Multi-dimensional scalar conservation laws with fluxes discontinuous in the unknown and the spatial variable

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MULTI-DIMENSIONAL SCALAR CONSERVATION LAWS WITH FLUXES DISCONTINUOUS IN THE UNKNOWN AND THE SPATIAL VARIABLE

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ABSTRACT. The paper deals with a scalar conservation law in an arbitrary dimension d with a discontinuous flux. The flux is supposed to be a discontinuous function in the spatial variable x and in an unknown function u. Under some additional hypothesis on the structure of possible discontinuities, we formulate an appropriate notion of entropy solution and establish its existence and uniqueness. The framework for proving the existence and uniqueness of entropy weak solutions is provided by the studies on entropy measure-valued solutions and may be viewed as a corollary of the uniqueness theorem for entropy measure-valued solutions.

1. INTRODUCTION

We focus on the Cauchy problem for a scalar hyperbolic conservation law

$$(1.1) \qquad \begin{aligned} u_{,t} + \operatorname{div} \mathbf{F}(x, u) &= 0 \quad \text{ in } \mathbb{R}_+^{d+1}, \\ u(0, \cdot) &= u_0 \quad \text{ in } \mathbb{R}^d, \end{aligned}$$

where $\mathbb{R}_+^{d+1} := (0, \infty) \times \mathbb{R}^d$, d denotes an arbitrary spatial dimension, $u : \mathbb{R}_+^{d+1} \to \mathbb{R}$ is an unknown and $\mathbf{F} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a given flux of the quantity u. In addition we assume that u vanishes as $|x| \to \infty$. Our main goal in the paper is to identify a class of fluxes $\mathbf{F}$ for which we can develop "well-posedness" of the problem (1.1). Indeed, we not only want to establish the proper notion of solution to (1.1) and its existence but also its uniqueness in an a priori chosen class. In particular, our primary motivation is that such class is equivalent to Kružkov entropy solution in case that $\mathbf{F}$ is a sufficiently smooth function. For brevity, we recall that for smooth $\mathbf{F}$ a weak solution to (1.1) is called the Kružkov entropy solution if it satisfies for all $k \in \mathbb{R}$ the following entropy inequality in the distributional sense in $\mathbb{R}_+^{d+1}$

$$(1.2) \qquad \begin{aligned} |u - k|_{,t} + \operatorname{div} \left(\operatorname{sgn}(u - k)(\mathbf{F}(x, u) - \mathbf{F}(x, k)) \right) \\ + \operatorname{sgn}(u - k) \operatorname{div} \mathbf{F}(x, k) \leq 0. \end{aligned}$$

More generally, it can be shown that (1.2) is equivalent to the following entropy inequality

$$(1.3) \qquad E(u)_{,t} + E'(u)\operatorname{div}_x\mathbf{F}(x,u) + \operatorname{div}\mathbf{Q}(x,u) - \operatorname{div}_x\mathbf{Q}(x,u) \le 0,$$

where E is an arbitrary smooth convex function (entropy) and $\mathbf{Q}$ (flux) satisfies

$$\partial_u\mathbf{Q}(x,u) = E'(u)\partial_u\mathbf{F}(x,u).$$

In (1.3) we use

$$\operatorname{div}_x\mathbf{F}(x,u) := \sum_{i=1}^d \frac{\partial\mathbf{F}_i(x,u)}{\partial x_i} \quad \operatorname{div}\mathbf{F}(x,u) := \sum_{i=1}^d \frac{\partial\mathbf{F}_i(x,u)}{\partial x_i} + \frac{\partial\mathbf{F}_i(x,u)}{\partial u}\frac{\partial u}{\partial x_i}.$$

Moreover, to simplify the presentation we frequently use the following notations: Small letters in italics always denote the scalar functions mapping $\mathbb{R}_+^{d+1}$, $\mathbb{R}^d$ respectively on $\mathbb{R}$, i.e., $v(t,x) : \mathbb{R}_+^{d+1} \to \mathbb{R}$ and $v(x) : \mathbb{R}^d \to \mathbb{R}$. The real function of one real variable will be always denoted by capital letter, i.e., $P : \mathbb{R} \to \mathbb{R}$. On the other hand vector-valued function of one real variable is denoted by capital bold letter, i.e., $\mathbf{Q} : \mathbb{R} \to \mathbb{R}^d$. Similarly vector-valued function of (t,x) or x respectively will be denoted by small bold letter, i.e., $\boldsymbol{v}(t,x) : \mathbb{R}_+^{d+1} \to \mathbb{R}^d$ and $\boldsymbol{v}(x) : \mathbb{R}^d \to \mathbb{R}^d$.

In the very beginning we proposed that we are interested in generalization of the notion of Kružkov entropy solution to more general class of possible fluxes $\mathbf{F}$. For this purpose we shall follow three recent results and attempt to combine them in a proper way to develop a unified theory. First fundamental result we refer to is the paper of Audusse and Perthame [4]. The authors dealt with $\mathbf{F}$ being possibly discontinuous with respect to x and proposed for $d = 1$ the generalized entropy inequality (1.2), where instead of taking constant k, they considered k being a function of x such that

$$(1.4) \qquad\qquad\qquad \operatorname{div}\mathbf{F}(x,k(x)) = 0.$$

In this case we see that the last term in (1.2), which is not well-defined in case of non-smooth $\mathbf{F}$, vanishes and they showed that for a sufficiently large class of k's satisfying (1.4) the uniqueness of solutions to (1.1)–(1.2) can be proved. More precisely, we recall here under which assumptions one is able to find a rich class of k's fulfilling (1.4). In [4] the authors assumed that there exist continuous functions f, g; $f(u) \to \infty$ as $|u| \to \infty$ such that[1]

$$(1.5) \qquad \mathbf{F}(x,u) \qquad\qquad\qquad \text{is Carathéodory,}$$

$$(1.6) \qquad f(u) \le |\mathbf{F}(x,u)| \le g(u),$$

$$(1.7) \qquad \mathbf{F}(x,u) \qquad\qquad\qquad \text{is for a.a. } x \text{ locally Lipschitz one to one.}$$

Under such conditions the authors were able to prove uniqueness (in a given class) of the solution to (1.1), that is equivalent to the Kružkov entropy solution in case that $\mathbf{F}$ is smooth.

Later Panov [13] generalized the method developed in [4] in the following way. He assumed that there exists a Carathéodory function $\theta(x,u) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ that is for almost all x strictly increasing with respect to u and for which there exist continuous functions f, g fulfilling $f(u) \to \infty$ as $|u| \to \infty$ such that

$$(1.8) \qquad\qquad\qquad f(u) \le |\theta(x,u)| \le g(u).$$

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[1]The assumption (1.7) can be slightly generalized, see p. 5 in [4] or the discussion below.

In addition, the author assumed that $\mathbf{F}$ is of the form

$$(1.9) \qquad \mathbf{F}(x, u) = \mathbf{G}(\theta(x, u)),$$

where $\mathbf{G} \in \mathcal{C}(\mathbb{R})$. In this setting it is shown in [13] that it is convenient (and in case $d = 1$ also equivalent) to replace (1.1)–(1.2) by the following: Let $\eta(x, v)$ be the inverse to θ, i.e., $\theta(x, \eta(x, v)) = v$, then for smooth fluxes $\mathbf{F}$ solving (1.1)–(1.2) is equivalent to finding v such that it solves for all $k \in \mathbb{R}$ the following system in the sense of distribution

$$(1.10) \qquad \eta(x, v)_{,t} + \operatorname{div} \mathbf{G}(v) = 0 \qquad \text{in } \mathbb{R}_+^{d+1},$$

$$(1.11) \qquad \eta(x, v(x, 0)) = u_0(x) \qquad \text{in } \mathbb{R}^d,$$

$$(1.12) \qquad |\eta(x, v) - \eta(x, k)|_{,t} + \operatorname{div}(\operatorname{sgn}(v - k)(\mathbf{G}(v) - \mathbf{G}(k))) \le 0, \qquad \text{in } \mathbb{R}_+^{d+1}.$$

The proof of existence and uniqueness of solutions is presented only for the dimension one, i.e., $d = 1$ in [13]. In addition, the author briefly mentions that the method can be generalized to the multi-dimensional case provided that the flux satisfies additional constrains on the character of continuity. However, such a generalization for $d > 1$ is not rigorously proved in [13].

Finally, in [6] the authors considered fluxes being independent of x but discontinuous with respect to u. They showed that for jump continuous fluxes, one can find nondecreasing U such that $\mathbf{F} \circ U$ is continuous and define a notion of entropy solution that is equivalent to the Kružkov entropy solution for smooth fluxes in the following way: A function v is an entropy weak solution to (1.10)–(1.12) if it solves for all $k \in \mathbb{R}$ the following system in the sense of distribution

$$(1.13) \qquad U(v)_{,t} + \operatorname{div} \mathbf{F}(U(v)) = 0 \qquad \text{in } \mathbb{R}_+^{d+1},$$

$$(1.14) \qquad U(v(x, 0)) = u_0(x) \quad \text{in } \mathbb{R}^d,$$

$$(1.15) \qquad |U(v) - U(k)|_{,t} + \operatorname{div}(\operatorname{sgn}(v - k)(\mathbf{F}(U(v)) - \mathbf{F}(U(k)))) \le 0, \qquad \text{in } \mathbb{R}_+^{d+1}.$$

The authors showed in [6] that for $\mathbf{F}$ being jump continuous function that is additionally Hölder continuous at 0 there exists just one entropy weak solution to (1.13)–(1.15).

Thus, our main goal in the paper and also our strategy is to combine methods developed in [4], [13] and [6] and to develop a theory that covers both possible discontinuities of a flux, i.e. discontinuities with respect to u and x. We rigorously treat the multi-dimensional case, which in particular in the dependence in space variable essentially requires new tools (e.g. kinetic formulation, cf. [8]). Thus, following above mentioned papers we assume here that $\mathbf{F}(x, u) = \mathbf{G}(\theta(x, u))$ where

(A1) $\mathbf{G}(v)$ is jump continuous[2]

(A2) θ is a Carathéodory strictly increasing function such that $\theta(x, 0) = 0$ and there exists a Carathéodory function $\eta(x, v)$ such that $\theta(x, \eta(x, v)) = v$ for all $v \in \mathbb{R}$ and a.a. $x \in \mathbb{R}^d$.

(A3) there exist continuous functions h_1 and h_2 such that for all $x \in \mathbb{R}^d$

$$h_1(u) \le |\theta(x, u)| \le h_2(u),$$

and such that for all $R > 0$ there exists C_R so that $h_2(u) \le C_R h_1(u)$ for all $|u| \le R$; in addition we require that $\lim_{|u| \to \infty} h_1(u) = \infty$.

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[2]A function $\mathbf{G}$ is called jump continuous if for all $s \in \mathbb{R}$ there exists $\lim_{t \to s_\pm} \mathbf{G}(s)$ and there is at most countable set where $\mathbf{G}$ is not continuous.

(A4) there exists $1 \le p \le \frac{d}{d-1}$ and constants $R_\infty > 0$ and $C_\infty > 0$ such that for all $x \in \mathbb{R}^d \setminus B_{R_\infty}(0)$

$$|\mathsf{G}(s)|^p \le C_\infty |\eta(x, s)|$$

Here, the assumption **(A1)** is inspired by [6]. The assumptions **(A2)** and **(A3)** were introduced in [13]. The requirement that $\theta(x, 0) = 0$ is technical. Although omitting it is possible, but generates additional terms in the proof of Lemma 2.2, which decreases the readability of the paper. Finally, the last assumption **(A4)** combines the requirements on the behavior of $\mathsf{F}(x, \cdot)$ near zero and behavior of $\mathsf{F}(\cdot, u)$ near infinity. Indeed, one could alternatively assume that G is $\alpha-$Hölder continuous in zero with an appropriate exponent α and that $|\theta(x, u)| \le C_\infty |u|$ for $|x| \to \infty$ and/or to introduce other assumptions on the behavior of F near ∞ for $u = 0$.

Besides the results described above we shall mention various other approaches to scalar conservation laws including discontinuities of fluxes. According to our knowledge the case of discontinuity in both variables x and u has not been considered. All of the studies on fluxes discontinuous in x confine to one-dimensional case. We recall here an overview paper of Risebro, cf. [16] concerned mostly with a front tracking method. The author describes various motivations for considering such problems.

In the works basing upon the classical Kružkov entropies the problem of giving sense to the last term in the entropy inequality (1.2) arises. Some approaches require introducing an interface entropy condition, see e.g. [1], which consequently entails the assumptions on the total variation of the flux function. Another approach is presented by Bachmann and Vovelle in [5]. The authors use the kinetic formulation of conservation laws. The very particular form of the flux function is considered. Both of the mentioned papers concern the one-dimensional setting and continuous dependence on u.

The separate issue are discontinuities of the flux with respect to the unknown function u. This topic was studied by Ammar and Wittbold, cf. [3] for the fluxes independent of x in a multi-dimensional case. The authors postulate the form of equation similar to the form obtained in [6] after the transformation. The proof bases upon the comparison principle and the entropy inequality involves semi Kružkov entropies, namely $E(u, k) = (u - k)^+$.

Among the variety of entropy conditions for the problems with $x-$discontinuous fluxes an interesting question arises which of them have a physical meaning. The classical framework for answering this question is derivation of target equations from primitive equations. With this strategy in mind, we shall be particularly interested in the entropy conditions of Audusse and Perthame (and hence also the ones of Panov), due to the rigorous derivation as a hydrodynamic limit from the particle system, see [7]. Also, it is important to mention at the very beginning that the entropy conditions considered here, are equivalent for smooth fluxes to the Kružkov entropies as will be demonstrated in Section 2.

We organize the paper as follows: Section 2 contains the discussion on relations between various notions of solutions. The one-dimensional case essentially varies from a multi-dimensional case, hence we discuss them separately to underline the differences. Section 3 concerns the existence of entropy measure-valued solutions.

We define the entropy measure-valued solutions (Definition 3.1) and state the theorems on existence and uniqueness of solutions (Theorems 3.1 and 3.2). The crucial tool for these results is the so-called averaged contraction property formulated in Lemma 3.1. The meaning of uniqueness of entropy measure-valued solutions requires the discussion on its appropriate interpretation. Two entropy measure-valued solutions are meant to be unique in the sense specified by (3.25)–(3.26), more precisely they are unique up to the level sets of the function U. The results on measure-valued solutions are essential for passing to entropy weak solutions. The main difference in comparison to the result on $x-$independent fluxes (cf. [6]) concentrates in showing the well-posedness of entropy measure-valued solutions and hence these steps are described in details. The passage from the level of measure-valued to weak solutions follows the same lines. We comment on this issue in Section 4, where the reader can find the definition of entropy weak solutions (Definition 4.1) and the statement of the theorem (Theorem 4.1).

2. Equivalent notions of entropy solutions

In this section, we define precisely what we mean by the Kružkov entropy solution to (1.1)–(1.2) or to (1.3) for smooth fluxes F and show several equivalent notions to such a solution for F satisfying in addition $(\mathbf{A1})$–$(\mathbf{A3})$. There is a straightforward connection between the Kružkov entropy solution and the entropy weak solution in the meaning of present considerations (see $(\mathbf{N4})$ below) in case of one dimension. Therefore we treat this situation separately. First, we formulate the main result of this section and then provide two separate proofs. The first one for dimension one, that is only the combination of the previous results and the second one for any dimension that is based on the kinetic formulation.

Lemma 2.1. *Let* F *satisfy* $(\mathbf{A1})$–$(\mathbf{A3})$ *and assume that* G, η *and* θ *are smooth. In addition let* $U : \mathbb{R} \to \mathbb{R}$ *be smooth, strictly increasing one-to-one mapping with a smooth inverse* G *and let* $u_0 \in L_{loc}^\infty(\mathbb{R}^d)$. *Assume that* $u \in L_{loc}^\infty(\mathbb{R}^{d+1})$ *be given and define*

$$(2.1) \qquad v(t,x) := \theta(x, u(t,x)),$$
$$(2.2) \qquad g(t,x) := G(v(t,x)).$$

Then the following statements are equivalent.
$(\mathbf{N1})$ *For all* $k \in \mathbb{R}$ *and all nonnegative* $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$ *there holds*

$$(2.3) \quad \begin{aligned} &\int_{\mathbb{R}_+^{d+1}} |u(t,x) - k| \psi_{,t}(t,x) - \operatorname{sgn}(u(t,x) - k) \operatorname{div} \mathsf{F}(x,k) \psi(t,x) \; dx \; dt \\ &+ \int_{\mathbb{R}_+^{d+1}} \operatorname{sgn}(u(x,t) - k)(\mathsf{F}(x,u(x,t)) - \mathsf{F}(x,k)) \cdot \nabla \psi(x,t) \; dx \; dt \\ &+ \int_{\mathbb{R}^d} |u_0(x) - k| \psi(0,x) \; dx \geq 0. \end{aligned}$$

(N2) *(Only valid for $d = 1$). For all $k_\alpha(x)$ satisfying $F(x, k_\alpha(x)) = \alpha$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^2)$ there holds*

$$\int_{\mathbb{R}_+^{d+1}} |u(t,x) - k_\alpha(x)|\psi_{,t}(t,x)\,dx\,dt$$

$$(2.4) \qquad + \int_{\mathbb{R}_+^{d+1}} \text{sgn}(u(t,x) - k)(\mathbf{F}(x, u(t,x)) - \mathbf{F}(x, k_\alpha(x))) \cdot \partial_x \psi(t,x)\,dx\,dt$$

$$+ \int_{\mathbb{R}^d} |u_0(x) - k|\psi(0,x)\,dx \geq 0.$$

(N3) *For all $k \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$ there holds*

$$\int_{\mathbb{R}_+^{d+1}} |\eta(x, v(t,x)) - \eta(x, k)|\psi_{,t}(t,x)\,dx\,dt$$

$$(2.5) \qquad + \int_{\mathbb{R}_+^{d+1}} \text{sgn}(v(t,x) - k)(\mathbf{G}(v(t,x)) - \mathbf{G}(k)) \cdot \nabla\psi(t,x)\,dx\,dt$$

$$+ \int |u_0(x) - \eta(x, k)|\psi(0,x)dx \geq 0.$$

(N4) *For all $k \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$ there holds*

$$\int_{\mathbb{R}_+^{d+1}} \text{sgn}(U(g(t,x)) - U(k))(\mathbf{G}(U(g(t,x))) - \mathbf{G}(U(k))) \cdot \nabla\psi(t,x)\,dx\,dt$$

$$(2.6) \qquad + \int_{\mathbb{R}_+^{d+1}} |\eta(x, U(g(t,x))) - \eta(x, U(k))|\psi_{,t}\,dx\,dt$$

$$+ \int_{\mathbb{R}^d} |u_0(x) - \eta(x, U(k))|\psi(0,x)\,dx \geq 0.$$

2.1. **One-dimensional case.** Here we present the proof of Lemma 2.1 for $d = 1$. According to [7, Proposition 3.1] the solution u in the sense of **(N1)** also satisfies the entropy inequality with adapted entropies introduced by Audusse and Perthame, namely **(N2)**. The solutions in the sense of Audusse-Perthame are equivalent to solutions defined by Panov **(N3)**, which is proved in [13, Theorem 1]. Then showing the equivalence of **(N3)** and **(N4)** presents no difficulties for any dimension $d \geq 1$. Indeed, if for any k we define $\tilde{k} := U(k)$, we can use $\tilde{k}$ in (2.5) and by using the definition of g we get exactly (2.6). The opposite implication is proved in the same way since U is one-to-one mapping.

2.2. **Multi-dimensional case.** Here we give the proof of Lemma 2.1 for the general case $d > 1$. The restriction of Audusse-Perthame solutions to one-dimensional setting prevents to follow the same lines of the equivalence proof between various notions of solutions for smooth fluxes in a multi-dimensional case. The problem of finding a solution to the stationary problem cannot be solved as obviously as in the one-dimensional case. We proceed here differently, namely we pass from the Kružkov entropy solutions to the kinetic formulation and then from the kinetic formulation we conclude the existence of solutions to **(N4)**.

2.2.1. *Kinetic formulation.* The concept of kinetic formulation for scalar conservation laws with the flux independent of x was introduced in [11, 12], see also [14, 15].

The extension for x−dependent fluxes was done in [8]. Define $\chi : \mathbb{R}^2 \to \{1, -1, 0\}$ as follows

$$(2.7) \qquad \chi(z, u) := \begin{cases} 1 & \text{if } 0 < z < u, \\ -1 & \text{if } u < z < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in the equivalence between Kružkov entropy solutions and kinetic formulation, which is provided by [8, Theorem 3]. Before recalling this result we introduce the notation and formulate the assumptions for the regularity of the flux function. Let $\boldsymbol{a} := (a_1, \ldots, a_{d+1})$ be defined as

$$(2.8) \qquad a_i(x, z) := \frac{\partial \mathbf{F}_i(x, z)}{\partial z},$$

$$(2.9) \qquad a_{d+1}(x, z) := -\operatorname{div}_x \mathbf{F}(x, z),$$

$$(2.10) \qquad a_{d+1}(x, 0) := 0 \quad \forall x \in \mathbb{R}^d.$$

Recall that to provide the existence of a solutions to (1.1) we need the following assumptions to be satisfied (see [10])

$$(2.11) \qquad a_i \in \mathcal{C}^1(\mathbb{R}^{d+1}), \ \partial_j a_i \in L^\infty_{\text{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^d)), \quad 1 \le i, j \le d+1,$$

$$(2.12) \quad \exists\, C > 0\ \forall (x, z) \in \mathbb{R}^{d+1}, \ |a_i(x, z)| \le C(1 + |x| + |z|), \quad 1 \le i \le d+1,$$

where $\partial_j := \partial_{x_j}$ for $1 \le j \le d$ and $\partial_{d+1} := \partial_z$.

Next, the key observation that relates Kružkov entropy solution with the kinetic formulation is the following.

Theorem 2.1. *Let $u \in \mathcal{C}([0, \infty); L^1(\mathbb{R}^d)) \cap L^\infty_{\text{loc}}((0, \infty); L^\infty(\mathbb{R}^d))$. Under assumptions (2.8)–(2.12), u is an entropy solution to (1.1) if and only if there exists a nonnegative measure $m(t, x, z)$ such that $m((0, T) \times \mathbb{R}^{d+1}) < \infty$ for all $T > 0$, and if $\chi(z, u(t, x))$ satisfies the following kinetic equation in the sense of distributions*

$$(2.13) \quad \begin{aligned} \partial_t \chi(z, u(t, x)) + \operatorname{div}_{x,z}[\boldsymbol{a}(x, z)\chi(z, u(t, x))] &= \partial_z m(t, x, z) \quad \text{in } \mathbb{R}^{d+1}_+ \times \mathbb{R}, \\ \chi(z, u(0, x)) &= \chi(z, u_0(x)) \quad \text{in } \mathbb{R}^d \times \mathbb{R}. \end{aligned}$$

Proof. See [8, Theorem 3]. $\qquad\square$

Finally, we prove the following statement that is the key tool for proving Lemma 2.1.

Lemma 2.2. *Assume that $\mathbf{F}$ satisfies (A1)–(A3) and that $\mathbf{G}$, θ and η are smooth. A function u is a kinetic solution to (1.1) if and only if there exists a function $v \in L^\infty_{loc}(\mathbb{R}^{d+1}_+)$ given by (2.1) that satisfies (N3).*

Having Lemma 2.2 enables proving Lemma 2.1. Indeed, by Theorem 2.1, the Kružkov and the kinetic solutions are equivalent. Next, by Lemma 2.2 we see that **(N3)** and kinetic solution is equivalent and consequently **(N1)** and **(N3)** are equivalent. To show also the equivalence of **(N3)** and **(N4)** is then straightforward.

Proof of Lemma 2.2. We only prove here that existence of a kinetic solution implies **(N3)**. The opposite direction can be proved similarly. First note that since $\theta(x, 0) = 0$ then also $\eta(x, 0) = 0$. We begin the proof by changing the variable z in (2.13). Hence, let $w := \theta(x, z)$ (which is equivalent to $z = \eta(x, w)$) and define $v(t, x) := \theta(x, u(t, x))$ (which also implies $u = \eta(x, v)$).

Assume that $E \in \mathcal{C}^\infty(\mathbb{R})$ and $\psi(t, x) \in \mathcal{D}(\mathbb{R}^{d+1})$ are arbitrary. Since $\theta(x, z)$ is smooth we know that $E'(\theta(x, z))$ is smooth as well. Denote $R := \|u\|_{L^\infty(\text{supp }\psi)}$

and let $\zeta \in \mathcal{D}(\mathbb{R})$ be such that $\zeta \equiv 1$ in $[-R, R]$. Finally, we set $\varphi(t, x, z) :=$
$E'(\theta(x, z))\zeta(z)\psi(t, x)$ as a test function in (2.13) (note that it is a smooth compactly
supported function) to obtain the following identity

$$
\begin{aligned}
(2.14) \quad & \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} \chi(z, u(t, x))\varphi_{,t}(t, x, z) \, dz \, dx \, dt \\
& + \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} \chi(z, u(t, x))\boldsymbol{a}(x, z) \cdot \nabla_{x,z}\varphi(t, x, z) \, dz \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} \partial_z \varphi(t, x, z) \, dm(t, x, z) - \int_{\mathbb{R} \times \mathbb{R}^d} \chi(z, u_0(x))\varphi(0, x, z) \, dz \, dx
\end{aligned}
$$

We shall evaluate all the terms appearing in (2.14). For the first term, we use the
definition of χ and φ, the Fubini theorem, the definition of ζ and the definition of
$v(t, x)$ to get

$$
\begin{aligned}
(2.15) \quad & \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} \chi(z, u(t, x))\varphi_{,t}(t, x, z) \, dz \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+} \int_0^{u(t,x)} E'(\theta(x, z))\zeta(z)\psi_{,t}(t, x) \, dz \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+} \int_0^{u(t,x)} E'(\theta(x, z))\psi_{,t}(t, x) \, dz \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+} \int_0^{v(t,x)} E'(w)\partial_w \eta(x, w)\psi_{,t}(t, x) \, dw \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+} (Q_1(x, v(t, x)) - Q_1(x, 0))\psi_{,t}(t, x) \, dx \, dt \\
& = \int_{\mathbb{R}^{d+1}_+} (Q_1(x, v(t, x))\psi_{,t}(t, x) \, dx \, dt + \int_{\mathbb{R}^d} Q_1(x, 0))\psi(0, x)dx,
\end{aligned}
$$

where we defined Q_1 by the relation

$$
(2.16) \qquad \partial_w Q_1(x, w) = E'(w)\partial_w \eta(x, w).
$$

With the last term on the right hand side of (2.14) we proceed similarly to get

$$
\begin{aligned}
(2.17) \quad & \int_{\mathbb{R} \times \mathbb{R}^d} \chi(z, u_0(x))\varphi(0, x, z) \, dz \, dx \\
& = \int_{\mathbb{R}^d} \int_0^{u_0(x)} E'(\theta(x, z))\psi(0, x) \, dz \, dx \\
& = \int_{\mathbb{R}^d} (Q_1(x, v_0(x)) - Q_1(x, 0))\psi(0, x) \, dx,
\end{aligned}
$$

where we defined

$$
v_0(x) := \theta(x, u_0(x)).
$$

To evaluate the second term on the left hand side of (2.14), we use (2.8)–(2.9), the fact that $\mathbf{F}(x,z) = \mathbf{G}(\theta(x,z))$ and the definition of φ to observe

$$\mathbf{a} \cdot \nabla_{x,z}\varphi = \sum_{i=1}^{d} \partial_z \mathbf{G}_i(\theta(x,z))\partial_{x_i}(E'(\theta(x,z))\psi(t,x))\zeta(z)$$

$$- \partial_z(E'(\theta(x,z))\zeta(z))\psi(t,x)\sum_{i=1}^{d}\partial_{x_i}\mathbf{G}_i(\theta(x,z))$$

$$= \sum_{i=1}^{d}\mathbf{G}_i'(\theta(x,z))\partial_z\theta(x,z)E'(\theta(x,z))\partial_{x_i}\psi(t,x)\zeta(z)$$

$$- \psi(t,x)\partial_z\zeta(z)\mathbf{G}_i'(\theta(x,z)\sum_{i=1}^{d}\partial_{x_i}\theta(x,z).$$

Thus, using this expression in (2.14), using the Fubini theorem and the definition of ζ we find that

$$\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}} \chi(z, u(t,x))\mathbf{a}(x,z) \cdot \nabla_{x,z}\varphi(t,x,z)\ dz\ dx\ dt$$

$$= \int_{\mathbb{R}_+^{d+1}} \int_0^{u(t,x)} \sum_{i=1}^{d}\mathbf{G}_i'(\theta(x,z))\partial_z\theta(x,z)E'(\theta(x,z))\partial_{x_i}\psi(t,x)\zeta(z)\ dz\ dx\ dt$$

$$- \int_{\mathbb{R}_+^{d+1}} \int_0^{u(t,x)} \psi(t,x)\partial_z\zeta(z)\mathbf{G}_i'(\theta(x,z)\sum_{i=1}^{d}\partial_{x_i}\theta(x,z)\ dz\ dx\ dt$$

(2.18)

$$= \int_{\mathbb{R}_+^{d+1}} \int_0^{u(t,x)} \mathbf{G}'(\theta(x,z))E'(\theta(x,z)) \cdot \nabla\psi(t,x)\partial_z\theta(x,z)\ dz\ dx\ dt$$

$$= \int_{\mathbb{R}_+^{d+1}} \int_0^{v(t,x)} \mathbf{G}'(w)E'(w) \cdot \nabla\psi(t,x)\ dw\ dx\ dt$$

$$= \int_{\mathbb{R}_+^{d+1}} \mathbf{Q}_2(v(t,x)) \cdot \nabla\psi(t,x)\ dx\ dt,$$

where $\mathbf{Q}_2$ is defined through the relation

$$(2.19) \qquad\qquad\qquad \mathbf{Q}_2'(w) = \mathbf{G}'(w)E'(w).$$

Finally, for the last term on the right hand side of (2.14), we have

$$\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}} \partial_z\varphi(t,x,z)\ dm(t,x,z)$$

$$(2.20) \quad = \int_{\mathbb{R}_+^{d+1}\times\mathbb{R}} (\zeta(z)E''(\theta(x,z))\partial_z\theta(x,z) + E'(\theta(x,z))\partial_z\zeta(z))\psi(t,x)\ dm(t,x,z)$$

$$= \int_{\mathbb{R}_+^{d+1}\times\mathbb{R}} E''(\theta(x,z))\partial_z\theta(x,z)\ dm(t,x,z)$$

where the last equality follows from the fact that the measure m is supported in the set $\mathbb{R}_+^{d+1} \times [-R, R]$ (cf. [8, relation (2), p. 480]). Therefore, substituting (2.15),

(2.17), (2.18) and (2.20) into (2.14), we deduce that

$$
\begin{aligned}
(2.21) \quad & \int_{\mathbb{R}^{d+1}_+} (Q_1(x, v(t,x))\psi_{,t}(t,x) \, dx \, dt + \int_{\mathbb{R}^{d+1}_+} \mathbf{Q}_2(v(t,x)) \cdot \nabla\psi(t,x) \, dx \, dt \\
& = -\int_{\mathbb{R}^d} Q_1(x, v_0(x))\psi(0,x) \, dx + \int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}} E''(\theta(x,z))\partial_z\theta(x,z) \, dm(t,x,z)
\end{aligned}
$$

for all $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$, all $E \in \mathcal{C}^\infty(\mathbb{R})$ with Q_1 and $\mathbf{Q}_2$ defined in (2.16), (2.19) respectively.

Since the measure m is supported on $\mathbb{R}^{d+1}_+ \times [-R, R]$, then only the behaviour of the entropy function on the set $[-R, R]$ influences the entropy inequality. Moreover, an arbitrary convex smooth function can be restricted to the set $[-R, R]$ and then extended to a smooth function having derivative with compact support in $\mathbb{R}$. Hence, we obtain for all smooth convex E that

$$
\begin{aligned}
(2.22) \quad & \int_{\mathbb{R}^{d+1}_+} (Q_1(x, v(t,x))\psi_{,t}(t,x) \, dx \, dt + \int_{\mathbb{R}^{d+1}_+} \mathbf{Q}_2(v(t,x)) \cdot \nabla\psi(t,x) \, dx \, dt \\
& \geq -\int_{\mathbb{R}^d} Q_1(x, v_0(x))\psi(0,x) \, dx.
\end{aligned}
$$

But since v is locally bounded, (2.22) holds for all convex $E \in \mathcal{C}(\mathbb{R})$. Consequently, a special choice $E(w) := |w - k|$ then leads to (2.5). Thus, the proof is complete. $\square$

3. ENTROPY MEASURE-VALUED SOLUTIONS

In this section, inspired by Lemma 2.1 we shall be concerned with entropy measure-valued solutions to (1.1) in case of non-smooth $\mathbf{F}$. More precisely, we focus here on fluxes $\mathbf{F}$ that satisfy (A1)–(A4). Hence following [6], we assume that $\mathbf{G} \in L^\infty_{loc}(\mathbb{R})^d$ is a jump continuous functions. Then we denote by $z_k \in \mathbb{R}$, $k \in \mathbb{N}$ the points such that $\lim_{s\to(z_k)_+} \mathbf{G}(s) \neq \lim_{s\to(z_k)_-} \mathbf{G}(s)$. Recall that the set of such points is countable. Next we construct a multi-valued mapping $\mathcal{G}$ by filling the jumps of $\mathbf{G}$ with intervals connecting $\lim_{s\to(z_k)_+} \mathbf{G}(s)$ and $\lim_{s\to(z_k)_-} \mathbf{G}(s)$. We remark that there is a variety of ways to fill up the jumps by more general curves. For such $\mathcal{G}$ we find functions U and $\mathbf{A}$ (not necessarily uniquely), such that $\mathcal{G}$ can be prescribed as the composition $\mathbf{A} \circ U^{-1}$, where U^{-1} is a maximal monotone operator, everywhere defined and possibly multi-valued[3]. To begin with, we set the properties of functions U and $\mathbf{A}$ and then propose how to construct these functions. Hence, let U and $\mathbf{A}$ satisfy the following conditions:

(3.1)
- the function $U \in \mathcal{C}(\mathbb{R})$ is nondecreasing and $\lim_{s\to\pm\infty} U(s) = \pm\infty$,
- let $\alpha_k := \inf_{\alpha;\, U(\alpha)=z_k} \alpha, \qquad \beta_k := \sup_{\beta;\, U(\beta)=z_k} \beta$,

  then for all $k \in \mathbb{N}$ there holds $\alpha_k < \beta_k < \alpha_{k+1}$,
- the function U is constant on $[\alpha_k, \beta_k]$ and strictly increasing

  on (β_k, α_{k+1}) for all $k \in \mathbb{N}$,
- the function $\mathbf{A} \in \mathcal{C}(\mathbb{R})^d$ satisfies $\mathbf{A}(s) \in \mathbf{G}(U(s))$,
- the function $\mathbf{A}$ is linear on $[\alpha_k, \beta_k]$ for all $k \in \mathbb{N}$.

---

[3] Our starting point can also be the problem where jump continuous function would be understood in a generalized sense, namely as a multi-valued mapping $\mathcal{G}$.

Note that if U and $\mathbf{A}$ satisfy (3.1) then necessarily

$$(3.2) \qquad \mathbf{A}(s) = \begin{cases} \mathbf{G}(U(s)) & \text{for } s \in (\beta_k, \alpha_{k+1}), \\ \dfrac{\mathbf{G}_+(z_k) - \mathbf{G}_-(z_k)}{\beta_k - \alpha_k}(s - \alpha_k) + \mathbf{G}_-(z_k) & \text{for } s \in [\alpha_k, \beta_k], \end{cases}$$

where $\mathbf{G}_\pm(z_k) := \lim_{s \to (z_k)_\pm} \mathbf{G}(s)$.

There are many ways how to construct U and $\mathbf{A}$ such that (3.1) holds. Since according to Theorem 3.1 the solution will not depend on the choice of U, we introduce here one possible choice, that shall also be used in the proof of the existence theorem. First, we define

$$\alpha_k = z_k + \sum_{n;\, z_n < z_k} \frac{1}{n^2} \qquad \text{and} \qquad \beta_k = z_k + \sum_{n;\, z_n \leq z_k} \frac{1}{n^2}$$

and define $\bar{U}$ as

$$(3.3) \qquad \bar{U}(g) = \begin{cases} g - \displaystyle\sum_{k:\, z_k < g} \frac{1}{k^2} & \text{if } g \notin (\alpha_m, \beta_m) \text{ for all } m \in \mathbb{N}, \\ z_k - \displaystyle\sum_{n:\, z_n < z_k} \frac{1}{n^2} & \text{otherwise.} \end{cases}$$

To normalize U to be zero at zero we set

$$(3.4) \qquad\qquad\qquad U(s) := \bar{U}(s) - \bar{U}(0).$$

It is an obvious observation that the function U is continuous, nondecreasing and $\operatorname{Im} U(\mathbb{R}) = \mathbb{R}$. Moreover, having U we can immediately find $\mathbf{A}$ such that (3.1) holds.

Then we define an entropy measure-valued solution to (1.1). If no assumption on smoothness of $\mathbf{F}$ and $\mathbf{G}$ is added, then in the identities (N1) and (N2) there appears a discontinuous function (discontinuous with respect to u) $\mathbf{F}$ and/or $\mathbf{G}$. While in (N4) we can identify $\mathbf{G} \circ U$ with $\mathbf{A}$ from (3.1) that is still continuous and we can define a proper notion of measure-valued solution corresponding to (N4) that is however in case of smooth fluxes equivalent to the standard notion of entropy measure-valued solution to (1.1), see e.g. [9, 17]. Moreover, we introduce a notion of a local entropy measure-valued solution, i.e., we do not prescribe any initial data and any behavior for $|x| \to \infty$. These conditions will be specified in the existence theorem under the additional assumption on the flux, i.e. (A4).

Definition 3.1. *Let $\mathbf{G}$ satisfy (A1)–(A3) and assume that U and $\mathbf{A}$ fulfill (3.1). We say that a Young measure $\nu : \mathbb{R}_+^{d+1} \to \operatorname{Prob}(\mathbb{R})$ is a local entropy measure valued solution to (1.1) if there exists $R(t,x) \in L_{loc}^\infty(\mathbb{R}_+^{d+1})$ such that*

$$(3.5) \qquad \operatorname{supp} \nu_{(t,x)} \subset [-R(t,x), R(t,x)] \qquad \text{for a.a. } (t,x) \in \mathbb{R}_+^{d+1}$$

and if for all $\mu \in \mathbb{R}$ and all nonnegative $\varphi \in \mathcal{D}(\mathbb{R}_+^{d+1})$ there holds

$$(3.6) \quad \begin{aligned} &\int_{\mathbb{R}_+^{d+1}} \langle |\eta(x, U(\lambda)) - \eta(x, U(\mu))|, \nu_{(t,x)}(\lambda)\rangle \varphi_{,t}(t,x) \, dx \, dt \\ &+ \int_{\mathbb{R}_+^{d+1}} \langle (\mathbf{A}(\lambda) - \mathbf{A}(\mu)) \operatorname{sgn}(\lambda - \mu), \nu_{(t,x)}(\lambda)\rangle \cdot \nabla\varphi(t,x) \, dx \, dt \geq 0. \end{aligned}$$

The key observation of the paper, on which all results presented here heavily rely, is the averaged contraction property, which is also the standard property for classical results with smooth fluxes $\mathbf{F}$.

Lemma 3.1. *Let $\mathbf{G}$ satisfy* **(A1)**–**(A3)** *and let $(\mathbf{A}_1, U_1)$, $(\mathbf{A}_2, U_2)$ be two different couples satisfying* (3.1). *Assume that ν and σ are two local entropy measure-valued solutions to* (1.1) *corresponding to $(\mathbf{A}_1, U_1)$, $(\mathbf{A}_2, U_2)$ respectively. Then*

$$(3.7) \quad \int_{\mathbb{R}^{d+1}_+} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi_{,t}(t,x) \, dx \, dt$$

$$+ \int_{\mathbb{R}^{d+1}_+} \langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \cdot \nabla \psi(t,x) \, dx \, dt \geq 0$$

for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^{d+1}_+)$. Here, we defined $\mathbf{Q}(\lambda, \mu)$ through

$$(3.8) \quad \mathbf{Q}(\lambda, \mu) := \begin{cases} (\mathbf{A}_1(\lambda) - \mathbf{A}_2(\mu)) \operatorname{sgn}\left(\dfrac{\lambda - \alpha_k^1}{\beta_k^1 - \alpha_k^1} - \dfrac{\mu - \alpha_k^2}{\beta_k^2 - \alpha_k^2} \right) \\ \qquad \textit{if there is } k \textit{ such that } \lambda \in [\alpha_k^1, \beta_k^1], \ \mu \in [\alpha_k^2, \beta_k^2], \\ (\mathbf{A}_1(\lambda) - \mathbf{A}_2(\mu)) \operatorname{sgn}(U_1(\lambda) - U_2(\mu)) \qquad \textit{otherwise.} \end{cases}$$

The numbers α_k^i, β_k^i, $i = 1, 2$ are defined in (3.1) *correspond to U_1, U_2 respectively.*

Proof. However the proof is similar to the one given in [6] and based on the idea of regularization of Young measures developed in [9, 17], but the possibly discontinuous dependence in x of the function η involves new difficulties. The main difference of the proof consists in using the smoothing kernel in the product form and then passing to the limit separately, first with the parameter of regularization with respect to x and then with the one with respect to t. It motivates us to conduct the whole proof rigorously in order to avoid any unclarity. Let $\omega \in \mathcal{D}(-1, 1)$ be a regularizing kernel, i.e., $\omega(x) = \omega(-x)$ and $\int_{-1}^{1} \omega(x) \, dx = 1$. Then, for any $\gamma > 0$, we define

$$\omega_1^\gamma(t) := \gamma^{-1}\omega(t/\gamma) \qquad\qquad \text{for all } t \in \mathbb{R},$$
$$\omega_2^\gamma(x) := \gamma^{-d}\omega(x_1/\gamma) \cdot \ldots \cdot \omega(x_d/\gamma) \qquad \text{for all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

For arbitrary $\varepsilon, \delta > 0$ we set $\omega^{\delta,\varepsilon}(t, x) := \omega_1^\delta(t) \cdot \omega_2^\varepsilon(x)$. Notice that for any Young measure $\nu \in L_w^\infty([0, T] \times \mathbb{R}^d; \mathcal{M}(\mathbb{R}))$ there exists a Young measure $\nu^\delta \in L_w^\infty(\mathbb{R}^d; \mathcal{C}^\infty([0, T]; \mathcal{M}(\mathbb{R})))$ with $\|\nu^\delta\|_{L_w^\infty([0,T] \times \mathbb{R}^d; \mathcal{M}(\mathbb{R}))} \leq 1$ such that for any $f \in \mathcal{C}(\mathbb{R})$ the following holds[4] $(\omega_1^\delta * \langle f, \nu \rangle) = \langle f, \nu^\delta \rangle$ for almost all $t \in \mathbb{R}$. Moreover, we can interchange the derivative as $\langle f, (\nu^\delta)_{,t} \rangle = \langle f, \nu^\delta \rangle_{,t}$ for all $t \in \mathbb{R}$. Similarly, there exists $\nu^\varepsilon \in L_w^\infty([0, T]; \mathcal{C}^\infty(\mathbb{R}^d_{loc}; \mathcal{M}(\mathbb{R})))$ with $\|\nu^\varepsilon\|_{L_w^\infty([0,T] \times \mathbb{R}^d; \mathcal{M}(\mathbb{R}))} \leq 1$ such that $\omega_2^\varepsilon * \langle f, \nu^\varepsilon \rangle = \langle f, \nu^\varepsilon \rangle$ and $\langle f, \partial_{x_i}\nu^\varepsilon \rangle = \partial_{x_i}\langle f, \nu^\varepsilon \rangle$ for all $x \in \mathbb{R}^d$, see [9].

We set

$$\varphi(t, x) := (\psi * \omega^{\delta,\varepsilon})(t, x) = \int_{\mathbb{R}^{d+1}_+} \psi(\tau, y) \omega^{\delta,\varepsilon}(t - \tau, y - x) \, dy \, d\tau$$

in (3.6) for the measure ν, where nonnegative $\psi \in \mathcal{D}((\varepsilon, \infty) \times \mathbb{R}^d)$ is arbitrary. Note that $\varphi \in \mathcal{D}(\mathbb{R}^{d+1}_+)$ is nonnegative and therefore such setting is possible. Using

---

[4]We extend the measure for $t < 0$ and $t > T$ by zero.

regularization (with respect to time), we have for all $\mu \in \mathbb{R}$

$$(3.9) \qquad \int_{\mathbb{R}_+^{d+1}} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_1(\mu))|, \nu_{(t,x)}(\lambda) \rangle \; (\psi * (\omega_1^\delta \cdot \omega_2^\varepsilon))_{,t} \; dx \; dt$$
$$= \int_{\mathbb{R}_+^{d+1}} \omega_2^\varepsilon * \langle |\eta(x, U_1(\lambda)) - \eta(x, U_1(\mu))|, \nu_{(t,x)}^\delta(\lambda) \rangle \, \psi_{,t} \; dx \; dt.$$

Similarly, using the regularization with respect to the spatial direction, we obtain for all $\mu \in \mathbb{R}$

$$(3.10) \qquad \int_{\mathbb{R}_+^{d+1}} \langle (\mathbf{A}_1(\lambda) - \mathbf{A}_1(\mu)) \operatorname{sgn}(\lambda - \mu), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla(\psi * (\omega_1^\delta \cdot \omega_2^\varepsilon)) \; dx \; dt =$$
$$= \int_{\mathbb{R}_+^{d+1}} \langle (\mathbf{A}_1(\lambda) - \mathbf{A}_1(\mu)) \operatorname{sgn}(\lambda - \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \cdot \nabla \psi \; dx \; dt.$$

Consequently, using (3.9) and (3.10) in (3.6), we deduce that for all $\mu \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}((\varepsilon, \infty) \times \mathbb{R}^d)$ there holds

$$(3.11) \qquad \int_{\mathbb{R}_+^{d+1}} \omega_2^\varepsilon * \langle |\eta(x, U_1(\lambda)) - \eta(x, U_1(\mu))|, \nu_{(t,x)}^\delta(\lambda) \rangle \psi_{,t} \; dx \; dt$$
$$+ \int_{\mathbb{R}_+^{d+1}} \langle (\mathbf{A}_1(\lambda) - \mathbf{A}_1(\mu)) \operatorname{sgn}(\lambda - \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \cdot \nabla \psi \; dx \; dt \geq 0,$$

which in particular implies that for all $\tilde{\mu} \in \mathbb{R}$ and all $(t,x) \in (\varepsilon, \infty) \times \mathbb{R}^d$ there holds

$$(3.12) \qquad \left(\omega_2^\varepsilon * \langle |\eta(x, U_1(\lambda)) - \eta(x, U_1(\tilde{\mu}))|, \nu_{(t,x)}^\delta(\lambda) \rangle \right)_{,t}$$
$$+ \operatorname{div} \left(\langle (\mathbf{A}_1(\lambda) - \mathbf{A}_1(\tilde{\mu})) \operatorname{sgn}(\lambda - \tilde{\mu}), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \right) \leq 0.$$

Similarly, for a Young measure σ and functions U_2 and $\mathbf{A}_2$, we can deduce that for any $\varepsilon > 0$, $\tilde{\lambda} \in \mathbb{R}$ and all $(t,x) \in (\varepsilon, \infty) \times \mathbb{R}^d$ we have

$$(3.13) \qquad \left(\omega_2^\varepsilon * \langle |\eta(x, U_2(\tilde{\lambda})) - \eta(x, U_2(\mu))|, \sigma_{(t,x)}^\delta(\mu) \rangle \right)_{,t}$$
$$+ \operatorname{div} \left(\langle (\mathbf{A}_2(\tilde{\lambda}) - \mathbf{A}_2(\mu)) \operatorname{sgn}(\tilde{\lambda} - \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \right) \leq 0.$$

Next, we show that (3.12) implies that for all $\mu \in \mathbb{R}$ the following inequality holds in a weak sense

$$(3.14) \qquad \left. \begin{array}{l} \left(\omega_2^\varepsilon * \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t,x)}^\delta(\lambda) \rangle \right)_{,t} \\[2mm] + \operatorname{div} \left(\langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \right) \leq 0 \end{array} \right\} \quad \text{in } (\varepsilon, \infty) \times \mathbb{R}^d.$$

Here, $\mathbf{Q}$ was defined in (3.8). Thus, we denote by α_k^i, β_k^i the numbers corresponding to $\mathbf{A}_i, U_i$ from (3.1) with $i = 1, 2$. Let $\mu \in \mathbb{R}$ be arbitrary and fixed. First, assume that there is k such that $\mu \in (\beta_k^2, \alpha_{k+1}^2)$. From (3.1), it follows that U_2 is strictly increasing on $(\beta_k^2, \alpha_{k+1}^2)$ and that the same holds for U_1 on $(\beta_k^1, \alpha_{k+1}^1)$. Moreover, since $U_2((\beta_k^2, \alpha_{k+1}^2)) = U_1((\beta_k^1, \alpha_{k+1}^1))$, we can find uniquely defined $\tilde{\mu} \in (\beta_k^1, \alpha_{k+1}^1)$ such that $U_1(\tilde{\mu}) = U_2(\mu)$. Note that $\tilde{\mu} := (U_1)^{-1}(U_2(\mu))$. Using $\tilde{\mu}$ in (3.12) we see that (3.14) follows, provided that for all $\lambda \in \mathbb{R}$ there holds

$$(3.15) \qquad (\mathbf{A}_1(\lambda) - \mathbf{A}_1(\tilde{\mu})) \operatorname{sgn}(\lambda - \tilde{\mu}) = \mathbf{Q}(\lambda, \mu).$$

Since **G** is continuous at $U_1(\tilde\mu)$, we see (by using (3.1)) that $\mathbf{A}_1(\tilde\mu) = \mathbf{A}_2(\mu)$. Moreover, since U_1 is nondecreasing, we have

$$\operatorname{sgn}(\lambda - \tilde\mu) = \operatorname{sgn}(\lambda - (U_1)^{-1}(U_2(\mu))) = \operatorname{sgn}(U_1(\lambda) - U_2(\mu))$$

for all $\lambda \in \mathbb{R}$ and (3.15) follows. Next, assume that there exists $k \in \mathbb{N}$ such that $\mu \in [\alpha_k^2, \beta_k^2]$. For such μ we define $\tilde\mu \in [\alpha_k^1, \beta_k^1]$ as

$$(3.16) \qquad \tilde\mu := (\mu - \alpha_k^2)\frac{\beta_k^1 - \alpha_k^1}{\beta_k^2 - \alpha_k^2} + \alpha_k^1.$$

Then, by using (3.2), it is easy to show that $\mathbf{A}_1(\tilde\mu) = \mathbf{A}_2(\mu)$ and that $U_1(\tilde\mu) = U_2(\mu)$. Moreover, it is evident that for all $\lambda \notin [\alpha_k^1, \beta_k^1]$ we have

$$\operatorname{sgn}(\lambda - \tilde\mu) = \operatorname{sgn}(U_1(\lambda) - U_2(\mu))$$

and for all $\lambda \in [\alpha_k^1, \beta_k^1]$ we have

$$\operatorname{sgn}(\lambda - \tilde\mu) = \operatorname{sgn}\left(\frac{\lambda - \alpha_k^1}{\beta_k^1 - \alpha_k^1} - \frac{\mu - \alpha_k^2}{\beta_k^2 - \alpha_k^2}\right).$$

Consequently, we see that inserting $\tilde\mu$ defined in (3.16) into (3.12) we obtain (3.14) for all $\mu \in \mathbb{R}$. Similarly, one can observe that for all $\lambda \in \mathbb{R}$, there holds

$$(3.17) \qquad \left.\begin{aligned} &\left(\omega_2^\varepsilon * \langle|\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \sigma_{(t,x)}^\delta(\mu)\rangle\right)_{,t} \\ &+ \operatorname{div}\left(\langle\mathbf{Q}(\lambda, \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\rangle\right) \le 0 \end{aligned}\right\} \quad \text{in } (\varepsilon, \infty) \times \mathbb{R}^d.$$

In order to deduce (3.7) we combine (3.14) and (3.17). For brevity, set $\zeta(x, \lambda, \mu) := |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|$. The main effort is directed to the function ζ, because of its dependence on the variable x. As a consequence of the Fubini theorem it holds (note that all expressions are well-defined)

$$(3.18) \qquad \begin{aligned} &\operatorname{div}\left(\langle\mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\rangle\right) \\ &= \left\langle\operatorname{div}\left(\langle\mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda)\rangle\right), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\right\rangle \\ &\quad + \left\langle\operatorname{div}\left(\langle\mathbf{Q}(\lambda, \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\rangle\right), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda)\right\rangle. \end{aligned}$$

We apply $\sigma_{(t,x)}^{\delta,\varepsilon}$ onto (3.14) (note that it is continuous function of μ), similarly we apply $\nu_{(t,x)}^{\delta,\varepsilon}$ onto (3.17). Summing the resulting expressions and using (3.18) we find that for all $(t, x) \in (2\varepsilon, \infty) \times \mathbb{R}^d$ there holds

$$(3.19) \qquad \begin{aligned} &\langle\omega_2^\varepsilon * \langle\zeta(x, \lambda, \mu), \nu_{(t,x)}^\delta(\lambda)\rangle_{,t}, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\rangle \\ &+ \langle\omega_2^\varepsilon * \langle\zeta(x, \lambda, \mu), \sigma_{(t,x)}^\delta(\mu)\rangle_{,t}, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda)\rangle \\ &+ \operatorname{div}\left(\langle\mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu)\rangle\right) \le 0. \end{aligned}$$

Thus, multiplying (3.19) by an arbitrary fixed nonnegative $\psi \in \mathcal{D}((2\varepsilon, \infty) \times \mathbb{R}^d)$, integrating the result over $\mathbb{R}^{d+1}_+$ and using integration by parts, we find that

$$
\begin{aligned}
(3.20) \quad &-\int_{\mathbb{R}^{d+1}_+} \left(\left\langle \omega_2^\varepsilon * \langle \zeta(x, \lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \rangle_{,t}, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \right\rangle \right. \\
&\qquad\qquad \left. + \left\langle \omega_2^\varepsilon * \langle \zeta(x, \lambda, \mu), \sigma_{(t,x)}^\delta(\mu) \rangle_{,t}, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \right\rangle \right) \psi \, dx \, dt \\
&\qquad + \int_{\mathbb{R}^{d+1}_+} \left\langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \right\rangle \cdot \nabla \psi \, dx \, dt \geq 0.
\end{aligned}
$$

First, we let $\varepsilon \to 0_+$. For this purpose we extract a subsequence, that we do not relabel, such that for any compact set $K \subset \mathbb{R}$

$$
\begin{aligned}
\omega_2^\varepsilon * \langle \zeta, \nu^\delta \rangle_{,t} &\to \langle \zeta, \nu^\delta \rangle_{,t} &&\text{strongly in} && L^1_{loc}(\mathbb{R}^{d+1}_+; \mathcal{C}(K)), \\
\omega_2^\varepsilon * \langle \zeta, \sigma^\delta \rangle_{,t} &\to \langle \zeta, \sigma^\delta \rangle_{,t} &&\text{strongly in} && L^1_{loc}(\mathbb{R}^{d+1}_+; \mathcal{C}(K)), \\
\sigma^{\delta,\varepsilon} &\rightharpoonup^* \sigma^\delta &&\text{weakly}^* \text{ in} && L^\infty_w(\mathbb{R}^{d+1}_+; \mathrm{Prob}(K)), \\
\nu^{\delta,\varepsilon} &\rightharpoonup^* \nu^\delta &&\text{weakly}^* \text{ in} && L^\infty_w(\mathbb{R}^{d+1}_+; \mathrm{Prob}(K)).
\end{aligned}
$$

Using these convergence results, we observe from (3.20) that

$$
\begin{aligned}
(3.21) \quad &-\int_{\mathbb{R}^{d+1}_+} \langle \langle \zeta(x, \lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \rangle_{,t}, \sigma_{(t,x)}^\delta(\mu) \rangle \psi \, dx \, dt \\
&\qquad - \int_{\mathbb{R}^{d+1}_+} \langle \langle \zeta(x, \lambda, \mu), \sigma_{(t,x)}^\delta(\mu) \rangle_{,t}, \nu_{(t,x)}^\delta(\lambda) \rangle \psi \, dx \, dt \\
&\qquad + \int_{\mathbb{R}^{d+1}_+} \langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \cdot \nabla \psi \, dx \, dt \geq 0.
\end{aligned}
$$

Similarly to (3.18) it is not difficult to observe that

$$
\begin{aligned}
(3.22) \quad &\langle \zeta, \nu_{(t,x)}^\delta \otimes \sigma_{(t,x)}^\delta \rangle_{,t} = \langle \langle \zeta, \nu_{(t,x)}^\delta \rangle, \sigma_{(t,x)}^\delta \rangle_{,t} = \langle \omega^\delta * \langle \zeta, \nu_{(t,x)} \rangle, \sigma_{(t,x)}^\delta \rangle_{,t} \\
&= \left\langle (\omega^\delta * \langle \zeta, \nu_{(t,x)} \rangle)_{,t}, \sigma_{(t,x)}^\delta \right\rangle + \left\langle (\omega^\delta * \langle \zeta, \sigma_{(t,x)} \rangle)_{,t}, \nu_{(t,x)}^\delta \right\rangle.
\end{aligned}
$$

Thus, using (3.21), (3.22) and integrating by parts with respect to t, we find that

$$
\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+} \langle \zeta(x, \lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \psi_{,t} \, dx \, dt \\
&\qquad + \int_{\mathbb{R}^{d+1}_+} \langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \cdot \nabla \psi \, dx \, dt \geq 0.
\end{aligned}
$$

Finally, letting $\delta \to 0_+$ we conclude (3.7) by the argument of weak* convergence results for measures ν^δ and σ^δ to ν and σ, respectively.

$\square$

In Lemma 3.1 we showed that any two local entropy measure-valued solutions satisfy the contraction property, which is the main tool for proving uniqueness. However, in order to get such a result, we have to specify in which sense an initial condition is attained and what is the behavior of the solution for $|x| \to \infty$. Indeed, if (possibly) two different solutions have different initial and "boundary" value, one cannot expect that they are identical. The next theorem provides sufficient conditions (that are also necessary) that provide the "uniqueness" of a solution.

Theorem 3.1 (**Uniqueness**). *Let* G *satisfy* (**A1**)–(**A4**) *and let* (A_1, U_1), (A_2, U_2) *be two different couples satisfying* (3.1). *Assume that* ν^1, ν^2 *are two local entropy measure-valued solutions to* (1.1) *corresponding to* (A_1, U_1) *and* (A_2, U_2) *respectively. Moreover, assume that*

- *There exists* $u_0 \in L^1_{loc}(\mathbb{R}^d)$ *(initial condition) such that for all compact* $K \subset \mathbb{R}^d$ *the following holds*

(3.23)
$$\liminf_{t \to 0_+} \int_K \langle |\eta(x, U_1(\lambda)) - u_0(x)|, \nu^1_{(t,x)}(\lambda) \rangle \, dx = 0,$$
$$\liminf_{t \to 0_+} \int_K \langle |\eta(x, U_2(\mu)) - u_0(x)|, \nu^2_{(t,x)}(\mu) \rangle \, dx = 0.$$

- *There exists* $\mathsf{F}_\infty \in L^1_{loc}(\mathbb{R}^{d+1}_+)$ *(behavior as* $|x| \to \infty$*) and* $1 \le p \le \frac{d}{d-1}$ *such that for any* $T > 0$ *the following holds*

(3.24)
$$\int_0^T \left(\int_{\mathbb{R}^d} \left| \langle \mathsf{A}_1(\lambda) - \mathsf{F}_\infty(t,x), \nu^1_{(t,x)}(\lambda) \rangle \right|^p \, dx \right)^{\frac{1}{p}} \, dt < \infty$$
$$\int_0^T \left(\int_{\mathbb{R}^d} \left| \langle \mathsf{A}_2(\mu) - \mathsf{F}_\infty(t,x), \nu^2_{(t,x)}(\mu) \rangle \right|^p \, dx \right)^{\frac{1}{p}} \, dt < \infty.$$

Then for a.a. $(t,x) \in \mathbb{R}^{d+1}_+$ *there exists* $\lambda_0 = \lambda_0(t,x)$ *such that*

(3.25)
$$\operatorname{supp} \nu^1_{(t,x)} \subset \{\lambda; \ U_1(\lambda) = U_1(\lambda_0)\},$$
$$\operatorname{supp} \nu^2_{(t,x)} \subset \{\mu; \ U_2(\mu) = U_1(\lambda_0)\}.$$

In particular, defining $u(t,x) := \int_{\mathbb{R}} \eta(x, U_1(\lambda)) \, d\nu^1_{(t,x)}(\lambda)$, *we have for all* $M \in \mathcal{C}(\mathbb{R})$ *and almost all* $(t,x) \in \mathbb{R}^{d+1}_+$ *that*

(3.26)
$$M(u(t,x)) = \int_{\mathbb{R}} M(\eta(x, U_1(\lambda))) \, d\nu^1_{(t,x)}(\lambda)$$
$$= \int_{\mathbb{R}} M(\eta(x, U_2(\mu))) \, d\nu^2_{(t,x)}(\mu).$$

Proof. Here, we again proceed rigorously in order to avoid any doubts about the correctness of the procedure. Let $0 < \varepsilon < t_0 < T < \infty$ be arbitrary. We define an affine ψ_1 as follows

$$\psi_1(t) := \begin{cases} 0 & t \in [0, t_0 - \varepsilon] \cup [T, \infty), \\ \dfrac{t - t_0 + \varepsilon}{\varepsilon} & t \in (t_0 - \varepsilon, t_0), \\ \dfrac{T - t}{T - t_0} & t \in (t_0, T). \end{cases}$$

Let $\psi_2^n \in \mathcal{D}(\mathbb{R}^d)$ be arbitrary such that $\|\psi_2^n\|_\infty \le 1$. Then we set $\psi(t,x) := \psi_1(t)\psi_2^n(x)$ in (3.7) (it is a possible test function since we can mollify ψ_1 and then

pass to the limit) to deduce that

$$\frac{1}{T - t_0} \int_{t_0}^{T} \int_{\mathbb{R}^d} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t,x)}^1(\lambda) \otimes \nu_{(t,x)}^2(\mu) \rangle \psi_2^n(x) \, dx \, dt$$

$$\leq \frac{1}{\varepsilon} \int_{t_0 - \varepsilon}^{t_0} \int_{\mathbb{R}^d} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t,x)}^1(\lambda) \otimes \nu_{(t,x)}^2(\mu) \rangle \psi_2^n(x) \, dx \, dt$$

$$+ \int_{t_0 - \varepsilon}^{T} \int_{\mathbb{R}^d} \langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^1(\lambda) \otimes \nu_{(t,x)}^2(\mu) \rangle \cdot \nabla \psi_2^n(x) \psi_1(t) \, dx \, dt$$

$$=: I_1(\varepsilon, t_0, n) + I_2(\varepsilon, t_0, T, n).$$

Assume that $\operatorname{supp} \psi_2^n \subset K$, where K is a compact subset of $\mathbb{R}^d$ and let $\varepsilon \to 0_+$. Then, it is easy to show that

$$\lim_{\varepsilon \to 0_+} I_2(\varepsilon, t_0, T, n) = I_2(t_0, T, n),$$

where

(3.27)
$$I_2(t_0, T, n) := \int_{t_0}^{T} \int_{\mathbb{R}^d} \langle \mathbf{Q}(\lambda, \mu), \nu_{(t,x)}^1(\lambda) \otimes \nu_{(t,x)}^2(\mu) \rangle \cdot \nabla \psi_2^n(x) \psi_1(t) \, dx \, dt$$

$$\leq \int_{0}^{T} \int_{\mathbb{R}^d} \langle |\mathbf{Q}(\lambda, \mu)|, \nu_{(t,x)}^1(\lambda) \otimes \nu_{(t,x)}^2(\mu) \rangle |\nabla \psi_2^n(x)| \, dx \, dt,$$

where we used the Jensen inequality for estimating the term on the right hand side. By the mean value theorem, we conclude that for almost all $t_0 \in (0, T)$ there holds

$$\lim_{\varepsilon \to 0_+} I_1(\varepsilon, t_0, n) = I_1(t_0, n),$$

where

$$I_1(t_0, n) := \int_{\mathbb{R}^d} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t_0,x)}^1(\lambda) \otimes \nu_{(t_0,x)}^2(\mu) \rangle \psi_2^n(x) \, dx$$

$$\leq \int_{K} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu_{(t_0,x)}^1(\lambda) \otimes \nu_{(t_0,x)}^2(\mu) \rangle \, dx.$$

We shall now show that I_1 tends to zero as $t_0 \to 0$. For this purpose we notice that ν^1 and ν^2 are for almost all $(t, x) \in \mathbb{R}_+^{d+1}$ probabilistic measures with compact support and we use the triangle inequality to obtain that for almost all $t_0 \in (0, T)$

$$I_1(t_0, n) \leq \int_{K} \langle |\eta(x, U_1(\lambda)) - u_0(x)| + |\eta(x, U_2(\mu)) - u_0(x)|,$$

$$\nu_{(t_0,x)}^1(\lambda) \otimes \nu_{(t_0,x)}^2(\mu) \rangle \, dx$$

$$= \int_{K} \langle |\eta(x, U_1(\lambda)) - u_0(x)|, \nu_{(t_0,x)}^1(\lambda) \rangle \, dx$$

$$+ C(\psi^n) \int_{K} \langle |\eta(x, U_2(\mu)) - u_0(x)|, \nu_{(t_0,x)}^2(\mu) \rangle \, dx.$$

Hence, using the assumption (3.23), we find that

(3.28) $$\liminf_{t_0 \to 0_+} I_1(t_0, n) = 0.$$

Consequently, we first let $\varepsilon \to 0_+$ and then $t_0 \to 0_+$ in (3.7) and with help of (3.28) and (3.27) we find that for arbitrary $\psi_2^n \in \mathcal{D}(\mathbb{R}^d)$ such that $\|\psi_2^n\|_\infty \leq 1$ and any

$T > 0$ there holds

$$
(3.29) \quad \int_0^T \int_{\mathbb{R}^d} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle \psi^n_2(x) \, dx \, dt
$$
$$
\leq T \int_0^T \int_{\mathbb{R}^d} \langle |\mathbf{Q}(\lambda, \mu)|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle |\nabla \psi^n_2(x)| \, dx \, dt.
$$

Using the definition of $\mathbf{Q}$ and the triangle inequality we observe that for almost all $(t, x) \in \mathbb{R}^{d+1}_+$

$$
\langle |\mathbf{Q}(\lambda, \mu)|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle \leq \langle |\mathbf{A}_1(\lambda) - \mathbf{F}_\infty(t, x)|, \nu^1_{(t,x)}(\lambda) \rangle
$$
$$
+ \langle |\mathbf{A}_2(\mu) - \mathbf{F}_\infty(t, x)|, \nu^2_{(t,x)}(\mu) \rangle.
$$

Hence, by (3.24) we conclude that

$$
(3.30) \quad \langle |\mathbf{Q}(\lambda, \mu)|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle \in L^1(0, T; L^p(\mathbb{R}^d)).
$$

Finally, we define a sequence $\psi^n_2 \nearrow 1$ of smooth nonnegative compactly supported functions as $\psi^n_2(x) := 1$ in $B(0, n)$, $\psi^n_2(x) := 0$ for $x \in \mathbb{R}^d \setminus B(0, 2n)$ such that $|\nabla \psi^n_2| \leq \frac{c}{n}$. One immediately observes that

$$
\int_{\mathbb{R}^d} |\nabla \psi^n_2|^q \, dx \leq C \qquad \text{for all } q \geq d.
$$

Consequently, we obtain that

$$
(3.31) \quad |\nabla \psi^n| \rightharpoonup^* 0 \text{ weakly}^* \text{ in } L^\infty(0, T; L^q(\mathbb{R}^d)) \qquad \text{for all } q \geq d.
$$

Hence, using (3.30) and the weak* convergence (3.31), we see that the right hand side of (3.29) tends to 0 as $n \to \infty$. With the monotone convergence theorem we conclude that

$$
(3.32) \quad \int_0^T \int_{\mathbb{R}^d} \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle \, dx \, dt \leq 0,
$$

which implies that for almost all $(t, x) \in \mathbb{R}^{d+1}_+$ we have

$$
(3.33) \quad \langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \rangle = 0.
$$

As a conclusion of (3.33) we shall recover relations (3.25). Assume that (t, x) is such a point for which (3.33) holds and let $\lambda_0 \in \text{supp} \, \nu^1_{(t,x)}$. Assume there is $\mu_0 \in \text{supp} \, \nu^2_{(t,x)}$ such that $U_1(\lambda_0) \neq U_2(\mu_0)$. Consequently, since η is for almost all $x \in \mathbb{R}^d$ strictly monotone with respect to the second variable (which is the consequence of (**A2**)), we get that $\eta(x, U_1(\lambda_0)) \neq \eta(x, U_2(\mu_0))$. Since η is for almost all x continuous with respect to the second variable (which is the consequence of being Carathéodory) there surely exist nonnegative $H_1, H_2 \in \mathcal{D}(\mathbb{R})$ such that $H_1(\lambda_0) = H_2(\mu_0) = 1$ and that

$$
\text{supp} \, H_1 \subset \{\lambda; \, |\eta(x, U_1(\lambda)) - \eta(x, U_1(\lambda_0))| \leq \frac{1}{4} |\eta(x, U_2(\mu_0)) - \eta(x, U_1(\lambda_0))|\},
$$
$$
\text{supp} \, H_2 \subset \{\mu; \, |\eta(x, U_2(\mu)) - \eta(x, U_2(\mu_0))| \leq \frac{1}{4} |\eta(x, U_2(\mu_0)) - \eta(x, U_1(\lambda_0))|\}.
$$

Hence, by using the triangle inequality, we get for all $(\lambda, \mu) \in \operatorname{supp} H_1 \times H_2$ that

$$|\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))| \geq |\eta(x, U_1(\lambda_0)) - \eta(x, U_2(\mu_0))|$$
$$- |\eta(x, U_1(\lambda)) - \eta(x, U_1(\lambda_0))| - |\eta(x, U_2(\mu)) - \eta(x, U_2(\mu_2))|$$
$$\geq \frac{1}{2}|\eta(x, U_1(\lambda_0)) - \eta(x, U_2(\mu_0))| > 0.$$

Therefore, we obtain

$$0 < \left\langle H_1(\lambda)H_2(\mu), \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \right\rangle$$
$$= \left\langle \frac{H_1(\lambda)H_2(\mu)}{|\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|}|\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \right\rangle$$
$$\leq C \left\langle |\eta(x, U_1(\lambda)) - \eta(x, U_2(\mu))|, \nu^1_{(t,x)}(\lambda) \otimes \nu^2_{(t,x)}(\mu) \right\rangle = 0,$$

which is a contradiction. The relation (3.26) then easily follows from (3.25).   □

Theorem 3.2 (**Existence**). *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let $\mathbf{G}$ satisfy (A1) – (A3) and be jump continuous. Then there exists a measure-valued solution ν to (1.1) in the sense of Definition 3.1. Moreover, this solution satisfies*

$$(3.34) \qquad \liminf_{t \to 0_+} \int_K \int_{\mathbb{R}} |\eta(x, U(\lambda)) - u_0(x)| \, d\nu_{(t,x)}(\lambda) \, dx = 0$$

for any compact $K \subset \mathbb{R}^d$.

If additionally $\mathbf{G}$ satisfies (A4), then there exists $\mathbf{F}_\infty \in L^1_{loc}(\mathbb{R}^{d+1}_+)$ (behavior as $|x| \to \infty$) and $1 \leq p \leq \frac{d}{d-1}$ such that for any $T > 0$ the following holds

$$(3.35) \qquad \int_0^T \left(\int_{\mathbb{R}^d} \left| \langle \mathbf{A}(\lambda) - \mathbf{F}_\infty(t,x), \nu_{(t,x)}(\lambda) \rangle \right|^p \, dx \right)^{\frac{1}{p}} dt < \infty.$$

Proof of Theorem 3.2. For function $\mathbf{G}$ there exist functions U and $\mathbf{A}$ as described in (3.1). We construct an approximative problem such that first, we approximate U and $\mathbf{A}$, and then mollify θ in the x variable. To be more precise, we define

$$(3.36) \qquad U_n(s) := U(s) + \frac{s}{n}.$$

Note that U_n is strictly increasing function and we denote U_n^{-1} its inverse. For the purpose of applying Lemma 2.2 and Theorem 2.13 we need to provide that the flux function is at least Lipschitz, so we find a sequence of smooth functions $\mathbf{A}^n$ such that for every compact set $K \subset \mathbb{R}$

$$(3.37) \qquad \mathbf{A}^n \to \mathbf{A} \text{ strongly in } \mathcal{C}(K)^d.$$

Note that such a construction is always possible due to the continuity of $\mathbf{A}$. Finally, we use a standard mollification procedure and introduce

$$\theta^{\frac{1}{n}}(x, s) := \omega^{\frac{1}{n}}(x) * \theta(x, s),$$

where $\omega^{\frac{1}{n}}$ is standard mollification kernel of radius $\frac{1}{n}$ and we denote $\eta^{\frac{1}{n}}(x, s)$ the inverse function to $\theta^{\frac{1}{n}}(x, s)$, i.e., $\eta^{\frac{1}{n}}(x, \theta^{\frac{1}{n}}(x, s)) = s$. Note that the inverse surely exists since θ is for almost all x strictly increasing with respect to s.

As a first level of approximation one could use the following problem

$$(3.38) \qquad \begin{aligned} u^n_{,t} + \operatorname{div} \mathbf{A}^n(U_n^{-1}(\theta(x, u^n))) &= 0 &&\text{in } \mathbb{R}^{d+1}_+, \\ u^n(0, x) &= u_0(x) &&\text{in } \mathbb{R}^d, \end{aligned}$$

and refer to [13] for the existence of an entropy solution. However, since such result is proved there rigorously only for $d = 1$ and not for $d > 1$, we proceed here differently (following the ideas introduced in [6]). Thus, our approximative problem we start with is the following

$$
\begin{aligned}
u_{,t}^n + \operatorname{div} \mathbf{A}^n(U_n^{-1}(\theta^{1/n}(x, u^n))) &= 0 && \text{in } \mathbb{R}_+^{d+1}, \\
u^n(0, x) &= u_0(x) && \text{in } \mathbb{R}^d.
\end{aligned}
$$
(3.39)

Due to the mollification of θ with respect to x, we can recall the results of Kružkov [10], where the existence of a unique entropy solution is shown.

Our goal now is to let $n \to \infty$ in (3.39). Due to the all introduced mollification, and since $\theta^{\frac{1}{n}}(x, 0) = 0$, we are now in position when we can apply Lemma 2.1. Hence, defining

$$
g^n(t, x) := U_n^{-1}(\theta^{\frac{1}{n}}(x, u(t, x))),
$$

we see from (2.6) that it satisfies

$$
\eta^{\frac{1}{n}}(x, U_n(g^n))_{,t} + \operatorname{div} \mathbf{A}^n(g^n) = 0 \qquad \text{in } \mathbb{R}_+^{d+1},
$$
(3.40)

$$
\eta^{\frac{1}{n}}(x, U_n(g^n(0, x))) = u_0(x) \qquad \text{in } \mathbb{R}^d,
$$
(3.41)

and in addition by Theorem 2.1 and Lemma 2.2 the following entropy inequality holds

$$
\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \operatorname{sgn}(U_n(g^n(t, x)) - U_n(k))(\mathbf{A}^n(g^n(t, x)) - \mathbf{A}^n(k)) \cdot \nabla \psi(t, x) \, dx \, dt \\
&\quad + \int_{\mathbb{R}_+^{d+1}} |\eta^{\frac{1}{n}}(x, U_n(g^n(t, x))) - \eta^{\frac{1}{n}}(x, U_n(k))| \psi_{,t} \, dx \, dt \\
&\quad + \int_{\mathbb{R}^d} |u_0(x) - \eta^{\frac{1}{n}}(x, U_n(k))| \psi(0, x) \, dx \geq 0.
\end{aligned}
$$
(3.42)

for any constant $k \in \mathbb{R}$ and for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$.

By using a standard comparison argument (see [10] or [4, 13]), we observe that

$$
|\eta^{\frac{1}{n}}(x, U_n(g^n(t, x)))| \leq \|\eta^{\frac{1}{n}}(x, U_n(g^n(0, x)))\|_\infty = \|u_0\|_\infty \text{ a.e. in } \mathbb{R}_+^{d+1}.
$$
(3.43)

Consequently, since $\eta^{\frac{1}{n}}(x, 0) = U_n(0) = 0$ and both functions are strictly increasing, we see that

$$
\|U_n(g^n)\|_\infty \leq \sup_{x \in \mathbb{R}^d} \theta^{\frac{1}{n}}(x, \|u_0\|_\infty) \leq h_2(\|u_0\|_\infty),
$$
(3.44)

where for the second inequality we used **(A3)**. Thus, finally we get

$$
\|g^n\|_\infty \leq U_n^{-1}(h_2(\|u_0\|_\infty)) \leq U^{-1}(h_2(\|u_0\|_\infty)) \leq C,
$$
(3.45)

where U^{-1} is understood as a maximal monotone operator. Note that for the second inequality we used the fact that $|U(s)| \leq |U_n(s)|$ and for the last inequality we used the fact that U maps any bounded interval onto a bounded interval.

Hence having (3.45), we can find $g \in L^\infty(\mathbb{R}_+^{d+1})$ and a Young measure $\nu_{(t,x)}$ corresponding to a (not relabeled) subsequence $\{g^n\}_{n=1}^\infty$, which is for almost all (t, x) compactly supported in a a ball $B(0, \min(U^{-1}(h_2(R))))$, with $R := \|u_0\|_\infty$ such that for any continuous f

$$
g^n \rightharpoonup^* g \qquad \text{weakly}^* \text{ in } L_{loc}^\infty(\mathbb{R}_+^{d+1}),
$$
(3.46)

$$
f(g^n) \rightharpoonup^* \overline{f} \qquad \text{weakly}^* \text{ in } L_{loc}^\infty(\mathbb{R}_+^{d+1}),
$$
(3.47)

where

$$(3.48) \qquad \overline{f}(t,x) = \int_{\mathbb{R}} f(\lambda)\, d\nu_{(t,x)}(\lambda).$$

Our goal is to show that the measure ν is an entropy measure valued solution in the sense of Definition 3.1. First, it directly follows from (3.45) that (3.5) holds. We let $n \to \infty$ in (3.42). For the first term we use the fact that U_n is strictly monotone and therefore

$$M^n(\xi) := \mathrm{sgn}(U_n(\xi) - U_n(k))(\mathbf{A}^n(\xi) - \mathbf{A}^n(k)) = \mathrm{sgn}(\xi - k)(\mathbf{A}^n(\xi) - \mathbf{A}^n(k)).$$

Moreover, it follows from (3.37) that for every compact set $K \subset \mathbb{R}$

$$M^n(\xi) \to M = \mathrm{sgn}(\xi - k)(\mathbf{A}(\xi) - \mathbf{A}(k)) \quad \text{strongly in } \mathcal{C}(K)^d.$$

Consequently, using (3.46) and (3.48) we conclude

$$(3.49) \qquad \begin{aligned} M^n(g^n) &\rightharpoonup^* \overline{M} \quad \text{weakly}^* \text{ in } L^\infty_{loc}(\mathbb{R}^{d+1}_+). \\ \overline{M}(t,x) &= \int_{\mathbb{R}} \mathrm{sgn}(\lambda - k)(\mathbf{A}(\lambda) - \mathbf{A}(k)) d\nu_{(t,x)}(\lambda). \end{aligned}$$

For the second and the the third term of (3.42), we recall the convergence properties of $\theta^{\frac{1}{n}}$ and $\eta^{\frac{1}{n}}$. Since θ is strictly increasing and continuous in u, then the same holds for $\theta^{\frac{1}{n}}$. The inverse function $\eta^{\frac{1}{n}}$ is also increasing and continuous with respect to u. The convergence of convolutions and the monotonicity of θ and $\theta^{\frac{1}{n}}$ provide that $\theta^{\frac{1}{n}}(x,u)$ converges pointwisely with respect to x and uniformly with respect to u on a bounded interval $[-R, R]$ to the function θ, see Proposition A.1. More precisely, we have for all $R > 0$

$$(3.50) \qquad \theta^{\frac{1}{n}} \to \theta \qquad \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{C}([-R, R])).$$

Consequently, by using Proposition A.2, we obtain that the inverse functions $\eta^{\frac{1}{n}}$ have the same convergence properties, namely for all $R > 0$ there holds

$$(3.51) \qquad \eta^{\frac{1}{n}} \to \eta \qquad \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{C}([-R, R])).$$

Moreover, using the definition of U_n we see that for all $R > 0$

$$(3.52) \qquad U_n \to U \qquad \text{strongly in } \mathcal{C}([-R, R]).$$

Consequently, it follows from (3.48), (3.51) and (3.52) that defining $\zeta^n(x,r) := |\eta(x, U_n(r)) - \eta^n(x, U_n(k))|\psi_{,t}$, we observe

$$(3.53) \qquad \begin{aligned} \lim_{n \to \infty} \int_{\mathbb{R}^{d+1}_+} \zeta^n(x, g^n)\, dx\, dt &= \lim_{n \to \infty} \langle \zeta^n, g^n \rangle \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^d} \int_{\mathbb{R}} |\eta(x, U(\lambda)) - \eta(x, U(k))| d\nu_{(t,x)}(\lambda)\psi_{,t}\, dx\, dt \end{aligned}$$

for all $\psi \in \mathcal{D}(\mathbb{R}^{d+1})$, where the duality pairing is understood between the spaces $L^1(\mathbb{R}^d; \mathcal{C}((-R, R); \mathbb{R}))$ and $L^\infty_w(\mathbb{R}^d; \mathcal{M}([-R, R]))$. In the same manner one can also identify the limit in the last term in (3.42). Hence using all above established convergence results, we can easily let $n \to \infty$ in (3.42) to obtain (3.6). Note that in fact, we get a stronger result, since we do not require $\psi(0, x) = 0$. Moreover, it also implies (3.34), which can be proved by following the scheme used in [17, proof of Theorem 3.2].

To finish the proof we need to show that condition (3.35) holds. We shall show that it holds with $\mathbf{F}_\infty \equiv 0$. This is due to the assumptions that we made for the

behavior of function $\mathbf{G}$ for $|x| \to \infty$, namely $(\mathbf{A4})$. However one may entertain more refined assumptions where the condition involving a nontrivial function $\mathbf{F}_\infty$ would be essential. Hence, we observe that $(\mathbf{A4})$ also implies that for all $x \in \mathbb{R}^d$, $|x| \geq R_\infty$

$$|\mathbf{G}(U(s))|^p \leq C_\infty |\eta(x, U(s))|.$$

By (3.2) the function $\mathbf{A}(s)$ is either equal to $\mathbf{G}(U(s))$ or is linear. In the first case we obtain that

$$|\mathbf{A}(s)|^p \leq C_\infty |\eta(x, U(s))|$$

and in the second case, namely for any $s_k \in [\alpha_k, \beta_k]$ we observe that for $r \in [0, 1]$

$$|\mathbf{A}(s_k)|^p = |r\mathbf{G}_+(z_k) + (1-r)\mathbf{G}_-(z)|^p \leq C_\infty |\eta(x, z_k)| \leq C_\infty C(s_k)|\eta(x, U(s_k))|,$$

where $C(s)$ is bounded on bounded intervals. Hence

$$
\begin{aligned}
\int_0^T \left(\int_{\mathbb{R}^d} |\langle \mathbf{A}, \nu_{(t,x)} \rangle|^p \, dx \right)^{\frac{1}{p}} dt &\leq \int_0^T \left(\int_{\mathbb{R}^d} \langle |\mathbf{A}|, \nu_{(t,x)} \rangle|^p \, dx \right)^{\frac{1}{p}} dt \\
&\leq \int_0^T \left(\int_{\mathbb{R}^d \setminus B_{R_\infty}(0)} \langle |\mathbf{A}|, \nu_{(t,x)} \rangle|^p \, dx \right)^{\frac{1}{p}} dt \\
&\quad + \int_0^T \left(\int_{B_{R_\infty}(0)} \langle |\mathbf{A}|, \nu_{(t,x)} \rangle|^p \, dx \right)^{\frac{1}{p}} dt \\
&\leq C_\infty \int_0^T \left(\int_{\mathbb{R}^d} \langle |\eta(x, U)|, \nu_{(t,x)} \rangle \, dx \right)^{\frac{1}{p}} dt + C \\
&\leq C_\infty \liminf_{n \to \infty} \int_0^T \left(\int_{\mathbb{R}^d} |\eta(x, U_n(g^n))| \, dx \right)^{\frac{1}{p}} dt + C,
\end{aligned}
$$

(3.54)

Next, since $\theta^{\frac{1}{n}}(x, s)$ is a convolution of $\theta(x, s)$ with respect to x, then for almost all $x \in \mathbb{R}^d$ and all $s \in \mathbb{R}$ we have

$$(3.55) \qquad \inf_{x \in \mathbb{R}^d} \theta(x, s) \leq \theta^{\frac{1}{n}}(x, s) \leq \sup_{x \in \mathbb{R}^d} \theta(x, s).$$

Consequently, using $(\mathbf{A3})$, we can deduce that for all $x \in \mathbb{R}^d$, all $R > 0$ and all $|s| \leq R$ that

$$(3.56) \qquad C_R^1 \theta(x, s) \leq \theta^{\frac{1}{n}}(x, s) \leq C_R^2 \theta(x, s).$$

Hence, by the strict monotonicity of θ we may conclude that the inverse $\eta^{\frac{1}{n}}$ satisfies

$$(3.57) \qquad C_R^2 \eta(x, s) \leq \eta^{\frac{1}{n}}(x, s) \leq C_R^4 \eta(x, s) \qquad \text{for all } |s| \leq R.$$

Hence, since g^n is bounded, wee see from (3.57) that

$$(3.58) \qquad I \leq C \liminf_{n \to \infty} \int_0^T \left(\int_{\mathbb{R}^d} |\eta^{\frac{1}{n}}(x, U_n(g^n))| \, dx \right)^{\frac{1}{p}} dt = \int_0^T \|u^n\|_1^{\frac{1}{p}} \, dt,$$

where u^n is the Kružkov entropy solution to (3.39) or equivalently the kinetic solution. However, using the standard stability result for the Kružkov solution, we have that

$$\|u^n(t)\|_1 \leq \|u_0\|_1 \leq C \qquad \text{for all } t > 0,$$

see e.g. [8, Proposition 2]. Note that this stability result formally follows from setting $k = 0$ in entropy inequality (1.2) and integration over $\mathbb{R}^d$. Here, the term

$\operatorname{sgn}(u^n - k) \operatorname{div} \mathbf{F}^n(x, 0)$ vanishes, since $\mathbf{F}^n(x, 0) = \mathbf{A}^n(U_n^{-1}(\theta^{\frac{1}{n}}(x, 0)))$ and $\theta^{\frac{1}{n}}(x, 0) = 0$. Consequently, we obtain

$$I \leq C$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Entropy weak solutions

In this section we introduce a notion of entropy weak solution for discontinuous $\mathbf{F}$ that will be according to Lemma 2.2 equivalent to the notion of entropy weak solution for continuous flux $\mathbf{F}$. Moreover, under the assumption **(A4)** we will establish the existence and uniqueness of such a solution.

Definition 4.1. *Let all assumptions of Theorem 3.2 be fulfilled. We say that $u \in L^\infty(0, \infty; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is an entropy weak solution to (1.1) if there exist $U : \mathbb{R} \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R} \to \mathbb{R}^d$ satisfying (3.1) and $g \in L^\infty(\mathbb{R}_+^{d+1})$ such that*

$$(4.1) \qquad \eta(x, U(g(t,x))) = u(t,x), \qquad \mathbf{A}(g(t,x)) \in \mathbf{G}(u(t,x)) \qquad \textit{a.e. in } \mathbb{R}_+^{d+1},$$

$$(4.2) \qquad u_{,t} + \operatorname{div} \mathbf{A}(g) = 0, \qquad\qquad\qquad \textit{in sense of distribution,}$$

$$(4.3) \qquad \liminf_{t \to 0} \int_K |u(t,x) - u_0(x)| \, dx = 0, \qquad \textit{for any compact } K \subset \mathbb{R}^d,$$

and for all smooth convex $\tilde{E}$, that are for all $k \in \mathbb{N}$ linear on (α_k, β_k), there holds

$$(4.4) \qquad Q_u(x, g)_{,t} + \operatorname{div} \mathbf{Q_A} \leq 0, \qquad\qquad \textit{in sense of distribution,}$$

with Q_u and $\mathbf{Q_A}$ given by

$$(4.5) \qquad \partial_s Q_u'(x, s) = \partial_s \eta(x, U(s))\tilde{E}'(s), \qquad \partial_s \mathbf{Q_A}(s) = \partial_s \mathbf{A}(s)\partial_s \tilde{E}(s),$$

where α_k, β_k are defined in (3.1).

The following theorem establishes the existence and uniqueness of weak entropy solution

Theorem 4.1. *Let $\mathbf{G}$ satisfy **(A1)**–**(A4)**. Then there exists a unique entropy weak solution u to (1.1).*

Once we showed the existence and uniqueness of entropy measure-valued solutions, then the existence of entropy weak solutions is proved exactly the same way as in [6] and hence we will not repeat the prove here.

Appendix A. Auxiliary results

Proposition A.1. *Let $[a, b] \subset \mathbb{R}$ and let f be continuous, f, f_n be monotone functions such that $f_n \to f$ pointwisely. Then $f_n \to f$ uniformly on $[a, b]$.*

The above fact in an elementary exercise. For the proof see e.g. [2].

Proposition A.2. *Let $[a, b] \subset \mathbb{R}$ and let $f, f_n : \mathbb{R} \to \mathbb{R}$, $\operatorname{Im}(f) = \mathbb{R}$, $\operatorname{Im}(f_n) = \mathbb{R}$, f, f_n be strictly monotone functions such that $f_n \to f$ pointwisely. Then the inverse functions converge locally uniformly to the inverse of the limit, namely $(f_n)^{-1} \to f^{-1}$ uniformly on every compact subset of $\mathbb{R}$.*

Proof. We provide the proof by contradiction. Assume that f_n converges uniformly to f and that $(f_n)^{-1}$ does not converge pointwisely to f^{-1}. Hence there exists $y, \varepsilon > 0$ and a subsequence $(f_{n_k})^{-1}$ such that

(A.1) $$(f_{n_k})^{-1}(y) \notin [f^{-1}(y) - \varepsilon, f^{-1}(y) + \varepsilon].$$

We only prove the case $(f_{n_k})^{-1}(y) > f^{-1}(y) + \varepsilon$. The second case follows analogously. Let $\bar{y}_{n_k} := f_{n_k}^{-1}(f(y))$. By (A.1) we have the estimate

$$\bar{y}_{n_k} > y + \varepsilon.$$

Using the strict monotonicity of f, monotonicity of f_{n_k} and the definition of $\bar{y}_{n_k}$ we conclude an existence of δ such that

(A.2) $$0 < \delta \le f(y + \varepsilon) - f(y) = f(y + \varepsilon) - f_{n_k}(\bar{y}_{n_k}) \le f(y + \varepsilon) - f_{n_k}(y + \varepsilon)$$

which contradicts the uniform convergence of f_n. Hence $(f_n)^{-1}$ converges pointwisely to f^{-1}. The uniform convergence of $(f_n)^{-1}$ can be concluded by Proposition A.1. □

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