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Abstract

We study a general class of nonlinear elliptic problems associated with the differential inclusion $\beta(x,u) - \mathrm{div}\ (a(x,\nabla u) + F(u)) \ni f$, where $f \in L^1(\Omega)$. A vector field $a(\cdot,\cdot)$ is monotone in the second variable and satisfies a non-standard growth condition described by an x-dependent convex function that generalizes both $L^{p(x)}$ and classical Orlicz settings. Using truncation techniques and a generalized Minty method in the functional setting of non reflexive spaces we prove existence of renormalized solutions for general L^1-data. Under an additional strict monotonicity assumption uniqueness of the renormalized solution is established. Sufficient conditions are specified which guarantee that the renormalized solution is already a weak solution to the problem.

AMS 2000 Classification:

Keywords: Orlicz spaces, renormalized solutions, generalized Minty method, maximal function

1 Introduction

Let Ω be bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence and uniqueness of renormalized solutions to the following nonlinear elliptic inclusion

$$\begin{aligned} \beta(x,u) - \mathrm{div}\,(a(x,\nabla u) + F(u)) &\ni f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{E,f}$$

with right-hand side $f \in L^1(\Omega)$. The function $F : \mathbb{R} \to \mathbb{R}^d$ is assumed to be locally Lipschitz and $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following assumptions:

(A1) $a(\cdot,\cdot)$ is a Carathéodory function.

(A2) there exist an $\mathcal{N}$-function $M : \Omega \times \mathbb{R}^d \to \mathbb{R}^+$ (see Definition 2.1 below), a constant $c_a \in (0,1]$ and a nonnegative function $a_0 \in L^1(\Omega)$ such that

$$a(x,\xi)\cdot\xi \geq c_a\{M^*(x,a(x,\xi)) + M(x,\xi)\} - a_0(x) \tag{1}$$

for a.a. $x \in \Omega$ and for every $\xi \in \mathbb{R}^d$, where M^* is the conjugate function to M (see relation (6)).

(A3) $a(\cdot, \cdot)$ is monotone, i.e.,

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq 0 \tag{2}$$

for a.a. $x \in \Omega$ and for every $\xi,\ \eta \in \mathbb{R}^d$.

Moreover, we assume that the conjugate function M^* satisfies the Δ_2-condition (see Section 2). Therefore there exist $c > 0$, $\nu > 0$, ξ_0, $|\xi_0| < \infty$ such that

$$M(x,\xi) \geq c|\xi|^{1+\nu} \tag{3}$$

for a.a. $x \in \Omega$ and for $\xi \in \mathbb{R}^d$, $|\xi| \geq |\xi_0|$ (see Proposition 2.1). However, no growth restriction is made on the $\mathcal{N}$-function M itself. As to the nonlinearity β in the problem (E, f) we assume that $\beta : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a set-valued mapping such that, for almost every $x \in \Omega$, $\beta(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a maximal monotone operator with $0 \in \beta(x, 0)$. Moreover, we assume that

$$\beta^0(\cdot, l) \in L^1(\Omega) \tag{4}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β.

There already exists a vast literature on problems of this type. Most of the literature has been devoted to the study of the case where the vector field a satisfies a polynomial growth (and coerciveness) condition. A model example of this type is the homogeneous Dirichlet boundary value problem for the p-Laplacian $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$, i.e. the equation

$$\beta(x, u) - \Delta_p(u) - \mathrm{div}F(u) \ni f.$$

It is well-known, even in this particular case, that for L^1-data a weak solution may not exist in general or may not be unique. In order to obtain well-posedness for this type of problems the notion of renormalized solution has been introduced by DiPerna and Lions for the Boltzmann equation in [25] and by Lions and Murat (see [19, 35]) for elliptic equations with integrable data. The equivalent notion of entropy solution has been introduced by Bénilan et al. in [12]. During the last two decades these solution concepts have been adapted to the study of various problems of partial differential equations. We refer to [3]-[8], [13], [16]-[19], [23, 24, 32, 38] among others.

More general problems involving vector fields satisfying variable growth and coerciveness condition of type

$$a(x,\xi) \cdot \xi \geq \lambda|\xi|^{p(x)} - c(x)$$
$$|a(x,\xi)| \leq d(x) + \mu|\xi|^{p(x)-1}$$

for a.a. $x \in \Omega$, for every $\xi \in \mathbb{R}^d$, where $\lambda, \mu > 0$, $p : \Omega \to \mathbb{R}$ is a measurable variable exponent with $1 < p^- < p(x) < p^+ < \infty$ for a.a. $x \in \Omega$, $c \in L^1(\Omega)$, $d \in L^{p'(x)}(\Omega)$ have been already considered. For results on existence of renormalized solutions of elliptic problems of type (E, f) with $a(\cdot,\cdot)$ satisfying a variable growth condition we refer to [20], [43] (for related results see also [9], [10], [42]). Note that vector fields satisfying this type of variable exponent growth and coerciveness condition fall into the scope of our study (with $M(x,\xi) = c_1|\xi|^{p(x)}$, $M^*(x,\xi) = c_2|\xi|^{p'(x)}$, where $p'(x) = p(x)/(p(x)-1)$, $c_1 = (1/p(x))(c(x))^{p(x)}$, $c_2 = 1/(p'(x)(c(x))^{p'(x)})$, $c : \Omega \to \mathbb{R}$ is measurable and $0 < c^- < c(x) < c^+ < \infty$). However, our setting is more general as we do not impose a growth restriction on M. Let us note that the functional setting for this type of problems involves variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ which, for range of exponents the authors considered, are separable, reflexive Banach spaces and thus standard monotonicity methods, adapted to the renormalized case, can be used in this case. The $L^{p(x)}$-spaces, in general, are not stable by convolution and smooth functions may fail to be dense in $W^{1,p(x)}(\Omega)$ (at least if $p(\cdot)$ is not log-Hölder continuous). This fact does not lead to further difficulties in the study of the above-mentioned works as the authors settle the problem in the energy space $W_0^{1,p(x)}(\Omega)$ which, by definition, is the norm closure of $C_c^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Taking into account anisotropic effects leads to the study of anisotropic elliptic problems of type (E, f) with constant exponents as in [11, 18] and also with variable exponents as in [37] (see also [33]), where the existence of a renormalized solution of an elliptic problem of type (E, f) with $\beta = 0, F = 0$ was provided. It was assumed that a vector field $a(x,\xi) = (a_1(x,\xi_1), \dots, a_d(x,\xi_d))$ with components $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following coerciveness and growth assumptions

$$a_i(x,r)r \geq \lambda |r|^{p_i(x)}$$
$$|a_i(x,r)| \leq d_i(x) + \mu |r|^{p_i(x)-1}$$

for a.a. $x \in \Omega$, for every $r \in \mathbb{R}$, where $\lambda, \mu > 0$, $p_i : \overline{\Omega} \to \mathbb{R}$, $i = 1, \dots, d$ are continuous variable exponents with $1 < p_i^- < p_i(x) < p_i^+ < d$ for all $x \in \overline{\Omega}$, $d_i \in L^{p'(x)}(\Omega)$. Moreover, the p_i^-, p_i^+, $i = 1, \dots, d$ satisfy some restrictive compatibility conditions. Choosing the $\mathcal{N}$-function $M(x,\xi) = \sum_{i=1}^d |\xi_i|^{p_i(x)}$ the two conditions above can be rewritten in the form of our general growth assumption (**A2**). Therefore our setting also includes and extends the anisotropic case. Let us note that the functional setting in the above mentioned papers involve the anisotropic Sobolev spaces $W_0^{1,\vec{p}}(\Omega)$ and the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(x)}(\Omega)$, $\vec{p} = (p_1, \dots, p_d)$ respectively. According to the restrictions on the exponents p_i, made by the authors, these Banach spaces are separable and reflexive, and the elliptic operator acts as a bounded monotone operator on this space into its dual. Therefore classical variational theory can be applied to prove existence of weak solutions in this case for, say, bounded data f. Moreover existence of renormalized solutions can be proved by approximation, using truncation techniques and Minty's monotonicity trick adapted to the renormalized setting.

Problems of type (E, f) involving vector fields with nonpolynomial (for instance, exponential) growth have also already been considered in the literature. Typically, the growth condition is expressed by a classical $\mathcal{N}$-function $M : \mathbb{R} \to \mathbb{R}$, not depending on the space variable x and only depending on the modulus $|\xi|$ of the vector ξ, as, for example, in [2] and [15]. The functional setting in these works involve the classical Orlicz spaces $L_M(\Omega)$ and Orlicz-Sobolev spaces $W^1L_M(\Omega)$ which fail to be reflexive if M and M^* do not satisfy the Δ_2-condition (see e.g. [1]). In this case, existence of approximate solutions follows from the theory of monotone operators in Orlicz-Sobolev spaces as developed by Gossez and Mustonen in [28]. The arguments used to prove the convergence of such approximate solutions to a renormalized solution of (E, f) are based on an approximation property in Orlicz-Sobolev spaces proved by Gossez in [27, Theorem 4]. The author says that it is possible to approximate the gradient of an $W_0^1L_M(\Omega)$-function in modular convergence by a sequence of gradients of smooth functions, compactly supported in Ω.

The setting considered in this paper includes and generalizes variable exponent, anisotropic and classical Orlicz settings (at least in the case when the latter is built on an M-function whose conjugate M^* satisfies the Δ_2-condition, what automatically implies that M satisfies the coerciveness condition (3)). The function M which describes the growth condition on the vector field a is a so-called *generalized* $\mathcal{N}$-function (see Definition 2.1 below). The corresponding generalized Orlicz spaces $L_M(\Omega; \mathbb{R}^d)$, often called Orlicz-Musielak spaces (see [36]) have been introduced in [40], [41]. In general, if M and M^* do not satisfy a Δ_2-condition these spaces fail to be separable or reflexive. In the setting of generalized Orlicz spaces, due to x-dependence of $\mathcal{N}$-function, the result similar to Gossez [27] can not be achived. As in the case of generalized Lebesgue spaces convolution with smooth compactly supported kernel may fail to be a bounded operator.

Our techniques to overcome these difficulties are inspired by former works [22, 30, 31, 45, 44]. The authors considered equations involving vector fields satisfying general non-standard growth conditions of type (**A2**) with a generalized $\mathcal{N}$-function $M(x, \xi)$. All these works are motivated by fluid dynamics.

Gwiazda et al. in [29] studied a steady and in [30] a dynamic model for non-Newtonian fluids under an additional strict monotonicity assumption on the vector field. The author used Young measure techniques in place of a monotonicity method. Additional assumption of strict monotonicity allows to conclude that the measure-valued solution is a Dirac delta and is a weak solution. A similar method is used in the variable exponent setting in [5].

A version of the Minty-Browder trick adapted to the setting of generalized Orlicz spaces was introduced in [44] (and later see [31]) in framework of non-Newtonian fluids. As we do not assume strict monotonicity of $a(\cdot, \cdot)$, we have to employ the generalized monotonicity method of [44]. Using the Galerkin method with smooth basis functions we can thereby prove existence of a weak solution u_ε of some approximate problem $(E_\varepsilon, f_\varepsilon)$ with $f_\varepsilon \in L^\infty(\Omega)$. In a second step we show that a subsequence of the approximate solutions u_ε converges to a renormalized solution of problem (E, f). In this step we combine truncation techniques and the generalized monotonicity method of [44] with an idea introduced in [22] which uses the maximal function of the gradient of $T_k(u_\varepsilon)$ to localize the critical terms that appear in the equation on sets where these gradients are bounded. Thereby, it is possible to overcome a difficulty that arises from the possible lack of reflexivity of $L_M(\Omega)^d$ and which consists in passing to the limit in expressions of the form $\int_\Omega f(x) \cdot g_\varepsilon(x)\, dx$ when $f \in L_{M^*}(\Omega; \mathbb{R}^d)$ and the sequence $\{g_\varepsilon\}_{\varepsilon>0}$ only converges weak-* in $L_M(\Omega; \mathbb{R}^d)$ to some function g.

The paper is organized as follows: in Section 2 we specify the functional setting and recall some basic facts for generalized Orlicz spaces. In Section 3 we introduce the notions of weak and also renormalized solution for problem (E, f). Our main result, existence of a renormalized solution to (E, f) for any L^1-data f, and the results on uniqueness of renormalized solutions and on existence of weak solutions, are collected in Section 4. The proof of existence of renomalized solution is in Section 5, the uniqueness is shown in Section 6 and existence of a weak solution proved in Section 7.

2 Functional setting and notation

2.1 Generalized Orlicz spaces

Definition 2.1 (generalized $\mathcal{N}$-function) *A function $M : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is said to be a generalized $\mathcal{N}$-function if it satisfies the following conditions:*

1. *M is a Carathéodory function, i.e. $M(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $M(x, \cdot)$ is continuous for a.a. $x \in \Omega$;*
2. *$M(x, \xi) = M(x, -\xi)$ for all $\xi \in \mathbb{R}^d$, $M(x, \xi) = 0$ if and only if $\xi = 0$;*
3. *$M(x, \cdot)$ is convex for a.a. $x \in \Omega$;*
4. *M has superlinear growth such that*

$$\lim_{|\xi|\to 0} \sup_{x\in\Omega} \frac{M(x,\xi)}{|\xi|} = 0, \qquad \lim_{|\xi|\to\infty} \inf_{x\in\Omega} \frac{M(x,\xi)}{|\xi|} = \infty. \tag{5}$$

For an $\mathcal{N}$-function M, we denote by M^* the *conjugate function* given by the Legendre–Fenchel transform

$$M^*(x, \eta) = \sup_{\xi\in\mathbb{R}^d} \left(\xi \cdot \eta - M(x, \xi)\right), \quad x \in \Omega, \ \eta \in \mathbb{R}^d. \tag{6}$$

The conjugate function M^* is also an $\mathcal{N}$-function (see [40]).

Then the Fenchel–Young inequality holds

$$|\xi \cdot \eta| \le M(x, \xi) + M^*(x, \eta) \qquad \text{for all } \xi, \eta \in \mathbb{R}^d. \tag{7}$$

The *generalized Orlicz class* $\mathcal{L}_M(\Omega; \mathbb{R}^d)$ is the set of all measurable functions $\xi : \Omega \to \mathbb{R}^d$ such that

$$\rho_{M,\Omega}(\xi) := \int_\Omega M(x, \xi(x))\, \mathrm{d}x < \infty.$$

Note that $\mathcal{L}_M(\Omega; \mathbb{R}^d)$ is a convex set and it may not be a linear space. The mapping $\rho_{M,\Omega}$ is a modular in the sense of [34, p. 208].

Since the function $M^* : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is convex with respect to its second argument and satisfies (5), we define an $\mathcal{N}$-functon m^* such that

$$m^*(r) = \inf_{x\in\Omega} \inf_{\xi\in\mathbb{R}^d, |\xi|=r} M^*(x,\xi)$$

then $m^* \le M^*(x,\xi)$. Consequently there exist a conjugate $\mathcal{N}$-function $m = m(|\xi|)$ to m^* such that $M(x,\xi) \le m(\xi)$. Therefore M maps bounded sets into bounded sets, which shows that

$$L^\infty(\Omega;\mathbb{R}^d) \subseteq \mathcal{L}_M(\Omega;\mathbb{R}^d). \tag{8}$$

The *generalized Orlicz space* $L_M(\Omega;\mathbb{R}^d)$ is defined as the linear hull of $\mathcal{L}_M(\Omega;\mathbb{R}^d)$. It is a Banach space with respect to the Luxemburg norm

$$\|\xi\|_{M,\Omega} := \inf\left\{\lambda > 0 : \int_\Omega M\left(x, \frac{\xi(x)}{\lambda}\right) \mathrm{d}x \le 1\right\};$$

the infimum is attained if $\xi \neq 0$. In general, $L_M(\Omega;\mathbb{R}^d)$ is neither separable nor reflexive. Finally, because of the superlinear growth of M (see (5)), there holds

$$L_M(\Omega;\mathbb{R}^d) \subseteq L^1(\Omega;\mathbb{R}^d). \tag{9}$$

Let us denote by $E_M(\Omega;\mathbb{R}^d)$ the closure of all bounded measurable functions defined on Ω with respect to the Luxemburg norm $\|\cdot\|_{M,\Omega}$. It turns out that $E_M(\Omega;\mathbb{R}^d)$ is the largest linear space contained in the Orlicz class $\mathcal{L}_M(\Omega;\mathbb{R}^d)$ such that

$$E_M(\Omega;\mathbb{R}^d) \subseteq \mathcal{L}_M(\Omega;\mathbb{R}^d) \subseteq L_M(\Omega;\mathbb{R}^d),$$

where inclusion is generally strict.

The space $E_M(\Omega;\mathbb{R}^d)$ is separable and $C_c^\infty(\Omega;\mathbb{R}^d)$ is dense in $E_M(\Omega;\mathbb{R}^d)$. The space $L_M(\Omega;\mathbb{R}^d)$ is the dual of $E_{M^*}(\Omega;\mathbb{R}^d)$, the duality pairing is given by

$$\langle \xi, \eta\rangle = \int_\Omega \xi\cdot\eta \,\mathrm{d}x, \quad \xi \in L_M(\Omega;\mathbb{R}^d),\ \eta \in E_{M^*}(\Omega;\mathbb{R}^d).$$

At this point, we may recall the *generalized Hölder inequality*

$$\int_\Omega |\xi\cdot\eta| \,\mathrm{d}x \le 2\,\|\xi\|_{M,\Omega}\,\|\eta\|_{M^*,\Omega} \qquad \text{for all } \xi \in L_M(\Omega;\mathbb{R}^d),\ \eta \in L_{M^*}(\Omega;\mathbb{R}^d).$$

If the $\mathcal{N}$-function M satisfies the so-called Δ_2-condition, i.e., if there exists $c > 0$ and a nonnegative integrable function $h : \Omega \to \mathbb{R}$ such that

$$M(x, 2\xi) \le cM(x,\xi) + h(x)$$

for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$, then $\mathcal{L}_M(\Omega;\mathbb{R}^d) = L_M(\Omega;\mathbb{R}^d) = E_M(\Omega;\mathbb{R}^d)$ (see [1, 34, 41]). The Δ_2-condition is rather restrictive. Nevertheless, for a measurable function $p : \Omega \to (1,\infty)$ the $L^{p(x)}$ spaces are included in the generalized Orlicz spaces framework with $M(x,\xi) = |\xi|^{p(x)}$ and with the classical assumption $1 < \operatorname{ess\,inf}_{x\in\Omega} p(x) \le p(x) \le \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$ both $|\cdot|^{p(x)}$ and $|\cdot|^{p'(x)}$, where $p'(x) = p(x)/(p(x)-1)$ a.e. in Ω, satisfy the Δ_2-condition.

Proposition 2.1 *Let M^* be an $\mathcal{N}$-function and let M^* satisfies Δ_2-condition. Then there exist $\nu > 0$ and $c > 0$ such that*

$$M(x,\xi) \geq c|\xi|^{1+\nu}$$

for all $\xi \in \mathbb{R}^d$ such that $|\xi| \geq |\xi_0|$.

Proof: Let

$$m^*(r) = \sup_{x\in\Omega} \sup_{\xi\in\mathbb{R}^d,\, |\xi|=r} M^*(x,\xi)$$

Obviously m^* is an $\mathcal{N}$-function and satisfies Δ_2 condition . Using the Collorary 5 from [39, Chapter II] we infer that there exist a conjugate $\mathcal{N}$-function $m = m(|\xi|)$ to m^* and constatns $\nu > 0$ and $c > 0$ such that $m(|\xi(x)|) \geq c|\xi|^{1+\nu}$ for $\xi \in \mathbb{R}^d$ s.t. $|\xi| \geq |\xi_0|$. According to definition of m^*, $M^*(x,\xi(x)) \leq m^*(|\xi(x)|)$ for a.a. $x \in \Omega$. Thus $m(|\xi|) \leq M(x,\xi)$ and for all measurable functions $\xi : \Omega \to \mathbb{R}^d$, we obtain

$$M(x,\xi) \geq c|\xi|^{1+\nu}$$

for all $\xi \in \mathbb{R}^d$ such that $|\xi| \geq |\xi_0|$.

Lemma 2.2 *Let M be an $\mathcal{N}$-function and $\{z_j\}_{j=1}^\infty$ a sequence of measurable functions $z_j : \Omega \to \mathbb{R}^d$ with $\sup\limits_{j\in\mathbb{N}} \int_\Omega M(x, z^j(x))\,\mathrm{d}x < \infty$. Then the sequence $\{z^j\}_{n=1}^\infty$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty} \left(\sup_{j\in\mathbb{N}} \int_{\{x:|z^j(x)|\geq R\}} |z^j(x)|\,\mathrm{d}x \right) = 0.$$

Proof can be found in [30, 31].

2.2 Function spaces and notation

2.2.1 The energy space

Let us introduce the linear space

$$V := \{\varphi \in L^1_{\mathrm{loc}}(\Omega);\ \exists\{\varphi_j\}_{j=1}^\infty \subset C_c^\infty(\Omega) \text{ such that } \nabla\varphi_j \overset{*}{\rightharpoonup} \nabla\varphi \text{ in } L_M(\Omega;\mathbb{R}^d) \text{ as } j \to \infty\}.$$

V endowed with the norm

$$\|\varphi\|_V = \|\nabla\varphi\|_{M,\Omega},\ \varphi \in V$$

is a Banach space. Moreover

$$V \hookrightarrow \{\varphi \in W_0^{1,1+\nu}(\Omega);\ \ \nabla\varphi \in L_M(\Omega;\mathbb{R}^d)\}$$

where $\hookrightarrow$ denotes continuous imbedding. If $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $h(0) = 0$ and $u \in V$, then also $h(u) \in V$. Note that if M^* satisfies the Δ_2-condition and if $g \in L^\infty(\Omega)$ and $\eta \in \mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)$, it follows that $g\varphi \in \mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)$.

2.2.2 Notation

For any $u : \Omega \to \mathbb{R}$ and $k \geq 0$, we denote $\{|u| \leq (<,>,\geq,=)k\}$ for the set $\{x \in \Omega : |u(x)| \leq (<,>,\geq,=)k\}$. For $r \in \mathbb{R}$ by $\mathrm{sign}_0(r)$ we mean the usual (single-valued) sign function, $\mathrm{sign}_0^+(r) = 1$ if $r > 0$ and $\mathrm{sign}_0^+(r) = 0$ if $r \leq 0$. Let $h_l(r) : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) = \min((l + 1 - |r|)^+, 1)$$

for each $r \in \mathbb{R}$. For any given $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(r) := \begin{cases} -k & \text{if } r \leq -k \\ r & \text{if } |r| < k \\ k & \text{if } r \geq k. \end{cases}$$

3 Notions of solution

3.1 Weak solutions

Definition 3.1 *A weak solution to (E, f) is a pair of functions $(u, b) \in V \times L^1(\Omega)$ satisfying $b(x) \in \beta(x, u(x))$ a.e. in Ω such that $a(x, \nabla u) \in L_{M^*}(\Omega; \mathbb{R}^d)$, $F(u) \in L_{M^*}(\Omega; \mathbb{R}^d)$ and*

$$b - \mathrm{div}(a(\cdot, \nabla u) + F(u)) = f \tag{10}$$

in $\mathcal{D}'(\Omega)$.

Corollary 3.1 *If (u, b) is a weak solution to (E, f) such that $u \in L^\infty(\Omega)$, it follows that $F(u) \in L^\infty(\Omega; \mathbb{R}^d)$ and therefore in $L_{M^*}(\Omega; \mathbb{R}^d)$. If moreover M satisfies the Δ_2-condition, then $a(x, \nabla u) \in L_{M^*}(\Omega; \mathbb{R}^d)$ is a direct consequence from the growth assumptions on $a(x, \nabla u)$.*

Indeed, from (1) it follows that

$$\frac{c_a}{2} a(x, \nabla u) \frac{2}{c_a} \nabla u \geq c_a \{M^*(x, a(x, \nabla u)) + M(x, \nabla u)\} - a_0(x) \tag{11}$$

for $c_a \in (0, 1]$ and $a_0 \in L^1(\Omega)$ nonnegative. Now, using the Fenchel-Young inequality (7) to estimate the left-hand side of (11) we arrive at

$$M^*(x, \frac{c_a}{2} a(x, \nabla u)) + M(x, \frac{2}{c_a} \nabla u) + a_0(x) \geq c_a \{M^*(x, a(x, \nabla u)) + M(x, \nabla u)\} \tag{12}$$

Now, since M^* is convex, $M^*(x, 0) = 0$ and $0 < \frac{c_a}{2} < 1$, from (12) we obtain

$$\frac{2}{c_a} \left(M(x, \frac{2}{c_a} \nabla u) + a_0(x) \right) \geq M^*(x, a(x, \nabla u)). \tag{13}$$

If M satisfies the Δ_2-condition, then $\nabla u \in L_M(\Omega; \mathbb{R}^d) = \mathcal{L}_M(\Omega; \mathbb{R}^d) = E_M(\Omega; \mathbb{R}^d)$ implies $\frac{2}{c_a} \nabla u \in \mathcal{L}_M(\Omega; \mathbb{R}^d)$ and the assertion follows by integrating (13). In general, $u \in V \cap L^\infty(\Omega)$ does not imply that

$$\int_\Omega M(x, \frac{2}{c_a} \nabla u) < \infty.$$

3.2 Renormalized solutions

Definition 3.2 *A renormalized solution to* (E, f) *is a function* u *satisfying the following conditions:*

(R1) $u : \Omega \to \mathbb{R}$ *is measurable,* $b \in L^1(\Omega)$ *and* $b \in \beta(x, u(x))$ *for a.a.* $x \in \Omega$.

(R2) *For each* $k > 0$, $T_k(u) \in V$, $a(x, \nabla T_k(u)) \in L_{M^*}(\Omega; \mathbb{R}^d)$ *and*

$$\int_\Omega bh(u)\varphi \,\mathrm{d}x + \int_\Omega (a(x, \nabla u) + F(u)) \cdot \nabla(h(u)\varphi) \,\mathrm{d}x = \int_\Omega fh(u)\varphi \,\mathrm{d}x \tag{14}$$

holds for all $h \in C_c^1(\mathbb{R})$ *and all* $\varphi \in V \cap L^\infty(\Omega)$.

(R3) $\int_{\{l<|u|<l+1\}} a(x, \nabla u) \cdot \nabla u \to 0$ *as* $l \to \infty$.

Remark 3.1 *Since* u *is only measurable,* ∇u *may not be defined as an element of* $\mathcal{D}'(\Omega)$. *However, it is possible to define a generalized gradient* ∇u *in the following sense: There exists a measurable function* $v : \Omega \to \mathbb{R}^d$, *such that* $v = \nabla T_k(u)$ *on* $\{|u| < k\}$ *for all* $k > 0$. *Therefore all the terms in* (14) *are well-defined (see [12] for more details).*

Remark 3.2 *If* (u, b) *is a renormalized solution to* (E, f), *then we get*

$$a(x, \nabla T_k(u)) \cdot \nabla T_k(u) \in L^1(\Omega)$$

for all $k > 0$ *by applying the generalized Hölder inequality. If* M *satisfies the* Δ_2*-condition,* $T_k(u) \in V$ *implies* $\nabla T_k(u) \in L_M(\Omega; \mathbb{R}^d) = \mathcal{L}_M(\Omega; \mathbb{R}^d) = E_M(\Omega; \mathbb{R}^d)$ *and using the same arguments as in Corollary 3.1 it follows that*

$$a(x, \nabla T_k(u)) \in L_{M^*}(\Omega; \mathbb{R}^d). \tag{15}$$

Hence if M *satisfies the* Δ_2*-condition, the assumption* (15) *in Definition 3.2 can be dropped.*

Remark 3.3 *If* (u, b) *is a renormalized solution of* (E, f) *such that* $u \in L^\infty(\Omega)$, *it is a direct consequence of Definition 3.2 that* u *is in* V *and since* (14) *holds with the formal choice* $h \equiv 1$, (u, b) *is a weak solution.*

Indeed, let us choose $\varphi \in C_c^\infty(\Omega)$ and plug $h_l(u)\varphi$ as a test function in (14). Since $u \in L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find that u solves (E, f) in the sense of distributions.

4 Main results

Our results are stated as follows: In this section we will state existence and uniqueness of renormalized solutions to (E, f) in the two following theorems. In Proposition 4.3 we give conditions on a_0 and f such that the renormalized solution to (E, f) is a weak solution. In the next sections of this paper we will present the proofs.

Theorem 4.1 *For $f \in L^1(\Omega)$ there exists at least one renormalized solution u to the problem (E, f).*

Theorem 4.2 *Let $\beta : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be such that $\beta(x, \cdot)$ is strictly monotone for almost every $x \in \Omega$. For $f \in L^1(\Omega)$ let (u, b), $(\tilde{u}, \tilde{b})$ be renormalized solutions to (E, f). Then $u = \tilde{u}$ and $b = \tilde{b}$.*

Proposition 4.3 *Let (u, b) be a renormalized solution to (E, f). Assume that* **(A2)** *is satisfied with $a_0 \in L^\infty(\Omega)$ and the right-hand side f is in $L^d(\Omega)$. Then $u \in V \cap L^\infty(\Omega)$ and thus, in particular, u is a weak solution to (E, f).*

5 Proof of Theorem 4.1

The following section will be devoted to the proof of Theorem 4.1 and we will divide it into several steps.

5.1 $(E_\varepsilon, f_\varepsilon)$ - approximation of the problem (E, f)

First we introduce the approximate problem to (E, f), namely

$$\begin{aligned} T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) - \operatorname{div}\big(a(x, \nabla u_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\big) = T_{1/\varepsilon}(f) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{aligned} \tag{$E_\varepsilon, f_\varepsilon$}$$

where for each $\varepsilon \in (0, 1]$, $\beta_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes the Moreau-Yosida approximation (see [21]) of β in the second variable. In particular $\beta_\varepsilon(\cdot, T_{1/\varepsilon}(\cdot))$ is a single-valued, monotone (with respect to the second variable and for a.a. $x \in \Omega$) Carathéodory function.

5.1.1 Existence for the problem $(E_\varepsilon, f_\varepsilon)$ - Galerkin approximation

We will show that there exists at least one weak solution u_ε to our approximate problem $(E_\varepsilon, f_\varepsilon)$ with $f_\varepsilon = T_{1/\varepsilon}(f) \in L^\infty(\Omega)$. Here, by a weak solution we mean $u_\varepsilon \in W_0^{1,1+\nu}(\Omega)$, $\nabla u_\varepsilon \in L_M(\Omega; \mathbb{R}^d)$ s.t.

$$\begin{aligned} \int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))\varphi \,\mathrm{d}x + \int_\Omega \varepsilon \arctan(u_\varepsilon)\varphi \,\mathrm{d}x + \int_\Omega \big(a(x, \nabla u_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\big) \cdot \nabla\varphi \,\mathrm{d}x \\ = \int_\Omega T_{1/\varepsilon}(f)\varphi \,\mathrm{d}x \end{aligned} \tag{16}$$

for every $\varphi \in C_c^\infty(\Omega)$.
We start with the Galerkin approximation. Let $\{\omega_i\}_{i=1}^\infty$ be a basis build by the eigenfunctions of the Laplace operator with zero Dirichlet boundary conditions. For $k \in \mathbb{N}$ let us look for an approximate solution of the form

$$u_\varepsilon^k := \sum_{i=1}^k c_i^k \omega_i \tag{17}$$

with $c_i^k \in \mathbb{R}$ such that

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^k)))\omega_i \,\mathrm{d}x + \int_\Omega \left(a(x, \nabla u_\varepsilon^k) + F(T_{1/\varepsilon}(u_\varepsilon^k))\right) \cdot \nabla \omega_i \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)\omega_i \,\mathrm{d}x \tag{18}$$

for $i = 1, \ldots, k$. Multiplying (18) by c_i^k and summing over $i = 1, ..., j$ with $j \leq k$ we obtain

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^k)))u_\varepsilon^j \,\mathrm{d}x + \int_\Omega \left(a(x, \nabla u_\varepsilon^k) + F(T_{1/\varepsilon}(u_\varepsilon^k))\right) \cdot \nabla u_\varepsilon^j \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^j \,\mathrm{d}x. \tag{19}$$

The existence of such an approximate solution to the Galerkin approximation u_ε^k can be obtained by the Lemma about Zeros of a Vector Field [26, Chapter 9]. Since $F(T_{1/\varepsilon}(\cdot))$ is a Lipschitz function, applying the Stokes Theorem it follows that for $j = k$ the term

$$\int_\Omega \left(F(T_{1/\varepsilon}(u_\varepsilon^k))\right) \cdot \nabla u_\varepsilon^k \,\mathrm{d}x = 0.$$

Hence for $j = k$ we have

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^k)))u_\varepsilon^k \,\mathrm{d}x + \int_\Omega a(x, \nabla u_\varepsilon^k) \cdot \nabla u_\varepsilon^k \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^k \,\mathrm{d}x. \tag{20}$$

We want to estimate the right-hand side of (20). Employing the Poincaré inequality, assumption (3) and the Young inequality we infer

$$\begin{aligned} \int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^k \,\mathrm{d}x &\leq c_d \|T_{1/\varepsilon}(f)\|_{L^\infty} \|\nabla u_\varepsilon^k\|_{L^1} \\ &\leq \gamma(c_d, c_a)\|T_{1/\varepsilon}(f)\|_{L^\infty} + \frac{c_a}{2}\left(\int_\Omega M(x, \nabla u_\varepsilon^k) \,\mathrm{d}x + c\right) \end{aligned} \tag{21}$$

where $c_d > 0$ is the constant from the Poincaré inequality and $\gamma(c_d, c_a) > 0$, $c > 0$ are bounded constants independent of $k > 0$. Combining (21) with (20), using the coercivity condition (1) on $a(\cdot, \cdot)$ and neglecting the nonnegative term $T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^k)))u_\varepsilon^k$ gives

$$\frac{c_a}{2} \int_\Omega M(x, \nabla u_\varepsilon^k) \,\mathrm{d}x + c_a \int_\Omega M^*(x, a(x, \nabla u_\varepsilon^k)) \,\mathrm{d}x \leq \gamma(c_d, c_a)\|T_{1/\varepsilon}(f)\|_{L^\infty} + \frac{c_a c}{2}. \tag{22}$$

Consequently, passing to a subsequence if necessary, from (22) we obtain

$$\nabla u_\varepsilon^k \overset{*}{\rightharpoonup} \nabla u_\varepsilon \text{ weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^d) \tag{23}$$

and

$$a(x,\nabla u_\varepsilon^k) \overset{*}{\rightharpoonup} \alpha \text{ weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^d) \text{ for some } \alpha\in L_{M^*}(\Omega;\mathbb{R}^d). \tag{24}$$

The condition (3) provides that $\{\nabla u_\varepsilon^k\}_{k=1}^\infty$ is uniformly bounded in the space $L^{1+\nu}(\Omega;\mathbb{R}^d)$, hence by the Poincaré inequality the sequence $\{u_\varepsilon^k\}_{k=1}^\infty$ is uniformly bounded in $W_0^{1,1+\nu}(\Omega)$. Therefore

$$\nabla u_\varepsilon^k \rightharpoonup \nabla u_\varepsilon \text{ weakly in } L^{1+\nu}(\Omega), \tag{25}$$

$$u_\varepsilon^k \to u_\varepsilon \text{ strongly in } L^{1+\nu}(\Omega) \tag{26}$$

and

$$u_\varepsilon^k \to u_\varepsilon \text{ a. e. in } \Omega. \tag{27}$$

Let us notice that for fixed $\varepsilon\in(0,1]$ and almost all $x\in\Omega$ the function $\beta_\varepsilon(x,\cdot)$ is a Carathéodory function and we have $|\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon^k))|\le\max(\beta^0(x,1/\varepsilon),-\beta^0(x,-1/\varepsilon))$, a.e. in Ω where, thanks to (4), β^0 is integrable. Then this together with (27) and the Lebesgue Dominated Convergence Theorem provides

$$T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon^k)))\to T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon))) \text{ strongly in } L^1(\Omega). \tag{28}$$

Since $F(\cdot)$ is continuous we obtain

$$F(T_{1/\varepsilon}(u_\varepsilon^k))\to F(T_{1/\varepsilon}(u_\varepsilon)) \text{ a.e. in } \Omega. \tag{29}$$

As $F(T_{1/\varepsilon}(u_\varepsilon^k))$ is uniformly bounded with respect to $k>0$, i.e.

$$\|F(T_{1/\varepsilon}(u_\varepsilon^k))\|_{L^\infty(\Omega;\mathbb{R}^d)}\le\sup_{\tau\in[-1/\varepsilon,1/\varepsilon]}|F(\tau)|<c \tag{30}$$

where the constant $c>0$ is independent of $k\in\mathbb{N}$ and as Ω is bounded, (27) and the continuity of $F(\cdot)$ together with the Lebesgue Dominated Convergence Theorem provides that

$$F(T_{1/\varepsilon}(u_\varepsilon^k))\to F(T_{1/\varepsilon}(u_\varepsilon)) \text{ in } L^1(\Omega;\mathbb{R}^d) \text{ as } k\to\infty. \tag{31}$$

Recall that if M is an $\mathcal{N}$-function, then M^* is also an $\mathcal{N}$-function. This and (22) allows us to apply Lemma 2.2 to M^* and conclude the uniform integrability of $\{a(\cdot,\nabla u_\varepsilon^k)\}_{k=1}^\infty$. Hence according to the Dunford-Pettis Theorem we have the weak precompactness of the sequence $\{a(x,\nabla u_\varepsilon^k)\}_{k=1}^\infty$ in $L^1(\Omega;\mathbb{R}^d)$. Therefore $\alpha\in L^1(\Omega;\mathbb{R}^d)$ and passing to a subsequence when necessary

$$a(\cdot,\nabla u_\varepsilon^k)\rightharpoonup\alpha \quad \text{weakly in} \quad L^1(\Omega;\mathbb{R}^d). \tag{32}$$

Using (28), (31), (32) and letting $k\to\infty$ in (19) gives

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon^j\,\mathrm{d}x+\int_\Omega\big(\alpha+F(T_{1/\varepsilon}(u_\varepsilon))\big)\cdot\nabla u_\varepsilon^j\,\mathrm{d}x=\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^j\,\mathrm{d}x. \tag{33}$$

Since (33) is also satisfied for all test functions from the basis $\{\omega_i\}_{i=1}^{\infty}$, density arguments give us that u_ε and α satisfy

$$T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) - \operatorname{div}\big(\alpha + F(T_{1/\varepsilon}(u_\varepsilon))\big) = T_{1/\varepsilon}(f)$$

in $\mathcal{D}'(\Omega)$. The last step is to identify the vector α. Let us notice that the convection term on the left-hand side of (33) vanishes when $j \to \infty$ by the Stokes Theorem. Since M, M^* are convex and nonnegative functions, the weak lower semicontinuity of M and M^* together with (22) imply that $\alpha \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d)$, $\nabla u_\varepsilon \in \mathcal{L}_M(\Omega; \mathbb{R}^d)$ respectively. Since M^* satisfies the Δ_2-condition it follows that $L_{M^*}(\Omega; \mathbb{R}^d) = \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d) = E_{M^*}(\Omega; \mathbb{R}^d)$ is a separable space. Therefore, $\alpha + F(T_{1/\varepsilon}(u_\varepsilon)) \in E_{M^*}(\Omega; \mathbb{R}^d)$ and since $(E_{M^*}(\Omega; \mathbb{R}^d))^* = L_M(\Omega; \mathbb{R}^d)$, using (23) and (26) we can pass to the limit with $j \to \infty$ in (33) and obtain

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon \,\mathrm{d}x + \int_\Omega \alpha \cdot \nabla u_\varepsilon \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f) u_\varepsilon \,\mathrm{d}x. \tag{34}$$

Now we apply the monotonicity trick for non reflexive spaces to obtain

$$\alpha = a(x, \nabla u_\varepsilon) \text{ a.e. in } \Omega.$$

First note that for $\zeta \in L^\infty(\Omega; \mathbb{R}^d)$ it follows that $a(x, \zeta) \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d)$. Indeed, with the same arguments as in Corollary 3.1 it follows that

$$\int_\Omega M^*(x, a(x, \zeta)) \,\mathrm{d}x \le \frac{2}{c_a} \int_\Omega M(x, \frac{2}{c_a}\zeta) + a_0 \,\mathrm{d}x \tag{35}$$

and for $\zeta \in L^\infty(\Omega; \mathbb{R}^d)$ the integral on the right-hand side of (35) is finite. Passing to a subsequence if necessary, for $k \to \infty$ from (20) we get

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega a(x, \nabla u_\varepsilon^k) \cdot \nabla u_\varepsilon^k \,\mathrm{d}x &= \lim_{k\to\infty} \left(\int_\Omega T_{1/\varepsilon}(f) u_\varepsilon^k \,\mathrm{d}x - \int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^k)))u_\varepsilon^k \,\mathrm{d}x \right) \\ &= \int_\Omega T_{1/\varepsilon}(f) u_\varepsilon \,\mathrm{d}x - \int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon \,\mathrm{d}x\end{aligned} \tag{36}$$

which together with (34) provides

$$\lim_{k\to\infty} \int_\Omega a(x, \nabla u_\varepsilon^k) \cdot \nabla u_\varepsilon^k \,\mathrm{d}x = \int_\Omega \alpha \cdot \nabla u_\varepsilon \,\mathrm{d}x. \tag{37}$$

Since $a(x, \cdot)$ is monotone

$$(a(x, \zeta) - a(x, \nabla u_\varepsilon^k)) \cdot (\zeta - \nabla u_\varepsilon^k) \ge 0 \tag{38}$$

a.e. in Ω and for all $\zeta \in L^\infty(\Omega; \mathbb{R}^d)$. Integrating (38), using $a(x, \zeta) \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d) = E_{M^*}(\Omega; \mathbb{R}^d)$ and (37) to pass to the limit with $k \to \infty$ we obtain

$$\int_\Omega (a(x, \zeta) - \alpha) \cdot (\zeta - \nabla u_\varepsilon) \,\mathrm{d}x \ge 0. \tag{39}$$

For $l > 0$ let

$$\Omega_l := \{x \in \Omega : |\nabla u_\varepsilon(x)| \le l \quad \text{a.e. in } \Omega\}.$$

Now let $0 < j < i$ be arbitrary, $z \in L^\infty(\Omega; \mathbb{R}^d)$ and $h > 0$. Plugging

$$\zeta = (\nabla u_\varepsilon)\mathbb{1}_{\Omega_i} + hz\mathbb{1}_{\Omega_j},$$

into (39) we get

$$-\int_{\Omega \setminus \Omega_i} (a(x,0) - \alpha) \cdot \nabla u_\varepsilon \, \mathrm{d}x + h \int_{\Omega_j} (a(x, \nabla u_\varepsilon + hz) - \alpha) \cdot z \, \mathrm{d}x \ge 0. \tag{40}$$

Note that by (1) $M^*(x, a(x,0)) \le a_0(x)$ a.e. in Ω and from (7) it follows that

$$\int_\Omega |a(x,0) \cdot \nabla u_\varepsilon| \, \mathrm{d}x \le \int_\Omega a_0 + M(x, \nabla u_\varepsilon) \, \mathrm{d}x. \tag{41}$$

Since $\nabla u_\varepsilon \in \mathcal{L}_M(\Omega; \mathbb{R}^d)$ the right-hand side of (41) is finite and consequently

$$a(x,0) \cdot \nabla u_\varepsilon \in L^1(\Omega).$$

As $\alpha \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d)$ and $\nabla u \in \mathcal{L}_M(\Omega; \mathbb{R}^d)$ it follows immediately by (7) that $\alpha \cdot \nabla u_\varepsilon$ is in $L^1(\Omega)$. Therefore, by the Lebesgue Dominated Convergence Theorem, the first term on the left-hand of (40) vanishes for $i \to \infty$. Passing to the limit with $i \to \infty$ in (40) and dividing by h we get

$$\int_{\Omega_j} (a(x, \nabla u_\varepsilon + hz) - \alpha) \cdot z \, \mathrm{d}x \ge 0.$$

Note that $a(x, \nabla u_\varepsilon + hz) \to a(x, \nabla u_\varepsilon)$ a.e. in Ω_j when $h \downarrow 0$. Moreover, for $0 < h < 1$

$$\int_{\Omega_j} M^*(x, a(x, \nabla u_\varepsilon + hz)) \, \mathrm{d}x \le \frac{2}{c_a} \sup_{0<h<1} \int_{\Omega_j} M(x, \frac{2}{c_a}(\nabla u_\varepsilon + hz)) + a_0(x) \, \mathrm{d}x \tag{42}$$

and the right-hand side of (42) is bounded since $\nabla u_\varepsilon + hz$ is uniformly (in h) bounded in $L^\infty(\Omega_j; \mathbb{R}^d)$ and according to (8) $M(x, \frac{2}{c_a}(\nabla u_\varepsilon + hz))$ is bounded. Hence it follows from Lemma 2.2 that $\{a(x, \nabla u_\varepsilon + hz)\}_h$ is uniformly integrable. Note that $|\Omega_j| < \infty$, hence by the Vitali Theorem it follows that

$$a(x, \nabla u_\varepsilon + hz) \to a(x, \nabla u_\varepsilon) \quad \text{in } L^1(\Omega_j; \mathbb{R}^d)$$

for $h \downarrow 0$ and therefore

$$\int_{\Omega_j} (a(x, \nabla u_\varepsilon + hz) - \alpha) \cdot z \, \mathrm{d}x \to \int_{\Omega_j} (a(x, \nabla u_\varepsilon) - \alpha) \cdot z \, \mathrm{d}x$$

for $h \downarrow 0$. Consequently,

$$\int_{\Omega_j} (a(x, \nabla u_\varepsilon) - \alpha) \cdot z \, \mathrm{d}x \ge 0$$

for all $z \in L^\infty(\Omega;\mathbb{R}^d)$. Substituting

$$z = \begin{cases} -\frac{a(x,\nabla u_\varepsilon)-\alpha}{|a(x,\nabla u_\varepsilon)-\alpha|} & \text{if } a(x,\nabla u_\varepsilon)-\alpha \neq 0 \\ 0 & \text{if } a(x,\nabla u_\varepsilon)-\alpha = 0 \end{cases}$$

into the above, we obtain

$$\int_{\Omega_j} |a(x,\nabla u_\varepsilon)-\alpha|\,\mathrm{d}x \leq 0.$$

Hence

$$a(x,\nabla u_\varepsilon) = \alpha \quad \text{a.e. in } \Omega_j. \tag{43}$$

Since j is arbitrary (43) holds a.e. in Ω.

5.2 A priori estimates

Lemma 5.1 *For $0 < \varepsilon \leq 1$ and $f \in L^1(\Omega)$ let $u_\varepsilon \in W_0^{1,1+\nu}(\Omega)$ and $\nabla u_\varepsilon \in L_M(\Omega;\mathbb{R}^d)$ satisfy $(E_\varepsilon, f_\varepsilon)$ in the distributional sense. Then*

$$\int_\Omega M(x,\nabla T_k(u_\varepsilon))\,\mathrm{d}x \leq k\|f\|_{L^1(\Omega)} \tag{44}$$

and

$$\int_\Omega M^*(x,a(x,\nabla T_k(u_\varepsilon)))\,\mathrm{d}x \leq k\|f\|_{L^1(\Omega)}. \tag{45}$$

Moreover,

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\mathrm{d}x \leq \int_{\{l<|u_\varepsilon|\}} |f|\,\mathrm{d}x \tag{46}$$

holds for all $\varepsilon \in (0,1]$.

Remark 5.1 *Using Lemma 2.2 and (44) and (45) we deduce that sequences*

$$\{a(x,\nabla T_k(u_\varepsilon))\}_{\varepsilon>0},\ \{\nabla T_k(u_\varepsilon)\}_{\varepsilon>0} \text{ are uniformly integrable w.r.t. } \varepsilon>0 \text{ in } L^1(\Omega;\mathbb{R}^d) \tag{47}$$

for any fixed $k \in \mathbb{N}$.

Proof:

Testing in $(E_\varepsilon, f_\varepsilon)$ by $T_k(u_\varepsilon)$ yields

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))T_k(u_\varepsilon)\,\mathrm{d}x + \int_\Omega \big(a(x,\nabla T_k(u_\varepsilon)) + F(T_{1/\varepsilon}(u_\varepsilon))\big)\cdot\nabla T_k(u_\varepsilon)\,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)T_k(u_\varepsilon)\,\mathrm{d}x.$$

As first term on the left-hand side is nonnegative and the integral over the convection term vanishes, by (1) and the Hölder inequality we get

$$c_a \int_\Omega \left(M^*(x, a(x, \nabla T_k(u_\varepsilon))) + M(x, \nabla T_k(u_\varepsilon))\right) \mathrm{d}x \le k\|f\|_{L^1(\Omega)},$$

where $c_a \in (0,1]$, and therefore (44) and (45) holds.

Let us define $g_l : \mathbb{R} \to \mathbb{R}$ by

$$g_l(r) := T_{l+1}(r) - T_l(r) = \begin{cases} -1 & \text{if } r \le -(l+1) \\ r+l & \text{if } -(l+1) < r \le -l \\ 0 & \text{if } |r| < l \\ r-l & \text{if } l \le r < l+1 \\ 1 & \text{if } l+1 \le r \end{cases}$$

Using $g_l(u_\varepsilon)$ as a test function in the problem $(E_\varepsilon, f_\varepsilon)$ we obtain

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))g_l(u_\varepsilon)\,\mathrm{d}x + \int_\Omega \left[a(x, \nabla u_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\right] \cdot \nabla g_l(u_\varepsilon)\,\mathrm{d}x$$
$$= \int_\Omega T_{1/\varepsilon}(f)g_l(u_\varepsilon)\,\mathrm{d}x.$$

As the first term on the left-hand side is nonnegative and the convection term vanishes, we find that

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon)\,\mathrm{d}x \le \int_{\{l<|u_\varepsilon|\}} |f|\,\mathrm{d}x \tag{48}$$

Let us notice that (46) is equal to (48).

Corollary 5.2 *There exists $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to 0^+} \gamma(r) = 0$ and*

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\,\mathrm{d}x \le \gamma(Cl^{-\nu}) \tag{49}$$

for any $\varepsilon \in (0,1]$, where C is independent of ε and l. Moreover

$$|\{|u_\varepsilon| \ge l\}| \le l^{-\nu}C \tag{50}$$

holds for $C(\nu, d, f)$ independently of ε.

Proof: Let us concentrate on (50). Note that

$$|\{|u_\varepsilon| \ge l\}| = |\{|T_l(u_\varepsilon)| \ge l\}|,$$

then by the Chebyshev, the Poincaré inequality and (3), (44) we obtain

$$|\{|u_\varepsilon| \ge l\}| \le \int_\Omega \frac{|T_l(u_\varepsilon)|^{1+\nu}}{l^{1+\nu}}\,\mathrm{d}x$$
$$\le C(\nu, d)l^{-(1+\nu)} \int_\Omega |\nabla T_l(u_\varepsilon)|^{1+\nu}\,\mathrm{d}x = C(\nu, d)\|f\|_{L^1(\Omega)}l^{-\nu}$$

Since $f \in L^1(\Omega)$, there exists $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to 0^+} \gamma(r) = 0$ and for any subset E of Ω $\int_E |f|\,\mathrm{d}x \le \gamma(|E|)$. Hence (48) provides (49).

5.3 Convergence results

The a priori estimates in Lemma 5.1 and Corollary 5.2 imply the following convergences as $\varepsilon \downarrow 0$:

Proposition 5.3 *For $\varepsilon \in (0,1]$ and $f \in L^1(\Omega)$ let $u_\varepsilon \in W_0^{1,1+\nu}(\Omega)$ and $\nabla u_\varepsilon \in L_M(\Omega;\mathbb{R}^d)$ satisfy $(E_\varepsilon, f_\varepsilon)$ in the distributional sense. Then there exists a Lebesgue measurable function $u : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ with $T_k(u) \in W_0^{1,1+\nu}(\Omega)$, $\nabla T_k(u) \in L_M(\Omega;\mathbb{R}^d)$ such that for a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ and any $k \in \mathbb{N}$,*

$$u_\varepsilon \to u \text{ a.e. in } \Omega, \tag{51}$$

where

$$|\{|u| > l\}| \leq Cl^{-\nu}. \tag{52}$$

Moreover

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } L^p(\Omega) \text{ for } p \in [1,\infty) \text{ and a.e. in } \Omega, \tag{53}$$

$$\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u) \text{ weakly in } L^{1+\nu}(\Omega;\mathbb{R}^d), \tag{54}$$

$$\nabla T_k(u_\varepsilon) \overset{*}{\rightharpoonup} \nabla T_k(u) \text{ weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^d), \tag{55}$$

and

$$a(x, \nabla T_k(u_\varepsilon)) \overset{*}{\rightharpoonup} a(x, \nabla T_k(u)) \text{ weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^d). \tag{56}$$

Proof: Applying directly the Lemma 5.1 and (3) together with the Sobolev Imbedding Theorem we obtain (53), (54), (55). Moreover there exists $\alpha_k \in L_{M^*}(\Omega;\mathbb{R}^d)$ such that

$$a(x, \nabla T_k(u_\varepsilon)) \overset{*}{\rightharpoonup} \alpha_k \text{ weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^d). \tag{57}$$

In (53) we choose by the diagonal method a subsequence such that the convergence in (53) holds for any $k \in \mathbb{N}$ (ε_i is still indicated by ε). Obviously the same subsequence can be taken in (54), (55) and (57).

Since (53) holds for any $k \in \mathbb{N}$ we obtain (51) where u is the Lebesgue measurable function which may take values $\pm\infty$. By (51)

$$\liminf_{\varepsilon\downarrow 0} |\{|u_\varepsilon| > l\}| \geq |\{|u| > l\}|$$

and using (50) we obtain (52).

We intend to show now that

$$\alpha_k = a(x, \nabla T_k(u)). \tag{58}$$

a.e. in Ω. The proof of (58) is divided in several steps.

Step 1:
Let $g \in L^1(\Omega;\mathbb{R}^d)$ and $x \in \Omega$ and let us introduce the maximal function

$$\mathcal{M}(g)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\tilde{g}(y)|\,\mathrm{d}y, \tag{59}$$

where $\tilde{g}$ is extention of g by zero outside of Ω. Let us recall some of its basic properties for the sake of completeness: Note that $|g| \le \mathcal{M}(g)$ holds for every Lebesgue point of g and therefore a.e. in Ω. Let us show that for any function $g \in L^1(\Omega;\mathbb{R}^d)$ the mapping $x \mapsto \mathcal{M}(g)(x)$ is lower semicontinuous on Ω. To this end we fix $x_0 \in \Omega$. For $\varepsilon > 0$ we define the function $x \mapsto M_g^\varepsilon(x)$, where

$$M_g^\varepsilon(x) := \frac{1}{|B(x,r_\varepsilon)|} \int_{B(x,r_\varepsilon)} |\tilde{g}(y)| \ \mathrm{d}y$$

and $r_\varepsilon > 0$ is chosen such that

$$\mathcal{M}(g)(x_0) - M_g^\varepsilon(x_0) < \varepsilon.$$

Note that

$$\mathcal{M}(g)(x) \ge M_g^\varepsilon(x) \tag{60}$$

a.e. in Ω for all $\varepsilon > 0$. Let us check that $x \mapsto M_g^\varepsilon(x)$ is continuous in $x_0 \in \Omega$. Since $|B(x,r_\varepsilon)| = |B(x_0,r_\varepsilon)|$, for $x \in \Omega$ we have

$$\left|M_g^\varepsilon(x) - M_g^\varepsilon(x_0)\right| \le \frac{1}{|B(x_0,r_\varepsilon)|}\left(\int_{B(x,r_\varepsilon)\setminus B(x_0,r_\varepsilon)} |\tilde{g}(y)|\,\mathrm{d}y + \int_{B(x_0,r_\varepsilon)\setminus B(x,r_\varepsilon)} |\tilde{g}(y)|\,\mathrm{d}y\right) \tag{61}$$

Note that $|B(x,r_\varepsilon) \setminus B(x_0,r_\varepsilon)|$ and $|B(x_0,r_\varepsilon) \setminus B(x,r_\varepsilon)|$ converge to zero as $x \to x_0$. Therefore

$$\left|M_g^\varepsilon(x) - M_g^\varepsilon(x_0)\right| \to 0 \tag{62}$$

for $x \to x_0$. Finally, using (60) and (62) we arrive at

$$\liminf_{x\to x_0} \mathcal{M}(g)(x) \ge \liminf_{x\to x_0} M_g^\varepsilon(x) = M_g^\varepsilon(x_0) \ge \mathcal{M}(g)(x_0) - \varepsilon, \tag{63}$$

passing to the limit with $\varepsilon \downarrow 0$ in (63) we obtain

$$\liminf_{x\to x_0} \mathcal{M}(g)(x) \ge \mathcal{M}(g)(x_0).$$

Now let us define

$$\Omega_m := \{x \in \Omega \,:\, \mathcal{M}(\nabla T_k(u))(x) < m\}.$$

Since $x \to \mathcal{M}(\nabla T_k(u))(x)$ is lower semicontinuous, Ω_m is an open set for all $m \in \mathbb{N}$. It is well-known that for $g \in L^1(\Omega;\mathbb{R}^d)$ and all $\lambda > 0$

$$|\{\mathcal{M}(g) > \lambda\}| \le \frac{5^d}{\lambda}\|g\|_1, \tag{64}$$

hence $\mathcal{M}(g)$ is finite a.e. in Ω and in particular $|\Omega \setminus \Omega_m| \to 0$ for $m \to \infty$. It follows directly from the properties of the maximal function that for $\Psi \in C_c^\infty(\Omega)$ with compact support in Ω_m we have $\nabla T_k(u)\Psi \in L^\infty(\Omega;\mathbb{R}^d)$.

Step 2:
In order to obtain (58) we show

$$\limsup_{\varepsilon\downarrow 0} \int_\Omega a(x,\nabla T_k(u_\varepsilon))\cdot\nabla T_k(u_\varepsilon)\Psi \,\mathrm{d}x \le \int_\Omega \alpha_k\cdot\nabla T_k(u)\Psi\,\mathrm{d}x \tag{65}$$

for all nonnegative $\Psi \in C_c^\infty(\Omega)$ with compact support in Ω_m. To this end we fix $k,l>0$, take $\varphi = h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi$ as a test function in $(E_\varepsilon, f_\varepsilon)$ and obtain:

$$\begin{aligned}
&\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))\,[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi]\,\mathrm{d}x\\
&+\int_\Omega a(x,\nabla u_\varepsilon)\cdot\nabla\,[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi]\,\mathrm{d}x\\
&+\int_\Omega F(T_{1/\varepsilon}(u_\varepsilon))\cdot\nabla\,[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi]\,\mathrm{d}x\\
&=\int_\Omega T_{1/\varepsilon}(f)\,[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi]\,\mathrm{d}x.
\end{aligned}\tag{66}$$

We denote (66) by

$$I_\varepsilon^0 + I_\varepsilon^1 + I_\varepsilon^2 = I_\varepsilon^3.$$

First we focus on easier terms - I_ε^0, I_ε^2 and I_ε^3. As

$$I_\varepsilon^0 = \int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{l+1}(u_\varepsilon)))\,[h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi]\,\mathrm{d}x$$

for $\varepsilon>0$ small enough, using (4) and the Lebesgue Dominated Convergence Theorem we get

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon^0 = 0.$$

Using integration by parts we obtain

$$I_\varepsilon^2 = \int_\Omega F'(T_{l+1}(u_\varepsilon))\nabla T_{l+1}(u_\varepsilon)\,[h_l(u_\varepsilon)\,(T_k(u_\varepsilon)-T_k(u))\,\Psi]\,\mathrm{d}x.$$

For fixed $l>0$, $\nabla T_{l+1}(u_\varepsilon)$ is uniformly integrable in $L^1(\Omega;\mathbb{R}^d)$ (see (47)), $F'(T_{l+1}(u_\varepsilon))$ is bounded (because F is locally Lipschitz continuous) and

$$|h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi| \le 2k\|\Psi\|_{L^\infty(\Omega)}. \tag{67}$$

Moreover, $F'(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi \to 0$ a.e. in Ω. Therefore the Egorov Theorem applied to $\{F'(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Psi\}_{\varepsilon>0}$ and uniform integrability of $\{F'(T_{l+1}(u_\varepsilon))\nabla T_{l+1}(u_\varepsilon)\,[h_l(u_\varepsilon)\,(T_k(u_\varepsilon)-T_k(u))\,\Psi]\}_{\varepsilon>0}$ imply $\lim_{\varepsilon\downarrow 0} I_\varepsilon^2 = 0$. Since

$|T_{1/\varepsilon}(f)| \leq |f|$ a.e. in Ω, by (67), (53) and the Lebesgue Dominated Convergence Theorem it follows that $I_\varepsilon^3 \to 0$ as $\varepsilon \downarrow 0$.

Finally we concentrate on the most difficult term I_ε^1.

$$\begin{aligned} I_\varepsilon^1 = I_\varepsilon^{1,1} + I_\varepsilon^{1,2} + I_\varepsilon^{1,3} = &\int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla h_l(u_\varepsilon) \left[(T_k(u_\varepsilon) - T_k(u))\Psi \right] \mathrm{d}x \\ &+ \int_\Omega a(x, \nabla u_\varepsilon) \cdot h_l(u_\varepsilon) \nabla [T_k(u_\varepsilon) - T_k(u)] \Psi \, \mathrm{d}x \\ &+ \int_\Omega a(x, \nabla u_\varepsilon) \cdot h_l(u_\varepsilon) (T_k(u_\varepsilon) - T_k(u)) \nabla \Psi \, \mathrm{d}x. \end{aligned} \tag{68}$$

Applying (49) we infer

$$\begin{aligned} \sup_{\varepsilon \in (0,1]} |I_\varepsilon^{1,1}| &= \sup_{\varepsilon \in (0,1]} \int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon) | \left[(T_k(u_\varepsilon) - T_k(u))\Psi \right] | \, \mathrm{d}x \\ &\leq \sup_{\varepsilon \in (0,1]} 2k \|\Psi\|_{L^\infty(\Omega)} \int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon) \, \mathrm{d}x \\ &\leq 2k \|\Psi\|_{L^\infty(\Omega)} \gamma(C l^{-\nu}) \end{aligned} \tag{69}$$

therefore

$$\lim_{l \to \infty} \sup_{\varepsilon \in (0,1]} |I_\varepsilon^{1,1}| = 0. \tag{70}$$

Let us consider now $I_\varepsilon^{1,3}$. Using

$$|h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \nabla \Psi| \leq 2k \|\nabla \Psi\|_{L^\infty(\Omega)},$$

employing again (53), (47) and the Vitali Theorem we obtain

$$\lim_{l \to \infty} \lim_{\varepsilon \downarrow 0} I_\varepsilon^{1,3} = 0$$

Then above considerations (66) provids

$$\limsup_{l \to \infty} \limsup_{\varepsilon \downarrow 0} \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot h_l(u_\varepsilon) \nabla (T_k(u_\varepsilon) - T_k(u)) \Psi \, \mathrm{d}x \leq 0 \tag{71}$$

Note that for $l > k$

$$\begin{aligned} &\int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla (T_k(u_\varepsilon) - T_k(u)) \Psi \, \mathrm{d}x - \int_{\{|u_\varepsilon| > k\}} |a(x, 0) \cdot \nabla T_k(u) \Psi| \, \mathrm{d}x \\ &\leq \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot h_l(u_\varepsilon) \nabla (T_k(u_\varepsilon) - T_k(u)) \Psi \, \mathrm{d}x. \end{aligned} \tag{72}$$

Since $u_\varepsilon \to u$ a.e. in Ω for $\varepsilon \downarrow 0$ it follows that $\mathbb{1}_{\{|u_\varepsilon| > k\}} \to \mathbb{1}_{\{|u| > k\}}$ a.e. on $\{|u| > k\}$ and on $\{|u| < k\}$ for $\varepsilon \downarrow 0$. Therefore the second member on the left-hand side vanishes for

$\varepsilon \downarrow 0$. Thanks to $\nabla T_k(u)\Psi \in L^\infty(\Omega;\mathbb{R}^d)$, we can now combine (71) with (72) and pass to the limit with $\varepsilon \downarrow 0$ to obtain (65).

Step 3:

Since $a(x,\cdot)$ is monotone and $\Psi \geq 0$, we have

$$\begin{aligned}\int_\Omega & a(x,\nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon)\Psi \,\mathrm{d}x \\ &\geq \int_\Omega a(x,\nabla T_k(u_\varepsilon)) \cdot \zeta\Psi \,\mathrm{d}x + \int_\Omega a(x,\zeta)\cdot(\nabla T_k(u_\varepsilon)-\zeta)\Psi \,\mathrm{d}x.\end{aligned} \tag{73}$$

for $\zeta \in L^\infty(\Omega;\mathbb{R}^d)$. Note that $a(x,\zeta) \in E_{M^*}(\Omega;\mathbb{R}^d)$. Letting $\varepsilon \downarrow 0$ in (73) and using (57), (55) and (65) we achieve

$$\int_\Omega (a(x,\zeta)-\alpha_k)\cdot(\zeta - \nabla T_k(u))\Psi \,\mathrm{d}x \geq 0. \tag{74}$$

Let $h>0$ and

$$\zeta = \nabla T_k(u)\mathbb{1}_{\Omega_m} + hz, \tag{75}$$

where $z \in L^\infty(\Omega;\mathbb{R}^d)$ is arbitrary. Plugging (75) into (74) using the assumption that $\operatorname{supp}\Psi \subset \Omega_m$ and dividing by h we arrive at

$$\int_{\Omega_m} (a(x,\nabla T_k(u)+hz)-\alpha_k)\cdot z\Psi \,\mathrm{d}x \geq 0. \tag{76}$$

Repeating the arguments from the Galerkin approximation case with Ω_m instead of Ω_j (see (42) for details), we get that the sequence $\{a(x,\nabla T_k(u)+hz)\}_{h>0}$ is uniformly integrable on Ω_m. Since $a(x,\nabla T_k(u)+hz) \to a(x,\nabla T_k(u))$ a.e. in Ω_m for $h \downarrow 0$, by the Vitali Theorem

$$a(x,\nabla T_k(u)+hz) \to a(x,\nabla T_k(u)) \quad \text{in } L^1(\Omega_m;\mathbb{R}^d)$$

and

$$\int_{\Omega_m} (a(x,\nabla T_k(u)+hz)-\alpha_k)\cdot z\Psi \,\mathrm{d}x \to \int_{\Omega_m} (a(x,\nabla T_k(u))-\alpha_k)\cdot z\Psi \,\mathrm{d}x$$

when $h \downarrow 0$. Consequently,

$$\int_{\Omega_m} (a(x,\nabla T_k(u))-\alpha_k)\cdot z\Psi \,\mathrm{d}x \geq 0$$

for all $z \in L^\infty(\Omega;\mathbb{R}^d)$. Substituting

$$z = \begin{cases} -\frac{a(x,\nabla T_k(u))-\alpha_k}{|a(x,\nabla T_k(u))-\alpha_k|} & \text{if } a(x,\nabla T_k(u))-\alpha_k \neq 0 \\ 0 & \text{if } a(x,\nabla T_k(u))-\alpha_k = 0 \end{cases}$$

and choosing $\Psi > 0$ on a set of positive measure in the above inequality, we obtain

$$\int_{\Omega_m} |a(x, \nabla T_k(u)) - \alpha| \Psi \,\mathrm{d}x \leq 0.$$

Hence

$$a(x, \nabla T_k(u)) = \alpha_k \quad \text{a.e. in } \Omega_m. \tag{77}$$

Since m was arbitrary and $|\Omega \setminus \Omega_m| \to 0$ as $m \to \infty$, (77) holds a.e. in Ω. Therefore $\alpha_k = a(x, \nabla T_k(u))$ a.e. in Ω.

5.4 Renormalized solutions for (E, f) with $f \in L^1$

Now we will show existence of renormalized solution and finish the proof of Theorem 4.1. From the Galerkin approximation of $(E_\varepsilon, f_\varepsilon)$ we can choose sequence $u_\delta = u^k_{\varepsilon(k)}$ with $\delta = \frac{1}{k} > 0$ such that

$$u_\delta \to u \text{ a.e. in } \Omega \tag{78}$$

$$\nabla T_k(u_\delta) \overset{*}{\rightharpoonup} \nabla T_k(u) \text{ weakly-}* \text{ in } L_M(\Omega; \mathbb{R}^d), \tag{79}$$

$$\nabla h(u_\delta) \overset{*}{\rightharpoonup} \nabla h(u) \text{ weakly-}* \text{ in } L_M(\Omega; \mathbb{R}^d) \tag{80}$$

for all $h \in C^1_c(\Omega)$ as $\delta \downarrow 0$.

Testing

$$T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) - \operatorname{div}\big(a(x, \nabla u_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\big) = T_{1/\varepsilon}(f)$$

by $h_l(u_\varepsilon)h(u_\delta)\phi$ where $\phi \in W^{1,\infty}_0(\Omega)$ and $h \in C^1_c(\Omega)$ we get

$$\begin{aligned}
&\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))h_l(u_\varepsilon)h(u_\delta)\phi \,\mathrm{d}x + \int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla \left[h_l(u_\varepsilon)h(u_\delta)\phi\right] \mathrm{d}x \\
&+ \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)) \cdot \nabla \left[h_l(u_\varepsilon)h(u_\delta)\phi\right] \mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f) \left[h_l(u_\varepsilon)h(u_\delta)\phi\right] dx
\end{aligned}$$

and we denote it by

$$I^0_{\varepsilon,\delta,l} + I^1_{\varepsilon,\delta,l} + I^2_{\varepsilon,\delta,l} = I^3_{\varepsilon,\delta,l}.$$

Note that in $I^0_{\varepsilon,\delta,l}$ u_ε can be replaced by $T_{l+1}(u_\varepsilon)$. Then for fixed l the sequence $\{(\beta_\varepsilon(x, T_{l+1}(u_\varepsilon))\}_{\varepsilon>0}$ is bounded a.e. in Ω by $\max(\beta^0(x, l+1), -\beta^0(x, -l-1))$ and, by (4), this function is in $L^1(\Omega)$. It follows that there exists b_l such that

$$\beta_\varepsilon(\cdot, (T_{l+1}(u_\varepsilon))) \rightharpoonup b_l \text{ weakly in } L^1(\Omega) \text{ for fixed } l \in \mathbb{R}. \tag{81}$$

Moreover also

$$T_{1/\varepsilon}(\beta_\varepsilon(\cdot, T_{l+1}(u_\varepsilon))) \rightharpoonup b_l \text{ weakly in } L^1(\Omega) \text{ for fixed } l \in \mathbb{R}.$$

Note that $h_l(u_\varepsilon)h(u_\delta)\phi$ is bounded uniformly (with respect to $\varepsilon > 0$) in $L^\infty(\Omega)$, hence by using (51) and the Egorov Theorem applied to $\{h_l(u_\varepsilon)\}_{\varepsilon>0}$ combined with uniform integrability of $\{T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))h_l(u_\varepsilon)h(u_\delta)\phi\}_{\varepsilon>0}$, we obtain

$$\lim_{\varepsilon\downarrow 0} I^0_{\varepsilon,\delta,l} = \int_\Omega b_l h_l(u)h(u_\delta)\phi \, \mathrm{d}x =: I^0_{\delta,l}. \tag{82}$$

Then Lebesgue Dominated Convergence Theorem provides

$$\lim_{\delta\downarrow 0} I^0_{\delta,l} = \int_\Omega b_l h_l(u)h(u)\phi \, \mathrm{d}x := I^0_l. \tag{83}$$

Since there exists $m > 0$ such that h has compact support in $[-m, m]$, for all $l > m$ we obtain

$$I^0_l = \int_\Omega b_l h(u)\phi \, \mathrm{d}x. \tag{84}$$

The investigation of $\lim\limits_{l\to\infty} I^0_l$ we continue in Section 5.5.

Observe that

$$\begin{aligned} I^1_{\varepsilon,l,\delta} &= \int_\Omega a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla h_l(u_\varepsilon)h(u_\delta)\phi \, \mathrm{d}x \\ &\quad + \int_\Omega a(x, \nabla T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla[h(u_\delta)\phi] \, \mathrm{d}x =: I^{1,1} + I^{1,2}, \end{aligned}$$

where

$$\sup_{\varepsilon\in(0,1]} |I^{1,1}| \le \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)} \sup_{\varepsilon\in(0,1]} \int_{\{l<|u_\varepsilon|<l+1\}} |a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon)| \, \mathrm{d}x \tag{85}$$

Using Corollary 5.2 from (85) it follows that

$$\lim_{l\to\infty} \limsup_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} |I^{1,1}| = 0. \tag{86}$$

By (45), (56) and Lemma 2.2 it follows that $a(x, \nabla T_{l+1}(u_\varepsilon)) \rightharpoonup a(x, \nabla T_{l+1}(u))$ in $L^1(\Omega; \mathbb{R}^d)$. Moreover, $h_l(u_\varepsilon) \to h_l(u)$ a.e. in Ω, $|h_l(u_\varepsilon)| \le 1$ and $\nabla(h(u_\delta)\phi) \in L^\infty(\Omega; \mathbb{R}^d)$. Applying the Egorov Theorem to $\{h_l(u_\varepsilon)\}_{\varepsilon>0}$ and using the uniform integrability of the sequence $\{a(x, \nabla T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla[h(u_\delta)\phi]\}_{\varepsilon>0}$ it follows that

$$\lim_{\varepsilon\downarrow 0} I^{1,2} = \int_\Omega a(x, \nabla T_{l+1}(u))h_l(u)\nabla(h(u_\delta)\phi) \, \mathrm{d}x. \tag{87}$$

Since $a(x, \nabla T_{l+1}(u))h_l(u) \in E_{M^*}(\Omega; \mathbb{R}^d)$, using (80) we can pass to the limit with $\delta \downarrow 0$ and obtain

$$\lim_{\delta\downarrow 0} I^{1,2} = \int_\Omega a(x, \nabla T_{l+1}(u))h_l(u)\nabla(h(u)\phi) \, \mathrm{d}x. \tag{88}$$

For $l > m$, where m is such that supp $h \subset [-m, m]$, from (88) we get

$$I^{1,2} = \int_\Omega a(x, \nabla u)\nabla(h(u)\phi)\,\mathrm{d}x. \tag{89}$$

For ε such that $1/\varepsilon \geq l+1$ we have

$$\begin{aligned} I^2_{\varepsilon,\delta,l} &= \int_\Omega F(T_{l+1}(u_\varepsilon)) \cdot \nabla h_l(u_\varepsilon) h(u_\delta)\phi\,\mathrm{d}x \\ &\quad + \int_\Omega F(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla[h(u_\delta)\phi]\,\mathrm{d}x =: I^{2,1} + I^{2,2}. \end{aligned} \tag{90}$$

Since $\nabla T_{l+1}(u_\varepsilon) \rightharpoonup \nabla T_{l+1}(u)$ weakly in $L^{1+\nu}(\Omega;\mathbb{R}^d)$ and $F(T_{l+1}(u_\varepsilon))h_l'(u_\varepsilon) \to F(T_{l+1}(u))h_l'(u)$ in $L^p(\Omega;\mathbb{R}^d)$ for $p = (1+\nu)'$ we have

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} I^{2,1} &= \lim_{\varepsilon\downarrow 0} \int_\Omega F(T_{l+1}(u_\varepsilon))h_l'(u_\varepsilon)\nabla T_{l+1}(u_\varepsilon)h(u_\delta)\phi\,\mathrm{d}x \\ &= \int_\Omega F(T_{l+1}(u))h_l'(u)\nabla T_{l+1}(u)h(u_\delta)\phi\,\mathrm{d}x. \end{aligned} \tag{91}$$

By Lebesgue Dominated Convergence Theorem

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} I^{2,1} = \int_\Omega F(T_{l+1}(u))h_l'(u)\nabla T_{l+1}(u)h(u)\phi\,\mathrm{d}x. \tag{92}$$

Choosing $m > 0$ such that $\operatorname{supp} h \subset [-m, m]$, T_{l+1} can be replaced by T_m in (92) and since $h_l'(u) = h_l'(T_m(u)) = 0$ for $l+1 > m$ it follows that

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} I^{2,1} = 0 \text{ for } l > m-1. \tag{93}$$

Since $F(T_{l+1}(\cdot))h_l(\cdot)$ is uniformly bounded, the a.e. convergence of $\{u_\varepsilon\}_{\varepsilon>0}$ and the Vitali Theorem provides that $F(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \to F(T_{l+1}(u))h_l(u)$ in $L^p(\Omega;\mathbb{R}^d)$ for any $p \in [1,\infty)$, thus

$$\lim_{\varepsilon\downarrow 0} I^{2,2} = \int_\Omega F(T_{l+1}(u))h_l(u) \cdot \nabla[h(u_\delta)\phi]\,\mathrm{d}x. \tag{94}$$

As $\nabla[h(u_\delta)\phi] \overset{*}{\rightharpoonup} \nabla[h(u)\phi]$ in $L_M(\Omega;\mathbb{R}^d)$ and F is locally Lipschitz continuous, we find that

$$\lim_{\delta\downarrow 0} I^{2,2} = \int_\Omega F(T_{l+1}(u))h_l(u) \cdot \nabla[h(u)\phi]\,\mathrm{d}x. \tag{95}$$

Again, for $m > 0$ such that $\operatorname{supp} h \subset [-m, m]$, T_{l+1} can be replaced by T_m in (95) and $h_l(u) = h_l(T_m(u)) = 1$ for $l > m$. Rewriting (95) we obtain

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} I^{2,2} = \int_\Omega F(u) \cdot \nabla[h(u)\phi]\,\mathrm{d}x \text{ for } l > m. \tag{96}$$

Applying the Lebesgue Dominated Convergence Theorem we get

$$\lim_{l\to\infty}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} I^3_{\varepsilon l,\delta} = \lim_{l\to\infty}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \int_\Omega T_{1/\varepsilon}(f)h_l(u_\varepsilon)h(u_\delta)\phi\,\mathrm{d}x = \int_\Omega fh(u)\phi\,\mathrm{d}x. \tag{97}$$

5.5 Subdifferential argument

Since $\beta(x,\cdot)$ is maximal monotone for almost all $x \in \Omega$, there exists $j : \Omega \times \mathbb{R} \to \mathbb{R}$, such that

$$\beta(x,r) = \partial_r j(x,r) \text{ for all } r \in \mathbb{R}, \text{ a.e. in } \Omega.$$

For $0 < \varepsilon \leq 1$ let us define $j_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$j_\varepsilon(x,r) = \inf_{s\in\mathbb{R}} \{j(x,s) + \frac{1}{2\varepsilon}|r-s|^2\}.$$

According to [21], j_ε has the following properties:

i.) j_ε is a Caratéodory function.

ii.) For any $0 < \varepsilon \leq 1$, $j_\varepsilon(x,r)$ is convex and differentiable with respect to $r \in \mathbb{R}$, moreover

$$\partial_r j_\varepsilon(x,r) = \beta_\varepsilon(x,r) \text{ for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \leq 1 \text{ and a.e. in } \Omega.$$

iii.) $j_\varepsilon(x,r) \uparrow j(x,r)$ pointwise in $\mathbb{R}$ as $\varepsilon \downarrow 0$ and a.e. in Ω.

From *ii.)* it follows that

$$j_\varepsilon(x,r) \geq j_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)) \tag{98}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω. Let $E \subset \Omega$ be an arbitrary measurable set and $\mathbb{1}_E$ its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (98) by $h_l(u_\varepsilon)\mathbb{1}_E$, integrating over Ω and using *iii.)*, we obtain

$$\int_E j(x,r)h_l(u_\varepsilon)\,\mathrm{d}x \geq \int_E j_{\varepsilon_0}(x, T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))\,\mathrm{d}x \tag{99}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min(\varepsilon_0, \frac{1}{l})$. Passing to the limit with $\varepsilon \downarrow 0$, and then with with $\varepsilon_0 \downarrow 0$ in (99) we obtain from (99) and by (81)

$$j(x,r) \geq j(x,u) + b_l(r-u) \tag{100}$$

for all $r \in \mathbb{R}$ almost everywhere in $\{|u| \leq l\}$ and therefore $b_l \in \beta(x,u)$ a.e. in $\{|u| \leq l\}$. Note that $b_l = b_m$ a.e. on $\{|u| \leq m\}$ for all $l \geq m > 0$. Moreover u is measurable and finite a.e. in Ω. Thus the function $b : \Omega \to \mathbb{R}$ defined by $b = b_l$ on $\{|u| \leq l\}$ is well-defined and measurable with $b \in \beta(x,u)$ a.e. in Ω. Next, we plug $h_l(u_\varepsilon)\frac{1}{k}T_k(u_\varepsilon)$ as a test function into $(E_\varepsilon, f_\varepsilon)$. Applying Corollary 5.2 on the diffusion term, the Stokes Theorem on the convection term and neglecting nonnegative terms we can pass to the limit with $\varepsilon \downarrow 0$ and obtain

$$\int_\Omega b_l \frac{1}{k} T_k(u)h_l(u)\,\mathrm{d}x \leq \int_\Omega |f|\,\mathrm{d}x. \tag{101}$$

Thanks to (100), $b_l \mathrm{sign}_0(u_\varepsilon)h_l(u) = |b_l|h_l(u)$ a.e. in Ω. Moreover, $|b_l|h_l(u) \to |b|$ a.e. in Ω for $l \to \infty$. Therefore, passing to the limit with $k \downarrow 0$ in (101) and using the Fatou Lemma we find

$$\int_\Omega |b| \,\mathrm{d}x \le \|f\|_{L^1(\Omega)}, \tag{102}$$

and $b \in L^1(\Omega)$.

5.6 Conclusion of Theorem 4.1

Gathering all convergence results from Subsection 5.4 it follows that u finally satisfies

$$\int_\Omega b_l h(u)\phi + (a(x,\nabla u) + F(u))\nabla(h(u)\phi)\,\mathrm{d}x + \limsup_{\delta\downarrow 0}\limsup_{\varepsilon\downarrow 0} I^{1,1} = \int_\Omega f h(u)\phi\,\mathrm{d}x \tag{103}$$

for all $l > m-1 > 0$, $\phi \in W_0^{1,\infty}(\Omega)$ and $h \in C_c^1(\Omega)$ such that $\mathrm{supp}\, h \subset [-m,m]$, where $I^{1,1}$ is defined in (90). Thanks to (86) and (102) we can pass to the limit in (103) and find (14) for all $\phi \in W_0^{1,\infty}(\Omega)$ and $h \in C_c^1(\Omega)$ arbitrary. Moreover, from (51) and (52) it follows that (u,b) satisfies **(R1)**. From (55) and (56) we have $T_k(u) \in V \cap L^\infty(\Omega)$ and $a(x,\nabla T_k(u)) \in L_{M^*}(\Omega;\mathbb{R}^d)$ for all $k > 0$. Using that the gradients of functions in V can be approximated by smooth functions in the weak-$*$ topology of $L_M(\Omega;\mathbb{R}^d)$ we finally arrive at

$$\int_\Omega b h(u)\phi\,\mathrm{d}x + (a(x,\nabla u) + F(u))\nabla(h(u)\phi)\,\mathrm{d}x = \int_\Omega f h(u)\phi\,\mathrm{d}x \tag{104}$$

for all $\phi \in V \cap L^\infty(\Omega)$ and $h \in C_c^1(\Omega)$, hence (u,b) satisfies (**R2**). Finally, from (49) with classical arguments we obtain (**R3**) and the proof of Theorem 4.1 is completed.

6 Proof of Theorem 4.2

We will need the following

Lemma 6.1 *For $f, \tilde{f} \in L^1(\Omega)$ let (u,b), $(\tilde{u},\tilde{b})$ be renormalized solutions to (E,f) and $(E,\tilde{f})$ respectively. Then*

$$\int_\Omega (b-\tilde{b})\mathrm{sign}_0^+(u-\tilde{u})\,\mathrm{d}x \le \int_\Omega (f-\tilde{f})\mathrm{sign}_0^+(u-\tilde{u})\,\mathrm{d}x. \tag{105}$$

Proof: The proof follows the same lines as in the classical L^p and the $L^{p(\cdot)}$ setting (see [43]). For $\delta > 0$, let H_δ^+ be a Lipschitz approximation of the sign_0^+ -function. Since (u,b), $(\tilde{u},\tilde{b})$ are renormalized solutions, it follows that $T_{l+1}(u), T_{l+1}(\tilde{u}) \in V \cap L^\infty(\Omega)$ for all $l > 0$. Hence $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ is in $V \cap L^\infty(\Omega)$ for all $\delta, l > 0$ and therefore an admissible test function. Now, we choose $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ as a test function in the renormalized formulation with $h = h_l$ for (u,b) and for $(\tilde{u},\tilde{b})$) respectively. Subtracting the resulting equalities, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 = I_{l,\delta}^6, \tag{106}$$

where $K := \{0 < T_{l+1}(u) - T_{l+1}(\tilde{u}) < \delta\}$ and

$$\begin{aligned}
I^1_{l,\delta} &= \int_\Omega \big(bh_l(u) - \tilde{b}h_l(\tilde{u})\big) H^+_\delta\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x, \\
I^2_{l,\delta} &= \int_\Omega \big(h_l'(u)a(x,\nabla u)\cdot\nabla u - h_l'(\tilde{u})a(x,\nabla\tilde{u})\cdot\nabla\tilde{u}\big) H^+_\delta\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x, \\
I^3_{l,\delta} &= \frac{1}{\delta}\int_K \big(h_l(u)a(x,\nabla u) - h_l(\tilde{u})a(x,\nabla\tilde{u})\big)\cdot\nabla\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x, \\
I^4_{l,\delta} &= \int_\Omega \big(h_l'(u)F(u)\cdot\nabla u - h_l'(\tilde{u})F(\tilde{u})\cdot\nabla\tilde{u}\big) H^+_\delta\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x, \\
I^5_{l,\delta} &= \frac{1}{\delta}\int_K \big(h_l(u)F(u) - h_l(\tilde{u})F(\tilde{u})\big)\cdot\nabla\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x, \\
I^6_{l,\delta} &= \int_\Omega \big(fh_l(u) - \tilde{f}h_l(\tilde{u})\big) H^+_\delta\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x.
\end{aligned}$$

Using the same arguments as in [43], i.e. neglecting the nonnegative part of $I^3_{l,\delta}$ and using that F is locally Lipschitz continuous, we can to pass to the limit with $\delta \downarrow 0$. Using the energy dissipation condition **(R3)** we can pass to the limit with $l \to \infty$ and find (105).

Now we are in the position to give the proof of Theorem 4.2:
Assuming $f = \tilde{f}$, from Lemma 6.1 we get

$$\int_\Omega (b - \tilde{b})\mathrm{sign}_0^+(u - \tilde{u})\,\mathrm{d}x \le 0, \tag{107}$$

hence $(b - \tilde{b})\mathrm{sign}_0^+(u - \tilde{u}) = 0$ almost everywhere in Ω. Now, let us write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 := \{x \in \Omega :\ \mathrm{sign}_0^+(u(x) - \tilde{u}(x)) = 0\}$, $\Omega_2 := \{x \in \Omega :\ (b(x) - \tilde{b}(x)) = 0\}$. Since $r \mapsto \beta(x,r)$ is strictly increasing for a.e. $x \in \Omega$, we can define the function $\beta_x^{-1} : \mathbb{R} \to \mathbb{R}$ such that $\beta_x^{-1}(r) = s$ for all $(r,s) \in \mathbb{R}^2$ such that $r \in \beta(x,s)$ for almost every $x \in \Omega$. For a.e. $x \in \Omega_2$ we have $b(x) = \tilde{b}(x)$, hence $u(x) = \beta_x^{-1}(b(x)) = \beta_x^{-1}(\tilde{b}(x)) = \tilde{u}(x)$. Therefore, $u(x) = \tilde{u}(x)$ a.e. in Ω_2 and $\mathrm{sign}_0^+(u - \tilde{u}) = 0$ a.e. in Ω. Interchanging the roles of u and $\tilde{u}$ and repeating the arguments, we get $\mathrm{sign}_0^+(\tilde{u} - u) = 0$ a.e. in Ω and we finally arrive at $u = \tilde{u}$ a.e. in Ω. Now, we write the renormalized formulation for (u,b) and $(\tilde{u},\tilde{b})$ respectively. Subtracting the resulting equalities, we obtain

$$\int_\Omega (b - \tilde{b})h(u)\varphi\,\mathrm{d}x = 0$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in C_c^\infty(\Omega)$. Choosing $h(u) = h_l(u)$ and passing to the limit with $l \to \infty$ we find $b = \tilde{b}$ a.e. in Ω.

7 Proof of Proposition 4.3

The proof of Proposition 4.3 follows the same lines as in [43]. For the sake of completeness we will repeat the arguments.

From Remark 3.3 it follows that it suffices to prove $u \in L^\infty(\Omega)$:

Note that for $\varepsilon, k > 0$, $h_l(u)\frac{1}{\varepsilon}T_\varepsilon(u - T_k(u))$ is an admissible test function in (14). Neglecting positive terms and passing to the limit with $l \to \infty$, we apply (1) to obtain

$$\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} c_a M(x,\nabla u)\,\mathrm{d}x \le \left(\|f\|_d\,(\phi(k))^{(d-1)/d} + \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\|a_0\|_\infty\right), \quad (108)$$

where $\phi(k) := |\{|u| > k\}|$ for $k > 0$. Now we use similar arguments as in [14]. We apply the continuous embedding of $W_0^{1,1}(\Omega)$ into $L^{d/(d-1)}(\Omega)$ and the Hölder inequality to get

$$\frac{1}{\varepsilon C_d}\|T_\varepsilon(u - T_k(u))\|_{\frac{d}{d-1}} \le \left(\frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\right)^{1/(1+\nu)'}\left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |\nabla u|^{1+\nu}\right)^{1/(1+\nu)}, \quad (109)$$

where $C_d > 0$ is the constant coming from the Sobolev embedding. From (3) it follows that

$$\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |\nabla u|^{1+\nu}\,\mathrm{d}x \le \frac{1}{c\cdot c_a\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} c_a M(x,\nabla u), \quad (110)$$

hence from (108), (109) and (110) we deduce

$$\frac{1}{\varepsilon C_d}\|T_\varepsilon(u - T_k(u))\|_{\frac{d}{d-1}} \le$$

$$\left(\frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\right)^{1/(1+\nu)'}\left(\frac{1}{c\cdot c_a}\left(\|f\|_d\,(\phi(k))^{(d-1)/d} + \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon}\|a_0\|_\infty\right)\right)^{1/(1+\nu)}. \quad (111)$$

From (111) and Young's inequality with $\alpha > 0$ it follows that

$$\frac{1}{C_d C}(\phi(k+\varepsilon))^{(d-1)/d} - \frac{\alpha^{1+\nu}}{(1+\nu)C\cdot c\cdot c_a}\left(\|f\|_d\,(\phi(k))^{(d-1)/d}\right) - \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon} \le 0, \quad (112)$$

where

$$C := \left(\frac{1}{\alpha^{(1+\nu)'}(1+\nu)'} + \frac{\|a_0\|_\infty}{c\cdot c_a}\frac{\alpha^{1+\nu}}{1+\nu}\right) > 0.$$

The mapping $(0,\infty) \ni k \to \phi(k)$ is non-increasing and therefore of bounded variation, hence it is differentiable almost everywhere on $(0,\infty)$ with $\phi' \in L^1_{\mathrm{loc}}(0,\infty)$. Since it is also continuous from the right, we can pass to the limit with $\varepsilon \downarrow 0$ in (112) to find

$$C''(\phi(k))^{(d-1)/d} + \phi'(k) \le 0 \quad (113)$$

for almost every $k > 0$ and $\alpha > 0$ chosen small enough such that

$$C'' := \left(\frac{1}{C_d C} - \frac{\alpha^{1+\nu}}{(1+\nu)C\cdot c\cdot c_a}\|f\|_d\right) > 0.$$

Now, the conclusion of the proof follows by contradiction. We assume that $\phi(k) > 0$ for each $k > 0$. For $k > 0$ fixed, we choose $k_0 < k$. Multiplying (113) by $\frac{1}{d}(\phi(k))^{-(d-1/d)}$ it follows that

$$\frac{1}{d}C'' + \frac{d}{ds}\left((\phi(s))^{(1/d)}\right) \leq 0 \tag{114}$$

for almost all $s \in (k_0, k)$. The left hand side of (114) is in $L^1(k_0, k)$, hence we integrate (114) over $[k_0, k]$. Moreover, since ϕ is non-increasing, integrating (114) over (k_0, k) we get

$$(\phi(k))^{1/d} \leq \phi(k_0)^{1/d} + \frac{1}{d}C''(k_0 - k). \tag{115}$$

Thanks to the second term on the right-hand side of (115), we conclude that there exists $k_1 > k_0$ such that $(\phi(k))^{1/d} \leq 0$ for all $k > k_1 > k_0$. Therefore $\phi(k) = 0$ for all $k > k_1 > k_0$ and the assertion follows.

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