]. The overview of these methods can be found in [21]. More recently, the theory for barotropic fluids was transferred into the heat conducting case. The question of existence of weak variational solutions has been addressed in [11] for evolutionary case with $\gamma \geq \frac{5}{3}$. This is the only known result including temperature dependence in the viscosity coefficients satisfying physically acceptable growth conditions. Analogous result for the stationary flow were presented in [18], [20] and then improved in [22], where the authors proved that if $\gamma > \frac{4}{3}$ then these solutions also fulfill the weak formulation of the pointwise total energy balance.

Much less is known about models that include chemical reactions. For the evolutionary case the existence of global in time solutions to system (1) coupled with the internal energy balance and supplemented by physically relevant constitutive relations was established by Giovangigli [13]. He assumed, however, that the initial conditions are sufficiently close to an equilibrium state.

Concerning large initial data, the first proof of existence of weak variational solutions to a system with arbitrary large number of reversible reactions is due to Feireisl, Petzeltová & Trivisa [12]. They considered temperature-dependent viscosity coefficients and the species diffusion fluxes given by the Fick law

$$\mathcal{F}_k = -\mathcal{D}_k \nabla Y_k, \quad k = 1, \ldots, n. \tag{12}$$

Unfortunately, it seems that their approach can not be applied to the case when pressure depends on the species concentration, mainly because of undetermined sign of entropy production rate in the associated entropy balance.

Regarding simplified models, the situation presents better, especially in case of one-dimensional models of irreversible reactions that were studied in a series of articles [15], [5], [8] and for the multidimensional combustion models. As far as the latter are concerned, the global existence of weak solutions with large initial data was obtained in [7] and then extended in [6] to treat dependence of pressure on the mass fraction of fuel. The case of one isothermal reversible reaction with pressure depending on concentration of all species with adiabatic exponent for the mixture γ greater than $\frac{7}{3}$ was srudied for the steady flow in [23].

The objective of this work is to investigate the issue of large data existence of solutions for the system (1). Let us emphasize that the model we consider is consistent with principles of continuum mechanics and does not violate the second law of thermodynamics when the heat conductivity is taken into account. In contrast, the presence of the species concentration in the state equation and approximation of the diffusion flux by the Fick law (12) would result in the entropy production rate which may fail to be non-negative. This, in turn, would contradict thermodynamic admissibility of the process. In consequence, to be phisically consistent, one has to deal with more general form of diffusion (4) leading to a new type of degeneration in the system (1) which involves the second space derivatives of ϱ. Therefore, more regularity for the density, than we can prove for the Navier-Stokes-type systems with constant viscosity coefficients, is needed. Here, the theory developed in [4], [17] is applied as a possible way to overcome this difficulty.

In the first part of present paper we establish the sequential stability of weak solutions to system (1) i.e. the closedness of the family of solutions bounded by a priori bounds in the framework of weak formulation. Then, we complement this result by constructing regular enough approximate solutions which preserve the mathematical structure of the system, but only when further restriction on the pressure is postulated.

The paper is organized as follows. In Section 2 we introduce the notion of weak solutions and formulate our first result– sequential stability of weak solutions. Then, in Section 3, we state a priori estimates which will be used throughout the proof of Theorem 1 presented in Section 4. We remark, that in order to avoid unnecessary repetitions, we will frequently refer to results known from the theory of single-component flows. Section 5 gives some insight into the scheme of construction of approximate solutions for the system with the cold component of the pressure modified close to vacuum.

2 Weak formulation

We consider system (1) with the initial conditions

$$\varrho(0,x) = \varrho^0(x), \quad \varrho\mathbf{u}(0,x) = \mathbf{m}^0(x), \quad \varrho Y_A(0,x) = \varrho_A^0(x) \quad \text{for all } x \in \Omega. \tag{13}$$

Then, the aim of this part of work is to prove the sequential stability of weak solutions to (1-11) and (13) specified by the following definition.

Definition 1 *A triple $(\varrho, \mathbf{u}, Y_A)$ is said to be a weak solution of (1-11) supplemented with the initial data (13) if the following conditions are satisfied:*

1.

$$
\begin{gathered}
\varrho \in L^\infty(0,T;L^1\cap L^\gamma(\Omega)),\\
\sqrt{\varrho} \in L^\infty(0,T;H^1(\Omega)),\\
\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega)),\\
\mu(\varrho)\mathbf{D}(\mathbf{u}) \in L^2(0,T;\big(W^{1,\infty}(\Omega)\big)^*_{\rm loc}),\quad \nu(\varrho)\operatorname{div}\mathbf{u} \in L^2(0,T;\big(W^{1,\infty}(\Omega)\big)^*_{\rm loc}),\\
\varrho \geq 0,\quad 0\leq Y_A\leq 1,\quad \textit{a.e. in } (0,T)\times\Omega,\\
\nabla(\varrho Y_A) \in L^2(0,T;\big(W^{1,\infty}(\Omega)\big)^*_{\rm loc}).
\end{gathered}
$$

2. The continuity equation

$$
\begin{cases}
\partial_t\varrho + \operatorname{div}(\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}) = 0\\
\varrho(0,x) = \varrho^0(x)
\end{cases}
$$

is satisfied in the sense of distributions.
3. The weak formulation of the momentum equation

$$
\begin{aligned}
&\int_\Omega \mathbf{m}^0\cdot\phi(0,x)\ \mathrm{d}x + \int_0^T\!\!\int_\Omega (\sqrt{\varrho}(\sqrt{\varrho}\mathbf{u})\cdot\partial_t\phi + \sqrt{\varrho}\mathbf{u}\otimes\sqrt{\varrho}\mathbf{u}:\nabla\phi)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad + \int_0^T\!\!\int_\Omega p(\varrho,Y_A,Y_B)\operatorname{div}\phi\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \langle 2\mu(\varrho)\mathbf{D}(\mathbf{u}),\nabla\phi\rangle\ \mathrm{d}t - \int_0^T \langle \nu(\varrho)\operatorname{div}\mathbf{u},\operatorname{div}\phi\rangle\ \mathrm{d}t = 0
\end{aligned}
$$

holds for any smooth, compactly supported test function $\phi(t,x)$ such that $\phi(T,\cdot)=0$. In this formula, the last two terms should be understood as

$$
\begin{aligned}
\langle 2\mu(\varrho)\mathbf{D}(\mathbf{u}),\nabla\phi\rangle = &-\int_\Omega \frac{\mu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_j\partial_{ii}\phi_j\ \mathrm{d}x - 2\int_\Omega \mu'(\varrho)\sqrt{\varrho}\mathbf{u}_j\partial_i\sqrt{\varrho}\partial_i\phi_j\ \mathrm{d}x\\
&-\int_\Omega \frac{\mu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_i\partial_{ji}\phi_j\ \mathrm{d}x - 2\int_\Omega \mu'(\varrho)\sqrt{\varrho}\mathbf{u}_i\partial_j\sqrt{\varrho}\partial_i\phi_j\ \mathrm{d}x
\end{aligned}
$$

and

$$
\langle \nu(\varrho)\operatorname{div}\mathbf{u},\operatorname{div}\phi\rangle = -\int_\Omega \frac{\nu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_i\partial_{ij}\phi_j\ \mathrm{d}x - 2\int_\Omega \nu'(\varrho)\sqrt{\varrho}\mathbf{u}_i\partial_i\sqrt{\varrho}\partial_j\phi_j\ \mathrm{d}x.
$$

4. The weak formulation of the mass balance equation for species A

$$
\begin{aligned}
&\int_\Omega \varrho_A^0\cdot\psi(0,x)\ \mathrm{d}x + \int_0^T\!\!\int_\Omega (\sqrt{\varrho}Y_A\sqrt{\varrho}\mathbf{u}\cdot\partial_t\psi + \sqrt{\varrho}Y_A\sqrt{\varrho}\mathbf{u}\cdot\nabla\psi)\ \mathrm{d}x\ \mathrm{d}t\\
&\qquad + \int_0^T \langle \mathcal{F}_A,\nabla\psi\rangle\ \mathrm{d}t = \int_0^T\!\!\int_\Omega \varrho\omega\psi\ \mathrm{d}x\ \mathrm{d}t
\end{aligned}
$$

is satisfied for any smooth, compactly supported test function $\psi(t,x)$ such that $\psi(T,\cdot)=0$, where the last term on the left hand side (l.h.s.) denotes

$$
\begin{aligned}
\langle \mathcal{F}_A,\nabla\psi\rangle = \frac{1}{m_A}\int_\Omega \varrho Y_A\Delta\psi\ \mathrm{d}x + \frac{2}{m_A}\int \sqrt{\varrho}Y_A\nabla\sqrt{\varrho}\cdot\nabla\psi + \left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \sqrt{\varrho}Y_A^2\nabla\sqrt{\varrho}\cdot\nabla\psi\ \mathrm{d}x\\
-\frac{1}{2}\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \varrho Y_A^2\Delta\psi\ \mathrm{d}x.
\end{aligned}
$$

We can now formulate our main result.

Theorem 1 *Let $\gamma>1$ and let $\mu(\varrho)$, $\nu(\varrho)$ be two $C^2(0,\infty)$ functions satisfying (9-11). Assume that $\{\varrho_n,\mathbf{u}_n,Y_{A,n}\}_{n\in\mathbb{N}}$ is a sequence of weak solutions to (1-11) satisfying energy-entropy inequalities (17), (19) and (25), with the initial data*

$$
\varrho_n(0,x)=\varrho_n^0(x),\quad \varrho_n\mathbf{u}_n(0,x)=\mathbf{m}_n^0(x)=\varrho_n^0(x)\mathbf{u}_n^0(x),\quad \varrho_nY_{A,n}(0,x)=\varrho_{A,n}^0(x)=\varrho_n^0(x)Y_{A,n}^0(x),
$$

satisfying

$$\varrho_n^0 \geq 0, \quad \varrho_n^0 \to \varrho^0 \quad in\ L^1(\Omega), \quad \varrho_n^0 \mathbf{u}_n^0 \to \varrho^0 \mathbf{u}^0 \quad in\ L^1(\Omega),$$
$$0 \leq Y_{A,n}^0 \leq 1, \quad \varrho_n^0 Y_{A,n}^0 \to \varrho^0 Y_A^0 \quad in\ L^1(\Omega),$$

together with the following bounds

$$\begin{gathered} \int_\Omega \left(\tfrac{1}{2}\varrho_n^0 \left|\mathbf{u}_n^0\right|^2 + \tfrac{1}{\gamma-1}\left(\varrho_n^0\right)^\gamma - \tfrac{1}{m_B}\varrho_n^0 \log \varrho_n^0 \right) \,\mathrm{d}x \leq C, \quad \int_\Omega \tfrac{1}{\varrho_n^0} \left|\nabla \mu\left(\varrho_n^0\right)\right|^2 \,\mathrm{d}x \leq C, \\ \int_\Omega \varrho_n^0 \left(Y_{A,n}^0\right)^2 \,\mathrm{d}x \leq C, \quad \int_\Omega \varrho_n^0 \left(1 + \left|\mathbf{u}_n^0\right|^2\right) \ln\left(1 + \left|\mathbf{u}_n^0\right|^2\right) \,\mathrm{d}x \leq C. \end{gathered} \tag{14}$$

Then, up to a subsequence, $\{\varrho_n, \sqrt{\varrho_n}\mathbf{u}_n, Y_{A,n}\}$ *converges strongly to the weak solution of the problem (1-11) in the sense of the above definition. More precisely, we have*

$$\begin{gathered} \varrho_n \to \varrho \quad strongly\ in\ C^0(0,T; L^{\frac{3}{2}}_{\mathrm{loc}}(\Omega)), \\ \sqrt{\varrho_n}\mathbf{u}_n \to \sqrt{\varrho}\mathbf{u} \quad strongly\ in\ L^2(0,T; L^2_{\mathrm{loc}}(\Omega)), \\ \mathbf{m}_n = \varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \quad strongly\ in\ L^2(0,T; L^1_{\mathrm{loc}}(\Omega)), \\ Y_{A,n} \to Y_A \quad strongly\ in\ L^p(0,T; L^p_{\mathrm{loc}}(\Omega)), \end{gathered}$$

for any p *finite and any* $T > 0$.

3 A priori estimates

In this section we state the a priori estimates, being derived under the hypothesis that all quantities in question are smooth enough to justify our manipulations.
We start with the conservation of mass. Integrating the continuity equation over Ω we deduce that

$$\frac{d}{dt}\int_\Omega \varrho \,\mathrm{d}x = 0,$$

i.e. knowing that $\int_\Omega \varrho^0(x)\,\mathrm{d}x = M$ we have $\int_\Omega \varrho(t,x)\,\mathrm{d}x = M$ for any $t \in [0,T]$.
Correspondingly, the sum of masses of both species must be conserved, in particular we have the following lemma (a kind of weak maximal principle).

Lemma 2 *For any smooth solution of (1) we have*

$$Y_A, Y_B \geq 0 \quad \text{on } \Omega \times (0,T), \tag{15}$$

and

$$Y_A + Y_B = 1. \tag{16}$$

Proof. Let ϕ_ε be a sequence of smooth functions such that

$$\begin{gathered} \operatorname{supp} \phi_\varepsilon \subset \Omega_T^-, \quad 0 \leq \phi_\varepsilon \leq 1 \\ \phi_\varepsilon(x) = 1 \quad \text{for} \quad \operatorname{dist}((t,x), \partial\Omega_T^-) \geq \varepsilon, \end{gathered}$$

where $\Omega_T^- = \{(t,x) \in ((0,T)\times\Omega) : Y_A(t,x) < 0\}$.[1]
Multiplying the species mass balance equation by ϕ_ε and integrating over $(0,T)\times\Omega$ we obtain

$$\begin{aligned} &-\int_{\Omega_T^-} \varrho Y_A \partial_t \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T^-} \varrho Y_A \mathbf{u}\cdot\nabla\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^-} \frac{1}{m_A} Y_A \nabla\varrho\cdot\nabla\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Omega_T^-} \frac{1}{m_A}\varrho\nabla Y_A\cdot\nabla\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T^-} Y_A \nabla p_M(\varrho, Y)\cdot\nabla\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T^-} \varrho\omega(Y)\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

[1] If Ω_T^- is not a regular domain, we take $\Omega_{T,\delta_n}^- = \{(t,x) \in ((0,T)\times\Omega) : Y_A(t,x) < \delta_n\}$ for $\delta_n > 0$ and pass with $\delta_n \to 0^+$.

Observe that when $\varepsilon \to 0^+$ then the four-component vector $(\partial_t\phi_\varepsilon, \nabla\phi_\varepsilon)$ approximates $-\mathbf{n} = -(n_t, \mathbf{n}_x)$, which is the inter normal vector to the boundary of Ω_T^-, so we get

$$\int_{\partial\Omega_T^-} \varrho Y_A n_t \, \mathrm{d}S_{t,x} + \int_{\partial\Omega_T^-} \varrho Y_A \mathbf{u}\cdot\mathbf{n}_x \, \mathrm{d}S_{t,x} - \int_{\partial\Omega_T^-} \frac{1}{m_A} Y_A \nabla\varrho \cdot \mathbf{n}_x \, \mathrm{d}S_{t,x}$$
$$- \int_{\partial\Omega_T^-} \frac{1}{m_A} \varrho \nabla Y_A \cdot \mathbf{n}_x \, \mathrm{d}S_{t,x} + \int_{\partial\Omega_T^-} Y_A \nabla p_M(\varrho, Y)\cdot \mathbf{n}_x \, \mathrm{d}S_{t,x} = \int_{\Omega_T^-} \varrho\omega(Y) \, \mathrm{d}x \, \mathrm{d}t.$$

Now, due to the fact that $Y_A|_{\partial\Omega_T^-} = 0$ all but the penultimate integral from the l.h.s. vanish and we are left only with

$$- \int_{\partial\Omega_T^-} \frac{1}{m_A} \varrho \nabla Y_A \cdot \mathbf{n}_x \, \mathrm{d}S_{t,x} = \int_{\Omega_T^-} \varrho\omega(Y) \, \mathrm{d}x \, \mathrm{d}t.$$

Due to assumption (8), the right hand side (r.h.s.) of the above equality is nonnegative. On the other hand, we know that $\left.\frac{\partial Y_A}{\partial \mathbf{n}}\right|_{\partial\Omega_T^-}$ is positive, hence the l.h.s. must be nonpositive. Therefore, the only possibility is that the Lebesgue measure of the set Ω_T^- is equal 0. In particular, in view of smoothness of Y_A we have (15) and then, the similar token applied to the continuity equation enables to verify (16).□

In the next step we present the usual energy approach to the second equation of system (1) which leads to the following equality.

Lemma 3 *The following equality holds for any smooth solution of (1)*

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\varrho^\gamma - \frac{1}{m_B}\varrho\log\varrho\right) \, \mathrm{d}x + \int_\Omega 2\mu(\varrho)|\mathbf{D}(\mathbf{u})|^2 \, \mathrm{d}x + \int_\Omega \nu(\varrho)|\operatorname{div}\mathbf{u}|^2 \, \mathrm{d}x$$
$$- \int_\Omega \varrho Y_A \left(\frac{1}{m_A} - \frac{1}{m_B}\right) \operatorname{div}\mathbf{u} \, \mathrm{d}x = 0. \quad (17)$$

Proof. We test the momentum equation by $\mathbf{u}$ and integrate by parts. □

Transforming the last term from the l.h.s. of (17), we can derive some useful bounds. First observe that due to Lemma 3 we may apply the Cauchy inequality (with ϵ) to estimate

$$\int_\Omega \varrho Y_A \left(\frac{1}{m_A} - \frac{1}{m_B}\right) \operatorname{div}\mathbf{u} \, \mathrm{d}x \le \int_\Omega \frac{\varrho^{\frac12}}{\mu(\varrho)^{\frac12}} \mu(\varrho)^{\frac12} |\operatorname{div}\mathbf{u}| \varrho^{\frac12} \, \mathrm{d}x \le \epsilon \int_\Omega \frac{\varrho}{\mu(\varrho)} \mu(\varrho) |\operatorname{div}\mathbf{u}|^2 \, \mathrm{d}x + C(\epsilon)\int_\Omega \varrho \, \mathrm{d}x.$$

The last term is controlled since $\varrho \in L_\infty(0,T;L^1(\Omega))$, while the first one is absorbed by the l.h.s. of (17) provided that

$$\mu(\varrho) \ge C\varrho^m \quad \text{for} \quad \varrho > 1, \; m \ge 1,$$
$$\mu(\varrho) \ge C\varrho^n \quad \text{for} \quad \varrho \le 1, \; n \le 1$$

and that ϵ is sufficiently small.
Therefore, assuming that the initial conditions satisfy

$$\int_\Omega \left(\frac{1}{2}\varrho^0 \left|\mathbf{u}^0\right|^2 + \frac{1}{\gamma-1}(\varrho^0)^\gamma - \frac{1}{m_B}\varrho^0 \log\varrho^0\right) \, \mathrm{d}x \le C,$$

and by the fact that $L^\infty(0,T;L^1(\Omega))$ norm of $\varrho\log\varrho$ may be estimated by the norm of density in $L^\infty(0,T;L^1(\Omega))$ and in $L^\infty(0,T;L^\gamma(\Omega))$, we get, due to (10), the following estimate

$$\|\sqrt{\varrho}\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\varrho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} + \|\sqrt{\mu(\varrho)}\mathbf{D}(\mathbf{u})\|^2_{L^2(0,T;L^2(\Omega))} \le C. \quad (18)$$

In order to proceed we need to find some better estimate of the norm of density than in $L^\infty(0,T;L^\gamma(\Omega))$. It will be a consequence of integrability of gradient of ϱ obtained by a modification of entropy inequality proved for the first time by Bresch & Desjardins [1]. We will roughly recall the most important steps from the original proof and focus on the new features of the system. More details can be found in the last section, in the proof of Lemma 12.

Lemma 4 *Let $\mu(\varrho)$, $\nu(\varrho)$ be two $C^2(0,\infty)$ functions satisfying (9) and (10). Then, any smooth solution of (1) satisfies*

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2+\frac{1}{\gamma-1}\varrho^\gamma-\frac{1}{m_B}\varrho\log\varrho\right)\,\mathrm{d}x+\int_\Omega\nabla\phi(\varrho)\cdot\nabla p(\varrho,Y)\,\mathrm{d}x$$
$$+\frac{1}{2}\int_\Omega\mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,\mathrm{d}x-\int_\Omega\varrho Y_A\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x=0 \quad (19)$$

for ϕ such that

$$\nabla\phi(\varrho)=2\frac{\mu'(\varrho)\nabla\varrho}{\varrho}.$$

Proof. We start with the following observation

$$\frac{d}{dt}\int_\Omega\varrho\mathbf{u}\cdot\nabla\phi(\varrho)\,\mathrm{d}x=\int_\Omega\nabla\phi(\varrho)\partial_t(\varrho\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x, \quad (20)$$

where the first term on the r.h.s. may be evaluated by multiplying the momentum equation by $\nabla\phi(\varrho)$ and integrating by parts

$$\int_\Omega\partial_t(\varrho\mathbf{u})\nabla\phi(\varrho)\,\mathrm{d}x=-\int_\Omega(2\mu(\varrho)+\nu(\varrho))\Delta\phi(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x+2\int_\Omega\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\mu(\varrho)\,\mathrm{d}x$$
$$-2\int_\Omega\nabla\phi(\varrho)\cdot\nabla\mu(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x-\int_\Omega\nabla\phi(\varrho)\cdot\nabla p(\varrho,Y)\,\mathrm{d}x-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x. \quad (21)$$

Next, multiplying continuity equation by $|\nabla\phi(\varrho)|^2$ we get the following "renormalized" version

$$\frac{d}{dt}\int_\Omega\frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\,\mathrm{d}x=-\int_\Omega\varrho\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\,\mathrm{d}x+\int_\Omega\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x$$
$$+\int_\Omega\varrho\left(\nabla\phi(\varrho)\right)^2\operatorname{div}\mathbf{u}\,\mathrm{d}x. \quad (22)$$

From (20), (21) and (22) we therefore deduce

$$\frac{d}{dt}\int_\Omega\left(\varrho\mathbf{u}\cdot\nabla\phi(\varrho)+\frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\right)\,\mathrm{d}x+\int_\Omega\nabla\phi(\varrho)\cdot\nabla p(\varrho,Y)\,\mathrm{d}x$$
$$=-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x. \quad (23)$$

Now, the r.h.s. may be transformed into the form

$$-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x=$$
$$\int_\Omega\nu(\varrho)(\operatorname{div}\mathbf{u})^2\,\mathrm{d}x+\int_\Omega2\mu(\varrho)|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega\mu(\varrho)|\nabla\mathbf{u}-\nabla^\perp\mathbf{u}|^2\,\mathrm{d}x$$

and thus (17) summed up with (23) implies (19). □

To make use of this lemma we should verify that all the negative contributions from the l.h.s. and the whole r.h.s. are bounded. Note that, for instance, the pressure term is equal to

$$\nabla\phi(\varrho)\cdot\nabla p(\varrho,Y)=\gamma\mu'(\varrho)\varrho^{\gamma-2}|\nabla\varrho|^2+\mu'(\varrho)\left(\frac{Y_A}{m_A}+\frac{Y_B}{m_B}\right)\varrho^{-1}|\nabla\varrho|^2+\mu'(\varrho)\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\nabla\varrho\cdot\nabla Y_A \quad (24)$$

where the first two parts have a positive sign on the l.h.s. of (19), while to control the last term we need the following result.

Lemma 5 *For any smooth solution of (1) we have*

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho Y_A^2 \,\mathrm{d}x + \frac{1}{\max\{m_A,m_B\}}\int_\Omega \varrho|\nabla Y_A|^2 \,\mathrm{d}x \\ \leq \int_\Omega \varrho|\omega(Y)|Y_A \,\mathrm{d}x + \frac{1}{4}\left(\frac{1}{\min\{m_A,m_B\}} - \frac{1}{\max\{m_A,m_B\}}\right)\int_\Omega |\nabla\varrho\cdot\nabla Y_A| \,\mathrm{d}x. \quad (25)$$

Proof. Multiplying the species mass balance equation by Y_A and integrating over Ω we deduce

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho Y_A^2 \,\mathrm{d}x + \int_\Omega \left(\frac{1-Y_A}{m_A} + \frac{Y_A}{m_B}\right)\varrho|\nabla Y_A|^2 \,\mathrm{d}x \\ = \left(\frac{1}{m_B} - \frac{1}{m_A}\right)\int_\Omega Y_A(1-Y_A)\nabla\varrho\cdot\nabla Y_A \,\mathrm{d}x + \int_\Omega \varrho\omega(Y)Y_A \,\mathrm{d}x.$$

Now, since $0 \leq Y_A \leq 1$ and we have $\frac{1-Y_A}{m_A} + \frac{Y_A}{m_B} \geq \frac{1}{\max\{m_A,m_B\}}$ and $Y_A(1-Y_A) \leq \frac{1}{4}$. □
To estimate the r.h.s. of (25) we use the Cauchy inequality

$$\int_\Omega |\nabla\varrho\cdot\nabla Y_A| \,\mathrm{d}x \leq C(\epsilon)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho} \,\mathrm{d}x + \epsilon\int_\Omega \varrho|\nabla Y_A|^2 \,\mathrm{d}x$$

with $\varepsilon < \frac{4\min\{m_A,m_B\}}{\max\{m_A,m_B\}-\min\{m_A,m_B\}}$. And thus, for the initial data satisfying

$$\int_\Omega \varrho^0\left(Y_A^0\right)^2 \,\mathrm{d}x \leq C,$$

we can integrate (25) with respect to time to get

$$\|\sqrt{\varrho}Y_A\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{\varrho}\nabla Y_A\|^2_{L^2(0,T;L^2(\Omega))} \\ \leq C\|Y_A\|_{L^\infty((0,T)\times\Omega)}\|\varrho\|_{L^\infty(0,T;L^1(\Omega))} + C(m_A,m_B)\left\|\frac{\nabla\varrho}{\sqrt{\varrho}}\right\|^2_{L^2(0,T;L^2(\Omega))}. \quad (26)$$

We can now return to the assertion of Lemma 4 giving rise to the following inequality

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{1}{\gamma-1}\varrho^\gamma - \frac{1}{m_B}\varrho\log\varrho\right) \mathrm{d}x + \int_\Omega \gamma\mu'(\varrho)\varrho^{\gamma-2}|\nabla\varrho|^2 \,\mathrm{d}x \\ + \int_\Omega \mu'(\varrho)\left(\frac{Y_A}{m_A} + \frac{Y_B}{m_B}\right)\varrho^{-1}|\nabla\varrho|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \mu(\varrho)|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2 \,\mathrm{d}x \\ \leq \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \varrho|\operatorname{div}\mathbf{u}| \,\mathrm{d}x + \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \mu'(\varrho)|\nabla\varrho||\nabla Y_A| \,\mathrm{d}x. \quad (27)$$

The first term from the r.h.s is bounded on account of Lemma 3. In order to estimate last term we use the Cauchy inequality (with ε) to show

$$\int_\Omega \mu'(\varrho)\nabla\varrho\cdot\nabla Y_A \,\mathrm{d}x \leq C_\epsilon\int_\Omega \frac{(\mu'(\varrho))^2}{\varrho}|\nabla\varrho|^2 \,\mathrm{d}x + \epsilon\int_\Omega \varrho|\nabla Y_A|^2 \,\mathrm{d}x.$$

So, the Gronwall-type argument applied to the first integral coupled with (26) applied to the second one yields boundedness of the l.h.s. of (27). In particular, under assumption that the initial data satisfy

$$\int_\Omega \frac{1}{\varrho^0}\left|\nabla\mu\left(\varrho^0\right)\right|^2 \,\mathrm{d}x \leq C,$$

we can integrate (27) with respect to time to obtain

$$\|\sqrt{\varrho}\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mu'(\varrho)\nabla\sqrt{\varrho}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\varrho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} + \|\sqrt{\mu'(\varrho)\varrho^{\gamma-2}}\nabla\varrho\|^2_{L^2(0,T;L^2(\Omega))} \\ + \|\sqrt{\mu(\varrho)}\mathbf{A}(\mathbf{u})\|^2_{L^2(0,T;L^2(\Omega))} \leq C, \quad (28)$$

where we denoted $\mathbf{A}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} - \nabla^T\mathbf{u}\right)$.
Now, one can check that via the Sobolev imbedding theorem we have

$$\frac{1}{C_S^2}\|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^6(\Omega))} \leq \|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;H^1(\Omega))} \leq \|\nabla\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^2)} + \|\varrho\|^{\gamma}_{L^\infty(0,T;L^\gamma(\Omega))} \tag{29}$$

where C_S is the constant from the Sobolev inequality. Moreover, applying the interpolation inequality we obtain

$$\|\varrho_n^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq \|\varrho_n^\gamma\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))}\|\varrho_n^\gamma\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))} \leq C. \tag{30}$$

Our ultimate goal before the limit passage is dedicated to better integrability of velocity.

Lemma 6 *Let assumptions (9), (10) be valid. Then for any $\delta \in (0,2)$ the smooth solution of (1) satisfies*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x + \frac{r}{2}\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x\\ &\leq C\left(\int_\Omega\left(\frac{p(\varrho,Y)^2\varrho^{-\frac{\delta}{2}}}{\mu(\varrho)}\right)^{\frac{2}{2-\delta}}\mathrm{d}x\right)^{\frac{2-\delta}{2}}\left(\int_\Omega \varrho(2+\ln(1+|\mathbf{u}|^2))^{\frac{2}{\delta}}\ \mathrm{d}x\right)^{\frac{\delta}{2}} + C\int_\Omega \mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x. \end{aligned}\tag{31}$$

Proof. We follow the same strategy as in the work of Mellet & Vasseur [17] (Lemma 3.2). Multiplying the momentum equation by $(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}$ and employing (10) we verify

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho\partial_t((1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2))\ \mathrm{d}x + \int_\Omega \frac{1}{2}\varrho\mathbf{u}\cdot\nabla(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x\\ &+r\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x \leq -\int_\Omega(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla p(\varrho,Y)\ \mathrm{d}x + C\int_\Omega\mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x.\end{aligned}\tag{32}$$

Multiplying continuity equation by $\frac{1}{2}(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)$ and integrating by parts

$$\int_\Omega \frac{1}{2}\partial_t\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x = \int_\Omega\frac{1}{2}\varrho\mathbf{u}\cdot\nabla(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x,$$

so the first two terms from the l.h.s. of (32) give $\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x$. To control the r.h.s. of (32) we first integrate by parts

$$\begin{aligned}&\left|\int_\Omega(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla p(\varrho,Y)\ \mathrm{d}x\right|\\ &\qquad\leq\left|\int_\Omega\frac{2u_iu_k}{1+|\mathbf{u}|^2}\partial_iu_kp(\varrho,Y)\ \mathrm{d}x\right| + \left|\int_\Omega(1+\ln(1+|\mathbf{u}|^2))\,\mathrm{div}\,\mathbf{u}\ p(\varrho,Y)\ \mathrm{d}x\right|,\end{aligned}$$

then using the Hölder and Cauchy inequalities we show the following estimate

$$\begin{aligned}&\int_\Omega(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla p(\varrho,Y)\ \mathrm{d}x\\ &\leq\int_\Omega\mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x + \frac{r}{2}\int_\Omega\mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x + C\int_\Omega(2+\ln(1+|\mathbf{u}|^2))\frac{(p(\varrho,Y))^2}{\mu(\varrho)}\ \mathrm{d}x.\end{aligned}$$

Hence (31) is obtained by applying to the last term from above the Hölder inequality with $p=\frac{2}{2-\delta}$, $q=\frac{2}{\delta}$ $(\frac{1}{p}+\frac{1}{q}=1)$ and $\delta\in(0,2)$. □
Observe that due to (28) the r.h.s. of (31) may be partially controlled, we know in particular that

$$\frac{d}{dt}\int_\Omega\frac{1}{2}\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x \leq C\left(\int_\Omega\left(\frac{(p(\varrho,Y))^2\varrho^{-\frac{\delta}{2}}}{\mu(\varrho)}\right)^{\frac{2}{2-\delta}}\mathrm{d}x\right)^{\frac{2-\delta}{2}} + C. \tag{33}$$

Next, since $\mu(\varrho) > r\varrho$, thus for initial conditions satisfying

$$\int_\Omega \varrho^0 \left(1 + |\mathbf{u}^0|^2\right) \ln \left(1 + |\mathbf{u}^0|^2\right) \, \mathrm{d}x \le C$$

we have boundedness of $\varrho(1 + |\mathbf{u}|^2)\ln(1 + |\mathbf{u}|^2)$ in $L^\infty(0,T;L^1(\Omega))$ if only $(p(\varrho,Y))^2\varrho^{-1-\frac{\delta}{2}}$ belongs to $L^1((0,T)\times\Omega)$. By virtue of Lemma 3 and estimate (30) this is true for $\gamma < 3$, otherwise the boundedness of the r.h.s. of (33) follows from the additional assumption (11).

4 Passage to the limit

In the previous we showed uniform estimates for the sequence of of smooth solutions $\{\varrho_n, \mathbf{u}_n, Y_n\}_{n\in\mathbb{N}}$ under assumption that the initial data satisfy (14). For convenience of the reader we list all of them once more

$$\|\varrho_n\|_{L^\infty(0,T;L^1(\Omega)\cup L^\gamma(\Omega))} \le C, \tag{34}$$
$$\|\varrho_n^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \le C, \tag{35}$$
$$\|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{36}$$
$$\|\varrho_n|\mathbf{u}_n|^2\ln(1+|\mathbf{u}_n|^2)\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{37}$$
$$\|Y_n\|_{L^\infty((0,T)\times\Omega)} \le C, \tag{38}$$
$$\|\nabla\sqrt{\varrho_n}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{39}$$
$$\|\sqrt{\varrho_n}\nabla\mathbf{u}_n\|_{L^2(0,T;L^2(\Omega))} \le C, \tag{40}$$
$$\|\sqrt{\varrho_n}\nabla Y_n\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{41}$$

In this section we present the proof of Theorem 1. It will be split into several steps.

1. Convergence of $\sqrt{\varrho_n}$

Lemma 7 *If $\mu(\varrho)$ satisfies (10), then for a subsequence we have*

$$\sqrt{\varrho_n} \to \sqrt{\varrho} \quad \textit{a.e. and } L^2_{\mathrm{loc}}((0,T)\times\Omega) \textit{ strongly.}$$

Moreover $\varrho_n \to \varrho$ strongly in $C(0,T;L^{\frac{3}{2}}_{\mathrm{loc}}(\Omega))$.

Proof. By (34) and (39) we see that $\sqrt{\varrho_n} \in L^\infty(0,T;H^1(\Omega))$. Next, from the renormalized continuity equation coupled with (36) and (40) we also get that $\partial_t\sqrt{\varrho_n}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Hence, the Aubin-Lions lemma implies strong convergence on every compact subset in $L^2((0,T)\times\Omega)$.
In order to proceed we observe that by the Sobolev imbedding theorem $\sqrt{\varrho_n} \in L^\infty(0,T;L^6(\Omega))$. Therefore, from the continuity equation $\partial_t\varrho_n \in L^\infty(0,T;W^{-1,\frac{3}{2}}(\Omega))$ which together with boundedness of $\nabla\varrho_n$ in $L^\infty(0,T;L^{\frac{3}{2}}(\Omega))$ establishes compactness of $\{\varrho_n\}$ in $C^0(0,T;L^{\frac{3}{2}}_{\mathrm{loc}}(\Omega))$.

2. Convergence of the pressure
In view of (35) and by the fact that ϱ_n^γ converges almost everywhere to ϱ^γ, we deduce that ϱ_n^γ converges strongly to ϱ^γ in $L^1_{\mathrm{loc}}((0,T)\times\Omega)$.
Concerning the molecular pressure, since $\varrho_n \in L^\infty(0,T;L^3(\Omega))$, thus (38) implies

$$\varrho_n Y_{A,n} \quad \text{is bounded in } L^\infty(0,T;L^p(\Omega))$$

for any $p \in [1,3]$. Additionally, note that the space gradient of $\varrho_n Y_{A,n}$ equals

$$\nabla(\varrho_n Y_{A,n}) = Y_{A,n}\nabla\varrho_n + \sqrt{\varrho_n}\sqrt{\varrho_n}\nabla Y_{A,n}$$

and is bounded in $L^2(0,T;L^q(\Omega))$ for $q \in [1,\frac{3}{2}]$, therefore $\varrho_n Y_{A,n} \in L^2(0,T;W^{1,\frac{3}{2}}(\Omega))$.
Now, let us verify that the time derivative

$$\partial_t(\varrho_n Y_{A,n}) = -\operatorname{div}(\varrho_n Y_{A,n}\mathbf{u}_n) - \operatorname{div}(\mathcal{F}_{A,n}) + \varrho_n\omega_n \quad \text{is bounded in } L^2(0,T;W^{-1,\frac{3}{2}}(\Omega)).$$

Indeed, as $\varrho_n \mathbf{u}_n Y_{A,n} = \sqrt{\varrho_n}\mathbf{u}_n \sqrt{\varrho_n} Y_{A,n}$ belongs to $L^\infty(0,T;L^q(\Omega))$ and

$$\mathcal{F}_{A,n} = \frac{1}{m_A}\nabla(\varrho_n Y_{A,n}) - \frac{Y_{A,n}}{m_A}\nabla(\varrho_n Y_{A,n}) - \frac{Y_{A,n}}{m_B}\nabla(\varrho_n(1-Y_{A,n}))$$

is bounded in $L^2(0,T;L^q(\Omega))$ for $q \in [1,\frac{3}{2}]$ we have, by the Aubin-Lions lemma, compactness of $\{\varrho_n Y_{A,n}\}$ in $L^2(0,T;L^p(\Omega))$ for $p \in (1,3)$.

3. Strong convergence of $Y_{A,n}$
As a consequence of the last result we have (up to a subsequence) that $\varrho_n Y_{A,n}$ converges a.e. to some ϱ_A and we define $Y_A = \frac{\varrho_A}{\varrho}$. Since we know as well that ϱ_n converges a.e. to ϱ it can be easily deduced that $Y_{A,n} = \frac{\varrho_n Y_{A,n}}{\varrho_n}$ converges a.e. to Y_A whenever $\{\varrho(t,x) \neq 0\}$. As a matter of fact this is also true in the set $\{\varrho(t,x) = 0\}$ on account of (38). In particular, we have a strong convergence of $Y_{A,n}$ in $L^p(0,T;L^p_{\rm loc}(\Omega))$ for any p finite.

4. Convergence of the convective term
Having proved strong convergence of density and the additional estimate for velocity (37), convergence in the convective and the viscosity terms can be shown identically as in the work of Mellet & Vasseur [17]. Below we recall their final result.

Lemma 8 *Let $p \in [1,\frac{3}{2})$, then up to a subsequence we have*

$$\begin{aligned} &\varrho_n \mathbf{u}_n \to \mathbf{m} \quad \text{a.e. in } (0,T)\times\Omega \text{ and strongly in } L^2(0,T;L^p_{\rm loc}(\Omega)),\\ &\sqrt{\varrho_n}\mathbf{u}_n \to \frac{\mathbf{m}}{\sqrt{\varrho}} \quad \text{strongly in } L^2_{\rm loc}((0,T)\times\Omega). \end{aligned}$$

In particular, we have $\mathbf{m}(t,x) = 0$ a.e. on $\{\varrho(t,x) = 0\}$ and there exists a function $\mathbf{u}(t,x)$ such that $\mathbf{m}(t,x) = \varrho(t,x)\mathbf{u}(t,x)$ and

$$\begin{aligned} \varrho_n \mathbf{u}_n \to \varrho\mathbf{u} \quad &\text{strongly in } L^2(0,T;L^p_{\rm loc}(\Omega)),\\ \sqrt{\varrho_n}\mathbf{u}_n \to \sqrt{\varrho}\mathbf{u} \quad &\text{strongly in } L^2_{\rm loc}((0,T)\times\Omega). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mu(\varrho_n)\mathbf{D}(\mathbf{u}_n) \to \mu(\varrho)\mathbf{D}(\mathbf{u}) \quad &\text{in } \mathcal{D}'(\Omega),\\ \nu(\varrho_n)\operatorname{div}\mathbf{u}_n \to \nu(\varrho)\operatorname{div}\mathbf{u} \quad &\text{in } \mathcal{D}'(\Omega). \end{aligned}$$

5 Remarks on construction of approximate solution

In this section we present a possible approach to the issue of solvability of system (1). As it was already announced the strategy requires either to consider additional friction of the form $\varrho|\mathbf{u}|\mathbf{u}$ or to modify the cold component of the pressure in the regime of small densities, i.e. for $\varrho \leq 1$

$$\tilde{p}_E(\varrho) \sim -\varrho^{-l} \tag{42}$$

and a positive constant l.

The second way seems more natural as ultimately we want to investigate the full system describing the motion of chemically reacting and heat conducting fluids for which it is not so evident that in the degenerated regimes (of low temperatures and densities) the medium behaves as a fluid. For further discussion on this topic we refer the reader to [3] and references therein.

By this modification the compactness of velocity can be obtained without Lemma 6, so to construct the approximate solution one should only care about preserving the structure (19). The basic idea is contained already in the work [2] and consists of introducing the smoothing operator $\delta\varrho\nabla\left(\mu'(\varrho)\Delta^{2s+1}\mu(\varrho)\right)$ with s sufficiently large, inspired from the capillarity forces [4]. In the next step we improve regularity of velocity using the biharmonic operator $\eta\Delta^2\mathbf{u}$. Finally, to get the estimate for the norm of $\Delta^{s+1}\varrho$ in $L^2((0,T)\times\Omega)$ at the level of Faedo-Galerkin approximation, we also need to regularize the the continuity equation by adding $\varepsilon\Delta\varrho$.

At the points when construction of approximate solution does not differ much from the case of single-component barotropic flow we present only main arguments and leave the details to the kind reader. For the sake of simplicity we assume $\Omega = \mathbb{T}^3$ with periodic boundary conditions and that $\mu(\varrho) = \varrho$, $\nu(\varrho) = 0$.
For the constant parameters $\varepsilon, \eta, \kappa, \delta > 0$ (we skip these subscripts when no confusion may arise) we will look for a triple $(\varrho, \mathbf{u}, Y_A)$ satisfying the following regularization of the original system.
1. Approximate continuity equation:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0, \tag{43}$$

with the initial condition

$$\varrho(0,x) = \varrho^0_\delta(x), \tag{44}$$

where

$$\varrho^0_\delta \in C^{2,l}(\Omega), \quad \inf_\Omega \varrho^0_\delta > 0. \tag{45}$$

2. The Faedo-Galerkin approximation for the weak formulation of the momentum balance:

$$\int_\Omega \varrho\mathbf{u}(T)\phi \,\mathrm{d}x - \int_\Omega \mathbf{m}^0\phi \,\mathrm{d}x + \eta\int_0^T\int_\Omega \Delta\mathbf{u}\cdot\Delta\phi \,\mathrm{d}x \,\mathrm{d}t - \int_0^T\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\phi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T\int_\Omega 2\varrho\mathbf{D}(\mathbf{u}) : \nabla\phi \,\mathrm{d}x \,\mathrm{d}t$$
$$- \int_0^T\int_\Omega \tilde{p}(\varrho, \varrho_A^+, \varrho_B^+)\operatorname{div}\phi \,\mathrm{d}x \,\mathrm{d}t - \delta\int_0^T\int_\Omega \varrho\nabla\Delta^{2s+1}\varrho\cdot\phi \,\mathrm{d}x \,\mathrm{d}t + \varepsilon\int_0^T\int_\Omega (\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\phi \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{46}$$

satisfied for any test function $\phi \in X_n$, where X_n is an n-dimensional Euclidean subspace of $L^2(\Omega)$, $X_n = span\{\phi_i\}_{i=1}^n$, with the scalar product

$$< \mathbf{u}, \mathbf{v} >= \int_\Omega \mathbf{u}\cdot\mathbf{v} \,\mathrm{d}x \quad \mathbf{u}, \mathbf{v} \in X_n.$$

The pressure $\tilde{p}(\varrho, \varrho_A, \varrho_B)$ consists of the molecular term $p_M(\varrho, \varrho_A^+, \varrho_B^+)$ specified in (3) and the modified cold component $\tilde{p}_E(\varrho)$ such that

$$\tilde{p}_E'(\varrho) = \begin{cases} 3\varrho^{-4} & \text{for } \varrho \le 1, \\ \frac{1}{\gamma}\varrho^{\gamma-1} & \text{for } \varrho > 1. \end{cases} \tag{47}$$

Furthermore, we set

$$\varrho_i^+ = \begin{cases} 0 & \text{if} \quad \varrho_i < 0, \\ \varrho_i & \text{if} \quad 0 \le \varrho_i < \varrho, \\ \varrho & \text{if} \quad \varrho \le \varrho_i, \end{cases} \qquad \text{for } i \in \{A, B\}. \tag{48}$$

3. Instead of the single equation of mass balance for the species A, we consider modified equations for both species in the form

$$\begin{aligned} \partial_t\varrho_A - \varepsilon\Delta\varrho_A + \operatorname{div}(\varrho_A\mathbf{u}) - \operatorname{div}\left(\left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_\kappa \nabla\varrho_A - \left(\frac{\varrho_A^+}{\varrho m_B}\right)_\kappa \nabla\varrho\right) = \varrho\left(\omega\left(\frac{\varrho_A}{\varrho}\right)\right)_\kappa, \\ \partial_t\varrho_B - \varepsilon\Delta\varrho_B + \operatorname{div}(\varrho_B\mathbf{u}) - \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_B} + \frac{\varrho_B^+}{\varrho m_A}\right)_\kappa \nabla\varrho_B - \left(\frac{\varrho_B^+}{\varrho m_A}\right)_\kappa \nabla\varrho\right) = -\varrho\left(\omega\left(\frac{\varrho_A}{\varrho}\right)\right)_\kappa, \end{aligned} \tag{49}$$

with the initial conditions

$$\begin{aligned} \varrho_A(0,x) = \varrho^0_{A,\delta}(x), \quad \varrho_B(0,x) = \varrho^0_{B,\delta}(x), \\ \varrho^0_{A,\delta}, \varrho^0_{B,\delta} \in C^{2,l}(\Omega), \quad \varrho^0_{A,\delta} + \varrho^0_{B,\delta} = \varrho^0_\delta. \end{aligned} \tag{50}$$

The above system of equations has just an auxiliary character, in the final result, instead of it, we will simply consider only the equation for species A.

The operator $f \to f_\kappa$, $\kappa = (\kappa_t, \kappa_x)$ is the standard smoothing operator, that applies to the variables x and t in the case of functions $\varrho, \mathbf{u}, (\varrho_A, \varrho_B)$. However, the regularization over time in (49) means that instead of $\varrho, \mathbf{u}, (\varrho_A, \varrho_B)$ we consider their continuous extensions respectively in the classes $V_\mathbb{R}, C(\mathbb{R}; X_n)$ and $(W_\mathbb{R}, W_\mathbb{R})$ that will be specified later on. We also assume that the supports of these extensions are contained in the time-space cylinder $(-2T, 2T) \times \Omega$, so that the integrals on the r.h.s. of the following exist

$$f_\kappa(t,x) = (f * \zeta_{\kappa_x}) * \psi_{\kappa_t} = \int_\mathbb{R} \psi_{\kappa_t}(t-s) \int_{\mathbb{T}^3} \zeta_{\kappa_x}(x-y) f(s,y) \, \mathrm{d}y \, \mathrm{d}s,$$

where

$$\zeta_{\kappa_x}(x) = \frac{1}{\kappa_x^3} \zeta\left(\frac{x}{\kappa_x}\right)$$

where $\zeta(x)$ is a regularizing kernel

$$\zeta \in C_c^\infty(\mathbb{T}^3), \quad \text{supp}\, \zeta \subset (-1,1)^3, \quad \zeta(x) = \zeta(-x) \geq 0, \quad \int_{\mathbb{T}^3} \zeta(x) \mathrm{d}x = 1.$$

Similarly, we define a regularizing kernel for the time coordinate

$$\psi \in C_c^\infty(\mathbb{R}), \quad \text{supp}\, \psi \subset (-1,1), \quad \psi(t) = \psi(-t) \geq 0, \quad \int_\mathbb{R} \psi(t) \mathrm{d}t = 1,$$

$$\psi_{\kappa_t} = \frac{1}{\kappa_t} \psi\left(\frac{t}{\kappa_t}\right).$$

We start with the proof of well posedness of the approximate system.

Theorem 9 *Let $\varepsilon, \kappa, \eta, \delta$ be fixed positive parameters. The approximate problem (43-50) admits a strong solution $\{\varrho, \mathbf{u}, \varrho_A, \varrho_B\}$ belonging to the regularity class*

$$\varrho \in C([0,T]; C^{2,l}(\Omega)), \quad \partial_t \varrho \in C([0,T]; C^{0,l}(\Omega)), \quad \inf_{[0,T]\times\Omega} \varrho > 0,$$

$$\mathbf{u} \in C^1([0,T], X_n),$$

$$\varrho_i \in C(0,T; W^{1,2}(\Omega)), \quad \partial_t \varrho_i, \Delta \varrho_i \in L^2((0,T)\times\Omega), \quad i \in \{A,B\}, \quad \varrho_A + \varrho_B = \varrho.$$

Proof. The proof splits into three main steps:

1. We look for the regular solution of the momentum equation $\mathbf{u}$ applying the fixed point argument to a suitable integral operator in the Banach space $C([0,T], X_n)$. To that purpose we first find the following mappings $\mathbf{u} \to \varrho(\mathbf{u})$ and $\mathbf{u} \to (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ determining the unique solution to the remaining equations in terms of $\mathbf{u}$.

2. For fixed $\mathbf{u}$ we solve the approximate continuity equation by use of the standard theory of linear parabolic equations.

3. We determine the partial densities ϱ_A, ϱ_B as solution of the system of semilinear parabolic equations, where $\mathbf{u}, \varrho(\mathbf{u})$ play the role of given data.

Continuity equation. Here we present the argument for existence of smooth, unique solution to the problem (43-45) in the situation when the vector field $\mathbf{u}(x,t)$ is given and belongs to $C(0,T;X_n)$. The following result can be proven by the Galerkin approximation and the well known statements about the regularity of parabolic systems (for the details of proof see [11], Lemma 3.1).

Lemma 10 *Let $\mathbf{u} \in C([0,T], X_n)$ for n fixed and let $\varrho_\delta^0 \in C^{2,l}(\Omega)$, $l \in (0,1)$ be such that*

$$0 < \underline{\varrho^0} \leq \varrho^0 \leq \overline{\varrho^0} < \infty.$$

Then there exists the unique classical solution to (43-45), i.e. $\varrho \in V_{[0,T]}$*, where*

$$V_{[0,T]} = \left\{ \begin{array}{c} \varrho \in C([0,T]; C^{2,l}(\Omega)), \\ \partial_t \varrho \in C([0,T]; C^{0,l}(\Omega)). \end{array} \right\} \tag{51}$$

Moreover, the mapping $\mathbf{u} \to \varrho$ *maps bounded sets in* $C([0,T], X_n)$ *into bounded sets in* $V_{[0,T]}$ *and is continuous with values in* $C^1([0,T] \times \Omega)$.
Finally,

$$\underline{\varrho}^0 e^{-\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \le \varrho(\tau, x) \le \overline{\varrho^0} e^{\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \quad \textit{for all } \tau \in [0,T],\ x \in \Omega.$$

Species mass balance equations. The existence of unique solution to the system (49-50) is guaranteed by the following result.

Lemma 11 *Let assumptions of Lemma 10 be satisfied. Suppose that* $\varrho^0_{A,\delta}, \varrho^0_{B,\delta} \in C^{2,l}(\Omega)$*, then the problem (49-50) with* $\mathbf{u}, \varrho$ *fixed, possesses the unique strong solution* (ϱ_A, ϱ_B) *belonging to the regularity class*

$$W_{(0,T)} = \left\{ \begin{array}{c} \varrho_i \in C(0,T; W^{1,2}(\Omega)), \\ \partial_t \varrho_i, \Delta \varrho_i \in L^2((0,T) \times \Omega) \end{array} \right\} \tag{52}$$

for $i \in \{A, B\}$*. Moreover, the mapping* $\mathbf{u} \to (\varrho_A, \varrho_B)$ *maps bounded sets in* $C([0,T], X_n)$ *into bounded sets in* W *and is continuous with values in* $L^2(0,T; W^{1,2}(\Omega)) \times L^2(0,T; W^{1,2}(\Omega))$*. In addition*

$$\varrho_A + \varrho_B = \varrho \quad \textit{a.e. in } (0,T) \times \Omega. \tag{53}$$

The precise proof of this fact, in the general setting of an arbitrary large number of species diffusing in accordance with (4), can be found in [19]. Its main idea is to consider first the system of two (independent) linear parabolic equations

$$\begin{aligned} \partial_t \varrho_A - \varepsilon \Delta \varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) - \operatorname{div}\left(\left(\frac{\tilde{\varrho}_B^+}{\varrho m_A} + \frac{\tilde{\varrho}_A^+}{\varrho m_B} \right)_\kappa \nabla \varrho_A - \left(\frac{\tilde{\varrho}_A^+}{\varrho m_B} \right)_\kappa \nabla \varrho \right) &= \varrho \left(\omega \left(\frac{\tilde{\varrho}_A}{\varrho} \right) \right)_\kappa, \\ \partial_t \varrho_B - \varepsilon \Delta \varrho_B + \operatorname{div}(\varrho_B \mathbf{u}) - \operatorname{div}\left(\left(\frac{\tilde{\varrho}_A^+}{\varrho m_B} + \frac{\tilde{\varrho}_B^+}{\varrho m_A} \right)_\kappa \nabla \varrho_B - \left(\frac{\tilde{\varrho}_B^+}{\varrho m_A} \right)_\kappa \nabla \varrho \right) &= -\varrho \left(\omega \left(\frac{\tilde{\varrho}_A}{\varrho} \right) \right)_\kappa, \end{aligned}$$

with the corresponding initial conditions, where all the quantities with $\tilde{}$ are fixed. The existence of the unique regular solution belonging to the Hölder space $C^{1,l/2;2,l}([0,T] \times \overline{\Omega})$ is then a consequence of the classic theory from the book of Ladyzhenskaya, Solonnikov and Uralceva [14]. Next, the original system (49) is recovered by the fixed point argument applied to the mapping

$$T : C([0,T]; W^{1,2}(\Omega)) \times C([0,T]; W^{1,2}(\Omega)) \to C([0,T]; W^{1,2}(\Omega)) \times C([0,T]; W^{1,2}(\Omega)),$$

$$T(\tilde{\varrho}_A, \tilde{\varrho}_B) = (\varrho_A, \varrho_B).$$

Remark 1 *It is possible to show that for* $\kappa > 0$ *the fixed point is smooth, hence is a classical solution of (49-50).*

The proof of (53) follows by subtracting both equations of (49) from the approximate continuity equation. The unique solution of the resulting system must be, due to the initial condition (50), equal to 0 a.e. in $(0,T) \times \Omega$.

Momentum equation. Now we prove that there exists $T = T(n)$ and $\mathbf{u} \in C([0,T], X_n)$ satisfying (46) for $\phi \in X_n$. For this purpose we apply the fixed point argument to the mapping

$$\mathcal{T} : C([0,T], X_n) \to C([0,T], X_n),$$

$$\mathcal{T}[\mathbf{u}](t) = \mathcal{M}\left[\varrho(t), P_n \varrho \mathbf{u}(0) + \int_0^t P_n \mathcal{N}(\varrho(\mathbf{u}), \varrho_A(\varrho, \mathbf{u}), \mathbf{u})(s) \mathrm{d}s \right], \tag{54}$$

where P_n is the orthogonal projection of $L^2(\Omega)$ onto X_n,

$$\mathcal{N}(\varrho(\mathbf{u}), \varrho_A(\varrho, \mathbf{u}), \mathbf{u}) = -\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \operatorname{div}(2\varrho \mathbf{D}(\mathbf{u})) + \nabla \tilde{p}(\varrho, \varrho_A^+, \varrho_B^+) - \delta \varrho \nabla \Delta^{2s+1} \varrho + \eta \Delta^2 \mathbf{u} + \varepsilon (\nabla \varrho \cdot \nabla) \mathbf{u}$$

and

$$\mathcal{M}\left[\varrho(t),\cdot\right]: X_n \to X_n, \int_\Omega \varrho(t)\mathcal{M}\left[\varrho(t),\mathbf{w}\right]\phi \; \mathrm{d}x = <\mathbf{w},\phi>, \quad \mathbf{w},\phi \in X_n.$$

Note, that $\mathcal{M}$ is bounded and continuous, since $\varrho(t,x) \geq \underline{\varrho} > 0$ for any $(t,x) \in ((0,T)\times\Omega)$ and since $\mathcal{N}(\varrho(\mathbf{u}),\varrho_A(\varrho,\mathbf{u}),\mathbf{u})(t)$ is bounded in X_n for $t \in (0,T)$.

Remark 2 *The last sentence is true since we know from the classical theory of parabolic equations that the regularity of the unique solution ϱ to the approximate continuity equation can be improved (see [14]) if the term $\operatorname{div}(\varrho\mathbf{u})$ is considered on the r.h.s. In fact, the bootstrap argument works, so we can justify that the regularizing term $\varrho\nabla\Delta^{2s+1}\varrho$ in the approximate momentum equation makes sense, i.e. that it is bounded in $L^1(X_n)$.*

Moreover, one can verify that $\mathcal{T}[\mathbf{u}]$ maps the ball

$$B_{R,\tau^0} = \left\{\mathbf{u} \in C([0,\tau^0],X_n) : \|\mathbf{u}\|_{C([0,\tau^0],X_n)} \leq R, \mathbf{u}(0,x) = P_n\left(\frac{\mathbf{m}^0}{\varrho^0_\delta}\right)\right\}$$

into itself and it is a contraction, for sufficiently small $\tau^0 > 0$. It therefore posses the unique fixed poin satisfying (46). Additionally, the time regularity of $\mathbf{u}$ may be improved directly by differentiating (54) with respect to time and estimating the norm of the resulting right hand side in X_n, so we get

$$\mathbf{u} \in C^1([0,\tau^0],X_n).$$

Provided the system enjoys the estimates independent of τ^0, we can iterate the local construction of solution described above to get the solution for any $T > 0$. The existence of such a bound is the main goal of the next step.

Uniform estimates and global in time existence of solutions.
From the previous step we justify that $\mathbf{u}$ can be used as a test function in the approximate momentum equation, so that we get

$$\partial_t \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho|^2 + \varrho\pi(\varrho)\right)\,\mathrm{d}x + \int_\Omega 2\varrho|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x$$
$$+\eta\int_\Omega|\Delta\mathbf{u}|^2\,\mathrm{d}x + \delta\varepsilon\int_\Omega|\Delta^{s+1}\varrho|^2\,\mathrm{d}x \leq \int_\Omega\left(\frac{\overset{+}{\varrho_A}}{m_A}+\frac{\overset{+}{\varrho_B}}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x, \quad (55)$$

where $\pi'(y) = \tilde{p}_E(y)/y^2$.
Applying the Cauchy inequality (with ϵ) we see that the r.h.s. may be bounded as follows

$$\left|\int_\Omega\left(\frac{\overset{+}{\varrho_A}}{m_A}+\frac{\overset{+}{\varrho_B}}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x\right| \leq C(m_A,m_B)\int_\Omega \varrho^{\frac{1}{2}}|\operatorname{div}\mathbf{u}|\varrho^{\frac{1}{2}}\,\mathrm{d}x \leq \epsilon\int_\Omega\varrho|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x + C(\epsilon,m_A,m_B)\int_\Omega\varrho\,\mathrm{d}x$$

for ϵ sufficiently small.
Hence, after assuming enough integrability on the initial data, we get uniform estimates with respect to maximal time of existence of solutions

$$\begin{array}{ll}\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), & \sqrt{\varrho}\nabla\mathbf{u} \in L^2(0,T;L^2(\Omega)),\\ \sqrt{\eta}\Delta\mathbf{u} \in L^2(0,T;L^2(\Omega)), & \sqrt{\varepsilon\delta}\Delta^{s+1}\varrho \in L^2(0,T;L^2(\Omega)),\\ \sqrt{\delta}\nabla^{2s+1}\varrho \in L^\infty(0,T;L^2(\Omega)), & \tilde{p}_E(\varrho) \in L^\infty(0,T;L^1(\Omega)).\end{array} \quad (56)$$

Let us emphasize, that the estimate of the norm of $\sqrt{\varepsilon\delta}\Delta^{s+1}\varrho$ in $L^2(0,T;L^2(\Omega))$ is crucial at this stage. In particular, it enables to pass to the limit with the dimension of Faedo-Galerkin approximation in the most regoristic regularizing term.
Additionally, by the Sobolev embedding we have that $\|\varrho^{-1}\|_{L^\infty(\Omega)} \leq C\|\varrho^{-1}\|_{H^k}$, so for $k > 3/2$ we have

$$\|\nabla^2\varrho^{-1}\|_{L^2(\Omega)} \leq (1+\|\varrho\|_{H^{k+2}(\Omega)})^2(1+\|\varrho^{-1}\|_{L^3(\Omega)})^3.$$

Therefore, taking $2s+1 \geq k+2$ and for the cold component of the pressure given by (47), we can use the bounds from (56) can to deduce that ϱ is a priori bounded away from zero for all time

$$\|\varrho\|_{L^\infty((0,T)\times\Omega)} \geq C(\delta) > 0 \quad \text{a.e. in}(0,T)\times\Omega. \tag{57}$$

In consequence we have uniform in time bound for $\mathbf{u}$, i.e.

$$\sup_{t\in[0,T_{max}]} \|\mathbf{u}(t)\|_{X_n} < C(\text{data}, \varepsilon, \kappa, \eta, \delta),$$

and as already pointed out, this is the final argument in favour of extending the solution obtained in the previous step for the whole interval $[0,T]$ for any $T>0$, and the proof of Theorem 9 is complete

Estimates independent of κ and dimension of Faedo-Galerkin approximation. Passage to the limit when $\kappa \to 0$ and $n \to \infty$.
Let us first observe that the estimates (56) and (57) are actually independent of n, ε and η. In order to deduce uniform estimates on the partial density ϱ_A we multiply the first equation of (49) by ϱ_A, so we get

$$\begin{aligned}\partial_t \int_\Omega \frac{\varrho_A^2}{2} \,\mathrm{d}x + \int_\Omega \left(\varepsilon + \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_\kappa\right) |\nabla \varrho_A|^2 \,\mathrm{d}x \\ = \int_\Omega \varrho_A \mathbf{u}\cdot\nabla\varrho_A \,\mathrm{d}x + \int_\Omega \left(\frac{\varrho_A^+}{\varrho m_B}\right)_\kappa \nabla\varrho\cdot\nabla\varrho_A \,\mathrm{d}x + \int_\Omega \varrho \left(\omega\left(\frac{\varrho_A}{\varrho}\right)\right)_\kappa \varrho_A \,\mathrm{d}x.\end{aligned} \tag{58}$$

The r.h.s. can be estimated by use of (56) and the definition of $\omega\left(\frac{\varrho_A}{\varrho}\right)$

$$\begin{aligned}\left|\int_\Omega \varrho_A \mathbf{u}\cdot\nabla\varrho_A \,\mathrm{d}x + \int_\Omega \frac{\varrho_A}{\varrho m_B}\nabla\varrho\cdot\nabla\varrho_A \,\mathrm{d}x + \int_\Omega \varrho\omega\left(\frac{\varrho_A}{\varrho}\right)\varrho_A \,\mathrm{d}x\right| \\ \leq \|\varrho\|_{H^{2s+1}}\|\mathbf{u}\|_2\|\nabla\varrho_A\|_2 + C(m_B)\|\nabla\varrho_A\|_2\|\nabla\varrho\|_2 + \overline{\omega}\|\varrho\|_{H^{2s+1}}\|\varrho_A\|_1.\end{aligned} \tag{59}$$

Therefore, repeating the same for ϱ_B, one can show that

$$\|\varrho_i(t)\| \in L^2(\Omega), \quad \|\nabla\varrho_i\| \in L^2((0,T)\times\Omega) \quad i\in\{A,B\}.$$

Moreover, estimating the r.h.s of (49) we get that also $\partial_t\varrho_i$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$ independently of κ and n.
Having these estimates, we are ready to let $\kappa\to 0$, $n\to\infty$ in the approximate system (43-48). In particular, the bounds from (56) together with estimate of the norm of $\partial_t\varrho$, obtained directly from the continuity equation, provide the strong convergence of the density. More precisely we have

$$\varrho_n \to \varrho \quad \text{strongly in } L^2(0,T;H^{2s+1}) \text{ and weakly in } L^2(0,T;H^{2s+2}),$$

which is sufficient to pass to the limit in the regularizing term of highest order $\varrho\nabla\Delta^{2s+1}\varrho$ in the approximate momentum equation. Passage to the limit in the rest of system (43-48) is an easy task, however it is important to note that the limit quantities $(\varrho, \mathbf{u}, \varrho_A, \varrho_B)$ satisfy equation (46) for any test function ϕ from $L^2(0,T;H^{2s+1})$ and the species mass balances equations (49) while tested by any function from $L^2(0,T;H^1)$.

Remark 3 *Note that, due to the weak lower semicontinuity of convex functions we can pass to the limit in (55). Indeed, by the strong convergence of density and velocity we check that*

$$\int_\Omega \left(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho_n|^2 + \varrho_n\pi(\varrho_n)\right)\,\mathrm{d}x \to \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho|^2 + \varrho\pi(\varrho)\right)\,\mathrm{d}x \tag{60}$$

in the sense of distributions on $(0,T)$ and for any smooth function $\psi\in C^\infty((0,T))$

$$\begin{aligned}\int_0^T \psi\int_\Omega 2\varrho|\mathbf{D}(\mathbf{u})|^2 \,\mathrm{d}x\,\mathrm{d}t + \eta\int_0^T\psi\int_\Omega |\Delta\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t + \delta\varepsilon\int_0^T\psi\int_\Omega|\Delta^{s+1}\varrho|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leq \liminf_{n\to\infty}\int_0^T\psi\int_\Omega 2\varrho_n|\mathbf{D}(\mathbf{u}_n)|^2\,\mathrm{d}x\,\mathrm{d}t + \eta\int_0^T\psi\int_\Omega|\Delta\mathbf{u}_n|^2\,\mathrm{d}x\,\mathrm{d}t + \delta\varepsilon\int_0^T\psi\int_\Omega|\Delta^{s+1}\varrho|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{61}$$

Remark 4 *By the assertion of Lemma 11 we deduce that the limit quantities* $\varrho_A, \varrho_B, \varrho$ *satisfy*

$$\varrho_A + \varrho_B = \varrho.$$

Moreover, after passage to the limit $\kappa \to 0$ *it is possible to show that* $\varrho_A, \varrho_B \geq 0$ *(for the proof see [19]), therefore we can replace* ϱ_i^+ *by* ϱ_i, $i \in \{A, B\}$, *in both equations of* (49) *and in their weak formulations.*

Estimates independent of $\varepsilon, \eta, \delta$**.**
By the above remark we verify that the estimates obtained in (56) are independent of ε, η, also on δ. This information is in a sense crucial, since it allows us to improve the information about density by repeating the Bresch-Desjardins entropy estimate. Indeed, as we know now that

$$\varrho \in L^2(H^{2s+2}) \quad \text{and} \quad \varrho(x,t) \geq C(\delta) > 0$$

we can test the momentum equation by the function $\nabla\phi = 2\frac{\nabla\varrho}{\varrho}$, so that we get the following equality:

Lemma 12 *We have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho|^2 + \varrho\pi(\varrho)\right)\,\mathrm{d}x + \int_\Omega \nabla\phi(\varrho)\cdot\nabla\tilde{p}(\varrho,\varrho_A)\,\mathrm{d}x\\
&\qquad + 2\delta\int_\Omega |\Delta^{s+1}\varrho|^2\,\mathrm{d}x + \delta\varepsilon\int_\Omega |\Delta^{s+1}\varrho|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2\,\mathrm{d}x + \eta\int_\Omega |\Delta\mathbf{u}|^2\,\mathrm{d}x\\
&= \varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x + \varepsilon\int_\Omega \Delta\varrho\frac{|\nabla\phi|^2}{2}\,\mathrm{d}x - \varepsilon\int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\,\mathrm{d}x - \eta\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\,\mathrm{d}x.
\end{aligned} \tag{62}$$

Proof. The basic idea of the proof is to find the explicit forms of the time derivative of the first integral:

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\mathbf{u}\cdot\nabla\phi(\varrho) + \varrho|\nabla\phi(\varrho)|^2\right)\,\mathrm{d}x.$$

For this purpose we first multiply the approximate continuity equation by $\frac{|\nabla\phi(\varrho)|^2}{2}$ and we obtain the following sequence of equalities

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\,\mathrm{d}x = \int_\Omega \varrho\partial_t\frac{|\nabla\phi(\varrho)|^2}{2}\,\mathrm{d}x - \int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}x + \varepsilon\int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\,\mathrm{d}x =\\
&\quad \int_\Omega \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\partial_t\varrho\right)\,\mathrm{d}x - \int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}x + \varepsilon\int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\,\mathrm{d}x =\\
&\quad \int_\Omega \left(-\varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho) + \varrho\mathbf{u}\otimes\nabla\phi(\varrho) : \nabla^2\phi(\varrho) - \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\varrho\,\mathrm{div}\,\mathbf{u}\right)\right)\,\mathrm{d}x\\
&\qquad - \int_\Omega \left(\frac{|\nabla\phi(\varrho)|^2}{2}\mathrm{div}(\varrho\mathbf{u}) - \varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x =\\
&= -\int_\Omega \varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\,\mathrm{d}x + \int_\Omega \mathrm{div}(\varrho\mathbf{u}\nabla\phi(\varrho)\otimes\phi(\varrho))\,\mathrm{d}x - \int_\Omega \varrho\mathbf{u}\Delta\phi(\varrho)\nabla\phi(\varrho)\,\mathrm{d}x - \int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}x\\
&\qquad + \int_\Omega \varrho^2\phi'(\varrho)\Delta\phi(\varrho)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x + \int_\Omega \varrho\left(\nabla\phi(\varrho)\right)^2\mathrm{div}\,\mathbf{u}\,\mathrm{d}x + \varepsilon\int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\,\mathrm{d}x\\
&= -\int_\Omega \varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\,\mathrm{d}x + \int_\Omega \varrho^2\phi'(\varrho)\Delta\phi(\varrho)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x + \int_\Omega \varrho\left(\nabla\phi(\varrho)\right)^2\mathrm{div}\,\mathbf{u}\,\mathrm{d}x + \varepsilon\int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\,\mathrm{d}x.
\end{aligned} \tag{63}$$

The mixed term is due to the continuity equation equal to

$$\frac{d}{dt}\int_\Omega \varrho\mathbf{u}\cdot\nabla\phi(\varrho)\,\mathrm{d}x = \int_\Omega \nabla\phi(\varrho)\partial_t(\varrho\mathbf{u})\,\mathrm{d}x + \int_\Omega (\mathrm{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x - \varepsilon\int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\,\mathrm{d}x, \tag{64}$$

and the first term on the r.h.s. may be evaluated by multiplying the approximate momentum equation by $\nabla\phi(\varrho)$ and integrating by parts

$$\int_\Omega \partial_t(\varrho\mathbf{u})\nabla\phi(\varrho)\ \mathrm{d}x = -\int_\Omega 2\varrho\Delta\phi(\varrho)\operatorname{div}\mathbf{u}\ \mathrm{d}x + 2\int_\Omega \nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\varrho\ \mathrm{d}x$$
$$-2\int_\Omega \nabla\phi(\varrho)\cdot\nabla\varrho\operatorname{div}\mathbf{u}\ \mathrm{d}x - \int_\Omega \nabla\phi(\varrho)\cdot\nabla\tilde{p}(\varrho,\varrho_A)\ \mathrm{d}x - \int_\Omega \nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\ \mathrm{d}x$$
$$+\delta\int_\Omega \varrho\nabla\Delta^{2s+1}\varrho\cdot\nabla\phi(\varrho)\ \mathrm{d}x - \eta\int_\Omega \Delta^2\mathbf{u}\cdot\nabla\phi(\varrho)\ \mathrm{d}x + \varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi(\varrho)\ \mathrm{d}x. \quad (65)$$

Recalling the form of $\phi(\varrho)$ it can be deduced that the combination of (63) with (64) and (65) yields

$$\frac{d}{dt}\int_\Omega \left(\varrho\mathbf{u}\cdot\nabla\phi(\varrho) + \frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\right)\ \mathrm{d}x + \int_\Omega \nabla\tilde{p}(\varrho,\varrho_A)\cdot\nabla\phi(\varrho)\ \mathrm{d}x + 2\delta\int_\Omega |\Delta^{s+1}\varrho|^2\ \mathrm{d}x$$
$$= -\int_\Omega \nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\ \mathrm{d}x + \int_\Omega (\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\ \mathrm{d}x - \varepsilon\int_\Omega \operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\ \mathrm{d}x - \eta\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\ \mathrm{d}x$$
$$+\varepsilon\int_\Omega \frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\ \mathrm{d}x + \varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi(\varrho)\ \mathrm{d}x. \quad (66)$$

It is then easy to check that the first two terms from the r.h.s of (66) can be rewritten as

$$-\int_\Omega \nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\ \mathrm{d}x + \int_\Omega (\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\ \mathrm{d}x = \int_\Omega 2\varrho|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x - \frac{1}{2}\int_\Omega \varrho|\nabla\mathbf{u} - \nabla^\perp\mathbf{u}|^2\ \mathrm{d}x,$$

and thus, the assertion of lemma follows by adding (55) to (66). □

The only nonpositive contribution to the l.h.s. of (62) is contained in the second integral, as we can not determine the sign of the part corresponding to molecular pressure. However, we have

$$\int_\Omega \nabla\phi\cdot\nabla p_M(\varrho,\varrho_A)\ \mathrm{d}x = \int_\Omega \left(\frac{2|\nabla\varrho|^2}{\varrho m_B} + \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\frac{2\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\right)\ \mathrm{d}x$$

moreover,

$$\left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \frac{\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\ \mathrm{d}x = \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \left(\frac{|\nabla\varrho|^2}{\varrho} + \nabla\varrho\cdot\nabla Y_A\right)\ \mathrm{d}x \quad (67)$$

and

$$\left|\int_\Omega \nabla\varrho\cdot\nabla Y_A\ \mathrm{d}x\right| \le C_\epsilon\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x + \epsilon\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x. \quad (68)$$

To control the second term we proceed by the same line as in the proof of Lemma 5. Mimicking the steps leading to (26), we multiply the species mass balance equation by Y_A and integrate by parts

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho Y_A^2\ \mathrm{d}x + \left(\varepsilon + \frac{1}{\max\{m_A,m_B\}}\right)\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x$$
$$\le \int_\Omega \varrho|\omega(Y)|Y_A\ \mathrm{d}x + \frac{1}{4}\left(\frac{1}{\min\{m_A,m_B\}} - \frac{1}{\max\{m_A,m_B\}}\right)\int_\Omega |\nabla\varrho\cdot\nabla Y_A|\ \mathrm{d}x. \quad (69)$$

Hence, by the Cauchy inequality, we can justify that the $L^1(\Omega)$ norm of $\varrho|\nabla Y_A|^2$ is controlled by the $L^1(\Omega)$ norm of $\frac{|\nabla\varrho|^2}{\varrho}$ independently of the approximation parameters, so we end up with

$$\int_\Omega |\nabla\phi\cdot\nabla p_M(\varrho,\varrho_A)|\ \mathrm{d}x \le C(m_A,m_B)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x.$$

Finally, the Gronwall-type argument can be applied to absorb this term by the l.h.s. of (62).
Concerning terms from the r.h.s of (62), the first of them can be estimated as follows

$$\left|\varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\right| \le 2\varepsilon\|\nabla\mathbf{u}\|_6\|\varrho^{-1}\|_\infty\|\varrho\|^2_{1,6/5}.$$

The Sobolev imbedding implies that for $C(s)\varepsilon < \eta$ and s sufficiently large we have

$$\left|\varepsilon \int_\Omega \nabla\varrho \cdot \nabla\mathbf{u} \cdot \nabla\phi \, \mathrm{d}x\right| \le \frac{\eta}{3}\|\Delta\mathbf{u}\|_2^2 + C(\varepsilon)\|\varrho^{-1}\|_\infty^2 \|\varrho\|_{H^{2s+1}}^4$$

and the last term is bounded uniformly in time due to (55) provided $\varepsilon = \varepsilon(\delta)$. For the second term we may write

$$\left|\varepsilon \int_\Omega \Delta\varrho \frac{|\nabla\phi|^2}{2} \, \mathrm{d}x\right| \le 4\varepsilon\|\varrho\|_{H^2}\|\varrho^{-1}\|_\infty^2\|\varrho\|_{H^1}^2 \le C(\varepsilon)\|\varrho\|_{H^{2s+1}}^3\|\varrho^{-1}\|_\infty^2$$

and the same argument leads to boundedness uniformly in time provided ε is sufficiently small with respect to δ.
By the definition of ϕ the third term equals

$$-\varepsilon \int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho \, \mathrm{d}x = -\varepsilon \int_\Omega (2\,\mathrm{div}\,\mathbf{u}\Delta\varrho + \mathbf{u}\cdot\nabla\phi\Delta\varrho) \, \mathrm{d}x,$$

hence we have

$$\left|-\varepsilon \int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho \, \mathrm{d}x\right| \le C\varepsilon \left(\|\mathbf{u}\|_{1,6}\|\varrho\|_{H^2} + \|\mathbf{u}\|_\infty\|\varrho^{-1}\|_\infty\|\varrho\|_{H^1}\|\varrho\|_{H^2}\right)$$

and for ε sufficiently small with respect to η the r.h.s. is bounded by

$$\frac{\eta}{3}\|\Delta\mathbf{u}\|_2^2 + C\varepsilon\left(\|\varrho\|_{H^{2s+1}}^2 + \|\varrho^{-1}\|_\infty^2\|\varrho\|_{H^{2s+1}}^4\right).$$

Finally, we estimate the last term in (62)

$$\left|\eta \int_\Omega \Delta\mathbf{u} \cdot \nabla\Delta\phi(\varrho) \, \mathrm{d}x\right| \le \sqrt{\eta}\|\Delta\mathbf{u}\|_{L^2(\Omega)}\sqrt{\eta}\|\nabla\Delta\phi(\varrho)\|_{L^2(\Omega)},$$

where

$$\nabla\Delta\phi(\varrho) = \frac{2\nabla\Delta\varrho}{\varrho} - \frac{2(\nabla\varrho\cdot\nabla)\nabla\varrho}{\varrho^2} - \frac{2(\nabla\varrho\cdot\nabla)\nabla\varrho}{\varrho^2} - \frac{2\Delta\varrho\nabla\varrho}{\varrho^2} + \frac{4|\nabla\varrho|^2\nabla\varrho}{\varrho^3}.$$

For s sufficiently large we may show that

$$\|\nabla\Delta\phi(\varrho)\|_{L^2(\Omega)} \le (1 + \|\varrho\|_{H^{2s+1}(\Omega)})^3(1 + \|\varrho^{-1}\|_{L^\infty(\Omega)})^3$$

and on account of (56), (57) both terms from the r.h.s. are bounded for all time.

Reassuming, from the Bresch-Desjardins relation we can additionally deduce that

$$\nabla\sqrt{\varrho} \in L^\infty(0,T;L^2(\Omega)) \quad \sqrt{\delta}\Delta^{s+1}\varrho \in L^2(0,T;L^2(\Omega))$$

uniformly withe respect to $\varepsilon, \eta, \delta$. Moreover, in view of (47) we can write

$$\begin{aligned}\int_\Omega \nabla\phi(\varrho)\cdot\nabla\tilde{p}_E(\varrho) \, \mathrm{d}x = 2\int_\Omega \tilde{p}_E'(\varrho)\frac{|\nabla\varrho|^2}{\varrho} \, \mathrm{d}x &= 6\int_{\{x\in\Omega:\varrho\le 1\}} \varrho^{-5}|\nabla\varrho|^2 \, \mathrm{d}x + 2\gamma\int_{\{x\in\Omega:\varrho>1\}} \varrho^{\gamma-2}|\nabla\varrho|^2 \, \mathrm{d}x \\ &\ge \int_{\{x\in\Omega:\varrho\le 1\}} |\nabla\xi(\varrho)^{-3/2}|^2 \, \mathrm{d}x + \int_{\{x\in\Omega:\varrho>1\}} |\nabla\varrho^{\gamma/2}| \, \mathrm{d}x,\end{aligned}$$

where ξ is smooth and such that $\xi(y) = y$ for $y \le 1/2$ and $\xi(y) = 0$ for $y > 1$. So, by the entropy equality (62) we obtain additionally that

$$\nabla\xi(\varrho)^{-3/2} \in L^2((0,T)\times\Omega), \quad \nabla\varrho^{\gamma/2} \in L^2((0,T)\times\Omega_2),$$

where $\Omega_2 = \{x \in \Omega : \varrho > 1\}$. Moreover, via the Sobolev imbedding theorem we show that

$$\frac{1}{C^2}\|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^6(\Omega_2))} \leq \|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;H^1(\Omega_2))} \leq \|\nabla\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^2)} + \|\varrho\|^{\gamma}_{L^\infty(0,T;L^\gamma(\Omega_2))}$$

where C is the constant from the Sobolev inequality.
Furthermore, by a simple interpolation one gets

$$\|\varrho^\gamma_{\varepsilon\eta}\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq \|\varrho^\gamma_{\varepsilon\eta}\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))}\|\varrho^\gamma_{\varepsilon\eta}\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))} \leq C.$$

Passage to the limit $\varepsilon, \eta \to 0$.
It turns out that the limit passages with ε and η can be done in one step. Indeed, by the previous estimates we can extract subsequences, such that

$$\eta\Delta\mathbf{u}, \varepsilon\nabla\varrho_\varepsilon \to 0 \text{ strongly in } L^2((0,T)\times\Omega),$$

and assuming suitable relation between ε and η also

$$\varepsilon\nabla\varrho\nabla\mathbf{u} \to 0 \text{ strongly in } L^1((0,T)\times\Omega).$$

After remarks from the previous section, the only questionable limit passage at this stage is in the convective term of momentum equation, since we need to justify the strong convergence of the velocity. The argument for this is that the lower bound on the density depends only on δ and is uniform with respect to ε, η. Therefore we have boundedness of $\nabla\mathbf{u}$ in $L^2((0,T)\times\Omega)$. To improve the time regularity observe that from the approximate continuity equation we can bound the norm of $\partial_t(\varrho\mathbf{u})$ in $L^p(0,T;H^{-k}(\Omega))$ for some $k = k(s) > 0$ and $p > 1$. Then, using the Aubin-Lions lemma we get that $\varrho_{\varepsilon\eta}\mathbf{u}_{\varepsilon\eta} \to m_\delta$ when $\varepsilon, \eta \to 0$ strongly in $L^2((0,T)\times\Omega)$, which, due to the convergence of ϱ and ϱ^{-1} almost everywhere, implies the strong convergence of $\mathbf{u}$.

Remark 5 *Note also that we can pass to the limit in the first energy estimate in the sense of distributions [similarly as in (60) and (61)], so we do not have to test by $\mathbf{u}$ to obtain the estimates uniform with respect to δ. However, testing by $\frac{\nabla\varrho}{\varrho}$ is still allowed.*

Passage to the limit $\delta \to 0$.
Here we lose the uniform bound from below for the density, so the strong convergence of velocity can not be deduced by the procedure described above. Nevertheless, we can still use the Hölder inequality to verify

$$\|\nabla\mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \leq C(\Omega)\left(1 + \|\nabla\xi(\varrho)^{-3/2}\|_{L^2((0,T)\times\Omega)}\right)\|\sqrt{\varrho}\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega)},$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{2\cdot 3/2\cdot 2}, \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{6\cdot 3/2\cdot 2}.$$

After applying the Sobolev imbedding we thus obtain

$$\mathbf{u} \in L^{3/2}(0,T;L^{9/2}(\Omega)). \tag{70}$$

This in turn implies that for $0 \leq \epsilon \leq 1/2$ we have the following estimate

$$\|\sqrt{\varrho}\mathbf{u}\|_{L^{p'}(0,T;L^{q'}(\Omega))} \leq \|\varrho\|^{1/2-\epsilon}_{L^\infty(0,T;L^\gamma(\Omega))}\|\sqrt{\varrho}\mathbf{u}\|^{2\epsilon}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}\|^{1-2\epsilon}_{L^{3/2}(0,T;L^{9/2}(\Omega))},$$

where p', q' are given by

$$\frac{1}{p'} = \frac{1-2\epsilon}{3/2}, \quad \frac{1}{q'} = \frac{1/2-\epsilon}{\gamma} + \frac{2\epsilon}{2} + \frac{1-2\epsilon}{9/2}.$$

Taking $\epsilon > 1/8$ we have $p', q' > 2$ and the argument for strong convergence of $\sqrt{\varrho_\delta}\mathbf{u}_\delta$ from previous section applies verbatim.

Remark 6 *The final information about the velocity we obtain from this procedure is* (70)*. Note that it could be improved by assuming faster growth of the barotropic pressure in the areas of small densities than* $-\varrho^{-3}$*. However, this still would not be sufficient to repeat the logharithmic estimate performed in the section dedicated to sequential stability of weak solutions.*

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