Transport equation with non-Radon velocity as embedding of system of ODEs

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Transport equation with non-Radon velocity as embedding of system of ODEs
Équation de transport à vitesse non radonienne comme plongement d'un système d'EDOs

Partial Differential Equations

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Abstract

We rewrite system of ODEs as transport equation with measure-valued solutions. The discrete states are obtained as zeroes of velocity with transmission conditions. The velocity itself is a non-Radon measure what accounts for infinite propagation speed. This communication is mediated through a "shadow" of higher order.

Résumé

On réécrit une système d'EDOs comme une equation de transport et ses solutions à valeurs dans mesures. On obtient les états discrets dans zéros de la vitesse avec conditions de la transmission. La vitesse elle-même est une mesure non radonienne et c'est pourquoi la vitesse de propagation des informations peut être infinie. Cette communication est transmise par une "ombre" d'ordre plus haut.

1 Motivations

The tendency to unification has been ever present in natural sciences as well as mathematics. The ability to describe various phenomena as aspects of a general theory is not only elegant but may also provide better insights in the structure of our physical or mathematical reality. A particularly spectacular example of unification is provided by the Standard Model of Particle Physics, where three of four fundamental interactions are reduced to aspects of a single theory. The original motivations of this note, however, are biological. Various biological phenomena may possess disparate behavior – in general discrete or continuous. Measure theory allows, in many cases, to put those modes into a single framework (see e.g. [5] for a unified description of (a)symmetric cell division).

The starting point for our discussion is the linear transport equation

$$\partial_t \mu + \partial_x (g_n(x)\mu) = 0 \tag{1}$$

with velocity $g_n(x) = n\mathbf{1}_{(0,1)}(x)$ (equal n on $(0,1)$ and 0 otherwise). The existence and uniqueness of measure-valued solutions under additional *transmission conditions* was shown in [3] while a detailed discussion of the nonuniqueness framework of transport equation developed on the base of cell differentiation processes can be found in [4] and [3]. Our long-term goal is to set every possible discrete-continuous dynamics in the continuous framework to enable a unified perspective on different systems (having in mind cellular systems in particular) and relations between them. Discrete states coupled with transport equation were analyzed in a special case in [2] and put into a continuous framework in [3]. The next step consists in embedding a *purely* discrete dynamics into a *purely continuous* transport equation. This apparent paradox is solvable.

2 Function g for two embedded ODEs

Let us, for clarity of presentation, consider the simplest possible conservative system of two linear ODEs:

$$\begin{cases} \frac{dw}{dt} = -w \\ \frac{dv}{dt} = w. \end{cases} \tag{2}$$

Conservativity means that the total mass $w + v$ is conserved. Transport equation has the same property and is our destination space. Let us define $\mu(\{0\})(t) := w(t)$ and $\mu(\{1\})(t) := v(t)$. This means that $\mu(t) = w(t)\delta_0 + v(t)\delta_1 +$ additional terms, which will be determined later.

Our hyperbolic-type transport equation, has *finite speed of information propagation*. The two ODEs, on the other hand, communicate with each other without delay. How to resolve this contradiction? When $n \to \infty$ the velocity in (1) tends to infinity. Although g_n has no limit, the solution of characteristic equation $\dot{x}_n = g_n$ *has* a limit. The graph of any nontrivial characteristic converges uniformly to the graph of Heavyside function with connected components. It is up to the connection at $t = s$ the solution of $\dot{x} = \delta_s(t)$. Yet this jump is allowed at any timepoint s. Hence, the δ has to be at *every* timepoint s. Obviously, still $g(t, x) = 0$ for $x \notin (0, 1)$. Based on the above discussion, we *postulate* the form of the limiting velocity g in $\partial_t\mu + \partial_x(g\mu) = 0$ as

$$g(t, x) := \#(dt)\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx).$$

Hash stands for the counting measure – this is exactly Dirac mass at every t. The term $\mathbf{1}_{(0,1)}\mathcal{L}^1(dx)$, on the other hand, reflects the mode of convergence of the characteristic. It stands for the "reciprocal of probability" of finding a particle at different points x at $t = 0$ as a limit of respective probabilities. It diffuses the point $t = 0$ and acts as a shadow of particles in a lower dimension.

3 Towards theory of transport equation

Let us begin with a very general building block of all solutions (compare [1]), an initial measure concentrated in one point. The interesting one is, of course, $x = 0$. To simplify notation, we define the initial condition at $t = -\infty$, namely $\mu(t = -\infty) = \delta_0(x)$. The solution is (extremely) nonunique, yet as a lemma, we prove that

$$\mu^s(t) = \delta_0(x)\mathbf{1}_{t<s} + \mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)\mathbf{1}_{t=s} + \delta_1(x)\mathbf{1}_{t>s}$$

satisfies the equation (note that we sometimes do not write $\mathcal{L}^1(dt)$ if there exists a density).

Proof. Take for simplicity $\mu = \mu^0$ and calculate: $\partial_t\mu = \delta_0(t)(\delta_1(x) - \delta_0(x))$ (in $\mathcal{D}'$). As to term $g\mu$, observe that g and μ are mutually $\mathcal{L}^1$ integrable (compare [1]), since $\mathcal{L}^1$-singular parts of both measures are on disjoint sets. The product of two measures is well defined:

$$g\mu = (\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx))^2\mathbf{1}_{t=0}\#(dt) = \mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)\delta_0(t).$$

As a result, $\partial_x(g\mu) = \delta_0(t)(\delta_1(x) - \delta_0(x))$. □

Now, let us consider again (2). A sample solution has the form $\begin{cases} w(t) = e^{-t} \\ v(t) = (1 - e^{-t}) \end{cases}$. Considering the solution of respective transport equation as a linear combination of building blocks, $\mu(t) = \int_0^\infty e^{-s}\mu^s(t)ds = e^{-t}\delta_0 + (1 - e^{-t})\delta_1$ is wrong. In the wake of integration we lose the shadow. Before, the shadow transported instantly a positive measure. Now, it transports a flux and as a result is diffuse and infinitesimal. Yet it cannot be omitted. We notice, that $\partial_t\mu = e^{-t}(\delta_1(x) - \delta_0(x))$ and $g\mu = 0$, what is bad. The point is that the composition, i.e. the integration should be done *after* multiplying by g and not *before*. Then

$$g\mu = \int_0^\infty e^{-s}g\mu^s ds = \int_0^\infty e^{-s}\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)\delta_s(t)ds = e^{-t}\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx).$$

We notice, that $\partial_x(g\mu)$ is equal to $-\partial_t\mu$. The function of building blocks is saved. The main question, however, is still not solved. How to describe μ?

4 Formal expansion in $\#(dt)$

Our problem is to define an object of zero measure in $\mathbb{R}$. This real line seems to be too sparse for anything reasonable, which multiplied by $\#(dt)$ would give a measure. To be precise, we would like to be able to write:

$$\mu(t) = \int_0^\infty e^{-s}\mu^s(t)ds = e^{-t}\delta_0 + (1-e^{-t})\delta_1 + \frac{e^{-t}\mathcal{L}^1(dt)}{\#(dt)}.$$

But why not? In this notation, μ is a sum of a term of order 0 concentrated in $\{0,1\}$ and a term of order 1 concentrated on $(0,1)$. On the other hand, $g = \#(dt)\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)$ is a term of order -1 concentrated on $(0,1)$. The product $g\mu$ contains terms of order 0 and 1. No terms of negative order. This saves the game, since we are left with a classical distribution, what allows testing it against classical smooth compactly supported functions. Are there reasonable situations with even lower or higher orders? We do not know for the time being. The calculus with orders is standard – in general they add up by multiplication and by summation the lowest is the resulting one (provided no cancelation occurs).

Let us stress once again: *eventually* only the lowest term counts. But *before* we conduct some calculations which result in canceling. This means that all the terms have to be kept until the last stage of calculation – that is until testing against a test function. Then only nonpositive orders in $\#(dt)$ play a role.

5 Existence and uniqueness in the sample case

We will say, to present a basic existence&uniqueness theory, that for a given g, having a given expansion in $\#(dt)$, μ is a solution provided $g\mu$ is well defined, of order 0; μ is of order 0 and

$$\partial_t\mu + \partial_x(g\mu) = 0 \quad (\text{here} \quad g = \#(dt)\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx))$$

in $\mathcal{D}'$. The time-continuity will be defined elsewhere, by use of a rotated setting, what mollifies the jumps.

We consider, for simplicity, the transmission condition $\frac{d\mu(t)(\{0\})}{dt} = -\mu(t)(\{0\})$. Compare the more "continuous" form ([3, 4]), which is very vague here. Set $\mu(0) = \delta_0$. The initial value at 1 is a minor modification, while initial condition with nontrivial measure of $(0,1)$ is unphysical and is not of our concern here.

Theorem 1. *The solution* $\mu(t) = e^{-t}\delta_0 + (1-e^{-t})\delta_1 + \frac{e^{-t}\mathcal{L}^1(dt)}{\#(dt)}\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)$ *is unique up to terms of higher, negligible order.*

Proof. First, we observe that in $\#(dt)$ expansion the only place where the 1–order term plays a role is interval $x \in (0,1)$, since g is of order -1 there. No nontrivial 0– or lower order terms are allowed on $x \in (0,1)$ since $\#(dt)$ fills the whole line. Thus, μ breaks into two parts of different orders: $\begin{cases} \mu\lfloor_{(0,1)} \text{ of order 1} \\ \mu\lfloor_{\mathbb{R}\setminus(0,1)} \text{of order 0.} \end{cases}$

Now, we prove, that for $x \notin [0,1]$, the solution vanishes. Let us consider a test function $\phi = \phi_1(t)\phi_2(x)$ such that $supp(\phi_2) \subset \mathbb{R}\setminus[0,1]$. Let us additionally assume that 0 is a left Lebesgue point of $\int \phi_2(x)d\mu(t)$ for *every* ϕ, i.e. weak Lebesgue point (in fact the solutions should be in $BV(\mathbb{R}, \text{a weak metric space})$) and we could assume left-continuity). Take a hat-type test function $\phi_1(t)$ around 0 and t_0 to prove that in all weak right Lebesgue points $\mu(t)$ is equal to $\mu(0)$, i.e. vanishes. Therefore, $\mu(t)$ is constant a.e. (the common Lebesgue set is of full measure; here suffices that it is dense). Now, by continuity of solution, we obtain that μ is constant.

The next step consists in proving uniqueness in $x = 0$. Here, in contrast to [3], the ODE in all but the last one zeroes of g is prescribed explicitly and it is enough to solve it. The solution $\mu(\{0\})(t) = e^{-t}$ is obviously unique. Next, a little more involved step on $(0,1)$. Take a test function $\phi = \phi_1(t)\phi_2(x)$ such that support of ϕ_2 is contained in $(0,1)$. Since there is no 0-order term in $supp(\phi)$, the term with time derivative vanishes and we are left with

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x\phi(t,x)\mathbf{1}_{(0,1)}(x)d(G^\mu(t))dt = 0,$$

where by G^μ we denote the shadow of μ, that is $\mu = \mu^{order0} + \frac{1}{\#(dt)}G^\mu$. In contrast to previous case ($x \notin [0,1]$), now the direction of propagation is horizontal. This is unusual and remarkable. Since we can take product functions, we rewrite this as

$$\int_{\mathbb{R}} \phi_1(t) \left[\int_{\mathbb{R}} \partial_x \phi_2(x) \mathbf{1}_{(0,1)}(x) d(G^\mu(t)) \right] dt = 0.$$

This means that $\int_{\mathbb{R}} \partial_x \phi_2(x) \mathbf{1}_{(0,1)}(x) dG^\mu(t) = 0$ for a.e. t.

Lemma 1. *Let $\nu \in \mathcal{M}((0,1))$ be such that for every $\phi \in C_c^\infty((0,1))$ $\int \partial_x \phi d\mu = 0$. Then $\nu = k\mathcal{L}^1$ for some k.*

The proof is simple and elementary. We conclude that $G^\mu(t) = k(t)1_{(0,1)}(x)\mathcal{L}^1(dx)$. How to calculate $k(t)$? We have the equation $\partial_t \mu + \partial_x(g\mu) = 0$ and so for $supp(\phi) \subset [0,\infty] \times (-\infty, 1)$ we obtain

$$\partial_t \mu + \partial_x(k(t)\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx)) = 0.$$

For this support of ϕ, however, there is no relevant part of order 0 in $\partial_t \mu$ for $x \in (0,1)$, and hence we end up (using $\mu(\{0\})(t) = e^{-t}$) with $\partial_t \mu\lfloor_{(-\infty,1)} = -e^{-t}\delta_0$. On the other hand, there is no free parameter in the other term and we simply calculate

$$\partial_x(k(t)\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx))\lfloor_{(-\infty,1)} = k(t)\delta_0(x).$$

Ergo, $k(t) = e^{-t}$, which finishes the proof of this part, since the function $k(t)$ has been determined explicitly.

Finally, to prove uniqueness in $x = 1$, we again do not take any test functions, but in general we will have to (for midpoints – compare [3]). Here, we take advantage of the fact that there is only one point in question left, namely $x = 1$. We know, that $\mu(t) = e^{-t}\delta_0(x) + \frac{e^{-t}\mathcal{L}^1(dt)}{\#(dt)}\mathbf{1}_{(0,1)}(x)\mathcal{L}^1(dx) + m(t)\delta_1(x)$ for some function m. Hence, $\partial_t \mu = -e^{-t}\delta_0(x) + \dot{m}(t)\delta_1(x)$. (we recall that during *distributional* derivation terms of higher order than the minimal are negligible). Similarly, differentiating the flux, we obtain $\partial_x(g\mu) = k(t)(\delta_0(x) - \delta_1(x)) = e^{-t}(\delta_0(x) - \delta_1(x))$. This means, that $\dot{m}(t) = e^{-t}$. Hence, $m(t) = 1 - e^{-t}$. □

The whole embedding can be done for any number of ODEs of any order (provided fluxes between neighbours are conservative). Some of the future extensions will include defining a proper notion of convergence in extended space, continuity, transmission conditions, characterization of orders and others. Yet the main goal here has been achieved: It *is* possible to reasonably embed a system of ODEs into a continuous setting, what promises a better insight into the relation between the two types of dynamics.

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