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Existence results for unsteady flows of nonhomogeneous non-Newtonian incompressible fluids - monotonicity methods in generalized Orlicz spaces

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Abstract

Our purpose is to show existence of weak solutions to unsteady flow of non-Newtonian incompressible nonhomogeneous fluids with non-standard growth conditions of the stress tensor. We are motivated by the fluids of anisotropic behaviour and characterised by rapid shear thickening. Since we are interested in flows with the rheology more general than power-law-type, we describe the growth conditions with the help of an x–dependent convex function and formulate our problem in generalized Orlicz spaces.

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1 Introduction and formulation of the problem

We wish to investigate and understand mathematical properties of the motion of incompressible, nonhomogeneous non-Newtonian fluid, which can be described by the system of equations:

$$\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0 \quad \text{in} \quad Q,\\
\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}_x \mathbf{S}(t, x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x P = \varrho \boldsymbol{f} \quad \text{in} \quad Q,\\
\mathrm{div}_x \boldsymbol{u} = 0 \quad \text{in} \quad Q,\\
\boldsymbol{u}(0, x) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega,\\
\varrho(0, x) = \varrho_0 \quad \text{in} \quad \Omega,\\
\boldsymbol{u}(t, x) = 0 \quad \text{on} \quad (0, T) \times \partial\Omega,
\end{aligned} \tag{1.1}$$

where $\varrho : Q \to \mathbb{R}$ is the mass density, $\boldsymbol{u} : Q \to \mathbb{R}^3$ denotes the velocity field, $P : Q \to \mathbb{R}$ the pressure, $\mathbf{S}$ the stress tensor, $\boldsymbol{f} : Q \to \mathbb{R}^3$ given outer sources. The set $\Omega \subset \mathbb{R}^3$ is a bounded domain with a regular boundary $\partial\Omega$ (of class, say $C^{2+\nu}$, $\nu > 0$, to avoid unnecessary technicalities connected with smoothness). We denote by $Q = (0, T) \times \Omega$ the time-space cylinder with some given $T \in (0, +\infty)$. The tensor $\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u})$ is a symmetric part of the velocity gradient.

It is supposed that initial density is bounded, i.e.,

$$\varrho(0, \cdot) = \varrho_0 \in L^\infty(\Omega) \tag{1.2}$$

and

$$0 < \varrho_* \le \varrho_0(x) \le \varrho^* < +\infty \quad \text{for a.a. } x \in \Omega. \tag{1.3}$$

There have been many studies concerning the mathematical analysis of time-dependent flows of nonhomogeneous, incompressible fluids flow depending on or independent of density.

Our interest is directed to the phenomena of viscosity increase under various stimuli: shear rate, magnetic or electric field. Particularly we want to focus on shear thickening fluids (STF) and magnetorheological (MR). Both types of fluid have the ability of transferring rapidly from liquid to solid-like state and this phenomenon is completely reversible, and the time scale for the transmission is of the order of a millisecond. The magnetorheological fluids [42] found their application for modern suspension system, clutches or crash-protection systems in cars and shock absorbers providing seismic protection.

In particular we are interested in fluids which viscosity increases dramatically with increasing shear rate or applied stress, i.e. we want to consider shear thickening fluids, which can behaves like a solid when it encounters mechanical stress or shear. STF moves like a liquid until an object strikes or agitates it forcefully. Then, it hardens in a few milliseconds and the process is completely reversible. Possible application for fluids with changeable viscosity appears in military armour or medic or sport protection. The so-called STF-fabric produced by simple impregnation process of e.g. Kevlar make it applicable to any high-performance fabric. The resulting material is thin and flexible, and provides protection against the risk of needle, knife or bullet contact that face police officers and medical personnel [9, 21, 25].

As follows from (1.1) we assume that the traceless part S of the Cauchy stress tensor depends on the density and due principle of objectivity the extra stress tensor depends on the velocity gradient only through the symmetric part $\mathsf{D}\boldsymbol{u}$. On the one hand it means that we want to be able to consider constitutive relation which are invariant w.r.t. translations and rotations perpendicular to one chosen direction. Where in this specific direction properties of the material can be different then w.r.t. to the others.

One of the example is magnethorheological fluid, which consist of the magnetic particles suspended within the carrier oil distributed randomly in suspension under normal circumstances. When a magnetic field is applied, however, the microscopic particles align themselves along the lines of magnetic flux. When the fluid is contained between two poles, the resulting chains of particles restrict the movement of the fluid, perpendicular to the direction of flux, effectively increasing its viscosity. Consequently mechanical properties of the fluid are anisotropic.

On the other hand we can consider the constitutive relation for fluids with dependence on outer field, in particular, we mean elektrorheological fluids.

In this case, from representation theorem it follows that the most general form for the stress tensor S (cf. [34]) is given by

$$\mathsf{S} = \alpha_1 \boldsymbol{E}\otimes\boldsymbol{E} + \alpha_2\mathsf{D} + \alpha_3\mathsf{D}^2 + \alpha_4(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}\boldsymbol{E}) + \alpha_5(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}^2\boldsymbol{E})$$

where α_i, $i = 1, \ldots, 5$ may be functions of invariants

$$|\boldsymbol{E}|^2,\ \operatorname{tr}\mathsf{D}^2,\ \operatorname{tr}\mathsf{D}^3,\ \operatorname{tr}(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E}),\ \operatorname{tr}(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E}).$$

Then it is easy to show that for $i = 1, 3, 5$, $\alpha_i = 0$ the stress tensor in the form

$$\mathsf{S} = |\operatorname{tr}\mathsf{D}^2|^3\mathsf{D} + |\operatorname{tr}(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E})|^6(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}\boldsymbol{E}) \tag{1.4}$$

is thermodynamically admissible (i.e. $\mathsf{S} : \mathsf{D} \geq 0$), satisfies a principle of material frame-indifference and is monotone. Moreover, without loosing of generality for $\boldsymbol{E} = (1, 0, 0)$ it can be calculated the standard growth conditions: $|\mathsf{S}(\mathsf{D}, \boldsymbol{E})| \leq c(1 + |\mathsf{D}|)^{p-1}$, $\mathsf{S}(\mathsf{D}, \boldsymbol{E}) : \mathsf{D} \geq c|\mathsf{D}|^p$ can not be satisfied, because the tensor S possess growth of different powers in various directions of D. From mechanical point of view minimal assumption are satisfied. From this reason we can not exclude constitutive relation of anisotropic behaviour like (1.4).

In our consideration we do not want to assume that S has only p-structure, i.e. $\mathsf{S} \approx \mu(\varrho)(\kappa + |\mathsf{D}\boldsymbol{u}|)^{p-2}\mathsf{D}\boldsymbol{u}$ or $\mathsf{S} \approx \mu(\varrho)(\kappa + |\mathsf{D}\boldsymbol{u}|^2)^{(p-2)/2}\mathsf{D}\boldsymbol{u}$ (where $\kappa > 0$ and μ is nonnegative bounded function). Standard growth conditions of the stress tensor, namely polynomial growth, see e.g. [13, 27]

$$\begin{aligned} |\mathsf{S}(x, \boldsymbol{\xi})| &\leq c(1 + |\boldsymbol{\xi}|^2)^{(p-2)/2}|\boldsymbol{\xi}| \\ \mathsf{S}(x, \boldsymbol{\xi}) : \boldsymbol{\xi} &\geq c(1 + |\boldsymbol{\xi}|^2)^{(p-2)/2}|\boldsymbol{\xi}|^2 \end{aligned} \tag{1.5}$$

can not suffice to describe our model. Motivated by this significant shear thickening phenomenon we want to investigate the processes where growth is faster than polynomial and possibly different in various directions of the shear rate. A viscosity of the fluid is not assumed to be constant and can depends on density and full symmetric part of the gradient. Therefore we formulate the growth conditions of the stress tensor using general convex function M called the N–function (the definition of N-function M and complementary function M^* apper in Section 2). Now we are able to describe the effect of rapidly shear thickening fluids.

We assume also that the stress tensor $\mathsf{S} : (0, T) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ satisfies ($\mathbb{R}^{3\times 3}_{\text{sym}}$ stands for the space of 3×3 symmetric matrices):

1. $\mathsf{S}(t, x, \varrho, \mathsf{K})$ is a Carathéodory function (i.e., measurable function of t, x for all $\varrho > 0$ and $\mathsf{K} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ and continuous function of ϱ and K for a.a. $x \in \Omega$) and $\mathsf{S}(t, x, \varrho, \mathbf{0}) = \mathbf{0}$.

2. There exist a positive constant c_c, N–functions M and M^* which denotes the complementary function to M such that for all $\mathsf{K} \in \mathbb{R}^{3\times 3}_{\text{sym}}$, $\varrho > 0$ and a.a. t, $x \in Q$ it holds
$$\mathsf{S}(t, x, \varrho, \mathsf{K}) : \mathsf{K} \geq c_c\{M(x, \mathsf{K}) + M^*(x, \mathsf{S}(t, x, \varrho, \mathsf{K}))\}. \tag{1.6}$$

3. S is monotone, i.e. for all $\mathsf{K}_1, \mathsf{K}_2 \in \mathbb{R}^{3\times 3}_{\text{sym}}$, $\varrho > 0$ and a.a. $x \in \Omega$
$$[\mathsf{S}(t, x, \varrho, \mathsf{K}_1) - \mathsf{S}(t, x, \varrho, \mathsf{K}_2)] : [\mathsf{K}_1 - \mathsf{K}_2] \geq 0.$$

We can observe that the case of stress tensors having convex potentials (additionally vanishing at 0 and symmetric w.r.t. the origin) significantly simplifies verifying condition 2. For finding N–functions M and M^* we take an advantage of the following relation

$$M(\boldsymbol{\xi}) + M^*(\nabla M(\boldsymbol{\xi})) = \boldsymbol{\xi} \cdot \nabla M(\boldsymbol{\xi}) \tag{1.7}$$

holding for all $\xi \in \mathbb{R}^{3\times3}_{\text{sym}}$, cf. [33]. This corresponds to the case when the Fenchel-Young inequality for N–functions becomes the equality. Once we have a given function $\mathbf{S}$, for simplicity consider it in the form $\mathbf{S}(\mathbf{D}\boldsymbol{u}) = 2\mu(|\mathbf{D}\boldsymbol{u}|^2)\mathbf{D}\boldsymbol{u}$ then choosing $M(x, \boldsymbol{\xi}) = M(\boldsymbol{\xi}) = \int_0^{|\boldsymbol{\xi}|^2} \mu(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}$ provides that (1.6) is satisfied with a constant $c = 1$. For such chosen M we only need to verify whether the $N-$function–conditions, i.e, behaviour in/near zero and near infinity, are satisfied. The monotonicity of $\mathbf{S}$ follows from the convexity of the potential.

Our assumptions can capture shear dependent viscosity function which includes power-low and Carreau-type models which are quite popular among rheologist, chemical engineering, colloidal mechanics (see [29] for more references). Neverdheless we want to investigate also more general constitutive relations like non-polynomial growth $\mathbf{S} \approx |\mathbf{D}\boldsymbol{u}|^p \ln(1 + |\mathbf{D}\boldsymbol{u}|)$ or of anisotropic behaviour $\mathbf{S}_{i,j} \approx |\cdot|^{p_{ij}}[\mathbf{D}\boldsymbol{u}]_{i,j}$, $i,\, j = 1, 2, 3$.

The appropriate spaces to capture such formulated problem are Orlicz spaces. We also allow the stress tensor to depend on x, this provides the possibility to consider magnetorheological fluids and significant influence of magnetic field on the increase of viscosity. Thus we use the generalized Orlicz spaces, often called Orlicz-Musielak spaces (see [30] for more details). For definitions and preliminaries of N-functions and Orlicz spaces see Section 2. Contrary to [30] we consider the N–function M not dependent only on $|\boldsymbol{\xi}|$, but on whole tensor $\boldsymbol{\xi}$. It results from the fact that the viscosity may differ in different directions of symmetric part of velocity gradient $\mathbf{D}\boldsymbol{u}$. Hence we want to consider the growth condition for the stress tensor dependent on whole tensor $\mathbf{D}\boldsymbol{u}$, not only on $|\mathbf{D}\boldsymbol{u}|$. The spaces with N–function dependent on vector-valued argument were investigated in [36, 37, 38].

The example of the generalized Orlicz space is generalized Lebesgue space, in this case $M(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p(x)}$. This kind of spaces were applied in [34] to description of flow of electrorheological fluid. The standard assumption in this work was $1 \le p_0 \le p(x) \le p_\infty < \infty$, where $p \in C^1(\Omega)$ was function of electric field E, i.e. $p = p(|E|^2)$, and $p_\infty > \frac{3d}{d+2}$ in case of steady flow, where

$d \geq 2$ is space dimension. For this reason there is satisfied the so-called $\triangle_2$–condition and consequently the space is reflexive and separable. One of the main problems in our model is that the $\triangle_2$–condition is not satisfied and we lose the above properties. Later in [4] the above result was improved by Lipschitz truncations methods for $L^{p(x)}$ setting for $\mathbf{S}$, where $\frac{2d}{d+2} < p(\cdot) < \infty$ was log-Hölder continuous and $\mathbf{S}$ was strongly monotone.

The mathematical analysis of homogeneous non-Newtonian fluids was initiated by Ladyzhenskaya [23, 24] where the global existence of weak solutions for $p \geq 1 + (2d)/(d+2)$ was proved for Dirichlet boundary conditions. Later the steady flow was considered by Frehse at al. in [15], where the existence of weak solutions was established for the constant exponent $p > \frac{2d}{d+2}$, $d \geq 2$ by Lipschitz truncation methods.

Wolf in [39] proved existence of weak solutions to non-stationary motion of an incompressible fluid with shear rate dependent viscosity for $p > 2(d+1)/(d+2)$ without assumptions on shape and size of Ω employing L^∞–test function and local pressure method. Finely the existence of global weak solutions with the Dirichlet boundary conditions for $p > (2d)/(d+2)$ was achieved in [5] by Lipshitz truncation and local pressure methods .

Most of the available results concerning nonhomogeneous incompressible fluids deal with the polynomial dependence between $\mathbf{S}$ and $|\mathbf{D}\boldsymbol{u}|$. The analysis of nonhomogeneous Newtonian ($p = 2$ in (1.5)) fluids was investigated by Antontsev, Kazhikhov and Monakhov [2] in the seventies. P.L. Lions in [26] presented the concept of renormalized solutions and obtained new convergence and continuity properties of the density.

The first result for unsteady flow of nonhomogenous non-Newtonian fluids goes back to Fernández–Cara [12], where existence of Dirichlet weak solutions was obtained for $p \geq 12/5$ if $d = 3$, later existence of space-periodic weak solution for $p \geq 2$ with some regularity properties of weak solutions whenever $p \geq 20/9$ (if $d = 3$) was obtained by Guilién-González in [17]. Frehse and Růžička showed in [14] existence of a weak solution for generalized Newtonian fluid of power-low type for $p > 11/5$. Authors needed also existence of the potential of $\mathbf{S}$. Resent results concerning fluids of where the growth condition is in (1.5) type for $p \geq 11/5$ belong to Frehse, Málek and Růžička [13]. The novelty of this paper is consideration of the full thermodynamic model for nonhomogeneous incompressible fluid. Particularly in [13, 14] the reader can find the concept of integration by parts formula, which we adapted to our case. Also more details concerning references can be found therein.

We can recall also some existing analytical results concerning the abstract parabolic problems in non-separable Orlicz spaces

$$
\begin{aligned}
\partial_t \boldsymbol{u} + A(t,x,\nabla_x \boldsymbol{u}) &= \boldsymbol{f} && \text{in} \quad Q,\\
\boldsymbol{u}(x,t) &= 0 && \text{on} \quad (0,T)\times\partial\Omega,\\
\boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x) && \text{in} \quad \Omega.
\end{aligned} \tag{1.8}
$$

Donaldson in [7] assumed that the operator A is an elliptic second-order operator in divergence form and monotone. The growth and coercivity conditions were more general than the standard growth conditions in L^p, namely the N–function formulation was stated. Under the assumptions on the N–function M: $\xi^2 << M(|\xi|)$ (i.e., ξ^2 grows essentially less rapidly than $M(|\xi|)$) and M^* satisfies a Δ_2–condition, the existence results to (1.8) was established. This restrictions on the growth of M were abandoned in [10].

The review paper [31] by Mustonen summarises the monotone-like mappings techniques in Orlicz and Orlicz–Sobolev spaces. The authors need the essential modifications of the notions like monotonicity, pseudomonotonicity, operators of type (M), (S_+), et al. The reason is that Orlicz–Sobolev spaces are not reflexive in general. Moreover, the nonlinear differential operators in divergence form with standard growth conditions are neither bounded nor everywhere defined. First results concerning non-Newtonian fluid with the assumption that $\mathbf{S}$ is strictly monotone and satisfies conditions 1.-2. were established by Gwiazda et al. [18]. The stronger assumption on $\mathbf{S}$ was crucial for the applied tools (Young measures). This restriction was abandoned in [20, 40], where authors used generalization of Minty trick for non-reflexive spaces. The above existence results were established for $p \geq 11/5$ in [20], but without including in the system the dependence on density.

We want to extend the existence theory to a more general class than polynomial growth conditions and to the system where density is not assumed to be constant.

One of the main problems in our model is that the $\triangle_2$–condition can be not satisfied and we lose many facilitating properties. An interesting obstacle here is the lack of the classical integration by parts formula, cf. [16, Section 4.1]. To extend it for the case of generalized Orlicz spaces we would essentially need that $C^\infty-$functions are dense in $L_M(Q)$ and $L_M(Q) = L_M(0,T;L_M(\Omega))$. The first one only holds if M satisfies Δ_2–condition. The second one is not the case in Orlicz and generalized Orlicz spaces. We recall the proposition from [7] (although it is stated for Orlicz spaces with $M = M(|\boldsymbol{\xi}|)$).

Proposition 1.1 *Let I be the time interval and $\Omega \subset \mathbb{R}^d$, $M = M(|\boldsymbol{\xi}|)$ an N–function, and $L_M(I \times \Omega), L_M(I; L_M(\Omega))$ the Orlicz spaces on $I \times \Omega$ and the vector valued Orlicz space on I respectively. Then*

$$L_M(I \times \Omega) = L_M(I; L_M(\Omega)),$$

if and only if there exist constants k_0, k_1 such that

$$k_0 M^{-1}(s) M^{-1}(r) \leq M^{-1}(sr) \leq k_1 M^{-1}(s) M^{-1}(r) \tag{1.9}$$

for every $s \geq 1/|I|$ and $r \geq 1/|\Omega|$.

One can conclude that (1.9) means that M must be equivalent to some power p, $1 \leq p \leq \infty$. Hence, if (1.9) should hold, very strong assumptions must be satisfied by M. Surely they would provide $L_M(Q)$ to be separable and reflexive.

Before we state our main theorem, let us denote some spaces. By $\mathcal{D}(\Omega)$ we mean the set of C^∞-functions with compact support contained in Ω. Let $\mathcal{V}$ be the set of all functions which belong to $\mathcal{D}(\Omega)$ and are divergence-free. Moreover, by $L^p, W^{1,p}$ we mean the standard Lebesgue and Sobolev spaces respectively, by H the closure of $\mathcal{V}$ w.r.t. the $\|\cdot\|_{L^2}$ norm and by $W^{1,p}_{0,\mathrm{div}}$ the closure of $\mathcal{V}$ w.r.t. the $\|\nabla(\cdot)\|_{L^p}$ norm. Let $W^{-1,p'} = (W^{1,p}_0)^*$, $W^{-1,p'}_{\mathrm{div}} = (W^{1,p}_{0,\mathrm{div}})^*$. By p' we mean the conjugate exponent to p, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

If X is a Banach space of scalar functions, then X^3 or $X^{3\times 3}$ denotes the space of vector- or tensor-valued functions where each component belongs to X. The symbols $L^p(0,T;X)$ and $C([0,T];X)$ mean the standard Bochner spaces. Finally, we recall that the Nikolsii space $N^{\alpha,p}(0,T;X)$ corresponding to the Banach space X and the exponents $\alpha \in (0,1)$ and $p \in [1,\infty]$ is given by

$$N^{\alpha,p}(0,T;X) := \{f \in L^p(0,T;X) \ : \ \sup_{0<h<T} h^\alpha \|\tau_h f - f\|_{L^p(0,T-h;X)} < \infty\},$$

where $\tau_h f(t) = f(t+h)$ for a.a. $t \in [0, T-h]$.

By (a,b) we mean $\int_\Omega a(x) \cdot b(x) dx$ and $\langle a, b\rangle$ denotes the duality pairing.

Definition 1.1 *We call the pair ϱ, $\boldsymbol{u}$ a* weak solution *to (1.1) if*

$$\begin{aligned}
&0<\varrho_*\le\varrho(t,x)\le\varrho^* \quad \text{for a.a. } (t,x)\in Q,\\
&\varrho\in C([0,T];L^q(\Omega)) \quad \text{for arbitrary } q\in[1,\infty),\\
&\partial_t\varrho\in L^{5p/3}(0,T;(W^{1,5p/(5p-3)})^*)\\
&\boldsymbol{u}\in L^\infty(0,T;H^3)\cap L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega)^3)\cap N^{1/2,2}(0,T;H^3)\\
&\mathbf{D}\boldsymbol{u}\in L_M(Q)^{3\times 3}_{\mathrm{sym}} \quad \text{and} \quad (\varrho\boldsymbol{u},\boldsymbol{\psi})\in C([0,T]) \text{ for all } \boldsymbol{\psi}\in H^3
\end{aligned}$$

$$\int_0^T \langle\partial_t\varrho, z\rangle-(\varrho\boldsymbol{u},\nabla_x z)dt=0 \tag{1.10}$$

for all $z\in L^r(0,T;W^{1,r}(\Omega))$ with $r=5p/(5p-3)$, i.e.

$$\int_{s_1}^{s_2}\int_\Omega \varrho\partial_t z+(\varrho\boldsymbol{u})\cdot\nabla_x z dxdt=\int_\Omega \varrho z(s_2)-\varrho z(s_1)dx$$

for all z smooth and s_1, $s_2\in[0,T]$, $s_1<s_2$ and

$$\begin{aligned}
&-\int_0^T\int_\Omega \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}-\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{\varphi}+\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}):\mathbf{D}\boldsymbol{\varphi}dxdt\\
&=\int_0^T\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{\varphi}dxdt+\int_\Omega \varrho_0\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0)dx \quad \text{for all } \boldsymbol{\varphi}\in\mathcal{D}((-\infty,T);\mathcal{V}),
\end{aligned} \tag{1.11}$$

and initial conditions are achieved in the following way

$$\lim_{t\to 0^+}\|\varrho(t)-\varrho_0\|_{L^q(\Omega)}+\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|^2_{L^2(\Omega)}=0 \quad \text{for arbitrary } q\in[1,\infty). \tag{1.12}$$

Theorem 1.1 *Let M be an N–function satisfying for some $\underline{c}>0$ and*

$$p\ge\frac{11}{5} \tag{1.13}$$

the condition

$$M(x,\boldsymbol{\xi})\ge\underline{c}|\boldsymbol{\xi}|^p \tag{1.14}$$

for a.a. $x\in\Omega$ and all $\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Moreover, let $\mathbf{S}$ satisfy conditions 1.-3. and $\boldsymbol{u}_0\in H(\Omega)^3$, $\varrho_0\in L^\infty(\Omega)$ with $0<\varrho_*\le\varrho_0(x)\le\varrho^*<+\infty$ for a.a. $x\in\Omega$ and $\boldsymbol{f}\in L^{p'}(0,T;L^{p'}(\Omega)^3)$. Then there exists a weak solution to (1.1).

In the following paper we consider the flow in the domain of space dimension $d = 3$, just for the brevity. The existence result can be easily extended to the case of arbitrary $d \geq 2$ and $p \geq \frac{3d+2}{d+2}$.

Our paper is organised as follows: Section 2 presents the notation and some properties of generalized Orlicz spaces. In Section 3 our main result of existence of weak solutions to the system (1.1) is proved. Section 4 (Appendix) contains some technical facts employed in Section 3.

2 Notation and properties of Orlicz spaces

Definition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^3$, a function $M : \Omega \times \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}_+$ is said to be an N–*function *if it satisfies the following conditions*

1. *M is a Carathéodory function such that $M(x, \mathsf{K}) = 0$ if and only if $\mathsf{K} = 0$, $M(x, \mathsf{K}) = M(x, -\mathsf{K})$ a.e. in Ω,*
2. *$M(x, \mathsf{K})$ is a convex function w.r.t. K,*
3. $$\lim_{|\mathsf{K}|\to 0} \sup_{x\in\Omega} \frac{M(x, \mathsf{K})}{|\mathsf{K}|} = 0, \tag{2.15}$$
4. $$\lim_{|\mathsf{K}|\to \infty} \inf_{x\in\Omega} \frac{M(x, \mathsf{K})}{|\mathsf{K}|} = \infty. \tag{2.16}$$

Definition 2.2 *The* complementary function *M^* to a function M is defined by*

$$M^*(x, \mathsf{L}) = \sup_{\mathsf{K}\in\mathbb{R}^{3\times3}_{\rm sym}} (\mathsf{K} : \mathsf{L} - M(x, \mathsf{K}))$$

for $\mathsf{L} \in \mathbb{R}^{3\times3}_{\rm sym}$, $x \in \Omega$.

The complementary function M^* is also an N-function (see [36]).

The *generalized Orlicz class $\mathcal{L}_M(Q)^{3\times3}_{\rm sym}$* is the set of all measurable functions $\mathsf{K} : Q \to \mathbb{R}^{3\times3}_{\rm sym}$ such that

$$\int_Q M(x, \mathsf{K}(t, x))dxdt < \infty.$$

The *generalized Orlicz space* $L_M(Q)^{3\times3}_{\mathrm{sym}}$ is defined as the set of all measurable functions $\mathsf{K} : Q \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ which satisfy

$$\int_Q M(x, \lambda\mathsf{K}(t,x))dxdt \to 0 \quad \text{as}\, \lambda \to 0.$$

The generalized Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|\mathsf{K}\|_M = \inf\left\{\lambda > 0 : \quad \int_Q M\left(x, \frac{\mathsf{K}(t,x)}{\lambda}\right) dxdt \leq 1\right\}.$$

Let us denote by $E_M(Q)^{3\times3}_{\mathrm{sym}}$ the closure of all measurable, bounded functions on Q in $L_M(Q)^{3\times3}_{\mathrm{sym}}$. The space $L_{M^*}(Q)^{3\times3}_{\mathrm{sym}}$ is the dual space of $E_M(Q)^{3\times3}_{\mathrm{sym}}$ (cf. [40]). It is easy to see that $E_M \subseteq \mathcal{L}_M \subseteq L_M$.

The functional

$$\varrho(\mathsf{K}) = \int_Q M(x, \mathsf{K}(x))dxdt$$

is a modular in the space of measurable functions $\mathsf{K} : Q \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$. A sequence $\{z^j\}_{j=1}^{\infty}$ converges modularly to z in $L_M(Q)^{3\times3}_{\mathrm{sym}}$ if there exists $\lambda > 0$ such that

$$\int_Q M\left(x, \frac{z^j - z}{\lambda}\right) dxdt \to 0 \quad \text{as } j \to \infty.$$

We will write $z^j \xrightarrow{M} z$ for the modular convergence in $L_M(Q)^{3\times3}_{\mathrm{sym}}$.

We say that an N-function M satisfies Δ_2*–condition* if for some nonnegative, integrable on Ω function g_M and a constant $C_M > 0$

$$M(x, 2\mathsf{K}) \leq C_M M(x, \mathsf{K}) + g_M(x) \quad \text{for all } \mathsf{K} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and a.a. } x \in \Omega. \quad (2.17)$$

If this condition fails we lose numerous properties of the space $L_M(Q)^{3\times3}_{\mathrm{sym}}$ like separability, density of C^∞-functions, reflexivity (even in simpler case for $M(x, \mathsf{K}) = M(|\mathsf{K}|)$ cf. [1]). In particular, if (2.17) holds, then $E_M(Q)^{3\times3}_{\mathrm{sym}} = L_M(Q)^{3\times3}_{\mathrm{sym}}$. More information can be found for the case of x-dependent generalized N-function in [36, 37, 41] and for less general N-functions [22, 30].

3 Proof of the Theorem 1.1

3.1 Uniform estimates

Let $\{\boldsymbol{\omega}^n\}_{n=1}^{\infty}$ be a basis of $W^{1,p}_{0,\mathrm{div}}(\Omega)^3$ constructed with help of eigenfunctions to the problem

$$((\boldsymbol{\omega}_i, \boldsymbol{\varphi}))_s = \lambda_i(\boldsymbol{\omega}_i, \boldsymbol{\varphi}) \quad \text{for all } \boldsymbol{\varphi} \in V_s,$$

where

$$V_s \equiv \text{the closure of } \mathcal{V} \text{ w.r.t. the } W^{s,2}(\Omega)\text{-norm}$$

and $((\cdot,\cdot))_s$ denotes the scalar product in V_s. We assume that $s > 3$ and then the Sobolev embedding theorem provides

$$W^{s-1,2}(\Omega) \hookrightarrow C(\Omega). \tag{3.18}$$

Moreover the basis is orthonormal in $L^2(\Omega)$ (see [29, Appendix]).

We denote $H_n := \mathrm{span}\{\boldsymbol{\omega}^1, ..., \boldsymbol{\omega}^n\}$ and define orthonormal projection $P^n : H \to H_n$ by $P^n\boldsymbol{u} := \sum_{i=1}^{n}(\boldsymbol{u}, \boldsymbol{\omega}^i)\boldsymbol{\omega}^i$ for every $n \in \mathbb{N}$. Let us seek for an approximate solution $\boldsymbol{u}^n$ of the system (1.1) in the following form of finite sums

$$\boldsymbol{u}^n(t,x) := \sum_{j=1}^{n} \alpha_j^n(t)\boldsymbol{\omega}^j(x) \tag{3.19}$$

for $n = 1, 2, \ldots$ with the unknown coefficients $\alpha_j^n \in C([0,T])$, $j = 1, 2, \ldots, n$, while ϱ^n is the solution of the continuos problem

$$\begin{aligned} \partial_t\varrho^n + \mathrm{div}_x(\varrho^n\boldsymbol{u}^n) &= 0, \\ \varrho^n(0) &= \varrho_0^n. \end{aligned} \tag{3.20}$$

with $\varrho_0^n \in C^1(\Omega)$ and $\boldsymbol{u}^n$ solve the Galerkin system

$$\begin{aligned} (\varrho^n\partial_t\boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\varrho^n[\nabla_x\boldsymbol{u}^n]\boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n, \mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j) &= (\varrho^n\boldsymbol{f}^n, \boldsymbol{\omega}^j) \\ \boldsymbol{u}^n(0) &= P^n(\boldsymbol{u}_0) \end{aligned} \tag{3.21}$$

for all $1 \le j \le n$ and a.a. $t \in [0,T]$. We assume additionally that

$$\begin{aligned} &\boldsymbol{u}_0^n \to \boldsymbol{u}_0 \quad \text{strongly in } H^3, \\ &\varrho_0^n \to \varrho_0 \quad \text{strongly in } L^\infty(\Omega), \\ &\varrho_0^n \in C^1(\Omega) \text{ and } \varrho_* \le \varrho_0^n \le \varrho^* \end{aligned} \tag{3.22}$$

and

$$\boldsymbol{f}^n \to \boldsymbol{f} \quad \text{strongly in } L^{p'}(0,T;L^{p'}(\Omega)^3). \tag{3.23}$$

Let us note that since our approximate solution $\boldsymbol{u}^n$ satisfies (3.20), $(3.21)_1$ for $1 \le j \le n$ is equivalent to

$$\begin{aligned}\left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{\omega}^j \right\rangle - (\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n, \nabla_x \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n, \mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j) \\ = (\varrho^n \boldsymbol{f}^n, \boldsymbol{\omega}^j)\end{aligned} \tag{3.24}$$

and consequently after integrating over time interval $(0,T)$ we have

$$\begin{aligned}\int_0^T \left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{\omega}^j \right\rangle - (\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n, \nabla_x \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n, \mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j) dt \\ = \int_0^T (\varrho^n \boldsymbol{f}^n, \boldsymbol{\omega}^j) dt\end{aligned} \tag{3.25}$$

for all $1 \le j \le n$ and (3.20) satisfies also

$$\int_0^T \langle \partial_t \varrho^n, z \rangle - (\varrho^n \boldsymbol{u}^n, \nabla_x z)\, dt = 0 \tag{3.26}$$

for all $z \in L^q(0,T;W^{1,q}(\Omega))$ with $q \in [1,\infty)$.

Before we will prove existence of the approximate solution we want to show that some uniform w.r.t. n a priori estimates are valid and to present some of their consequences which we will use later.

In the first step we concentrate on equations (3.20). Since (3.18) holds, we will use standard techniques for transport equation and apply the method of characteristics. We notice (3.20) is an equation of the first order w.r.t. $\varrho^n(t,x)$, we solve the Cauchy problem

$$\begin{aligned}\frac{dy^n(t,x)}{dt} &= \boldsymbol{u}^n(t, y^n(t,x)) \\ y^n(0,x) &= x,\end{aligned} \tag{3.27}$$

with help of Caratheodory's theory. The system (3.27) defines so-called characteristics associated with (3.20). Note that for every $t \in [0,T]$ the map $x \to y^n(t,x)$ is a diffeomorphism of $\bar{\Omega}$ onto $\bar{\Omega}$. Using this fact and $\operatorname{div}_x \boldsymbol{u}^n = 0$ we can see that the solution of (3.20) is given by

$$\varrho^n(t, y^n(t,x)) = \varrho_0^n(x). \tag{3.28}$$

Since (3.28) is satisfied and according to assumptions on ϱ_0^n we obtain that

$$0 < \varrho_* \le \varrho^n(t,x) \le \varrho^* < +\infty \quad \text{for all}\,(t,x) \in Q. \tag{3.29}$$

For later consideration let us note that the Alaoglu-Banach theorem provides existence of subsequence such that

$$\begin{aligned} \varrho^n &\rightharpoonup \varrho \quad \text{weakly in}\, L^q(Q)\, \text{for any}\, q \in [1,\infty), \\ \varrho^n &\overset{*}{\rightharpoonup} \varrho \quad \text{weakly-(*) in } L^\infty(Q). \end{aligned} \tag{3.30}$$

If we multiply (3.21) by α_j^n, sum up over j and use (3.20), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho^n|\boldsymbol{u}^n|^2 dx + (\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n),\mathbf{D}\boldsymbol{u}^n) = (\varrho^n\boldsymbol{f}^n,\boldsymbol{u}^n) \tag{3.31}$$

Using the Hölder, the Korn-Poincaré and the Young inequalities, assumption (1.14) and (3.29) we are able to estimate the right-hand side of (3.31) in following way

$$|(\varrho^n\boldsymbol{f}^n,\boldsymbol{u}^n)| \le C_1(\Omega,c_c,\underline{c},\varrho^*,p)\|\boldsymbol{f}^n\|^{p'}_{L^{p'}(\Omega)} + \frac{c_c}{2}\int_\Omega M(x,\mathbf{D}\boldsymbol{u}^n)dx. \tag{3.32}$$

Integrating (3.31) over the time interval $(0,s_0)$, using estimate (3.32), (3.29), the coercivity conditions (1.6) on $\mathbf{S}$, continuity of P^n uniformly w.r.t. n and strong convergence $\boldsymbol{f}^n \to \boldsymbol{f}$ in $L^{p'}(0,T;L^{p'}(\Omega))$ we obtain

$$\begin{aligned} \int_\Omega \frac{1}{2}\varrho^n(s_0)|\boldsymbol{u}^n(s_0)|^2dx + \int_0^{s_0}\int_\Omega \frac{c_c}{2}M(x,\mathbf{D}\boldsymbol{u}^n) + c_c M^*(x,\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n))dxdt \\ \le C_2(\Omega,c_c,\underline{c},\varrho^*,p,\|\boldsymbol{f}\|_{L^{p'}(0,T;L^{p'}(\Omega))}) + \frac{1}{2}\varrho^*\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.33}$$

where C_2 is a nonnegative constant independent of n and dependent on given data. Noticing that $E_{M^*}(Q)^{d\times d}_{\mathrm{sym}}$ is separable, $(E_{M^*})^* = L_M$ and using the Alaoglu-Banach theorem we obtain for suitable subsequence, as a direct consequence of (3.33), that

$$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \quad \text{weakly-(*) in } L_M(Q)^{3\times3}_{\mathrm{sym}}. \tag{3.34}$$

Moreover, the condition (1.14) provides that $\{\mathbf{D}\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in the space $L^p(Q)^{3\times3}$ for $p \ge \frac{11}{5}$

$$\int_0^T \|\mathbf{D}\boldsymbol{u}^n\|^p_{L^p(\Omega)}dt \le C \tag{3.35}$$

and hence there exists a subsequence such that

$$\mathbf{D}\boldsymbol{u}^n \rightharpoonup \mathbf{D}\boldsymbol{u} \quad \text{weakly in} \quad L^p(Q)^{3\times 3}. \tag{3.36}$$

According to the Korn inequality we also obtain

$$\int_0^T \|\nabla_x \boldsymbol{u}^n\|_{L^p(\Omega)}^p dt \le C \tag{3.37}$$

and

$$\boldsymbol{u}^n \rightharpoonup \boldsymbol{u} \quad \text{weakly in} \quad L^p(0,T,W^{1,p}_{0,\mathrm{div}}(\Omega)^3). \tag{3.38}$$

Using (3.33) we deduce that

$$\|\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\|_{L^1(Q)} \le C. \tag{3.39}$$

moreover, we get that the sequence $\{\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\}_{n=1}^\infty$ is uniformly bounded in Orlicz class $\mathcal{L}_{M^*}(Q)^{3\times 3}$. Consequently for a subsequence we infer that

$$\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \quad \text{weak-(*) in } L_{M^*}(Q)^{3\times 3}_{\mathrm{sym}}. \tag{3.40}$$

Applying Lemma 4.2 we conclude the uniform integrability of the sequence Consequently there exists a tensor $\overline{\mathbf{S}} \in L^1(Q)^{3\times 3}$ and subsequence $\{\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\}_{n=1}^\infty$ such that

$$\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in} \quad L^1(Q)^{3\times 3}. \tag{3.41}$$

Furthermore (3.33) and (3.29) provide

$$\begin{aligned} &\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 \le C, \\ &\sup_{t\in[0,T]} \|\varrho^n(t)|\boldsymbol{u}^n(t)|^2\|_{L^1(\Omega)} \le C, \end{aligned} \tag{3.42}$$

where C is a positive constant dependent on the size of data, but independent of n. It follows immediately that for some subsequence

$$\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^\infty(0,T;H(\Omega)^3). \tag{3.43}$$

In particular, from $(3.42)_1$ follows that there exists constant C_B s.t.

$$\|\boldsymbol{u}^n\|_{L^q(0,T;H(\Omega))} \le C_B \quad \text{for } q \ge 1. \tag{3.44}$$

Since the sequence $\{\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in $L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega)^3)$ the Galiardo-Nirenberg-Sobolev inequalities provides also uniform boundedness in the space $L^p(0,T;L^{3p/(3-p)})$. The standard interpolation (see e.g. [32, Proposition 1.41]) of $L^\infty(0,T;L^2)$ and $L^p(0,T;L^{3p/(3-p)})$ (this particular argument deals with the case $p<3$, the case $p\geq 3$ can be treated easier e.g. with the Poincaré or the Morrey inequality) gives us

$$\int_0^T \|\boldsymbol{u}^n\|_{L^r(\Omega)}^r dt \leq C_B \quad \text{for } 1\leq r\leq 5p/3 \tag{3.45}$$

for some constant C_B, therefore (3.29) and (3.45) infer also

$$\int_0^T \|\varrho^n \boldsymbol{u}^n\|_{L^{5p/3}(\Omega)}^{5p/3} dt \leq C. \tag{3.46}$$

Consequently we can take subsequences satisfying

$$\boldsymbol{u}^n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^{5p/3}(0,T;L^{5p/3}(\Omega)^3) \tag{3.47}$$

and there exist subsequence $\{\varrho^n\boldsymbol{u}^n\}_{n=1}^\infty$ and $\overline{\varrho\boldsymbol{u}}\in L^{5p/3}(0,T;L^{5p/3}(\Omega)^3)$ such that

$$\varrho^n\boldsymbol{u}^n \rightharpoonup \overline{\varrho\boldsymbol{u}} \quad \text{weakly in} \quad L^{5p/3}(0,T;L^{5p/3}(\Omega)^3). \tag{3.48}$$

Using (3.29), (3.37) and (3.45) and applying the Hölder inequality, we obtain

$$\int_0^T |(\varrho^n\boldsymbol{u}^n\otimes\boldsymbol{u}^n, \nabla_x\boldsymbol{u}^n)|dt \leq C \quad \iff \quad p\geq \frac{11}{5} \tag{3.49}$$

(here is the restriction for the exponent p stated in (1.13)).

Using (3.46) it follows from (3.26) that

$$\int_0^T \|\partial_t\varrho^n\|_{(W^{1,5p/(5p-3)})^*}^{5p/3} dt \leq C. \tag{3.50}$$

Hence the Alaoglu-Banach theorem provides existence of a subsequence such that

$$\partial_t\varrho^n \rightharpoonup \partial_t\varrho \quad \text{weakly in } L^{5p/3}(0,T;(W^{1,5p/(5p-3)})^*). \tag{3.51}$$

3.2 Existence of approximate solution

On the basis of estimates proved in above Subsection 3.1 we will show the existence of solutions of (3.21) and (3.20) using Schauder's fixed point theorem for the operator

$$\Lambda : B\subset Y\to B : \tilde{\boldsymbol{u}}^n\to\boldsymbol{u}^n$$

where $Y := L^q(0,T;L^q(\Omega)^3) \cap L^q(0,T;H_n)$, $q = 2p'$ is equipped with the norm of the $L^q(0,T;L^q(\Omega)^3)$ and B is the closed ball which will be defined later. For given $\tilde{\boldsymbol{u}}^n \in B$ the element $\Lambda\tilde{\boldsymbol{u}}^n = \boldsymbol{u}^n$ is a solution of the problem

$$\begin{aligned} \partial_t \tilde{\varrho}^n + \mathrm{div}_x(\tilde{\varrho}^n \tilde{\boldsymbol{u}}^n) = 0, \\ \tilde{\varrho}^n(0) = \varrho_0^n, \end{aligned} \tag{3.52}$$

$$\begin{aligned} (\tilde{\varrho}^n \partial_t \boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\tilde{\varrho}^n [\nabla_x \boldsymbol{u}^n] \tilde{\boldsymbol{u}}^n, \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\tilde{\varrho}^n, \mathbf{D}\tilde{\boldsymbol{u}}^n), \mathbf{D}\boldsymbol{\omega}^j) = (\tilde{\varrho}^n \boldsymbol{f}^n, \boldsymbol{\omega}^j), \\ \boldsymbol{u}^n(0) = P^n(\boldsymbol{u}_0). \end{aligned} \tag{3.53}$$

It means that in the first step we find solution $\tilde{\varrho}^n$ of the linear problem (3.52) and next we look for the vector $\boldsymbol{u}^n$, solution of the linearization (3.53) of the system (3.21).

The system (3.53) can be rewritten as a system of ordinary differential equations (details the reader can find in [2, 27, 28]). We obtain local in time solvability according to Peano's existence theorem for the system of ordinary differential equations. The global solvability is provided by the a'priori estimates (3.33) where $\boldsymbol{u}^n$ is replaced by $\tilde{\boldsymbol{u}}^n$ in suitable places.

Let us take $\tilde{\boldsymbol{u}}^n \in B := \overline{B_{C_B}(0)}$, where $B_{C_B}(0)$ is a ball and C_B is a constant from (3.45). Inequalities $2p' \leq 5p/3$ for $p \geq 11/5$ assure that $Y \supset B$. Previous estimates (3.44) and (3.45) provide that Λ maps B into B. Using $(3.42)_1$ and (3.37) we deduce that $\boldsymbol{u}^n \in L^\infty(0,T;H_n) \cap L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega)^3)$. The continuity of the operator Λ results from the theorem on a continuous dependence of the solutions of the Cauchy problem (3.53) on the coefficients and right-hand side. Now the main difficulty is to show compactness of the operator Λ. Similarly like in [2, 14] our plan is to prove that

$$\int_0^{T-\delta} \|\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)\|^2_{L^2(\Omega)} ds \to 0 \quad \text{as } \delta \to 0 \tag{3.54}$$

is satisfied. According to [35, Theorem 5] and parabolic embedding theorem $\Lambda(B)$ is a compact subset of Y. Applying Schauder's fixed point theorem we deduce that there exists the fixed point $\tilde{\boldsymbol{u}}^n$ and the corresponding density $\tilde{\varrho}^n$ which solve the system (3.21), (3.20).

To show (3.54) we will follow [2, Chap.3. Lemma 1.2] with some modification concerning change from L^2-structure for L^p-structure and additional one concerning the nonlinear term controlled by nonstandard condition (1.6).

Let us fix δ and s, $0 < \delta < T$, $0 \le s \le T - \delta$. Next we test (3.53) at time t by $\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)$ and integrate the equation over time interval $(s, s+\delta)$ w.r.t. time t. Using the integration by parts formula w.r.t. time, the equality $\partial_t \tilde{\varrho}^n = -\mathrm{div}_x(\tilde{\varrho}^n \tilde{\boldsymbol{u}}^n)$ and obvious identity

$$\tilde{\varrho}^n(s+\delta)\boldsymbol{u}^n(s+\delta) - \tilde{\varrho}(s)\boldsymbol{u}^n(s) = \tilde{\varrho}^n(s+\delta)[\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)] + [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}^n(s)]\boldsymbol{u}^n(s)$$

we get

$$\begin{aligned}
&\int_\Omega \tilde{\varrho}^n(s+\delta)|\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)|^2 + [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}^n(s)]\boldsymbol{u}^n(s) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dx \\
&+ \int_s^{s+\delta} \int_\Omega \mathrm{div}_x(\tilde{\varrho}^n(t)\tilde{\boldsymbol{u}}^n(t))\boldsymbol{u}^n(t) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dxdt \\
&+ \int_s^{s+\delta} \int_\Omega \tilde{\varrho}^n(t)[\nabla_x \boldsymbol{u}^n(t)]\tilde{\boldsymbol{u}}^n(t) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dxdt \\
&+ \int_s^{s+\delta} \int_\Omega \mathbf{S}(t, x, \tilde{\varrho}^n(t), \mathbf{D}\tilde{\boldsymbol{u}}^n(t)) : \mathbf{D}[\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dxdt \\
&= \int_s^{s+\delta} \int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dxdt.
\end{aligned} \tag{3.55}$$

Now, let us test (3.52) at time t by $\boldsymbol{u}^n(s) \cdot (\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s))$ and integrate the equation over time interval t on $(s, s+\delta)$ then we obtain.

$$\begin{aligned}
&\int_\Omega [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}^n(s)]\boldsymbol{u}^n(s) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dx \\
&= -\int_s^{s+\delta} \int_\Omega \mathrm{div}_x(\tilde{\varrho}^n(t)\tilde{\boldsymbol{u}}^n(t))\boldsymbol{u}^n(s) \cdot [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]dxdt.
\end{aligned}$$

Substituting above relation into (3.55) some obvious manipulations, i.e.

$$\begin{aligned}
&(\mathrm{div}_x(\tilde{\varrho}^n \tilde{\boldsymbol{u}}^n)\boldsymbol{u}^n(s), [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]) = \\
&- (\tilde{\varrho}^n(t)[\nabla_x \boldsymbol{u}^n(s)]\tilde{\boldsymbol{u}}^n(t), [\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)]) \\
&- (\tilde{\varrho}^n(t)\boldsymbol{u}^n(s) \otimes \tilde{\boldsymbol{u}}^n(t), \nabla_x[\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)])
\end{aligned} \tag{3.56}$$

and (3.29) infers

$$
\begin{aligned}
&\|\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)\|^2_{L^2(\Omega)}dx \le \\
&\frac{1}{\varrho_*}\{|-\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(s)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]dxdt \\
&+\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(t)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]dxdt \\
&-\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)[\nabla_x\boldsymbol{u}^n(s)]\tilde{\boldsymbol{u}}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]dxdt \\
&-\int_s^{s+\delta}\int_\Omega \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t)):\mathbf{D}[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]dxdt \\
&+\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]dxdt|\}.
\end{aligned}
\tag{3.57}
$$

Next we integrate over $(0,T-\delta)$ w.r.t. time s and we intend to show that for any of the ten addends $I_k(s)$, $k=1,2,\dots,10$ in the right-hand side of (3.57), the following inequalities are valid

$$
\int_0^{T-\delta} I_k(s)ds \le \kappa_k\theta(\delta) \quad \text{for } k=1,2,\dots,10, \tag{3.58}
$$

where $\theta(\delta)\to 0$ as $\delta\to 0$ and constant κ_k is independent of δ. To estimate first six integrals let us employ (3.29), the Hölder inequality, the assumption that $q=2p'$ and the fact that Λ maps B into B. Employing additionally the Young and Jensen inequality and following obvious relation $\int_0^{T-\delta}\frac{1}{\delta}\int_s^{s+\delta}a(t)dtds \le \int_0^T a(s)ds$ for $a(t)\ge 0$ for one of representative terms we obtain

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(s)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x\boldsymbol{u}^n(s+\delta)dxdtds| \\
&\le \varrho^*\int_0^{T-\delta}\int_s^{s+\delta}\|\boldsymbol{u}^n(s)\|_{L^q(\Omega)}\|\tilde{\boldsymbol{u}}^n(t)\|_{L^q(\Omega)}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|_{L^p(\Omega)}dtds \\
&\le \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\boldsymbol{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\left|\frac{1}{\delta}\int_s^{s+\delta}\|\tilde{\boldsymbol{u}}^n(t)\|_{L^q(\Omega)}dt\right|^q+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}ds \\
&\le \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\boldsymbol{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\frac{1}{\delta}\int_s^{s+\delta}\|\tilde{\boldsymbol{u}}^n(t)\|^q_{L^q(\Omega)}dt+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}ds \\
&\le \delta\varrho^*\left(\frac{1}{q}\|\boldsymbol{u}^n(s)\|_{L^q(0,T;L^q(\Omega))}+\frac{1}{q}\|\tilde{\boldsymbol{u}}^n(s)\|_{L^q(0,T;L^q(\Omega))}+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n\|_{L^p(0,T:L^p(\Omega))}\right) \\
&\le \kappa_1\delta
\end{aligned}
\tag{3.59}
$$

Next we deal with nonlinear viscous term. Using the Fubini theorem, the Fenchel-Young inequality (Proposition 4.1) and the Jensen inequality we get the following estimates

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t)):\mathbf{D}\boldsymbol{u}^n(s+\delta)dxdtds| \\
&= \delta\int_0^{T-\delta}\int_\Omega\left\{\left|\frac{1}{\delta}\int_s^{s+\delta}\mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))dt\ \cdot\mathbf{D}\boldsymbol{u}^n(s+\delta)\right|\right\}dxds \\
&\le \delta\int_0^{T-\delta}\int_\Omega\left\{M^*\left(x,\frac{1}{\delta}\int_s^{s+\delta}\mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))dt\right)+M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)\right\}dxds \\
&\le \delta\int_\Omega\int_0^{T-\delta}\left\{\frac{1}{\delta}\int_s^{s+\delta}M^*\left(x,\mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))\right)dt+M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)\right\}dsdx \\
&\le \int_\Omega\left\{\int_0^T M^*\left(x,\mathbf{S}(t,x,\tilde{\varrho}^n(s),\mathbf{D}\tilde{\boldsymbol{u}}^n(s))\right)ds+\int_0^{T-\delta}M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)ds\right\}dx \\
&\le \kappa_2\delta,
\end{aligned}
\tag{3.60}
$$

where κ_2 is uniform w.r.t. n.

Using assumptions on $\boldsymbol{f}^n$ and (3.37) we deduce

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t)\cdot\boldsymbol{u}^n(s+\delta)dxdtds| \\
&\leq \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{p'}\left|\frac{1}{\delta}\int_s^{s+\delta}\|f(t)\|_{L^{p'}(\Omega)}ds\right|^{p'}+\|\boldsymbol{u}^n(s+\delta)\|_{L^p(\Omega)}\right\}ds \qquad (3.61)\\
&\leq \delta\varrho^*\left(\frac{1}{p'}\|\boldsymbol{u}^n(s+\delta)\|^{p'}_{L^{p'}(0,T;L^{p'}(\Omega))}+\frac{1}{p}\|\boldsymbol{u}^n\|^p_{L^p(0,T;L^p(\Omega))}\right)\leq\kappa_3\delta
\end{aligned}
$$

We proceed with the second source term in similar way. Summarising all of the above estimates for integrals on the right-hand side of (3.57) we prove (3.54) and existence of approximate solution $\boldsymbol{u}^n$.

Remark: Since we already know that $\{\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;H^3)$ and above considerations shows that for any approximate solution of (3.20), (3.21) we obtain

$$\int_0^{T-\delta}\|\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)\|^2_{L^2(\Omega)}ds\leq\kappa\delta$$

where κ is independent of n and δ. Therefore, as a byproduct, we obtain that $\{u^n\}_n$ is uniformly bounded in Nikolsii space $N^{1/\delta,2}(0,T;H^3)$.

3.3 Strong convergence of ϱ^n and $\boldsymbol{u}^n$

Since in this moment we have existence of approximate solution to (3.20 - 3.21) and the previous consideration shows (3.54) uniformly w.r.t. n, we get by [35, Theorem 1] that

$$\boldsymbol{u}^n\to\boldsymbol{u}\quad\text{strongly in } L^2(Q). \qquad (3.62)$$

Moreover due to [35, Theorem 3], $(3.42)_1$ and (3.38)

$$\boldsymbol{u}^n\to\boldsymbol{u}\quad\text{strongly in } C(0,T;L^2(\Omega)). \qquad (3.63)$$

Using (3.29), (3.30) and (3.51) the Aubin-Lions Lemma provides that

$$\varrho^n\to\varrho\quad\text{strongly in } C([0,T];W^{-1,5p/3}). \qquad (3.64)$$

If we employ the same methods like Lions in [6] and [26, Chapter 2], we are able to deduce that

$$\varrho^n\to\varrho\quad\text{strongly in } C([0,T];L^q(\Omega))\text{ for all } q\in[1,\infty)\text{ and a.e. in } Q, \qquad (3.65)$$

and also

$$\lim_{t\to 0^+} \|\varrho(t)-\varrho_0\|_{L^q(\Omega)} = 0 \quad \text{for all}\, q\in[1,\infty), \tag{3.66}$$

which is the first part of initial condition (1.12). To give to the rider a view of main steps we list some of them.

Using the fact that $\mathrm{div}_x \boldsymbol{u}^n = 0$ we see that the so-called strong and weak form of transport equation coincide, i.e. equation (3.20) is equivalent to $\partial_t \varrho^n - \boldsymbol{u}_n \nabla_x \varrho^n = 0$ in a weak sense. Consequently with the concept of renormalized solution to the equation (3.26), it is possible to strengthen (3.64). First, we need the time-space version of the Friedrichs commutator lemma (see [11, Corollary 10.3],[6]). Since $\varrho \in L^q(0,T;L^q(\Omega))$ for $q\in[1,\infty)$ and $\boldsymbol{u}\in L^p(0,T;W^{1,p}_{\mathrm{div}}(\Omega)^3)$ then

$$\mathrm{div}_x\left(\sigma_\epsilon * (\varrho^n \boldsymbol{u}^n)\right) - \mathrm{div}_x\left((\sigma_\epsilon * \varrho^n)\boldsymbol{u}^n\right) \to 0 \quad \text{in } L^r(Q) \tag{3.67}$$

for r such that $\frac{1}{q}+\frac{1}{p}=\frac{1}{r}\in(0,1]$, where σ_ϵ is the standard mollifying operator acting on the space variable.

Additionally since $\varrho^n \geq \varrho_*$ and continuity equation (3.20) is satisfied, then ϱ^n satisfies renormalized continuity equation, namely

$$\partial_t b(\varrho^n) + \mathrm{div}_x\left(b(\varrho^n)\boldsymbol{u}^n\right) = 0 \tag{3.68}$$

in a weak sense for $b\in C^1([0,\infty))\cap W^{1,\infty}(0,\infty)$ which vanishes near zero (see [6], [11, Appendix]). Next we are able to prove that

$$\varrho^n \in C([0,T],L^q(\Omega)) \quad \text{for } q\in[1,\infty)$$

With above information at hand following [6] or [26] we can prove (3.65).

The task now is to show that

$$\varrho^n \boldsymbol{u}^n \rightharpoonup \varrho\boldsymbol{u} \quad \text{weakly in}\, L^q(0,T;L^q(\Omega)^3)\, \text{for all}\, q\in[1,5p/3]. \tag{3.69}$$

Indeed, since (3.65) provides, in particular, that ϱ^n converges strongly to ϱ in $L^{5p/3+\gamma}(0,T;L^{5p/3+\gamma}(\Omega)^3)$, where $\gamma\in[0,\infty)$. This together with (3.47) implies

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_\Omega \varrho^n\boldsymbol{u}^n\cdot\boldsymbol{\varphi}\,dxdt &= \lim_{n\to\infty}\int_0^T(\varrho^n,\boldsymbol{u}^n\cdot\boldsymbol{\varphi})dt = \int_0^T(\varrho,\boldsymbol{u}\cdot\boldsymbol{\varphi})dt\\ &= \int_0^T\int_\Omega\varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\,dxdt\end{aligned}$$

for every $\boldsymbol{\varphi} \in (L^{5p/6+\varepsilon}(0,T;L^{5p/6+\varepsilon}(\Omega)^3))^*$, where $\varepsilon(\gamma) \in [0, \frac{5p}{3})$. Therefore (3.48) infers that (3.69) holds. Finally from (3.51) and (3.69) we conclude that ϱ and $\boldsymbol{u}$ satisfy (1.10).

Additionally previous considerations imply, by using test function of the form $\mathbb{1}_{(t_1,t_2)}h$, $h \in W^{1,5p/(5p-3)}$ in (1.10), partial integration w.r.t. time and the density $W^{1,5p/(5p-3)}$ in L^1, that $\varrho \in C([0,T]; L^\infty_{\text{weak}})$, i.e. for all $h \in L^1$ and all $0 \le t_0 \le T$ we have

$$\lim_{t \to t_0} (\varrho(t), h) = (\varrho(t_0), h). \tag{3.70}$$

Using (3.62) and (3.45) we infer by interpolation inequalities that

$$\boldsymbol{u}^n \to \boldsymbol{u} \quad \text{strongly in } L^r(Q)^3 \text{ for all } r \in [1, 5p/3) \text{ and a.e. in } Q. \tag{3.71}$$

Summarising (3.71), (3.29) and (3.47), (3.65),

$$\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n \rightharpoonup \varrho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } L^{r'}(0,T;W^{-1,r'}) \text{ for } r \text{ sufficiently large,}$$

i.e. $\frac{1}{q} + \frac{6}{5p} + \frac{1}{r} < 1$, where arbitrary $q \in [1, \infty)$. Density argument and (3.38) provides

$$\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n \rightharpoonup \varrho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } L^{p'}(0,T;W^{-1,p'}_{\text{div}}) \text{ for } p \ge 11/5. \tag{3.72}$$

particularly we obtain

$$\lim_{n\to\infty} \int_0^T \int_\Omega \varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n : \boldsymbol{\varphi} = \int_0^T \int_\Omega \varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n : \boldsymbol{\varphi} \quad \text{for } \boldsymbol{\varphi} \in \mathcal{D}((-\infty,T);\mathcal{V}). \tag{3.73}$$

3.4 Integration by parts

For any function z (for which integrals below have sense) and for $h > 0$ from above we denote

$$(\tilde{\sigma}_h^+ * z)(t,x) := \frac{1}{h}\int_0^h z(t+\tau,x)d\tau,$$
$$(\tilde{\sigma}_h^- * z)(t,x) := \frac{1}{h}\int_{-h}^0 z(t+\tau,x)d\tau.$$

where $*$ denoes convolution over time variable and let us define

$$D^{+h}z := \frac{z(t+h,x)-z(t,x)}{h},$$
$$D^{-h}z := \frac{z(t,x)-z(t-h,x)}{h}.$$

It is easy to observe that

$$\partial_t(\tilde{\sigma}_h^+ * z) = D^{+h}z \qquad \text{and} \qquad \partial_t(\tilde{\sigma}_h^- * z) = D^{-h}z. \tag{3.74}$$

Let us take $h>0$ and $0<s_0<s<T$ such that $h \le \min\{s_0, T-s\}$. We multiply each equation in system (3.24) by

$$\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \alpha_j^i(t))\ \mathbb{1}_{(s_0,s)}),$$

next we sum up over $j=1,\ldots,i$ where $i \le n$ and integrate this sum over time interval $(0,T)$. Noticing that $\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u}^i)\ \mathbb{1}_{(s_0,s)}) = \sum_{j=1}^i \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \alpha_j^i(t))\ \mathbb{1}_{(s_0,s)})\boldsymbol{\omega}^j(x)$ let

$$\boldsymbol{u}^{h,i} \stackrel{\text{def}}{=} \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u}^i)\ \mathbb{1}_{(s_0,s)})$$

with $h \le \min\{s_0, T-s\}$. Since

$$\int_0^T \left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{u}^{h,i} \right\rangle dt = \int_0^T \left\langle \partial_t(\tilde{\sigma}_h^- * (\varrho^n \boldsymbol{u}^n)), ((\tilde{\sigma}_h^- * \boldsymbol{u}^i)\mathbb{1}_{(s_0,s)}) \right\rangle dt,$$

and $i \le n$ we get in the limit as $n \to \infty$

$$\begin{aligned}\int_{s_0}^s \left\langle (\partial_t(\tilde{\sigma}_h^- * \varrho\boldsymbol{u})), (\tilde{\sigma}_h^- * \boldsymbol{u}^i) \right\rangle dt &= \int_0^T \int_\Omega (\varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x \boldsymbol{u}^{h,i} dxdt \\ &\quad - \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}^{h,i} dxdt + \int_0^T \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u}^{h,i} dxdt.\end{aligned} \tag{3.75}$$

Indeed, let us notice that for fixed h and j $\boldsymbol{u}^{h,i}$, $\mathbf{D}\boldsymbol{u}^{h,i} \in L^\infty$. Then the convergence process in the first term on the left-hand side of (3.75) is provided by the fact that $\tilde{\sigma}_h^- * \boldsymbol{u}^i$ is locally Lipschitz w.r.t time variable and (3.69) holds. In terms on the left hand side we use respectively (3.72), (3.40) (obviously $L^\infty \subset E_M$) and (3.23) with (3.65).

Our aim now is to use a test function in (3.75)

$$\boldsymbol{u}^h \stackrel{\text{def}}{=} \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u})\ \mathbb{1}_{(s_0,s)})$$

with $0 < h < \min\{s_0, T-s\}$. For this purpose define the truncation operator $\bar{\mathcal{T}}_m : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ such that

$$\bar{\mathcal{T}}_m(\mathbf{K}) = \begin{cases} \mathbf{K} & |\mathbf{K}| \le m, \\ m\frac{\mathbf{K}}{|\mathbf{K}|} & |\mathbf{K}| > m. \end{cases}$$

Observe the following identity

$$\begin{aligned}\int_{s_0}^{s} \left\langle (\partial_t(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u})), (\tilde{\sigma}_h^- * \boldsymbol{u}^i)\right\rangle dt = &\int_0^T\int_\Omega (\varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x \boldsymbol{u}^{h,i} dxdt \\ &+ \int_0^T\int_\Omega (\bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}^{h,i} dxdt \\ &- \int_0^T\int_\Omega \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}^{h,i} dxdt \\ &+ \int_0^T\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}^{h,i} dxdt.\end{aligned} \tag{3.76}$$

Let us concentrate now on the right-hand side of (3.76) and investigate the first and the last terms as less complicated then the second one.

The sequence $\{\boldsymbol{u}^{h,i}\}$ is weakly convergent to $\boldsymbol{u}^h$ in $L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega))$ as $i\to\infty$. Note that if $p \ge \frac{11}{5}$, then since ϱ is bounded we infer that the term $\int_0^T\int_\Omega (\varrho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\nabla_x\boldsymbol{u}^{h,i}dxdt \to \int_0^T\int_\Omega (\varrho\boldsymbol{u}\otimes\boldsymbol{u})\cdot\nabla_x\boldsymbol{u}^{h}dxdt$ as $i\to\infty$.

Since $\boldsymbol{f}\in L^{p'}(0,T;L^{p'}(\Omega)^3)$ we treat in the same way the source term to obtain that $\int_0^T\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}^{h,i}dxdt \to \int_0^T\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}^{h}dxdt$ as $i\to\infty$.

Before we show convergence process in the second term on the right-hand side of (3.76) let us remind that (1.14) holds. Then it is easy to see that there exists some $\tilde{c} > 0$ such that

$$M^*(x,\xi) \le \tilde{c}|\xi|^{p'}.$$

Hence M^* satisfies Δ_2–condition. We fix $k\in\mathbb{N}$ and using the Fenchel-Young inequality, the convexity of M and that M^* satisfies Δ_2-condition we estimate the integral

$$\begin{aligned}\int_0^T\int_\Omega |(\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}})\cdot \mathbf{D}\boldsymbol{u}^{h,i}|dxdt &\le \int_0^T\int_\Omega M^*(x,2^k(\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}}))dxdt\\ &+\int_0^T\int_\Omega M(x,\frac{1}{2^k}\mathbf{D}\boldsymbol{u}^{h,i})dxdt\\ &\le C^k_{M^*}\int_0^T\int_\Omega M^*(x,\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}})dxdt\\ &+k\int_0^T\int_\Omega g_{M^*}(x)\mathbb{1}_{\{|\bar{\mathbf{S}}(t,x)|>m\}}dxdt\\ &+\frac{1}{2^k}\int_0^T\int_\Omega M(x,\mathbf{D}\boldsymbol{u}^{h,i})dxdt.\end{aligned} \tag{3.77}$$

Inequality (3.33) and Proposition 4.4 provide that for each $0<h\le \min\{s_0,T-s\}$ holds that

$$\sup_h\sup_{i\in\mathbb{N}}\int_0^T\int_\Omega M(x,\mathbf{D}\boldsymbol{u}^{h,i})dxdt<C,$$

where C is nonnegative constant independent of i and h. Consequently we infer that

$$\lim_{k\to\infty}\frac{1}{2^k}\sup_h\sup_{i\in\mathbb{N}}\int_0^T\int_\Omega M(x,\mathbf{D}\boldsymbol{u}^{h,i})dxdt=0. \tag{3.78}$$

Due to the convexity of M^*, symmetry and that $M^*(x,0)=0$ a.e. it holds

$$M^*(x,\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}})\le M^*(x,\bar{\mathbf{S}}).$$

Since M^* satysfy Δ_2-condition, $\bar{\mathbf{S}}$ is an element of $\mathcal{L}_{M^*}(Q)^{3\times 3}_{\mathrm{sym}}$ and above inequality yields by the Lebesgue convergence theorem that $\int_Q M^*(x,\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}})dxdt$ converges to zero as $m\to\infty$. Hence

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{s_0}^s\int_\Omega C^k_{M^*}M^*(x,\bar{\mathcal{T}}_m(\bar{\mathbf{S}})-\bar{\mathbf{S}})+kg_{M^*}(x)\mathbb{1}_{\{|\bar{\mathbf{S}}(t,x)|>m\}}dxdt=0. \tag{3.79}$$

Then we can pass to the limits in the second and the third term on the right-hand side of (3.76) (together with (3.77)) consecutively with $i\to\infty$, $m\to\infty$ and $k\to\infty$.

Now we will concentrate on the left hand-side term of (3.75). Let us notice that as $\varrho\boldsymbol{u}\in L^\infty(0,T;L^2(\Omega))$, $\tilde{\sigma}_h^-*\varrho\boldsymbol{u}$ is Lipschitz function w.r.t.

time variable hence $\partial_t(\tilde{\sigma}_h^- * \varrho\boldsymbol{u}) \in L^\infty(0,T;L^2(\Omega))$. By (3.43) letting $i \to \infty$ we obtain

$$\begin{aligned}
L_h &:= \int_{s_0}^{s}\int_\Omega (\partial_t(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u}))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})dxdt \\
&= \int_{s_0}^{s}\int_\Omega \Big((\varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{u}^h - \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}^h + \varrho\boldsymbol{f}\cdot\boldsymbol{u}^h\Big)\, dxdt =: R_h
\end{aligned}$$

Using (3.74) we get

$$L_h = \int_{s_0}^{s}\int_\Omega (D^{-h}(\varrho\boldsymbol{u}))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})dxdt. \tag{3.80}$$

Moreover notice that

$$\begin{aligned}
L_h &= \int_{s_0}^{s}\int_\Omega (\varrho D^{-h}\boldsymbol{u})\cdot(\tilde{\sigma}_h^- * \boldsymbol{u}) + ((D^{-h}\varrho)\boldsymbol{u}(t-h))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})dxdt \\
&= \int_{s_0}^{s}\int_\Omega \varrho\cdot\frac{1}{2}\partial_t|\tilde{\sigma}_h^- * \boldsymbol{u}|^2 + \big(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u})\big)\cdot\big(\nabla_x\big(\boldsymbol{u}(t-h)\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\big)\big)\, dxdt,
\end{aligned} \tag{3.81}$$

where we used (3.74) and relation $D^{-h}\varrho = -\mathrm{div}_x(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u}))$, which is provided by the fact that the couple $(\varrho, \boldsymbol{u})$ solves continuity equation $\partial_t\varrho + \mathrm{div}_x(\varrho\boldsymbol{u}) = 0$ in a weak sense. Inserting $z = \frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}|^2$ into weak formulation of the continuity equation, which means that for all s_0, $s \in [0,T]$, $s_0 < s$

$$\int_{s_0}^{s}\int_\Omega (\varrho(\tau)\cdot\partial_t z(\tau) + \varrho(\tau)\boldsymbol{u}(\tau)\cdot\nabla_x z(\tau))\, dxd\tau = \int_\Omega \varrho(s)\cdot z(s) - \varrho(s_0)\cdot z(s_0)dx$$

(for all $z \in L^r(0,T;W^{1,r})$ with $r = 5p/(5p-3)$ and $\partial_t z \in L^{1+\delta}(0,T;L^{1+\delta})$) we obtain

$$\begin{aligned}
L_h = &\int_\Omega \varrho(s)\cdot(\frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}(s)|^2)dx - \int_\Omega \varrho(s_0)\cdot(\frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}(s_0)|^2)dx \\
&- \int_{s_0}^{s}\int_\Omega (\varrho\boldsymbol{u})\cdot(\frac{1}{2}\nabla_x|\tilde{\sigma}_h^- * \boldsymbol{u}|^2)dxdt \\
&+ \int_{s_0}^{s}\int_\Omega \big(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u})\big)\cdot\big(\nabla_x\left[\boldsymbol{u}(t-h)\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\right]\big)\, dxdt
\end{aligned} \tag{3.82}$$

Let us notice that $\tilde{\sigma}_h^- * \boldsymbol{u}$ converges strongly (locally in time) to $\boldsymbol{u}$ in $L^2(0,T;L^2(\Omega)^3)$ and in $L^{5p/3}(0,T;L^{5p/3}(\Omega)^3)$ and $\nabla_x\tilde{\sigma}_h^- * \boldsymbol{u}$ converges strongly (locally in time) to $\nabla_x \boldsymbol{u}$ in $L^p(0,T;L^p(\Omega)^{3\times 3})$. The same arguments are valid for translation $\tau_{-h}\boldsymbol{u} = \boldsymbol{u}(t-h)$. Then by Hölder inequality letting $h \to 0^+$ in above we obtain for almost all s_0 and s in $(0,T)$

$$
\begin{aligned}
\lim_{h\to 0^+} L_h = & \int_{s_0}^{s}\int_\Omega (\varrho \boldsymbol{u})\cdot(\frac{1}{2}\nabla_x|\boldsymbol{u}|^2)dxdt \\
& + \frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2dx - \frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2dx \\
= & \int_{s_0}^{s}\int_\Omega \varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{u}dxdt \\
& + \frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2dx - \frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2dx.
\end{aligned}
\tag{3.83}
$$

Next we consider the right-hand side of (3.75) and pass with $j \to \infty$. At first we investigate the convergence of the term $\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{u}^h)dxdt$. Since condition (1.14) provides that $\mathbf{D}\boldsymbol{u} \in L^p(0,T;L^p(\Omega)^{3\times 3})$ and due to the Korn's inequality $\nabla_x\boldsymbol{u} \in L^p(0,T;L^p(\Omega)^{3\times 3})$ also the sequence $\nabla_x\boldsymbol{u}^h = \nabla_x\left(\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u})\mathbb{1}_{(s_0,s)})\right)$ is uniformly bounded in $L^p(0,T;L^p(\Omega)^{3\times 3})$. Hence we obtain, for subsequence if needed,

$$
\lim_{h\to 0^+}\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{u}^h)dxdt = \int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{u})dxdt. \tag{3.84}
$$

Since $\boldsymbol{f} \in L^{p'}(0,T;L^{p'}(\Omega))$ and ϱ satisfies (3.29) in the same way we conclude

$$
\lim_{h\to 0^+}\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{f})\cdot\boldsymbol{u}^h dxdt = \int_{s_0}^{s}\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}dxdt. \tag{3.85}
$$

Let us concentrate now on the term

$$
\int_0^T\int_\Omega \overline{\mathbf{S}} : (\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u})\mathbb{1}_{(s_0,s)}))dxdt = \int_{s_0}^{s}\int_\Omega(\tilde{\sigma}_h^- * \overline{\mathbf{S}}) : (\tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u})dxdt.
$$

Sequences $\{\tilde{\sigma}_h^- * \overline{\mathbf{S}}\}$ and $\{\tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u}\}$ converge in measure on Q due to Proposition 4.3. Moreover, since M and M^* are convex nonnegative functions, then the weak lower semicontinuity and estimate (3.33) provide that the integrals

$$
\int_0^T\int_\Omega M(x,\mathbf{D}\boldsymbol{u})dxdt \quad \text{and} \quad \int_0^T\int_\Omega M^*(x,\overline{\mathbf{S}})dxdt \tag{3.86}
$$

are finite. Hence Proposition 4.4 implies that the sequences $\{\tilde{\sigma}_h^- * \overline{\mathbf{S}}\}$ and $\{\tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u}\}$ are uniformly integrable and hence according to Lemma 4.1 we have

$$\begin{aligned} \tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u} &\xrightarrow{M} \mathbf{D}\boldsymbol{u} \quad \text{modularly in } L_M(Q)^{3\times 3}_{\mathrm{sym}}, \\ \tilde{\sigma}_h^- * \overline{\mathbf{S}} &\xrightarrow{M^*} \overline{\mathbf{S}} \qquad \text{modularly in } L_{M^*}(Q)^{3\times 3}_{\mathrm{sym}}. \end{aligned} \tag{3.87}$$

Applying Proposition 4.2 allows to conclude

$$\lim_{h\to 0^+} \int_{s_0}^{s} \int_\Omega (\tilde{\sigma}_h^- * \overline{\mathbf{S}}) : (\tilde{\sigma}_h^- * \mathbf{D}\boldsymbol{u}) dxdt = \int_{s_0}^{s} \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u} dxdt. \tag{3.88}$$

Summarising arguments (3.83), (3.88) and (3.85) we are able to pass to the limit in (3.75) and we obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2 dx &+ \int_{s_0}^{s}\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u} dxdt \\ &= \int_{s_0}^{s}\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u} dxdt + \frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2 dx. \end{aligned} \tag{3.89}$$

3.5 Continuity w.r.t. time in weak topology and initial condition

Using already proved properties of the density and the velocity field, namely $\varrho \in C([0,T], L^q(\Omega))$ for $q \in [1,\infty)$ and $\boldsymbol{u} \in C(0,T;H^3)$, it leads to conclusion that $(\varrho(\cdot)\boldsymbol{u}(\cdot), \tilde{\boldsymbol{\varphi}})$ is continuous at $s_1 \in (0,T)$ for all $\tilde{\boldsymbol{\varphi}} \in V_s$, in other words, $\varrho\boldsymbol{u} \in C(0,T;(V_s)^*_{\mathrm{weak}})$ or

$$\lim_{s_2\to s_1} (\varrho(s_2)\boldsymbol{u}(s_2) - \varrho(s_1)\boldsymbol{u}(s_1), \tilde{\boldsymbol{\varphi}}) = 0.$$

Since $\boldsymbol{u} \in L^\infty(0,T;H^3)$, $\varrho \in C([0,T];L^q(\Omega))$ for $q \in [1,\infty)$ and V_s is dense in H, we observe that $\varrho\boldsymbol{u} \in C([0,T];H_{\mathrm{weak}})$. As a consequence we have

$$\lim_{s_1\to 0} (\varrho(s_1)\boldsymbol{u}(s_1) - \varrho_0\boldsymbol{u}_0, \tilde{\boldsymbol{\varphi}}) = 0 \quad \text{for all } \tilde{\boldsymbol{\varphi}} \in H. \tag{3.90}$$

Integrating (3.31) over time interval $(0,s_1)$, using the fact that and the fact that $(\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{u}^n)$ is nonnegative (because of monotonicity) and taking the limit $n\to\infty$ we obtain

$$(\varrho(s_1), |\boldsymbol{u}(s_1)|^2) - (\varrho(0), |\boldsymbol{u}(0)|^2) \le 2\int_0^{s_1} (\varrho\boldsymbol{f}, \boldsymbol{u}) dt. \tag{3.91}$$

If we employ obvious identity

$$\|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1)-\boldsymbol{u}_0)\|^2_{L^2(\Omega)} = (\varrho(s_1),|\boldsymbol{u}(s_1)|^2)-2(\varrho(s_1)\boldsymbol{u}(s_1),\boldsymbol{u}_0)+(\varrho(s_1),|\boldsymbol{u}_0|^2),$$

then the second part of property (1.12) is an easy consequence of (3.91) and the following operations

$$\begin{aligned}
&\|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1)-\boldsymbol{u}_0)\|^2_{L^2(\Omega)} = (\varrho(s_1),|\boldsymbol{u}(s_1)|^2)-2(\varrho(s_1)\boldsymbol{u}(s_1),\boldsymbol{u}_0)+(\varrho(s_1),|\boldsymbol{u}_0|^2)\\
&= (\varrho(s_1),|\boldsymbol{u}(s_1)|^2)-(\varrho_0,|\boldsymbol{u}_0|^2)-2(\varrho(s_1)\boldsymbol{u}(s_1)-\varrho_0\boldsymbol{u}_0,\boldsymbol{u}_0)+(\varrho(s_1)-\varrho_0,|\boldsymbol{u}_0|^2)\\
&\leq 2\int_0^{s_1}(\varrho\boldsymbol{f},\boldsymbol{u})dt-2(\varrho(s_1)\boldsymbol{u}(s_1)-\varrho_0\boldsymbol{u}_0,\boldsymbol{u}_0)+(\varrho(s_1)-\varrho_0,|\boldsymbol{u}_0|^2).
\end{aligned} \tag{3.92}$$

Letting $s_1 \to 0^+$ in (3.92) using (3.90), (3.70) and $(\varrho\boldsymbol{f},\boldsymbol{u}) \in L^1(0,T;L^1(\Omega))$ we can conclude

$$\lim_{s_1\to 0}\|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1)-\boldsymbol{u}_0)\|^2_{L^2(\Omega)} = 0. \tag{3.93}$$

Hence this implies together with (3.29) the second part of (1.12). Above arguments and (3.92), (3.93) provide also the fact which we will use later

$$\lim_{s_1\to 0}(\varrho(s_1),|\boldsymbol{u}(s_1)|^2) = (\varrho_0,|\boldsymbol{u}_0|^2). \tag{3.94}$$

3.6 Monotonicity method

Using the property (3.94) and letting $s_0 \to 0$ in (3.89) we obtain

$$\begin{aligned}
\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2dx + \int_0^s\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}dxdt&\\
= \int_0^s\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}dxdt + \frac{1}{2}\int_\Omega \varrho_0(x)|\boldsymbol{u}_0(x)|^2dx.&
\end{aligned}$$

Additionaly integrating (3.31) over the interval $(0,s)$ allows to conclude that

$$\limsup_{n\to\infty}\int_0^s\int_\Omega \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n dxdt = \int_0^s\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}dxdt. \tag{3.95}$$

By Q^s we will mean the set $(0,s)\times\Omega$. Since $\mathbf{S}$ is monotone, then we have

$$\int_{Q^s}(\mathbf{S}(t,x,\varrho^n,\mathbf{w})-\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w}-\mathbf{D}\boldsymbol{u}^n)dxdt \geq 0 \tag{3.96}$$

for all $\mathbf{w} \in L^\infty(Q)^{3\times 3}$. Observe that also $\mathbf{S}(x, \varrho^n, \mathbf{w}) \in L^\infty(Q)^{3\times 3}$. We prove it by contradiction, i.e. let us assume that $\mathbf{S}(x, \varrho^n, \mathbf{w})$ is unbounded. Then, since M is nonnegative, by (1.6), it holds

$$|\mathbf{w}| \geq \frac{M^*(x, \mathbf{S}(t, x, \varrho^n, \mathbf{w}))}{|\mathbf{S}(t, x, \varrho^n, \mathbf{w})|}.$$

The right-hand side tends to infinity as $|\mathbf{S}(t, x, \varrho^n, \mathbf{w})| \to \infty$ by (2.16), which contradicts that $\mathbf{w} \in L^\infty(Q)^{3\times 3}$. Now employing continuity of $\mathbf{S}$ w.r.t. the second variable and (3.29) we obtain uniform boundedness of $\{\mathbf{S}(t, x, \varrho^n, \mathbf{w})\}$ w.r.t n. Together with boundedness of Q^s this gives uniform integrability of $\{M^*(\mathbf{S}(t, x, \varrho^n, \mathbf{w}))\}$. Lemma 4.1 and (3.65) provides modular convergence of the sequence. Since M^* satisfies Δ_2 condition modular and strong convergence in L_{M^*} coincides (see [22]) and hence $\mathbf{S}(t, x, \varrho^n, \mathbf{w}) \to \mathbf{S}(t, x, \varrho^n, \mathbf{w})$ strongly in L_{M^*}. Therefore by (3.34) we deduce

$$\lim_{n\to\infty} \int_{Q^s} \mathbf{S}(t, x, \varrho^n, \mathbf{w}) : \mathbf{D}\boldsymbol{u}^n dxdt = \int_{Q^s} \mathbf{S}(t, x, \varrho, \mathbf{w}) : \mathbf{D}\boldsymbol{u}^n dxdt. \tag{3.97}$$

Before passing to the limit with $n \to \infty$, we rewrite (3.96)

$$\begin{aligned} &\int_{Q^s} \mathbf{S}(t, x, \varrho^n, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n dxdt \\ &\qquad \geq \int_{Q^s} \mathbf{S}(t, x, \varrho^n, \mathbf{D}\boldsymbol{u}^n) : \mathbf{w} dxdt + \int_{Q^s} \mathbf{S}(t, x, \varrho^n, \mathbf{w}) : (\mathbf{D}\boldsymbol{u}^n - \mathbf{w}) dxdt. \end{aligned} \tag{3.98}$$

hence (3.36), (3.41), (3.95), (3.97) infer

$$\int_{Q^s} \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u} dxdt \geq \int_{Q^s} \overline{\mathbf{S}} : \mathbf{w} dxdt + \int_{Q^s} \mathbf{S}(t, x, \varrho, \mathbf{w}) : (\mathbf{D}\boldsymbol{u} - \mathbf{w}) dxdt \tag{3.99}$$

and consequently

$$\int_{Q^s} (\mathbf{S}(t, x, \varrho, \mathbf{w}) - \overline{\mathbf{S}}) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}) dxdt \geq 0. \tag{3.100}$$

Let $k > 0$ and denote by

$$Q_k = \{(t, x) \in Q^s : |\mathbf{D}\boldsymbol{u}(t, x)| \leq k \ \text{ a.e. in } Q^s\}$$

and let $0 < j < i$ be arbitrary and $h > 0$

$$\mathbf{w} = (\mathbf{D}\boldsymbol{u})\mathbb{1}_{Q_i} + h\mathbf{v}\mathbb{1}_{Q_j},$$

where $\mathbf{v} \in L^\infty(Q)^{3\times 3}$ is arbitrary. By (3.100)

$$-\int_{Q^s\setminus Q_i} (\mathbf{S}(t,x,\varrho,\mathbf{0}) - \overline{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}dxdt + h\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v}dxdt \geq 0. \tag{3.101}$$

Note that $\mathbf{S}(t,x,\varrho,\mathbf{0}) = \mathbf{0}$. Obviously

$$\int_{Q^s\setminus Q_i} \overline{\mathbf{S}} \cdot \mathbf{D}\boldsymbol{u}dxdt = \int_Q (\overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u})\mathbb{1}_{Q^s\setminus Q_i}dxdt.$$

By Proposition 4.1 and (3.86) we obtain

$$\int_Q \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}dxdt < \infty.$$

Then as $i \to \infty$ we have

$$(\overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u})\mathbb{1}_{Q^s\setminus Q_i} \to 0 \quad \text{ a.e. in } Q.$$

Hence by the Lebesgue dominated convergence theorem

$$\lim_{i\to\infty} \int_{Q^s\setminus Q_i} \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}dxdt = 0.$$

Letting $i \to \infty$ in (3.101) and dividing by h, we get

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v}dxdt \geq 0.$$

Since $\mathbf{D}\boldsymbol{u} + h\mathbf{v} \to \mathbf{D}\boldsymbol{u}$ a.e. in Q_j when $h \to 0^+$ and $\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v})$ is uniformly bounded in $L^\infty(Q_j)^{3\times 3}$, $|Q_j| < \infty$, hence by the Vitali lemma we conclude

$$\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v}) \to \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) \text{ in } L^1(Q_j)^{3\times 3}$$

and

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v}dxdt \to \int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}) : \mathbf{v}dxdt$$

when $h \to 0^+$. Consequently,

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}) : \mathbf{v} dxdt \geq 0$$

for all $\mathbf{v} \in L^\infty(Q)^{3\times 3}$. The choice

$$\mathbf{v} = \begin{cases} -\frac{\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}}{|\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}|} & \text{for} \quad \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) \neq \overline{\mathbf{S}}, \\ 0 & \text{for} \quad \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) = \overline{\mathbf{S}}, \end{cases}$$

yields

$$\int_{Q_j} |\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}| dxdt \leq 0.$$

Hence

$$\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) = \overline{\mathbf{S}} \quad \text{a.e. in } Q_j. \tag{3.102}$$

Since j was arbitrary, (3.102) holds a.e. in Q^s. Since it holds for almost all s such that $0 < s < T$, hence $\overline{\mathbf{S}} = \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u})$ a.e. in Q.

4 Appendix

For completeness of the paper let us recall some general properties of Orlicz spaces, see e.g. [30] and technical facts which are rewritten from [18, 20]. In our considerations the general space dimension $d = 3$.

Proposition 4.1 (Fenchel-Young Inequality) *Let M be an N–function and M^* the complementary to M. Then the following inequality is satisfied*

$$|\boldsymbol{\xi} \cdot \boldsymbol{\eta}| \leq M(x,\boldsymbol{\xi}) + M^*(x,\boldsymbol{\eta})$$

for all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\text{sym}}$ and a.a. $x \in \Omega$.

We recall an analogue to the Vitali's lemma, however for the modular convergence instead of the strong convergence in L^p.

Lemma 4.1 *Let $\mathbf{z}^j : Q \to \mathbb{R}^{d\times d}$ be a measurable sequence. Then $\mathbf{z}^j \xrightarrow{M} \mathbf{z}$ in $L_M(Q)^{d\times d}$ modularly if and only if $\mathbf{z}^j \to \mathbf{z}$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda\mathbf{z}^j)\}$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty} \left(\sup_{j\in\mathbb{N}} \int_{\{(t,x):|M(x,\lambda\mathbf{z}^j)|\geq R\}} M(x,\lambda\mathbf{z}^j) dxdt \right) = 0.$$

Proof: Note that $\mathbf{z}^j \to \mathbf{z}$ in measure if and only if $M\left(\cdot, \frac{\mathbf{z}^j - \mathbf{z}}{\lambda}\right) \to 0$ in measure for all $\lambda > 0$. Moreover the convergence $\mathbf{z}^j \to \mathbf{z}$ in measure implies that for all measurable sets $A \subset Q$ it holds

$$\liminf_{j\to\infty} \int_A M(x, \mathbf{z}^j) dxdt \geq \int_A M(x, \mathbf{z}) dxdt.$$

Note also that the convexity of M implies

$$\int_A M\left(x, \frac{\mathbf{z}^j - \mathbf{z}}{\lambda}\right) dxdt \leq \int_A M\left(x, \frac{\mathbf{z}^j}{2\lambda}\right) dxdt + \int_A M\left(x, \frac{\mathbf{z}}{2\lambda}\right) dxdt.$$

Hence by the classical Vitali's lemma for $f^j(x) = M\left(x, \frac{\mathbf{z}^j - \mathbf{z}}{\lambda}\right)$ we obtain that $f^j \to 0$ strongly in $L^1(Q)$. □

Lemma 4.2 *Let M be an N–function and for all $j \in \mathbb{N}$ let $\int_Q M(x, \mathbf{z}^j) dxdt \leq c$. Then the sequence $\{\mathbf{z}^j\}_{j=1}^{\infty}$ is uniformly integrable.*

Proof: Let us define $\delta(R) = \min_{|\boldsymbol{\xi}|=R} \inf_{x\in\Omega} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|}$. Then for all $j \in \mathbb{N}$ it holds

$$\int_{\{(t,x):|\mathbf{z}^j(t,x)|\geq R\}} M(x, \mathbf{z}^j(t,x)) dxdt \geq \delta(R) \int_{\{(t,x):|\mathbf{z}^j(t,x)|\geq R\}} |\mathbf{z}^j(t,x)| dxdt.$$

Since the left-hand side is bounded, then we obtain

$$\sup_{j\in\mathbb{N}} \int_{\{(t,x):|\mathbf{z}^j(t,x)|\geq R\}} |\mathbf{z}^j(t,x)| dxdt \leq \frac{c}{\delta(R)}.$$

Using condition (2.16) we obtain uniform integrability. □

Proposition 4.2 *Let M be an N–function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\psi}^j : Q \to \mathbb{R}^{d\times d}$ and $\boldsymbol{\phi}^j : Q \to \mathbb{R}^{d\times d}$ are uniformly bounded in $L_M(Q)^{d\times d}$ and $L_{M^*}(Q)^{d\times d}$ respectively. Moreover $\boldsymbol{\psi}^j \xrightarrow{M} \boldsymbol{\psi}$ modularly in $L_M(Q)^{d\times d}$ and $\boldsymbol{\phi}^j \xrightarrow{M^*} \boldsymbol{\phi}$ modularly in $L_{M^*}(Q)^{d\times d}$. Then $\boldsymbol{\psi}^j \cdot \boldsymbol{\phi}^j \to \boldsymbol{\psi} \cdot \boldsymbol{\phi}$ strongly in $L^1(Q)$.*

Proof: Due to Lemma 4.1 the modular convergence of $\{\boldsymbol{\psi}^j\}$ and $\{\boldsymbol{\phi}^j\}$ implies the convergence in measure of these sequences and consequently also the convergence in measure of the product. Hence it is sufficient to show the uniform integrability of $\{\boldsymbol{\psi}^j \cdot \boldsymbol{\phi}^j\}$. Notice that it is equivalent with the uniform integrability of the term $\left\{\frac{\boldsymbol{\psi}^j}{\lambda_1} \cdot \frac{\boldsymbol{\phi}^j}{\lambda_2}\right\}$ for any $\lambda_1, \lambda_2 > 0$. The assumptions of the proposition provide there exist some $\lambda_1, \lambda_2 > 0$ such that the sequences

$$\left\{M\left(x, \frac{\boldsymbol{\psi}^j}{\lambda_1}\right)\right\} \quad \text{and} \quad \left\{M^*\left(x, \frac{\boldsymbol{\phi}^j}{\lambda_2}\right)\right\}$$

are uniformly integrable. Hence let us use the same constants and estimate with the help of Fenchel-Young inequality

$$\left|\frac{\boldsymbol{\psi}^j}{\lambda_1} \cdot \frac{\boldsymbol{\phi}^j}{\lambda_2}\right| \leq M\left(x, \frac{\boldsymbol{\psi}^j}{\lambda_1}\right) + M^*\left(x, \frac{\boldsymbol{\phi}^j}{\lambda_2}\right).$$

Obviously the uniform integrability of the right-hand side provides the uniform integrability of the left-hand side and this yields the assertion. $\square$

Remark: The same proofs for Proposition 4.3 and 4.4 works if instead of a standard mollifier ϱ^j we will take

$$\tilde{\sigma}_h^+ = \frac{1}{h}\mathbb{1}(\tau)_{[0,h]} \quad \text{or} \quad \tilde{\sigma}_h^- = \frac{1}{h}\mathbb{1}(\tau)_{[-h,0]}$$

with $h > 0$.

Proposition 4.3 *Let ϱ^j be a standard mollifier, i.e., $\varrho \in C^\infty(\mathbb{R})$, ϱ has a compact support and $\int_{\mathbb{R}} \varrho(\tau)d\tau = 1, \varrho(t) = \varrho(-t)$. We define $\varrho^j(t) = j\varrho(jt)$. Moreover let $*$ denote a convolution in the variable t. Then for any function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in L^1(Q)$ it holds*

$$(\varrho^j * \psi)(t,x) \to \psi(t,x) \quad \textit{in measure.}$$

Proof: For a.a. $x \in \Omega$ the function $\psi(\cdot, x) \in L^1(0,T)$ and $\varrho^j * \psi(\cdot, x) \to \psi(\cdot, x)$ in $L^1(0,T)$ and hence $\varrho^j * \psi \to \psi$ in measure on the set $[0,T] \times \Omega$.$\square$

Proposition 4.4 *Let ϱ^j be defined as in Proposition 4.3. Given an $N-$function M and a function $\psi : Q \to \mathbb{R}^d$ such that $\boldsymbol{\psi} \in \mathcal{L}_M(Q)^{d\times d}$. Then the sequence $\{M(x, \varrho^j * \boldsymbol{\psi})\}$ is uniformly integrable.*

Proof: We start with an abstract fact concerning the uniform integrability. Namely, the following two conditions are equivalent for any measurable sequence $\{z^j\}$

(a) $\forall \varepsilon > 0 \quad \exists \delta > 0: \quad \sup\limits_{j\in\mathbb{N}} \sup\limits_{|A|\le\delta} \int_A |z^j(x)| dxdt \le \varepsilon,$

(b) $\forall \varepsilon > 0 \quad \exists \delta > 0: \quad \sup\limits_{j\in\mathbb{N}} \int_Q \left| |z^j(x)| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \le \varepsilon,$

where we use the notation

$$|\xi|_+ = \max\{0, \xi\}.$$

The implication $(a) \Rightarrow (b)$ is obvious. To show that also $(b) \Rightarrow (a)$ holds let us estimate

$$\begin{aligned}\sup_{j\in\mathbb{N}} \sup_{|A|\le\delta} \int_A |z^j| dxdt &\le \sup_{|A|\le\delta} |A| \cdot \frac{1}{\sqrt{\delta}} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \\ &\le \sqrt{\delta} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ dxdt.\end{aligned}$$

Notice that since M is a convex function, then the following inequality holds for all $\delta > 0$

$$\int_Q \left| M(x, \boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}} \right|_+ dxdt \ge \int_Q \left| M(x, \varrho^j * \boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}} \right|_+ dxdt. \qquad (4.103)$$

Finally, since $\boldsymbol{\psi} \in \mathcal{L}_M(Q)^{d\times d}$, then also $\int_Q |M(x,\boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}}|_+ dxdt$ is finite and hence taking supremum over $j \in \mathbb{N}$ in (4.103) we prove the assertion. $\square$

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