Asymptotic Behaviour in One Dimensional Model of Interacting Particles

Rafał Celiński

Preprint no. 2011 - 008

Ph.D. Programme: Mathematical Methods in Natural Sciences (MMNS)
e-mail: mmns@mimuw.edu.pl
http://mmns.mimuw.edu.pl

ASYMPTOTICS BEHAVIOUR IN ONE DIMENSIONAL MODEL OF INTERACTING PARTICLES

RAFAŁ CELIŃSKI

Abstract. We consider the equation $u_t = \varepsilon u_{xx} + (u\ K' * u)_x$ for $x \in \mathbb{R}$, $t > 0$ and with $\varepsilon \geq 0$, supplemented with a nonnegative, integrable initial datum. We present a class of interaction kernels K' such that the large time behaviour of solutions to this initial value problem is described by a compactly supported self-similar profile.

1. Introduction

We study the asymptotic behaviour of solutions to the one-dimensional initial value problem

$$u_t = \varepsilon u_{xx} + \left(u\ K' * u\right)_x \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{1.2}$$

where the *interaction* kernel K' is a given function, an initial datum $u_0 \in L^1(\mathbb{R})$ is nonnegative and $\varepsilon \geq 0$.

Equation (1.1) arises in study of an animal aggregation as well as in some problems in mechanics of continuous media. The unknown function $u = u(x,t)$ represents either the population density of a species or, in the case of materials applications, a particle density. The kernel K' in (1.1) can be understood as the derivative of a certain function K, that is, K' stands for $\mathrm{d}K/\mathrm{d}x$. We use this notation to emphasise that the cell interaction described by equation (1.1) takes place by means of a potential K. Moreover, our assumptions on interaction kernel K' imply that equation (1.1) describe particles interacting according to a repulsive force (this will be clarified bellow).

Let us first notice that the one-dimensional parabolic-elliptic system of chemotaxis

$$u_t = \varepsilon u_{xx} - (uv_x)_x, \qquad -v_{xx} + v = u, \qquad x \in \mathbb{R}, \quad t > 0 \tag{1.3}$$

Date: May 12, 2011.

2000 *Mathematics Subject Classification.* 35K15, 35B40, 92C17.

Author was supported by the International Ph.D. Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences) and by the MNiSzW grant No. N N201 418839.

can be written as equation (1.1). Indeed, if we put $K(x) = -\frac{1}{2}e^{-|x|}$ into the (1.1), which is the fundamental solution of the operator $\partial_x^2 - \mathrm{Id}$, one can rewrite the second equation of (1.3) as $v = -K * u$. Here, however, we should emphasise that, below we consider repulsive phenomena, where the interaction kernel has the opposite sign, see Remark 2.3 for more details.

This work is motivated by the recent publication by Karch and Suzuki [8] where the authors study the large time asymptotics of solutions to (1.1)-(1.2) under the assumption $K' \in L^1(\mathbb{R})$. They showed that either the fundamental solution of the heat equation or a nonlinear diffusion wave appear in the asymptotic expansion of solutions as $t \to \infty$. Analogous results on the solutions to the one dimensional chemotaxis model (1.3) can be found in [12, 13]. Here, we would like to point out that, in all those results, a diffusion phenomena play a pivotal role in the large time behaviour of solutions to problem (1.1)-(1.2).

The main goal of this work is to show that for a large class of interaction kernels $K' \in L^\infty(\mathbb{R})\backslash L^1(\mathbb{R})$, the diffusion is completely negligible in the study of the large time asymptotics of solutions. Let us be more precise. Our assumption on the interaction kernel imply that $K'(x)$ is sufficiently small perturbation of the function $-\frac{A}{2}H(x)$, where, $A \in (0, \infty)$ is a constant and H is the classical sign function given by the formula: $H(x) = -1$ for $x < 0$ and $H(x) = 1$ for $x > 0$ (*cf.* Remark 2.2). Under these assumptions, we show that for large values of time, a solution of problem (1.1)-(1.2) looks as a compactly supported self-similar profile, defined as the space derivative of a rarefaction wave, *i.e.* the solution of the Riemann problem for the nonviscous Burgers equation $u_t + Auu_x = 0$ (see Corollary 2.6 for the precise statement).

In our reasoning, first, we consider $\varepsilon > 0$, and our result on the large time behaviour are, in some sense, independent of ε. Next, we pass to the limit $\varepsilon \to 0$ to obtain an analogous result for the inviscid aggregation equation $u_t - (u\ K' * u)_x = 0$. In particular, our assumptions imply that weak, nonnegative solutions to the initial value problem for this inviscid equation exists for all $t > 0$.

To conclude this introduction, we would like to recall, that the multidimensional inviscid aggregation equation $u_t - \nabla \cdot (u \nabla K * u) = 0$ was derived as a macroscopic equation from the so-called "individual cell-based mode" [4, 15], namely, as a continuum limit for a system of particles $X_k(t)$ placed at the point k in time t and evolving by the system of differential equations:

$$\frac{dX_k(t)}{dt} = -\sum_{i \in \mathbb{Z}\backslash\{k\}} \nabla K(X_k(t) - X_i(t)), \quad k \in \mathbb{Z}$$

where K is the potential. Results on the local and global existence as well as the blow-up of solutions of this inviscid aggregation equation one can find in [1, 2, 3, 11] and in references therein.

Notation. In this work, the usual norm of the Lebesgue space $L^p(\mathbb{R})$ with respect to the spatial variable is denoted by $\|\cdot\|_p$ for any $p \in [1,\infty]$ and $W^{k,p}(\mathbb{R})$ is the corresponding Sobolev space. The set $C_c^\infty(\mathbb{R})$ consist of smooth and compactly supported functions. Moreover $(f*g)(x)$ denotes the usual convolution, *i.e.* $(f*g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y$. The letter C corresponds to a generic constants (always independent of x and t) which may vary from line to line. Sometimes, we write, *e.g.* $C = C(\alpha,\beta,\gamma,...)$ when we want to emphasise the dependence of C on parameters $\alpha,\beta,\gamma,....$

2. Main results

We begin our study of large time behaviour of solution by recalling that, for $\varepsilon > 0$, the initial value problem (1.1)-(1.2) is known to have a unique and global-in-time solution for a large class of initial conditions u_0 and interaction kernels K'. Such results are more-or-less standard and the detailed reasoning can be found in [9]. In particular, our assumptions (see Theorem 2.1 below) imply that $K' \in L^\infty(\mathbb{R})$, hence the kernel K' is mildly singular in the sense stated in [9, Thm 2.5]. In this case, results from [9] can be summarised as follows: for every $u_0 \in L^1(\mathbb{R})$ such that $u_0 \geq 0$, there exists the unique global-in-time solution u of problem (1.1)-(1.2) satisfying

$$u \in C\left([0,+\infty),\ L^1(\mathbb{R})\right) \cap C\left((0,+\infty),\ W^{1,1}(\mathbb{R})\right) \cap C^1\left((0,+\infty),\ L^1(\mathbb{R})\right).$$

In addition, the condition $u_0(x) \geq 0$ implies $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. Moreover we obtain the conservation of the L^1-norm of nonnegative solutions:

$$\|u(t)\|_{L^1} = \int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = \|u_0\|_{L^1}. \tag{2.1}$$

In Theorem 2.5 below, we pass to the limit $\varepsilon \to 0$, to obtain nonnegative weak solutions of problem (1.1)-(1.2) with $\varepsilon = 0$, for which the conservation of mass (2.1) holds true, as well.

The goal of this work is to study the large time behaviour of solution to (1.1)-(1.2). First, we state conditions under which these solutions decay as $t \to \infty$.

Theorem 2.1 (Decays of L^p norm)**.** *Assume that $u = u(x,t)$ is a nonnegative solution to problem (1.1)-(1.2) with $\varepsilon > 0$, where the interaction kernel has the form $K'(x) = -\frac{A}{2}H(x) + V(x)$, where H is the sign function, $A > 0$ is a constant, and the function V*

satisfies

$$V \in W^{1,1}(\mathbb{R}) \text{ with } \|V_x\|_{L^1} < A. \tag{2.2}$$

Suppose also that $u_0 \in L^1(\mathbb{R})$ *is nonnegative. Then for every* $p \in [1, \infty]$ *the following inequality hold true*

$$||u(t)||_p \leq (A - ||V_x||_1)^{\frac{1-p}{p}} \, ||u_0||_1^{1/p} \, t^{\frac{1-p}{p}} \tag{2.3}$$

for all $t > 0$.

Remark 2.2. Notice that, under assumption (2.2), we have $V(x) = \int_{-\infty}^{x} V_y(y)\,\mathrm{d}y$. Hence, we get immediately that $V \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $\lim_{|x|\to\infty} V(x) = 0$, and the following estimate, $\|V\|_\infty \leq \|V_x\|_1 < A$ hold true. Consequently, our assumption on the interaction kernel K' imply that $K' + \frac{A}{2}H \in C_0(\mathbb{R})$ (continuous and decaying at infinity functions). This means that the kernel K' has to jump at zero exactly as the rescaled sign function $-\frac{A}{2}H$ and has to converge to the constants $\pm\frac{A}{2}$ as $x \to \mp\infty$, respectively. In some sense, this means that the potential $K(x)$ looks as $-\frac{A}{2}|x|$ at $x = 0$ and as $|x| \to +\infty$.

Remark 2.3. Our assumptions on the kernel $K'(x)$ imply that interactions between particles are similar as in the chemorepulsion motion in chemotaxis phenomena, namely, when regions of high chemical concentrations have a repulsive effect on particles. Such a model was studied for example in [5].

In the next step of this work, we derive an asymptotic profile as $t \to \infty$ of solutions (1.1)-(1.2). First, notice that if the large time behaviour of a solution to problem (1.1)-(1.2) is described by the heat kernel or the nonlinear diffusion wave (as *e.g.* in [8]) then we expect the following decay rate $||u(t)||_p \leq C\, t^{\frac{1-p}{2p}}$ for all $t > 0$. Observe, that the function u from Theorem 2.1 decays faster, hence, its asymptotic behaviour as $t \to \infty$ should be different.

From now on, without loss of generality, we assume that $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 1$. Indeed, due to the conservation of mass (2.1), it suffices to replace u in equation (1.1) by $\frac{u}{\int_{\mathbb{R}} u_0\,\mathrm{d}x}$ and K' by $K' \int_{\mathbb{R}} u_0\,\mathrm{d}x$.

Next, let us put

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y - \frac{1}{2}, \tag{2.4}$$

where $u(x,t)$ is the solution of (1.1)-(1.2). Since $u = U_x$, using the explicit form of the kernel K' (cf. Lemma 3.1 below), we obtain that the primitive $U = U(x,t)$ satisfy the following equation

$$U_t = \varepsilon U_{xx} - AUU_x + U_x\; V * U_x, \tag{2.5}$$

which can also be considered as a nonlinear and nonlocal perturbation of the viscous Burgers equation.

Our main result says that the large time behaviour of U is described by a self-similar profile, given by a rarefaction wave, namely, the unique entropy solution of the Riemann problem for the scalar conservation law

$$W_t^R + AW^R W_x^R = 0 \tag{2.6}$$

$$W^R(x,0) = \frac{1}{2}H(x). \tag{2.7}$$

It is well-known (see *e.g.* [6]) that this rarefaction wave is given by the explicit formula

$$W^R(x,t) := \begin{cases} -\dfrac{1}{2} & \text{for} \quad x \le -\dfrac{At}{2}, \\ \dfrac{x}{At} & \text{for} \quad -\dfrac{At}{2} < x < \dfrac{At}{2}, \\ \dfrac{1}{2} & \text{for} \quad x \ge \dfrac{At}{2}. \end{cases} \tag{2.8}$$

Theorem 2.4 (Convergence towards rarefaction waves)**.** *Let the assumptions of Theorem 2.1 hold true. Assume, moreover, that a nonnegative initial datum $u_0(x)$ satisfies*

$$\int_{\mathbb{R}} u_0(x)\, dx = 1, \quad \textit{and} \quad \int_{\mathbb{R}} u_0(x)|x|\, dx < \infty. \tag{2.9}$$

Then, there exist a constant $C > 0$ independent of ε such that for every $t > 0$ and each $p \in (1, \infty]$ the following estimate hold true

$$\|U(\cdot,t) - W^R(\cdot,t)\|_p \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left(\log(2+t)\right)^{\frac{1}{2}(1+\frac{1}{p})}, \tag{2.10}$$

where $U = U(x,t)$ is the primitive of solution of problem (1.1)-(1.2) *given by* (2.4) *and $W^R = W^R(x,t)$ is the rarefaction wave given by* (2.8)*.*

Next, we show that the asymptotic formula (2.10) holds also true for weak solutions of problem (1.1)-(1.2) with $\varepsilon = 0$.

Theorem 2.5. *Assume that the kernel K' has properties stated in Theorem 2.1 and the nonnegative initial condition $u_0 \in L^1(\mathbb{R})$ satisfies* (2.9)*. Then the initial value problem*

$$U_t = -AUU_x + U_x V * U_x \tag{2.11}$$

$$U(x,0) = U_0(x) = \int_{-\infty}^{x} u_0(y)\, dy - \frac{1}{2} \tag{2.12}$$

has a weak solution $U \in C\big(\mathbb{R}\times(0,\infty)\big)$ *such that* $U_x \in L^\infty_{loc}\big((0,\infty), L^\infty(\mathbb{R})\big)$ *that satisfies problem* (2.11)-(2.12) *in the following integral sense*

$$-\int_0^\infty\int_{\mathbb{R}} U\varphi_t\,dx\,dt-\int_{\mathbb{R}} U_0(x)\varphi(x,0)\,dx=\frac{A}{2}\int_0^\infty\int_{\mathbb{R}} U^2\varphi_x\,dx\,dt+\int_0^\infty\int_{\mathbb{R}} U_x\left(V_x{*}U\right)\varphi\,dx\,dt$$

for all $\varphi\in C_c^\infty(\mathbb{R}\times[0,+\infty))$. *This solution satisfies*

$$\|U(\cdot,t)-W^R(\cdot,t)\|_p \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+t)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)}, \tag{2.13}$$

for a constant $C>0$, *for all* $t>0$, *and each* $p\in(1,\infty]$.

Next, we use the result from Theorems 2.4 and 2.5 to describe the large time asymptotics of solutions to problem (1.1)-(1.2).

Corolary 2.6. *Let the assumptions either of Theorem 2.1 or Theorem 2.5 hold true. For the solution* $u=u(x,t)$ *of problem* (1.1)-(1.2) *with* $\varepsilon\ge 0$ *we define its rescaled version* $u^\lambda(x,t)=\lambda u(\lambda x,\lambda t)$ *for* $\lambda>0$, $x\in\mathbb{R}$ *and* $t>0$. *Then, for every test function* $\varphi\in C_c^\infty(\mathbb{R})$ *and each* $t_0>0$

$$\int_{\mathbb{R}} u^\lambda(x,t_0)\varphi(x)\,dx \to -\int_{\mathbb{R}} W^R(x,t_0)\varphi_x(x)\,dx \quad as \quad \lambda\to+\infty.$$

In other words, for each $t_0>0$, the family of rescaled solutions $u^\lambda(x,t_0)=\lambda u(\lambda x,\lambda t_0)$ to problem (1.1)-(1.2) with $\varepsilon\ge 0$ converges weakly as $\lambda\to\infty$ to the compactly supported self-similar profile defined as

$$\left(W^R\right)_x(x,t_0) := \begin{cases} \dfrac{1}{At} & \text{for} \quad |x|<\dfrac{At}{2}, \\ 0 & \text{for} \quad |x|\ge\dfrac{At}{2}. \end{cases} \tag{2.14}$$

3. Large time asymptotics

In this section, we prove all results stated in Section 2. We begin by an elementary result.

Lemma 3.1. *Let* H *be the sign function. For all* $\varphi\in W^{1,1}(\mathbb{R})$ *the following inequality hold true:* $H*\varphi_x=2\varphi$.

Proof. First, we assume that $\varphi\in C_c^\infty(\mathbb{R})$. Then

$$H*\varphi_x=\int_{\mathbb{R}} H(x-y)\varphi_y(y)\,\mathrm{d}y=\int_{-\infty}^x \varphi_y(y)\,\mathrm{d}y-\int_x^\infty \varphi_y(y)\,\mathrm{d}y=2\varphi(x).$$

The proof for general $\varphi\in W^{1,1}(\mathbb{R})$ is completed by a standard approximation argument.

□

Now, we are in a position to prove Theorem 2.1 concerning the decay of solution in the L^p-spaces.

Proof of Theorem 2.1. Note, that, by (2.1), we have $\|u(t)\|_1 = \|u_0\|_1$ which implies (2.3) for $p = 1$. Hence, we can assume that $p > 1$.

We multiply equation (1.1) by pu^{p-1} (recall that u is nonnegative), integrate with respect to x over $\mathbb{R}$, and integrate by parts to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} u^p \, \mathrm{d}x = -\frac{4(p-1)\varepsilon}{p}\int_{\mathbb{R}} \left[\left(u^{p/2}\right)_x\right]^2 \mathrm{d}x + (p-1)\int_{\mathbb{R}} u^p K' * u_x \, \mathrm{d}x.$$

First term on the right-hand side (containing $\varepsilon > 0$) is obviously nonpositive, hence, we skip it in our estimates. Using the explicit form of the kernel $K' = -\frac{A}{2}H + V$ and Lemma 3.1, we rewrite the second term as follows:

$$(p-1)\int_{\mathbb{R}} u^p K' * u_x \, \mathrm{d}x = (p-1)\left(-A\int_{\mathbb{R}} u^{p+1}\, \mathrm{d}x + \int_{\mathbb{R}} u^p \; V_x * u \, \mathrm{d}x\right). \tag{3.1}$$

Notice, that a simple computation involving the Hölder and the Young inequalities leads to the estimates

$$\left|\int_{\mathbb{R}} u^p \; V_x * u \, \mathrm{d}x\right| \le ||V_x * u||_{p+1} ||u^p||_{\frac{p+1}{p}} \le ||V_x||_1 ||u||_{p+1}^{p+1}. \tag{3.2}$$

Hence, using (3.1) and (3.2) we get

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)^p \, \mathrm{d}x \le (p-1)\left(-A + ||V_x||_1\right) ||u(t)||_{p+1}^{p+1}. \tag{3.3}$$

Moreover, it follows from the Hölder inequality (with the exponents p and $\frac{p}{p-1}$) that

$$\int_{\mathbb{R}} u^p \, \mathrm{d}x = \int_{\mathbb{R}} u^{\frac{1}{p}} \; u^{\frac{p^2-1}{p}} \, \mathrm{d}x \le \Big(\int_{\mathbb{R}} u \, \mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}} u^{p+1} \, \mathrm{d}x\Big)^{\frac{p-1}{p}},$$

which means

$$\int_{\mathbb{R}} u^{p+1} \, \mathrm{d}x \ge ||u_0||_1^{\frac{-1}{p-1}} \Big(\int_{\mathbb{R}} u^p \, \mathrm{d}x\Big)^{\frac{p}{p-1}}, \tag{3.4}$$

because $\|u(t)\|_1 = \|u_0\|_1$. Applying estimate (3.4) to (3.3), we obtain the following differential inequality for $\int_{\mathbb{R}} u^p \, \mathrm{d}x$:

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)^p \, \mathrm{d}x \le (p-1)\left(-A + ||V_x||_1\right) ||u_0||_1^{-\frac{1}{p-1}} \left(\int_{\mathbb{R}} u(x,t)^p \, \mathrm{d}x\right)^{\frac{p}{p-1}}. \tag{3.5}$$

It is easy to prove that any nonnegative solution of the differential inequality

$$\frac{d}{dt} f(t) \le -D f(t)^{\frac{p}{p-1}},$$

with a constant $D > 0$, satisfies

$$f(t) \le \left(\frac{D}{p-1}\right)^{1-p} t^{1-p}.$$

Hence, it follows from (3.5) and from the assumption $\|V_x\|_1 < A$ that

$$||u(t)||_p \le (A - ||V_x||_1)^{\frac{1-p}{p}} \, ||u_0||_1^{1/p} \, t^{\frac{1-p}{p}} \tag{3.6}$$

for all $t > 0$. Finally, passing to the limit $p \to \infty$ in (3.6) we obtain

$$||u(t)||_\infty \le (A - ||V_x||_1)^{-1} \ t^{-1}$$

for all $t > 0$. This completes the proof of Theorem 2.1. □

Let us now recall some result on smooth approximations of rarefaction waves, more precisely, the solution of the following Cauchy problem:

$$\begin{aligned} &Z_t - \varepsilon Z_{xx} + AZZ_x = 0, \\ &Z(x,0) = Z_0(x) = \tfrac{1}{2}H(x). \end{aligned} \tag{3.7}$$

where $A > 0$.

Lemma 3.2 (Hattori-Nishihara [7]). *Problem* (3.7) *has a unique, smooth, global-in-time solution* $Z(x,t)$ *satisfying*

i) $-1/2 < Z(x,t) < 1/2$ *and* $Z_x(x,t) > 0$ *for all* $(x,t) \in \mathbb{R} \times (0,\infty)$*;*

ii) for every $p \in [1,\infty]$*, there exists a constant* $C = C(p) > 0$ *independent of* $\varepsilon > 0$ *such that*

$$\|Z_x(t)\|_p \le Ct^{-1+1/p}$$

and

$$\|Z(t) - W^R(t)\|_p \le Ct^{-(1-1/p)/2}$$

for all $t > 0$*, where* $W^R(x,t)$ *is the rarefaction wave given by formula* (2.8)*.*

Sketch of the proof. All results stated in Lemma 3.2 can be found in [7] with some additional improvements contained in [10, sect. 3], and they are deduced from an explicit formula for smooth approximation of rarefaction waves. Here however, we should emphasise that the authors of [7] consider equation (3.7) with $\varepsilon = 1$ but, by a simple scaling argument, we can extend those results for all $\varepsilon > 0$. Indeed, we check that the function $f(x,t) = Z(\varepsilon x, \varepsilon t)$ satisfies $f_t - f_{xx} + Aff_x = 0$. Hence, by the result from [7] we have

$$\|f_x(t)\|_p \le Ct^{\frac{1-p}{p}} \qquad \text{and} \qquad \|f(t) - W^R(t)\|_p \le Ct^{-(1-1/p)/2}.$$

Now, coming back to original variables, we have

$$\varepsilon^{\frac{p-1}{p}} \|Z_x(\cdot, \varepsilon t)\|_p \le C \ (\varepsilon t)^{\frac{1-p}{p}} \varepsilon^{\frac{p-1}{p}}$$

and so, defining the new variable $\tilde{t} = \varepsilon t$, we obtain $\|Z_x(\tilde{t})\|_p \le C\ \tilde{t}^{\frac{1-p}{p}}$ with a constant C independent of ε. A similar reasoning should be applied in the case of the second inequality in Lemma 3.2.ii. □

Next, we study the large time asymptotics of $U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y - \frac{1}{2}$, which satisfy equation (2.4). Recall that $u = U_x$. In the proof of Theorem 2.4, we need the following auxiliary result.

Lemma 3.3. *Let u_0 satisfy conditions* (2.9)*. Assume that $U = U(x,t)$, defined by* (2.4)*, is the solution of equation* (2.5) *supplemented with the initial condition $U_0(x) = \int_{-\infty}^{x} u_0(y)\,dy - 1/2$ and $Z = Z(x,t)$ is the smooth approximation of the rarefaction wave, namely, the solution of problem* (3.7)*. Then, for every $t_0 > 0$ we have*

$$\sup_{t>t_0} \frac{1}{\log(2+t)} \|U(t) - Z(t)\|_1 < \infty$$

Proof. At the beginning, let us notice that assumption (2.9) on u_0 imply that $U_0(x) \in L^1(-\infty, 0)$ and $U_0(x) - 1 \in L^1(0,\infty)$. Hence, we have that $U_0 - Z_0 \in L^1(\mathbb{R})$.

Denoting $R = U - Z$ and using equations (2.5) and (3.7), we see that this new function satisfies

$$R_t = \varepsilon R_{xx} - \frac{A}{2}(U^2 - Z^2)_x + U_x\ V * U_x.$$

We multiply this equation by $\operatorname{sgn} R$ (in fact, by a smooth approximation of $\operatorname{sgn} R$) and we integrate with respect to x to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} |R|\,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} R_{xx}\operatorname{sgn} R\,\mathrm{d}x - \frac{A}{2}\int_{\mathbb{R}} (U^2 - Z^2)_x \operatorname{sgn} R\,\mathrm{d}x + \int_{\mathbb{R}} U_x\ V * U_x \operatorname{sgn} R\,\mathrm{d}x.$$

The first term on the right-hand side of the above equation is nonpositive because this is the well-known Kato inequality. The second term is equal to 0 because of the following calculations:

$$\begin{aligned}
\int_{\mathbb{R}} (U^2 - Z^2)_x \operatorname{sgn} R\,\mathrm{d}x &= \int_{\mathbb{R}} \left(R^2 + 2RZ\right)_x \operatorname{sgn} R\,\mathrm{d}x \\
&= \int_{\mathbb{R}} 2R_x|R|\,\mathrm{d}x + \int_{\mathbb{R}} 2ZR_x \operatorname{sgn} R\,\mathrm{d}x + \int_{\mathbb{R}} 2Z_x|R|\,\mathrm{d}x \\
&= -2\int_{\mathbb{R}} Z_x|R|\,\mathrm{d}x + 2\int_{\mathbb{R}} Z_x|R|\,\mathrm{d}x = 0
\end{aligned}$$

since $\int_{\mathbb{R}} R_x|R|\,\mathrm{d}x = 0$. Moreover, using the Young inequality, we have

$$\left|\int_{\mathbb{R}} U_x\ V * U_x\ \operatorname{sgn} R\,\mathrm{d}x\right| \le \|U_x\ V * U_x\|_1 \le \|U_x\|_\infty \|V\|_1 \|U_x\|_1.$$

Hence, by the fact that $U_x(t) = u(t)$ and using the decay estimates from Theorem 2.1 for $p = 1$ and $p = \infty$ we get the following differential inequality

$$\frac{d}{dt}\|R(t)\|_1 \leq Ct^{-1}$$

which completes the proof of Lemma 3.3. □

Now, we are in a position to prove our main result about convergence the primitive of u towards a rarefaction wave.

Proof of Theorem 2.4. Let $Z = Z(x,t)$ be the smooth approximation of the rarefaction wave from Lemma 3.2. Denote $R = Z - U$. Hence, by Lemma 3.2 and Theorem 2.1, we have

$$\|R_x(t)\|_\infty = \|U_x(t) - Z_x(t)\|_\infty \leq \|u(t)\|_\infty + \|Z_x(t)\|_\infty \leq C\, t^{-1}$$

for a constant $C > 0$. Moreover, using the Sobolev-Gagliardo-Nirenberg inequality

$$\|R\|_p \leq C\|R_x\|_\infty^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|R\|_1^{\frac{1}{2}(1+\frac{1}{p})},$$

valid for every $p \in (1,\infty]$ and Lemma 3.3 we have

$$\|U(t) - Z(t)\|_p \leq Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+t)\right)^{\frac{1}{2}(1+\frac{1}{p})}$$

for all $t > 0$.

Finally, to complete the proof, we use Lemma 3.2 to replace the smooth approximation $Z(x,t)$ by the rarefaction wave $W^R(x,t)$. □

The proof of Theorem 2.5 relies on a form of Aubin-Simon's compactness result that we recall below.

Theorem 3.4 ([14, Theorem 5]). *Let X, B and Y be Banach spaces satisfying $X \subset B \subset Y$ with compact embedding $X \subset B$. Assume, for $1 \leq p \leq +\infty$ and $T > 0$, that*

- *F is bounded in $L^p(0,T;X)$,*
- *$\{\partial_t f\ :\ f \in F\}$ is bounded in $L^p(0,T;Y)$.*

Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p = +\infty$).

Proof of Theorem 2.5. We denote U^ε as a solution of equation (2.5) with $\varepsilon > 0$ supplemented with a initial condition (2.12). The proof follows three steps: first we show that the family

$$\mathcal{F} \equiv \{U^\varepsilon : \varepsilon \in (0,1]\},$$

is relative compact in $C([t_1,t_2],C[-R,R])$ for every $0 < t_1 < t_2 < \infty$ and every $R > 0$. Next, we show that there exist a function $\bar{U} = \lim_{\varepsilon\to 0} U^\varepsilon$ which is a weak solution of problem (2.11)-(2.12). Finally we prove that $\bar{U}$ satisfy estimate (2.13).

Step 1. Compactness. We apply Theorem 3.4 with $p = \infty$, $F = \mathcal{F}$, and

$$X = C^1([-R, R]), \qquad B = C([-R, R]), \qquad Y = W^{-1,1}([-R, R]),$$

where $R > 0$ is fixed and arbitrary, and Y is the dual space of $W_0^{1,1}([-R, R])$. Obviously, the embedding $X \subseteq B$ is compact by the Arzela-Ascoli theorem.

First, we show that the sets $\mathcal{F}$ and $\{\partial_x U^\varepsilon : \varepsilon \in (0, 1]\}$ are bounded subsets of $L^\infty([t_1, t_2], C([-R, R]))$. Indeed, it follows from definition of function U^ε, namely from (2.4), that

$$|U^\varepsilon(x, t)| \le \|(U^\varepsilon)_x(\cdot, t)\|_1 + \frac{1}{2} = \|u_0\|_1 + \frac{1}{2}. \tag{3.8}$$

Moreover, using Theorem 2.1 we have

$$\|(U^\varepsilon)_x(\cdot, t)\|_\infty \le (A - \|V_x\|_1)^{-1}\, t^{-1}. \tag{3.9}$$

To check the second condition of Aubin-Simon's compactness criterion, it is suffices to show that there is a positive constant C which independent of $\varepsilon \in (0, 1]$ such that $\sup_{t\in[t_1,t_2]} \|\partial_t U^\varepsilon\|_Y \le C$. Let us show this estimate by a duality argument. For every $\varphi \in C_c^\infty((-R, R))$ and $t \in [t_1, t_2]$, by (3.8), (3.9) and Theorem 2.1, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} \partial_t U^\varepsilon(t)\varphi \,\mathrm{d}x\right| &\le \left|\int_{\mathbb{R}} \varepsilon U_x^\varepsilon(t)\varphi_x \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} A U^\varepsilon(t) U_x^\varepsilon(t)\varphi \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} U_x^\varepsilon(t) V * U_x^\varepsilon(t)\varphi \,\mathrm{d}x\right| \\
&\le \|\varphi_x\|_\infty \int_{\mathbb{R}} |U_x^\varepsilon(t)| \,\mathrm{d}x + A\|U^\varepsilon(t)\|_\infty \|\varphi\|_\infty \int_{\mathbb{R}} |U_x^\varepsilon(t)| \,\mathrm{d}x + \|U_x^\varepsilon(t)\|_\infty^2 \|V\|_1 \|\varphi\|_1 \\
&\le \|\varphi_x\|_\infty \|u_0\|_1 + A\|u_0\|_1(\|u_0\|_1 + 1/2)\|\varphi\|_\infty + (A - \|V_x\|_1)^{-2} t_1^{-2} \|V\|_1 \|\varphi\|_1.
\end{aligned}$$

Hence, the proof of Step 1 is completed.

Step 2. Limit function. By Step 1, for every $0 < t_1 < t_2 < +\infty$, the family $\{U^\varepsilon : \varepsilon \in (0, 1]\}$ is relatively compact in $C([t_1, t_2], C(-R, R))$. Consequently, by a diagonal argument, there exists a sequence of $\{U^{\varepsilon_n} : \varepsilon_n \in (0, 1]\}$ and a function $\bar{U} \in C((0, +\infty), C(\mathbb{R}))$ such that

$$U^{\varepsilon_n} \to \bar{U} \quad \text{as} \quad \varepsilon_n \to 0 \quad \text{in} \quad L^\infty_{loc}\big(\mathbb{R} \times (0, +\infty)\big). \tag{3.10}$$

Moreover, by the Banach-Alaoglu Theorem, it follows from the estimate (3.9) that

$$U_x^{\varepsilon_n} \to \bar{U}_x \quad \text{as} \quad \varepsilon_n \to 0$$

weak-$*$ in $L^\infty_{loc}\big((0, \infty), L^\infty(\mathbb{R})\big)$.

Now, multiplying equation (2.5) by a test function $\varphi \in C_c^\infty(\mathbb{R}\times[0,+\infty))$ and integrating the resulting equation over $\mathbb{R}\times[0,\infty)$, we obtain the identity

$$\begin{aligned}(3.11)\qquad -\int_0^\infty\int_{\mathbb{R}} U^{\varepsilon_n}\varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x = \varepsilon_n \int_0^\infty\int_{\mathbb{R}} U^{\varepsilon_n}\varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{A}{2}\int_0^\infty\int_{\mathbb{R}} (U^{\varepsilon_n})^2\varphi_x\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * U^{\varepsilon_n}\right)\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

It is easy to pass to the limit $\varepsilon_n \to 0$ in left-hand side of (3.11), using the Lebesgue dominated convergence theorem. To deal with term in the right-hand side we make the following decomposition:

$$(3.12)\qquad \int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * U^{\varepsilon_n}\right)\varphi\,\mathrm{d}x = \int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * (U^{\varepsilon_n} - \bar{U})\right)\varphi\,\mathrm{d}x + \int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * \bar{U}\right)\varphi\,\mathrm{d}x.$$

We can estimate the first term on the right-hand side of (3.12) as follows:

$$(3.13)\qquad \left|\int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * (U^{\varepsilon_n} - \bar{U})\right)\varphi\,\mathrm{d}x\right| \le \|U_x^{\varepsilon_n}(t)\|_\infty \int_{\mathbb{R}} \left|V_x * (U^{\varepsilon_n} - \bar{U})\varphi\right|\,\mathrm{d}x$$

Let us notice, that $V_x * (U^{\varepsilon_n} - \bar{U})$ tends to zero as $\varepsilon_n \to 0$ by Lebesgue dominated convergence theorem and it is bounded independently of ε_n. Hence, using the Lebesgue dominated convergence theorem and Theorem 2.1, we deduce that the right-hand side of (3.13) converge to zero. The second term on the right-hand side of (3.12) obviously converge to $\int_{\mathbb{R}} \bar{U}_x\left(V_x * \bar{U}\right)\varphi\,\mathrm{d}x$ by the weakly-$*$ convergence of $U_x^{\varepsilon_n}$ in $L^\infty(\mathbb{R})$ since $(V_x * \bar{U})\varphi \in L^1(\mathbb{R})$. This completes the proof of Step 2.

Step 3. Convergence towards rarefaction wave. To prove (2.13), we use the Fatou Lemma and (3.10), to obtain

$$\|\bar{U}(t) - W^R(t)\|_p \le \liminf_{\varepsilon_n\to 0} \|U^{\varepsilon_n}(t) - W^R(t)\|_p$$

for all $t > 0$.

Now, it is enough to use Theorem 2.4 to estimate the quantity on right-hand side, since constant C in (2.10) is independent of ε. Hence the proof of Theorem 2.5 is finished. □

At last, we prove Corollary 2.6.

Proof of Corollary 2.6. First, we express the result stated in Theorems 2.4 and 2.5 in another way. We consider the rescaled family of function $U^\lambda(x,t) = U(\lambda x, \lambda t)$ for all $\lambda > 0$. Let us also notice that $W^R(x,t)$ is self-similar in the sense that $\left(W^R\right)^\lambda(x,t) = W^R(x,t)$ for all $x \in \mathbb{R}$, $t > 0$, $\lambda > 0$. Hence, changing the variables and using Theorem 2.4

and Theorem 2.5 for the case $\varepsilon = 0$, we obtain

$$\|U^\lambda(\cdot, t_0) - \left(W^R\right)^\lambda(\cdot, t_0)\|_p = \lambda^{-1/p}\|U(\cdot, \lambda t_0) - W^R(\cdot, \lambda t_0)\|_p \leq C\lambda^{-1/p}(\lambda t_0)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+\lambda t_0)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)} \to 0$$

as $\lambda \to \infty$. It means that the family of functions U^λ converge in $L^p(\mathbb{R})$ as $\lambda \to \infty$ towards $W^R(x,t)$ for every $t_0 > 0$ and $p \in (1, \infty]$.

This scaling argument allows us to express the convergence of solutions to original problem (1.1)-(1.2) towards a self-similar profile. Indeed, let us note that since $u = U_x$, it follows immediately that $u^\lambda(x,t) = \lambda u(\lambda x, \lambda t) = \partial_x U^\lambda(x,t)$. Hence, the weak convergence of u^λ towards the distributional derivative of the rarefaction wave $\partial_x W^R$ is the immediate consequence of the Lebesgue dominated convergence theorem and of Theorem 2.4 for $p = \infty$ since $|U^\lambda(x, t_0)| \leq \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x + \frac{1}{2}$ □

References

[1] A.L. Bertozzi, J.A. Carrillo, and T. Laurent, *Blowup in multidimensional aggregation equations with mildly singular interaction kernels*, Nonlinearity **22**, (2009), pp. 683–710.

[2] A.L. Bertozzi and Y. Huang, *Self-similar blowup solutions to an aggregation equation in* $\mathbb{R}^n$, SIAM J. Appl. Math. **70**, (2010), pp. 2582–2603.

[3] A.L. Bertozzi and T. Laurent, *Finite-time blow-up of solutions of an aggregation equation in* $\mathbb{R}^n$, Commun. Math. Phys. **274**, (2007), pp. 717–735.

[4] M. Bodnar and J.J.L Velázquez, *Derivation of macroscopic equations for individual cell-based models: A formal approach*, Math. Meth. Appl. Sci. **28**, (2005), pp. 1757–1779.

[5] T. Cieślak, P. Laurençot and C. Morales-Rodrigo, *Global existence and convergence to steady states in a chemorepulsion system*, Parabolic and Navier-Stokes equations. Part 1, 105–117, Banach Center Publ., 81, Part 1, Polish Acad. Sci. Inst. Math., Warsaw, 2008.

[6] L. C. Evans, Partial Differential Equation, AMS, *Rhode Island*, 1998.

[7] Y. Hattori and K. Nishihara, *A note on the stabibity of the rarefaction wave of the Burgers equation*, Japan J. Indust. Appl. Math. **8**, (1991), pp. 85–96.

[8] G. Karch and K. Suzuki, *Spikes and diffusion waves in one dimensional model of chemotaxis*, Nonlinearity **23**, (2010), pp. 3119–3137.

[9] G. Karch and K. Suzuki, *Blow-up versus global existence of solutions to aggregation equations*, (2009), arXiv:1004.4021.

[10] S. Kawashima and Y. Tanaka, *Stability of rarefaction waves for a model system of radiating gas*, Kyushu J. Math. **58**, (2004), pp. 211–250.

[11] T. Laurent, *Local and global existence for an aggregation equation*, Commun. Partial Diff. Eqns **32**, (2007), pp. 1941–1964.

[12] T. Nagai, R. Syukuinn and M. Umesako, *Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in* $\mathbb{R}^N$, Funkcial. Ekvac. **46**, (2003), pp. 383–407.

[13] T. Nagai and T. Yamada, *Large time behaviour of bounded solutions to a parabolic system of chemotaxis in the whole space*, J. Math. Anal. Appl. **336**, (2007), pp. 704–726.

[14] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pure Appl. **146**, (1987), pp. 65–96.

[15] A. Stevens, *A stochastic cellular automaton modeling gliding and aggregation of myxobacteria*, SIAM J. Appl. Math. **61**, (2000), pp. 172–182.

Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, POLAND

E-mail address: Rafal.Celinski@math.uni.wroc.pl