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Models of discrete and continuous cell differentiation in the framework of transport equation

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Abstract

We introduce a class of structured population models describing cell differentiation that consists of discrete and continuous transitions. The model is defined in a framework of measure-valued solutions of a nonlinear transport equation with a growth term. To obtain ODE-type quasistationary node points we exploit the idea of non-Lipschitz zeroes in the velocity. This, in combination with the so-called measure-transmission conditions, allows us to prove the existence and uniqueness of solutions. Since the analysis has biological motivations, we provide examples of its application.

Keywords: structured population model, cell differentiation, measure-valued solution, transport equation, transmission conditions.

AMS Subject Classification: 28A33, 35F16, 35F31, 92D25.

1 Introduction

1.1 Discrete versus continuous models of cell differentiation

Mathematical models have a long history of use to understand processes of cell differentiation and tissue regeneration. Tissues are composed of cells of different maturation stages and it is an open question whether maturation is a discrete or a continuous process. One established method of modeling of hierarchical cell systems is to use a discrete collection of ordinary differential equations describing dynamics of cells at n different maturation stages and transitions between the stages [18, 19, 21, 24]. Another class of models is based on partial differential equations of the transport type or delay differential equations describing cell differentiation as a continuous process [1, 2, 3, 6, 8, 11, 12].

In this paper, to describe a differentiation process that consists of discrete and continuous transitions we propose a nonlinear structured population model in the form of a transport equation defined on the space of nonnegative Radon measures. Dynamics of cell population is based on the birth and death process as well as on the transitions of cells between subsequent differentiation stages. The framework proposed by us is an extension of the existing approaches and allows for coupling of both discrete and continuous aspects of the transitions between different stages of the process.

The point of departure for this work is a structured population model of cell differentiation, which has been recently proposed in [12] in the form of two ordinary differential equations coupled with a transport equation,

$$\begin{array}{rcl} \frac{d}{dt}w(t) & = & \alpha(v(t))p_w w(t), \\ \partial_t u(t,x) + \partial_x[g(v(t),x)u(t,x)] & = & p(v(t),x)u(t,x), \\ g(v(t),0)u(t,0) & = & (1-\alpha(v(t)))p_w w(t), \\ \frac{d}{dt}v(t) & = & g(x^*,v(t))u(t,x^*) - \mu v(t), \end{array} \tag{1}$$

with initial data

$$w(0) = w_0 \geq 0, \qquad u(0,x) = u_0(x) \geq 0, \qquad v(0) = v_0 \geq 0.$$

The model describes time evolution of a population of cells, where the numbers of undifferentiated stem cells and fully differentiated mature cells, are modeled using functions $w(t)$ and $v(t)$, respectively, whereas the concentration of cells at the intermediate stages of differentiation is modeled by the function $u(t,x)$ structured by the stage of cell differentiation $x \in [0,x^*]$. The function $\alpha(v(t))p_w$, with p_w being the proliferation rate, describes growth and differentiation of stem cells, modeled explicitly in Ref. [12]. It is assumed that the progenitor cells differentiate at the rate g, which depends on their maturity stage and is also regulated by the feedback from mature cells. Furthermore, $p(v(t),x)$ denotes the proliferation rates of precursor cells. Mature cells do not divide and die at the rate μ. The model is based on the hypothesis that cell differentiation may take place between cell divisions and hence a continuum of differentiation stages can be defined. The rationale for such assumption is provided by the fact that differentiation is controlled by intracellular biochemical processes, which are continuous in time, at least when averaged over a large number of cells. An example of such behavior has been observed in neurogenesis [16].

As it is shown in [12] this model may exhibit different dynamics than the corresponding discrete model, proposed in [21], and analyzed in [22, 24]. The discrete, so-called multi-compartmental, model takes the form of n ordinary differential equations.

$$\begin{array}{rcl} \frac{d}{dt}u_1 &=& p_1(s)u_1 - g_1(s)u_1, \\ \frac{d}{dt}u_i &=& p_i(s)u_i + g_{i-1}(s)u_{i-1} - g_i(s)u_i, \quad \text{for } i = 2, ..., n-1 \\ \frac{d}{dt}u_n &=& g_{n-1}(s)u_{n-1} - d_n u_n, \end{array} \tag{2}$$

with initial data

$$u_i(0) = u_0, \quad \text{for } i = 1, ..., n,$$

where $g_i(s)u_i$ denotes the flux of cells from subpopulation i differentiating to subpopulation $i+1$. The terms $p_i(s)u_i$ and $d_i u_i$ describe cell fluxes due to proliferation and death, respectively. In the general case, proliferation or differentiation may depend on signal intensity s, which in the situation under consideration is a function of u_n. This model is based on a classical assumption that in each lineage of cell precursors there exists a discrete chain of maturation stages, which are sequentially traversed, [20], and the transitions between compartments are coordinated by the divisions of cells.

Comparing both models shows that they have different structure of steady states. The discrete compartmental model admits semi-trivial steady states of the form $(0, .., 0, \bar{u}_i, .., \bar{u}_n)$, which do not exist in the continuous differentiation model [12]. In fact, as it was shown in [12], the structured population model (1) is not a limit of the discrete model (2). To pass from discrete to continuous model, one needs to assume, for a proper time scaling, that commitment and maturation of cell progenitors do not proceed according to the division clock (one division equals one maturation step) but are continuous processes and advance between divisions and therefore, the models are not equivalent. Nevertheless, the models may exhibit the same dynamics for a suitable choice of the maturation rate function, [12].

Motivated by these observations, in the present paper we develop a new structured population model of cell differentiation, which may exhibit the dynamics of the multi-compartmental model, in particular, it admits semi-trivial steady states (see Section 2.1). The model has a form of a transport equation with a nonlinear and non-Lipschitz velocity coefficient, complemented by additional transmission conditions. Main result of this work consists in showing how to describe dynamics of a transport process involving discrete and continuous transitions in the framework of structured population models. Building the model requires taking into consideration biological assumptions as well as analytical aspects related to the model structure.

The paper is organized as follows. In Section 2 we propose the model and demonstrate its relation to models (1) and (2). In Section 3 we define the framework and introduce the measure-transmission conditions, which lead to the uniqueness of solutions. We compare our approach to that using the notion of superposition solution, see, e.g. [4]. Section 4 concerns the linear problem with a constant feedback. We explicitly define a solution by the transport of measure formula and ramification of characteristics at the node points. In a series of technical lemmas we prove that it indeed is the measure-transmission solution defined in Section 3. Section 5 is concerned with uniqueness in the linear case. We use the dual equation method extended to account for inhomogeneity of the domain. As a consequence, an explicit solution is obtained. In Section 6 we extend the above results to time-dependent feedback. Sections 7 and 8 are devoted to the problem of existence and uniqueness in the nonlinear case formulated as a fixed point question. Schauder's Fixed Point Theorem is used to prove existence and the contraction principle is applied to prove uniqueness.

2 New model of discrete-continuous differentiation

We consider a generic model of cell differentiation and proliferation given in the form of the transport equation defined in the space of nonnegative Radon measures (for definitions and basic properties of Radon measures as well as closely related BV functions we refer to e.g. [7], [9], [13]),

$$\begin{aligned} \partial_t \mu(t) + \partial_x \left(g(v(t), x)\mu(t) \right) &= p(v(t), x)\mu(t), \\ \mu(0) &= \mu_0, \end{aligned} \tag{3}$$

where $x \in [0, x^*]$ denotes the state of an individual and, for every Borel set $A \in \mathcal{B}(\mathbb{R})$, $\mu(t)(A) = \int_{\{A\}} d\mu(t)$ is the measure of cells being in state $x \in A$ at time t. $v(t) = \int_{\{x^*\}} d\mu(t)$ denotes the mass of the last point (corresponding to the concentration of mature cells). To describe the dynamics containing discrete and continuous transitions, we assume that the function g vanishes at finitely many points x_i, $i = 0, \ldots, N$, and its reciprocal, $1/g$, is integrable in $\mathbb{R}^+$. The latter condition reflects the biological assumption that all differentiation stages can be achieved in a finite time, what is the case in healthy cell systems. Mathematically, it implies that g is non-Lipschitz in these points and it is related to the Osgood's uniqueness condition for the ODE generating characteristics. Zeros of the function g define the *node points* in the domain of differentiation stages, x_i, for $i = 0, ..., N$ such that $0 = x_0 < x_1 < ... < x_N = x^*$. Henceforth, the domain is a union of the node points and intervals between them, i.e., $x \in [x_0, x_N] = \{x_0\} \cup (x_0, x_1) \cup \{x_1\} \cup (x_1, x_2) ... \cup \{x_N\}$. Note that here the term $p(v(t), x)\mu(t)$ describes both the proliferation and the death process and therefore, the function $p(v(t), x)$ may be negative.

Further, we assume that g is a function with separated variables, i.e., $g(v(t), x) = g_1(v(t))g_2(x)$, with g_2 vanishing at the node points. This assumption allows to transform the variables in the equation so that we obtain equation (3) with $g(v(t), x) = g_1(v(t))\mathbf{1}_{x\neq x_i}$ for $i = 0, ..., N$, where the characteristic function of the set of node points is defined by

$$\mathbf{1}_{x\neq x_i} := \begin{cases} 1 & \text{for } x \notin \{x_0, \dots, x_N\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We note that additional conditions have to be imposed at the node points x_i, $i = 0, \dots, N$ to provide uniqueness of solutions. We propose to supplement the model with *transmission conditions* defined by

$$g_1(v(t))\frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) = c_i(v(t)) \int_{\{x_i\}} d\mu(t), \tag{5}$$

where $\frac{D\mu}{D\mathcal{L}^1}$ denotes the density of the measure μ with respect to one-dimensional Lebesgue measure. These conditions determine the outflux from the node points, with c_i being the outflux rates. They are the natural consequence of a biological assumption that the fate of cells at some differentiation stages is determined by external signaling cues, in our case dependent on the concentration $v(t)$. Consequently, for a given signaling intensity the distribution of the fate is unique. This reasoning also agrees with the mathematical structure of the model. In the model without transmission conditions solutions are non-unique as we can see in the following example.

Example 2.1. Let us consider the equation with no feedback and such that $x \in [0, \infty)$ with $x_0 = 0$. Assuming transport velocity having the form $g(x) = \mathbf{1}_{x\neq x_0}$, and a growth constant equal 0, we obtain

$$\partial_t\mu + \partial_x\mathbf{1}_{x\neq x_0}\mu = 0. \tag{6}$$

For initial condition $\mu_0 = \delta_0$,

- $\mu^{null}(t) = \delta_0(x)$,
- $\mu^{\infty}(t) = \delta_t(x)$,
- $\mu^1(t) = e^{-t}\delta_0(x) + e^{x-t}\mathbf{1}_{(0,t]}(x)$

are solutions of the above problem. Here, we adopt the notation, where Dirac delta is also treated as a function, i.e., the equality means the density is prescribed with respect to one-dimensional Lebesgue measure.

Proof. μ^{null} is obviously a solution. As for μ^{∞}, we take a test function $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R})$ and calculate

$$\begin{aligned}
&\big(\partial_t\mu^{\infty} + \partial_x(\mathbf{1}_{x\neq 0}\mu^{\infty}), \varphi(t,x)\big)\\
= & -\int\int D_1\varphi(t,x)d\delta_t(x)dt - \int \varphi(0,x)d\delta_0(x) - \int\Big(\int D_2\varphi(t,x)\mathbf{1}_{x\neq 0}d\delta_t(x)\Big)\,dt\\
= & -\int D_1\varphi(t,t)dt - \varphi(0,0) + \int D_2\varphi(t,t)\mathbf{1}_{t>0}dt = -\int \frac{d}{dt}\varphi(t,t)dt - \varphi(0,0) = 0.
\end{aligned}$$

Now we can observe that any convex combination of solutions is a solution as well. For instance, a nontrivial solution is given by $\frac{1}{2}\left(\delta_0(x)+\delta_t(x)\right)$.

Another class of solutions is constituted by μ^{null} concatenated at time $t=s$ with μ^{∞}. We define

$$\mu^s(t) := \begin{cases}\delta_0(x), & t\in[0,s],\\ \delta_{t-s}(x), & t>s.\end{cases}$$

Considering the following convex combination,

$$\begin{aligned}
\int_0^{\infty} e^{-s}\mu^s(t)ds &= \int_0^t e^{-s}\delta_{t-s}(x)ds + \int_t^{\infty} e^{-s}\delta_0(x)ds\\
&= \int_0^t e^{-(t-s)}\delta_s(x)ds + e^{-t}\delta_0(x) = e^{-(t-x)}\mathbf{1}_{(0,t]}(x) + e^{-t}\delta_0(x) = \mu^1(t),
\end{aligned}$$

proves that μ^1 is a distributional solution as well. ☐

This example shows that distributional solutions are in general non-unique. To be more precise, $c_0(t)$ in the transmission condition (5) completing the model (6) is equal $c_0=0$ for μ^{null} and $c_0=1$ for μ^1. In our biological application μ^{null} corresponds to dormant cells, μ^{∞} to the differentiation of all cells at the same time point. μ^1 is a solution in a typical situation, where in a given interval of time only a finite proportion of the stem cells differentiates.

2.1 Link to the previous models

The models of the form (3) with the transmission condition (5) allow coupling of discrete and continuous aspects of the transitions between different stages of the process, such as finite speed of transmission between two stages and admissibility of semitrivial stationary solutions. In particular, the model of continuous cell differentiation (1) can be directly embedded into (3)–(5) by taking $N=1$, $x_0=0$, $x_1=x^*$ and

$$\mu(t) = w(t)\delta_0(x) + u(t,x)\mathcal{L}^1(dx) + v(t)\delta_{x^*}(x).$$

The identification of the model parameters is straightforward,

$$g(v,x) := \begin{cases} \hat{g}(v,x) & \text{for } x \in (0,x^*), \\ 0 & \text{otherwise.} \end{cases}$$

$$p(v,x) := \begin{cases} p_w & \text{for } x = 0 \\ \hat{p}(v,x) & \text{for } x \in (0,x^*) \\ -\mu & \text{for } x = 1, \end{cases}$$

where g and p denote the parameters of the model (3)-(5), and $\hat{g}$ and $\hat{p}$ correspond to the model (1). Finally, $c_0(v) = (1-\alpha(v))p_w$ and $c_1(v) = 0$.

The multi-compartmental model of discrete cell differentiation can be defined in the new framework as a singular limit when the speed of transmission between the stages tends to infinity. We take

$$\begin{aligned} \partial_t \mu(t) + \partial_x \left(\frac{1}{\varepsilon} g\left(v(t), x\right) \mu(t) \right) &= p(v(t),x)\mu(t), \\ \frac{1}{\varepsilon} g\left(v(t), x_i^+\right) \frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) &= c_i(v(t)) \int_{\{x_i\}} d\mu(t), \\ \mu(0) &= \mu_0, \end{aligned}$$

where the support of the initial measure μ_0 and of the proliferation rate function $p(\cdot, x)$ is contained in the set $\{x_1, x_2, ..., x_N\}$; and $p(v(t), x_i) = p_i(v(t))$. Letting ε tend to zero, the system of ODEs (2) is obtained.

The next example shows that in our setting multiple semi-trivial steady states may exist, provided the rates c_i are functions depending on the mass of mature cells. Such steady states appear in the multicompartmental model of type (2), but do not exist in the structured population model (1). Appearance of such steady states may have an interesting biological interpretation describing stemness of the cells as a feature that depends on environmental conditions and is not necessarily linked to a unique cell population [24]. Biological consequences of such scenario were recently discussed in [17].

We consider a discrete-continuous model of cell differentiation with three node points, corresponding to three different differentiation stages, and a continuous differentiation between them. Taking the model

$$\partial_t \mu + \partial_x (\mathbf{1}_{(x \neq x_i)} \mu) = (\mathbf{1}_0(x) + \mathbf{1}_1(x) - \mathbf{1}_2(x))\mu,$$

with a function $c(v)$ such that $c_0(1)$ is arbitrary, $c_1(1) = 1$, $c_0(3) = 1$ and $c_1(3) = 1.5$, we observe that

$$\mu = \delta_1(x) + \mathbf{1}_{(1,2)}(x) + \delta_2(x)$$

is a semi-trivial steady state, whereas

$$\mu = \delta_0(x) + \mathbf{1}_{(0,1)}(x) + 2\delta_1(x) + 3 \cdot \mathbf{1}_{(1,2)}(x) + 3\delta_2(x)$$

is also a steady state.

3 Mathematical framework of the model

In the remainder of this paper we consider the problem

$$\begin{aligned} \partial_t \mu(t) + \partial_x(g_1(v(t))\mathbf{1}_{x\neq x_i}(x)\mu(t)) &= p(v(t),x)\mu(t), & (7)\\ g_1(v(t))\frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) &= c_i(v(t))\int_{\{x_i\}} d\mu(t), & (8)\\ \mu(0) &= \mu_0, & (9)\end{aligned}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, x_i with $i = 0, 1, \ldots, N$ are given points in $\mathbb{R}$, $\mathbf{1}_{x\neq x_i}$ is a characteristic function of the set of node points as defined in (4), and $\frac{D\mu}{D\mathcal{L}^1}$ denotes the density of the measure μ with respect to one-dimensional Lebesgue measure.

Remark 3.1. $\mu(t)$ is a positive Radon measure for every $t \in \mathbb{R}^+$ and, in general, it does not have to be absolutely continuous with respect to one-dimensional Lebesgue measure $\mathcal{L}^1$. However, we stipulate that the limit $\lim_{x\to x_i^+} \frac{D\mu}{D\mathcal{L}^1}$ is well defined (see Definition 3.3).

$v(t) = \int_{\{x_N\}} d\mu(t)$ denotes the mass of the last point (mature cells). The initial data are defined on the interval $[x_0, x_N]$. Note that we define the model on the whole $\mathbb{R}$, although solutions are nonzero only on the interval $[x_0, x_N]$. Assumptions on the model parameters are the following:

Assumptions 3.2. *(i)* $g_1(v) \in W^{1,\infty}(\mathbb{R})$, *and* $g_1 > 0$,

(ii) $p = p(v(t), x) = p_1(v(t))p_2(x)$,

(iii) $p_1(v) \in W^{1,\infty}(\mathbb{R})$,

(iv) $p_2(x) \in L^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}\backslash\{x_0, \ldots x_N\})$,

(vi) $c_i = c_i(v) \in W^{1,\infty}(\mathbb{R})$, $\quad i = 0, 1, \ldots, N$,

(vii) $c_i \geq 0$, $\quad i = 0, 1, \ldots, N$

(viii) $c_N = 0$,

where $W^{1,\infty}(\mathbb{R})$ denotes the standard Sobolev space with norm $\|\psi\|_{W^{1,\infty}(\mathbb{R})} := \max\{\|\psi\|_\infty, \|\partial_x\psi\|_\infty\}$.

Our approach is based on solutions in the spaces of positive Radon measures with metrics appropriate for nonprobabilistic measures, such as flat metric or generalized Wasserstein metric. They metrize both weak* and narrow topologies on each tight subset of Radon measures with uniformly bounded total variation. They are modifications of the Wasserstein distance for the spaces of probability measures, which proved to be useful for the analysis of equations given in a conservative form, e.g. conservative transport equation [5]. Since the nonlinear structured population model is not conservative, we employ a flat metric, also known as the bounded Lipschitz distance [23]. The flat metric corresponds to the dual norm of $W^{1,\infty}(\mathbb{R}^+)$ and is defined as a function $\rho_F : \mathcal{M}(\mathbb{R}^+) \times \mathcal{M}(\mathbb{R}^+) \to [0,\infty]$ such that

$$\rho_F(\mu,\nu) := \sup\Big\{ \int_{\mathbb{R}^+} \psi d(\mu-\nu) \big|\, \psi \in C^1(\mathbb{R}^+), \|\psi\|_{W^{1,\infty}} \leq 1\Big\}, \tag{10}$$

where $\mu,\nu \in \mathcal{M}^+(\mathbb{R}^+)$.

The framework studied in papers [14, 15] was developed for the structured population models with Lipschitz continuous parameter functions $F_i\colon \mathcal{M}(\mathbb{R}^+) \to W^{1,\infty}(\mathbb{R}^+)$, where $F_i(\mu_t)(x) = f(x)G\left(\int_0^\infty \varphi d\mu_t\right)$, with $f \in W^{1,\infty}$, G being a Lipschitz continuous function and $\varphi \in W^{1,\infty}$. Such nonlinearities were proposed for a generic structured population model by Diekmann and Getto in [10]. In case of proposed model (3), the function g has the form $g(v(t),x) = g_1(v(t))\mathbf{1}_{x\neq x_i} = f(x)G\left(\int_0^\infty \varphi d\mu_t\right)$ for $\varphi = \mathbf{1}_{x_N}$ and $f(x) = \mathbf{1}_{x\neq x_i}$, which are not $W^{1,\infty}$ functions. Therefore, the framework developed in [14, 15] cannot be directly applied and well-posedness of the problem requires a new proof.

Definition 3.3 (Measure-transmission solution)**.** *A measure-valued function $\mu \in C([0,\infty),(\mathcal{M},\rho_F))$ with $v(t) = \int_{\{x_N\}} d\mu(t) \in BV_{loc}[0,\infty)$ is called a* measure-transmission *solution of problem (7)–(9), if*

(i) for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$

$$\begin{aligned} -\int_{\mathbb{R}^+}\int_{\mathbb{R}} \partial_t\varphi(t,x)d\mu(t)(x)dt - \int_{\mathbb{R}^+}\int_{\mathbb{R}} g_1(v(t))\mathbf{1}_{x\neq x_i}(x)\partial_x\varphi(t,x)d\mu(t)(x)dt \\ = \int_{\mathbb{R}^+}\int_{\mathbb{R}} p_1(v(t))p_2(x)\varphi(t,x)d\mu(t)(x)dt + \int_{\mathbb{R}} \varphi(0,x)d\mu(0)(x) \end{aligned} \tag{11}$$

(ii) for every $t^* > 0$ there exists $\varepsilon(t^*)$ such that for every $t > t^*$ $\mu(t)$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ for $x \in (x_i, x_i+\varepsilon)$ and for $\mathcal{L}^1$ a.e. $t \in (0,\infty)$

$$\lim_{x\to x_i^+} g_1(v(t))\frac{D\mu(t)}{D\mathcal{L}^1}(x) = c_i(v(t))\int_{\{x_i\}} d\mu(t)$$

(iii) for every i, $i = 0, .., N$, it holds $\int_{\{x_i\}} d\mu(t) \to \int_{\{x_i\}} d\mu(0)$ for $t \to 0$.

In next sections we show existence and uniqueness of the measure-valued solutions of (7)–(9). The proof consists of the following steps:

(1) Demonstration of existence and uniqueness of measure-transmission solutions, in the sense of Definition 3.3, to a linear problem obtained from the original model by freezing variable v. This step includes:

  (1.1) Constructing a solution.

  (1.2) Proving that the constructed solution is a measure-transmission solution.

  (1.3) Proving that transmission conditions provide uniqueness.

(2) Applying a change of variables to extend the existence result to the model with time-dependent velocity function g.

(3) Proving existence of solutions to the nonlinear problem using Schauder's Fixed Point Theorem and Helly's Compactness Theorem.

(4) Proving uniqueness of solutions to the nonlinear problem using the contraction principle.

3.1 Relation to the notion of superposition solution

Our approach is related to the framework developed by Ambrosio and Crippa [4] linking solutions of the linear conservative first order transport equation (continuity equation)

$$\begin{aligned} \partial_t \mu + \mathrm{div}_x(b\mu) &= 0, \\ \mu(0) &= \mu_0 \end{aligned} \tag{12}$$

with solutions of the ordinary differential equation

$$\begin{aligned} \dot\gamma(t) &= b(t, \gamma(t)), \\ \gamma(0) &= x \end{aligned} \tag{13}$$

in case of non-Lipschitz b, leading to the nonuniqueness of solutions of the equation of the characteristic (13). In such case the well-posedness of the continuity equations can be provided by fixing a family of measures determining the probability of the choice of a specific trajectory. To be more specific, we recall the definition of a superposition solution.

Definition 3.4 (after [4]). *Assume that η_x is a bounded Radon measure defined on the space $\Gamma_T = C([0,T];\mathbb{R}^d)$ and the support of η_x contains absolutely continuous functions, which solve the equation* (13). *The superposition solution induced by the family $\eta_{x\in\mathbb{R}^d}$ is a family of measures $\mu_t^\eta \in \mathcal{M}(\mathbb{R}^d)$, for $t\in[0,T]$, defined as follows*

$$\langle \mu_t^\eta, \varphi\rangle = \int_{\mathbb{R}^d}\left(\int_{\Gamma_T}\varphi(\gamma(t))d\eta_x(\gamma)\right)d\mu_0(x), \quad \forall\varphi\in C_c(\mathbb{R}^d).$$

In brief, the superposition solution is a superposition of solutions obtained via transport of measures along nonunique characteristics. It is a unique solution of the problem (12) in a class of problems with a given family of measures η_x.

Remark 3.5. In the linearized case, i.e. $g_1(v(t)) = 1$, $p_1(v(t)) = p_2(x) = 1$, the measure-transmission solution can be proved to be also the superposition solution with a parametrized measure η_x, which is uniquely induced by the transmission condition.

4 Linear problem

First we consider the following linear problem

$$\begin{aligned}
\partial_t\mu(t) + \partial_x\left(\mathbf{1}_{x\neq x_i}\mu(t)\right) &= p(t,x)\mu(t), && (14)\\
\frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) &= c_i(t)\int_{\{x_i\}}d\mu(t), \quad i\in\{0,...,N\}, && (15)\\
\mu(0) &= \mu_0, && (16)
\end{aligned}$$

where $p(t,x) = p_1(t)p_2(x)$ (compare Assumptions 3.2 ii). $p_1(t)$ is an arbitrary BV function of t since the composition of a Lipschitz function with a BV function belongs to BV. We construct an explicit solution of (14)–(16) and show that it is a measure-transmission solution of (14)–(16) in sense of Definition 3.3.

First we define the measure in x_0 and the outflow from x_0

$$\int_{\{x_0\}}d\mu(t) = \left(\int_{\{x_0\}}d\mu(0)\right)\exp\left(\int_0^t(p(s,x_0)-c_0(s))ds\right) \tag{17}$$

$$f_0(t) = c_0(t)\left(\int_{\{x_0\}}d\mu(0)\right)\exp\left(\int_0^t(p(s,x_0)-c_0(s))ds\right),$$

and in any of the inner quasi-stationary points x_i (which have both inflow and outflow) or x_N (where $c_N = 0$)

$$\frac{d}{dt}\int_{\{x_i\}}d\mu(t) = (p(t,x_i)-c_i(t))\int_{\{x_i\}}d\mu(t) + h_i(t),$$

where h_i is defined below. Solving this equation yields the formula for the measure in point x_i

$$\int_{\{x_i\}} d\mu(t) = e^{\int_0^t (p(s,x_i)-c_i(s))ds} \left[\int_{\{x_i\}} d\mu(0) + \int_{(0,t]} h_i(dr) e^{-\int_0^r (p(s,x_i)-c_i(s))ds} \right]. \tag{18}$$

Consequently, the outflow from the point x_i is given by $f_i(t) = c_i(t) \int_{\{x_i\}} d\mu(t)$.

For any Borel set $A \subset (x_{i-1}, x_i))$ it holds that $A = A_1 \cup A_2$, where $A_1 = A \cap (x_{i-1}, x_{i-1}+t)$, $A_2 = A \cap [x_{i-1}+t, x_i)$, where both A_1 and A_2 could be empty (see Fig.1). Projecting A_1 on the line $\{x = x_{i-1}\}$ and A_2 on $\{t = 0\}$, we define the measure of A as

$$\begin{aligned} \mu(t)(A) &:= \mu(t)(A_1) + \mu(t)(A_2) \\ &= \int_{t+x_{i-1}-A_1} e^{\int_\tau^t p(s,x_{i-1}+s-\tau)ds} f_{i-1}(\tau)d\tau + \int_{A_2-t} e^{\int_0^t p(s,x+s)ds} d\mu(0), \end{aligned} \tag{19}$$

where the integrals are calculated on the sets defined in the following way: $A - a := \{y - a \mid y \in A\}$ and $a - A := \{a - y \mid y \in A\}$.

We notice that the first term has regularizing properties and since, after some time, the second term vanishes, the solution becomes more regular than the initial Radon measure.

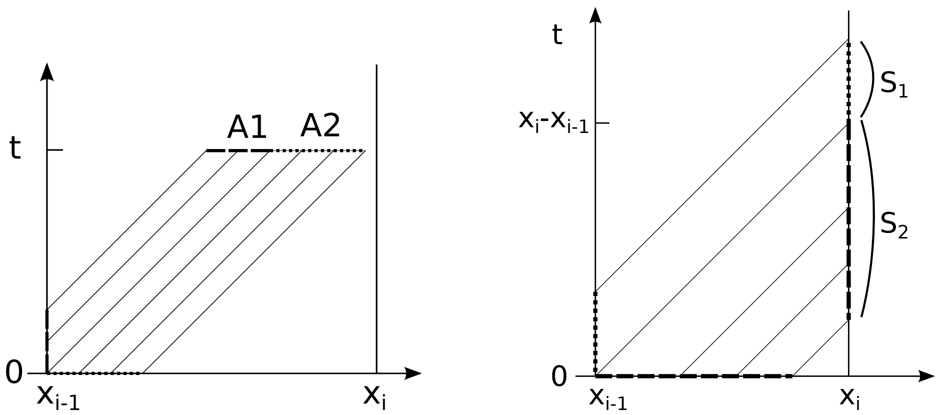

Figure 1: Schematic presentation of the definition of sets A_i and S_i.

Next we compute h_i, i.e. the total inflow to the next quasistationary point. Similarly as above, we define $h_i(t)$ as a measure on the time axis for fixed $x = x_i$. For a bounded Borel set $S \subset (0, \infty)$, it holds that $S = S_1 \cup S_2$, see Fig. 1, with $S_1 = S \cap [x_i - x_{i-1}, \infty)$ and $S_2 = S \cap (0, x_i - x_{i-1})$. We obtain,

$$\begin{aligned} h_i(S) = h_i(S_1) + h_i(S_2) &= \int_{S_1 - x_i + x_{i-1}} e^{\int_\tau^{(\tau + x_i - x_{i-1})} p(s,x_{i-1}+s)ds} f_{i-1}(\tau)d\tau \\ &+ \int_{x_i - S_2} e^{\int_0^{(x_i - x)} p(s,x+s)ds} d\mu(0). \end{aligned} \tag{20}$$

Repeating the above procedure finitely many times provides the solution at any point of the interval $[x_0, x_N]$, i.e.,

$$\mu(t)(A) := \mu(t)(A \cap \{x_0\}) + \mu(t)(A \cap (x_0, x_1)) + \mu(t)(A \cap \{x_1\}) + ... + \mu(t)(A \cap \{x_N\}) \quad (21)$$

By assumption, $c_N = 0$ and therefore, prolongation by 0 for $x < x_0$ and $x > x_N$ allows us to consider measures $\mu(t)$ as defined on the whole $\mathbb{R}$ for every t.

Remark 4.1. Observe that the form of transmission conditions has a regularizing effect, while a solution is traversing a node point.

Showing the properties of μ we use the following property of BV functions.

Proposition 4.2. *(see [7] Proposition 13.12) The product of two BV_{loc} functions is in BV_{loc}.*

Lemma 4.3. *For every $t \in \mathbb{R}^+$ $\mu(t)$ is a Radon measure.*

Proof. Follows by induction. f_0 is an everywhere defined right-continuous BV_{loc} function and h_1 is a measure by (20). The function $t \mapsto \int_{\{x_1\}} d\mu(t)$ is in $BV_{loc}[0, \infty)$ by (18). And this, in turn, yields BV regularity of $f_1(t)$. Repeating this reasoning we prove that $t \mapsto \int_{\{x_i\}} d\mu(t)$ is in BV_{loc} and h_i is a Radon measure for every i. Formulas (18) and (19) define the measure $\mu(t)$. □

Lemma 4.4. *For every $T > 0$ there exist constants $M(T)$ and $K(T)$ such that for any $t \in [0, T]$*

$$\int_{\mathbb{R}} d\mu(t) \leq M(T) e^{K(T)t} \int_{\mathbb{R}} d\mu(0)$$

Remark 4.5. In fact we can prove the local version of this inequality,

$$\int_{[a,b]} d\mu(t) \leq M(T) e^{K(T)t} \int_{[a-T,b]} d\mu(0)$$

Proof of Lemma 4.4. We define:
$C_p(t) = \sup_{[0,t]\times\mathbb{R}} |p(s, y)|$, $C_i(t) = \sup_{[0,t]} |p(s, x_i) - c_i(s)|$, $i = 0, ..., N$, and $C_i^c(t) = \sup_{[0,t]} |c_i(s)|$.
First, we calculate the measure of $\{x_0\}$

$$\int_{\{x_0\}} d\mu(t) = \left(\int_{\{x_0\}} d\mu(0) \right) e^{\int_0^t (p(s,x_0) - c_0(s))ds} \leq \left(\int_{\{x_0\}} d\mu(0) \right) e^{C_0(T)t}.$$

For the first interval we obtain

$$\begin{aligned}
\int_{(x_0,x_1)} d\mu(t) &\leq \left(\int_{(0,t)} f_0(s) ds \right) e^{C_p(T)t} + \left(\int_{(x_0,x_1)} d\mu(0) \right) e^{C_p(T)t} \\
&\leq t \sup_{(0,t)} c_0(s) \sup_{(0,t)} \int_{\{x_0\}} d\mu(s) e^{C_p(T)t} + e^{C_p(T)t} \int_{(x_0,x_1)} d\mu(0) \\
&\leq T C_0^c(t) e^{C_0(T)t} \int_{\{x_0\}} d\mu(0) + e^{C_p(T)t} \int_{(x_0,x_1)} d\mu(0) \\
&\leq M_1(T) e^{K_1(T)t} \int_{[x_0,x_1)} d\mu(0).
\end{aligned}$$

And in the next point x_1 we obtain

$$\begin{aligned}\int_{\{x_1\}} d\mu(t) &\leq e^{C_1(T)t}\int_{\{x_1\}} d\mu(0) + \int_{(0,t]} h_1(dr)e^{C_1(T)t}\\ &\leq e^{C_1(T)t}\int_{\{x_1\}} d\mu(0) + e^{C_1(T)t}\int_{(0,t]} h_1(dr).\end{aligned}$$

Estimating the last integral results in

$$\begin{aligned}\int_{(0,t]} h_1(dr) &\leq e^{C_p(t)t}\left(\int_{(x_0,x_1)} d\mu(0) + \int_{(0,t)} f_0(s)ds\right)\\ &\leq e^{C_p(t)t}\left(\int_{(x_0,x_1)} d\mu(0)\right) + C_1^c\left(\sup_{(0,t)}\int_{\{x_0\}} d\mu(s)\right)\\ &\leq e^{K_2(T)t}\left(\int_{[x_0,x_1]} d\mu(0)\right),\end{aligned}$$

what completes the estimate for x_1. The next terms (on (x_{i-1},x_i) and $\{x_i\}$) are defined recursively and estimated likewise, what proves the lemma.

Lemma 4.6. *For every $t>0$ and for every $i\in\{0,1,...,N\}$ there exists $\varepsilon>0$ such that $\mu(t)$ is absolutely continuous with respect to one-dimensional Lebesgue measure on $(x_i,x_i+\varepsilon)$. Furthermore, $\gamma^t(x):=\frac{D\mu}{D\mathcal{L}^1}(x)$ has a right limit in x_i for $t>0$.*

Proof. f_i in formula (19) is an everywhere defined right continuous BV_{loc} function (in particular locally bounded) and for $\varepsilon<t$ only the first term of this formula is non-vanishing and defines an absolutely continuous Radon measure. To prove the existence of the right limit of $\gamma^t(x)$, we calculate

$$\begin{aligned}\gamma^t(x_i+\delta) &= Jf_i(t-\delta)e^{\int_{t-\delta}^t p(s,x_i+(s-(t-\delta)))ds}\\ &= Jf_i(t-\delta) + Jf_i(t-\delta)\left(e^{\int_{t-\delta}^t p(s,x_i+(s-(t-\delta)))ds}-1\right),\end{aligned}$$

where $J=\frac{\partial x}{\partial t}=1$ is the Jacobian of transformation from x to t. The first term has a limit due to the fact that $f_i\in BV$ and the second term tends obviously to 0. □

Lemma 4.7. *$\mu(t)$ defined in (21) is a measure-transmission solution of problem (14)–(16).*

Proof. The initial condition is satisfied in the sense of Lipschitz-narrow continuity, see Lemma 4.9. Transmission conditions are satisfied as a consequence of Lemma 4.6 with $f_i(t)=c_i(t)\int_{\{x_i\}} d\mu(t)$. To show that (14) holds in sense of Definition 3.3, we need to prove that

$$\begin{aligned}-\int_{\mathbb{R}^+}\int_{\mathbb{R}}\partial_t\varphi(t,x)d\mu(t)dt &- \int_{\mathbb{R}^+}\int_{\mathbb{R}}\mathbf{1}_{x\neq x_i}(x)\partial_x\varphi(t,x)d\mu(t)dt\\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} p(t,x)\varphi(t,x)d\mu(t)dt, \qquad (22)\end{aligned}$$

where a test function $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$.

First, we simplify the setting by assuming $\varphi(t,x) = \varphi_1(t)\varphi_2(x)$. Such product functions are linearly dense in all smooth compactly supported functions with respect to $W^{1,\infty}$ norm. Then, the first integral in (22) reduces to

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\varphi_1'(t)\varphi_2(x)d\mu(t)dt &= \int_{\mathbb{R}^+}\varphi_1'(t)\left(\int_{\mathbb{R}}\varphi_2(x)d\mu(t)\right)dt\\ &= -\int_{\mathbb{R}^+}\varphi_1(t)\frac{d}{dt}\left(\int_{\mathbb{R}}\varphi_2(x)d\mu(t)\right)dt,\end{aligned}$$

where in the last equality we have used the Lipschitz continuity of the inner integral (see Lemma 4.9), what has allowed us to integrate by parts. Consequently, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}}\varphi_1'(t)\varphi_2(x)d\mu(t)dt + \int_{\mathbb{R}^+}\int_{\mathbb{R}}\varphi_1(t)\mathbf{1}_{x\neq x_i}(x)\varphi_2'(x)d\mu(t)dt\\ &+\int_{\mathbb{R}^+}\int_{\mathbb{R}}p(t,x)\varphi_1(t)\varphi_2(x)d\mu(t)dt =\\ &\int_{\mathbb{R}^+}\varphi_1(t)\left(-\frac{d}{dt}\int_{\mathbb{R}}\varphi_2(x)d\mu(t) + \int_{\mathbb{R}}\mathbf{1}_{x\neq x_i}(x)\varphi_2'(x)d\mu(t) + \int_{\mathbb{R}}p(t,x)\varphi_2(x)d\mu(t)\right)dt.\end{aligned}$$

Our goal is to prove that the above expression vanishes for every φ_1, φ_2.

By partition of the unity we can assume that $supp(\varphi_2) \subset (x_{i-1}, x_{i+1})$. Using formulas (19) and (20) we express the solutions in terms of their values on $\{(t,x) : t = 0, x \geq x_i\} \cup \{(t,x) : x = x_i, t \geq 0\}$ (see Fig. 2) and obtain

$$\begin{aligned}&-\frac{d}{dt}\int_{\mathbb{R}}\varphi_2(x)d\mu(t) + \int_{\mathbb{R}}\mathbf{1}_{x\neq x_i}(x)\varphi_2'(x)d\mu(t) + \int_{\mathbb{R}}p(t,x)\varphi_2(x)d\mu(t)\\ =\ &-\frac{d}{dt}\int_{\{x<x_i\}}\varphi_2(x)d\mu(t) + \int_{\{x<x_i\}}\varphi_2'(x)d\mu(t) + \int_{\{x<x_i\}}p(t,x)\varphi_2(x)d\mu(t)\\ &-\frac{d}{dt}\int_{\{x_i\}}\varphi_2(x)d\mu(t) + \int_{\{x_i\}}p(t,x)\varphi_2(x)d\mu(t)\\ &-\frac{d}{dt}\int_{\{x>x_i\}}\varphi_2(x)d\mu(t) + \int_{\{x>x_i\}}\varphi_2'(x)d\mu(t) + \int_{\{x>x_i\}}p(t,x)\varphi_2(x)d\mu(t)\\ =:\ & I_1^< + I_2^< + I_3^< + I_1^= + I_3^= + I_1^> + I_2^> + I_3^>\end{aligned}$$

Note that the terms on the right hand-side containing the time derivatives are well defined as one dimensional measures.

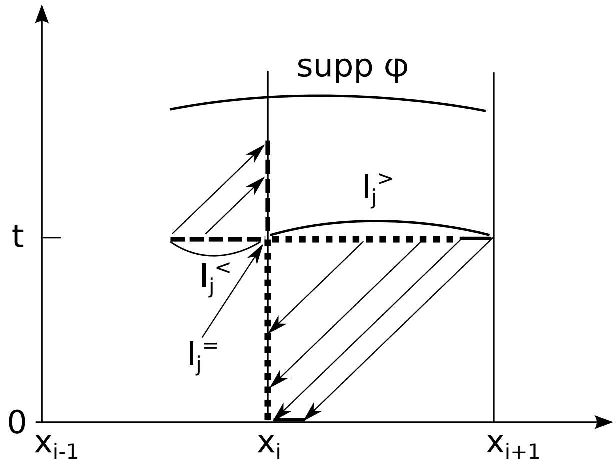

Figure 2: Schematic illustration of the definition of integration intervals for $I_j^=$, $I_j^<$ and $I_j^>$.

Now we transform all the terms and show that their sum vanishes.

$$\begin{aligned}
I_1^> &= - \frac{d}{dt}\int_0^t e^{\int_{t_b}^t p(s,x_i+(s-t_b))ds} f_i(t_b)\varphi_2(x_i+(t-t_b))dt_b \\
&- \frac{d}{dt}\left(\int_{(x_i,x_{i+1})} e^{\int_0^t p(s,x+s)ds}\varphi_2(x+t)d\mu(0)\right) \\
&= - \int_0^t \left(p(t,x_i+(t-t_b))\varphi_2(x_i+(t-t_b)) + \varphi_2'(x_i+(t-t_b))\right) \times \\
&\times\ e^{\int_{t_b}^t p(s,x_i+(s-t_b))ds} f_i(t_b)dt_b \\
&- f_i(t)\varphi_2(x_i) - \int_{(x_i,x_{i+1})} \left(p(t,x+t)\varphi_2(x+t) + \varphi_2'(x+t)\right) e^{\int_0^t p(s,x+s)ds}d\mu(0).
\end{aligned}$$

Since p has only BV regularity with respect to time, the last integral is well defined almost everywhere, i.e., only in the points of continuity of p.

Further terms read

$$\begin{aligned}
I_2^> &= \int_0^t \varphi_2'(x_i+(t-t_b))e^{\int_{t_b}^t p(s,x_i+(s-t_b))ds} f_i(t_b)dt_b \\
&+ \int_{(x_i,x_{i+1})} e^{\int_0^t p(s,x+s)ds}\varphi_2'(x+t)d\mu(0), \\
I_3^> &= \int_0^t e^{\int_{t_b}^t p(s,x_i+(s-t_b))ds} f_i(t_b)\varphi_2(x_i+(t-t_b))p(t,x_i+(t-t_b))dt_b \\
&+ \int_{(x_i,x_{i+1})} e^{\int_0^t p(s,x+s)ds}\varphi_2(x+t)p(t,x+t)d\mu(0),
\end{aligned}$$

where the last term cancels out with the appropriate term in $I_1^>$.

Calculating the integrals at $\{x_i\}$ results in

$$I_1^= = -\varphi_2(x_i)\frac{d}{dt}\int_{\{x_i\}} d\mu(t) = -\varphi_2(x_i)\left[(p(t,x_i) - c_i(t))\int_{\{x_i\}} d\mu(t) + h_i(t)\right].$$

The last equality follows by formula (18) and holds in the sense of equality of measures,

$$I_3^= = p(t,x_i)\varphi_2(x_i)\int_{\{x_i\}} d\mu(t).$$

Finally, $I_1^= + I_3^= = c_i(t)\varphi_2(x_i)\int_{\{x_i\}} d\mu(t) - \varphi_2(x_i)h_i(t)$. Calculating $I^<$ involves integrating with respect to a singular measure $h_i(dt_b) \equiv h_i(t_b)dt_b$ what leads to the expressions for derivatives holding almost everywhere with respect to time. Note that we have to reverse the sign in the exponent.

$$\begin{aligned}
I_1^< &= -\frac{d}{dt}\int_{(t,\infty)} e^{-\int_t^{t_b} p(s,x_i+(s-t_b))ds}\varphi_2(x_i+(t-t_b))h_i(t_b)dt_b \\
&= -\int_{(t,\infty)} \big(p(t,x_i+(t-t_b))\varphi_2(x_i+(t-t_b)) \\
&\quad +\varphi_2'(x_i+(t-t_b))\big)e^{-\int_t^{t_b} p(s,x_i+(s-t_b))ds}h_i(t_b)dt_b + h_i(t)\varphi_2(x_i)
\end{aligned}$$

$$\begin{aligned}
I_2^< &= \int_{(t,\infty)} e^{-\int_t^{t_b} p(s,x_i+(s-t_b))ds}\varphi_2'(x_i+(t-t_b))h_i(t_b)dt_b \\
I_3^< &= \int_{(t,\infty)} e^{-\int_{t_b}^{t} p(s,x_i+(s-t_b))ds}\varphi_2(x_i+(t-t_b))p(t,x_i+(t-t_b))h_i(t_b)dt_b
\end{aligned}$$

To finish the proof we add all obtained expressions in the space of measures and use the definition of f_i. □

Remark 4.8. If $\varphi_1 \in C_c^\infty([0,\infty))$, we may decompose $\varphi_1 = \varphi_{11}^\varepsilon + \varphi_{12}^\varepsilon$ in such a way that $supp(\varphi_{11}) \subset [0,2\varepsilon)$ and $supp(\varphi_{12}) \subset [\varepsilon,\infty)$. Then, φ_{12}^ε satisfies equation (11). Taking the limit $\varepsilon \to 0$ in the weak formulation, we obtain that the only non-vanishing term for φ_{11}^ε contains its time derivative. By narrow-continuity of solutions (see Lemma 4.9), it converges to $-\varphi_1(0)\int\varphi_2(x)d\mu(0)$. This term cancels with the term $\int_{\mathbb{R}}\varphi(0,x)d\mu(0)(x)$ from Definition 3.3.

Lemma 4.9. *For every $T > 0$ the mapping $t \mapsto \mu(t)$ is in $Lip([0,T],(\mathcal{M},\rho_F))$.*

Proof. Take $t_s, t \in [0,T]$. Without loss of generality we can assume that $t_s = 0$. This is a consequence of the semigroup property, i.e., the fact that the solution at time t with initial condition at t_s defined by formulas (17), (18) and (19) is the same as the solution at time t with initial conditions at 0. Denoting by $d = \min_i |x_i - x_{i-1}|$, we assume, without loss of generality, that $t < d$. We want to show that $\rho_F(\mu(t), \mu(0)) \leq C(T)t$. Taking a test function $\psi \in C^1(\mathbb{R})$ such that $\|\psi\|_{W^{1,\infty}} \leq 1$ we calculate (see Fig. 3)

$$\int_{\mathbb{R}} \psi(x) d(\mu(t) - \mu(0)) = D_0 + I_1 + D_1 + I_2 + \ldots + I_N,$$

where

$$\begin{aligned}
D_0 &:= \int_{(x_0, x_0+t]} \psi(x) d\mu(t) + \int_{\{x_0\}} \psi(x) d\mu(t) - \int_{\{x_0\}} \psi(x) d\mu(0), \\
I_1 &:= \int_{(x_0+t, x_1)} \psi(x) d\mu(t) - \int_{(x_0, x_1-t)} \psi(x) d\mu(0), \\
D_1 &:= \int_{\{x_1\}} \psi(x) d\mu(t) + \int_{(x_1, x_1+t]} \psi(x) d\mu(t) - \int_{[x_1-t, x_1)} \psi(x) d\mu(0) - \int_{\{x_1\}} \psi(x) d\mu(0), \\
I_2 &:= \int_{(x_1+t, x_2)} \psi(x) d\mu(t) - \int_{(x_1, x_2-t)} \psi(x) d\mu(0)
\end{aligned}$$

and the further terms are defined analogously. Let us estimate the subsequent terms

$$\begin{aligned}
|D_0| &\leq \left| \int_{[0,t)} e^{\int_{t_b}^t p(s, x_0+(s-t_b)) ds} f_0(t_b) \psi(x_0 + (t - t_b)) dt_b \right| \\
&\quad + \left| \psi(x_0) \left(\int_{\{x_0\}} d\mu(0) \right) (e^{\int_0^t (p(s,x_0) - c_0(s)) ds} - 1) \right| =: D_{01} + D_{02},
\end{aligned}$$

and, since $|\psi| \leq 1$, we obtain

$$\begin{aligned}
D_{01} &= \left| \int_{(0,t)} e^{\int_{t_b}^t p(s, x_0+(s-t_b)) ds} f_0(t_b) \psi(x_0 + (t - t_b)) dt_b \right| \\
&\leq t e^{C_p(T)T} C_0^c(T) \left(\sup_{[0,T]} \int_{\{x_0\}} d\mu(t) \right) \|\psi\|_\infty \leq L_{01}^D t,
\end{aligned}$$

where L_{01}^D is a constant. $D_{02} \leq \left(\int_{\{x_0\}} d\mu(0) \right) C_0(T) e^{C_0(T)T} t = L_{02}^D t$ with a new constant L_{02}^D.

Now let us consider the first "interval" integral at time t, which is compared with its counterpart at time 0.

$$\begin{aligned}
|I_1| &= \left| \int_{(x_0+t, x_1)} \psi(x) d\mu(t) - \int_{(x_0, x_1-t)} \psi(x) d\mu(0) \right| \\
&= \left| \int_{(x_0, x_1-t)} \psi(x+t) e^{\int_0^t p(s, x+s) ds} d\mu(0)(x) - \int_{(x_0, x_1-t)} \psi(x) d\mu(0) \right| \\
&\leq \int_{(x_0, x_1-t)} \left| \psi(x+t) e^{\int_0^t p(s, x+s) ds} - \psi(x) \right| d\mu(0) \\
&\leq \int_{(x_0, x_1-t)} \left(|\psi(x)| |e^{\int_0^t p(s,x+s) ds} - 1| + \sup |\psi'| t e^{TC_p(T)} \right) d\mu(0) \\
&\leq \left(\int_{(x_0, x_1)} d\mu(0) \right) e^{TC_p(T)} \left(C_p(T) + 1 \right) t = L_1^I t
\end{aligned}$$

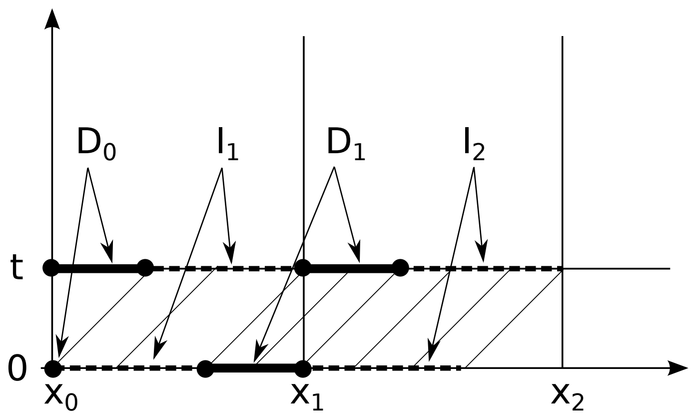

Figure 3: Schematic presentation of terms D_i and I_j.

The same estimate is valid for I_i with $i = 2, \dots, n$.

$$
\begin{aligned}
D_1 &= \int_{\{x_1\}} \psi(x) d\mu(t) + \int_{(x_1, x_1+t]} \psi(x) d\mu(t) - \int_{\{x_1\}} \psi(x) d\mu(0) - \int_{[x_1-t, x_1)} \psi(x) d\mu(0) \\
&= \int_{\{x_1\}} \psi(x) d\mu(0) (e^{\int_0^t (p(s,x_1) - c_1(s))ds} - 1) + \psi(x_1) \int_{(0,t]} h_1(r) e^{\int_r^t (p(s,x_i) - c_i(s))ds} dr \\
&\quad - \int_{[x_1-t, x_1)} \psi(x) d\mu(0) + \int_{(x_1, x_1+t]} \psi(x) d\mu(t).
\end{aligned}
$$

To estimate D_1 we split it into the sum of three terms $D_1 := D_{11} + D_{12} + D_{13}$, where

$$
\begin{aligned}
D_{11} &= \int_{\{x_1\}} \psi(x) d\mu(0) \left(e^{\int_0^t (p(s,x_1) - c_1(s))ds} - 1 \right), \\
D_{12} &= \psi(x_1) \int_{(0,t]} h_1(r) e^{\int_r^t (p(s,x_i) - c_i(s))ds} dr - \int_{[x_1-t, x_1)} \psi(x) d\mu(0), \\
D_{13} &= \int_{(x_1, x_1+t]} \psi(x) d\mu(t).
\end{aligned}
$$

We calculate (see Fig. 4)

$$
|D_{11}| \leq \left(\int_{\{x_1\}} d\mu(0) \right) C_1(T) e^{TC_1(T)} t = L_{11}^D t,
$$

using the following estimate

$$
\sup_{t \in [0,T]} \left| e^{\xi(t)} - 1 \right| = \sup_{t \in [0,T]} \left| \int_0^{\xi(t)} e^s ds \right| \leq \sup_{t \in [0,T]} |\xi(t)| \, e^{\sup_{t \in [0,T]} |\xi(t)|}.
$$

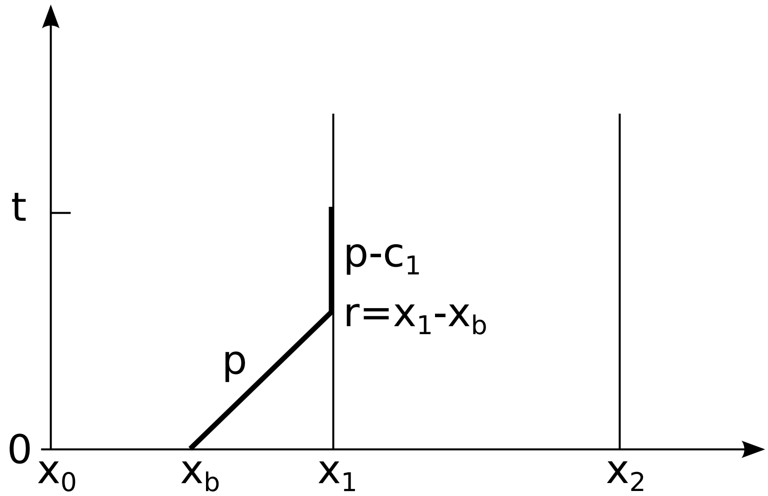

Figure 4: Plot of a path of integration in the term D_{12}. p and $p - c_1$ denote the coefficients of the growth of the part of solution corresponding to D_{12}.

Also

$$\begin{aligned} |D_{12}| &= \Big|\psi(x_1)\int_{[x_1-t,x_1)} e^{\int_{x_1-x_b}^{t}(p(s,x_1)-c_1(s))ds} e^{\int_0^{x_1-x_b} p(s,x_1+(s-t))ds} d\mu(0)(x_b) \\ &\quad - \int_{[x_1-t,x_1)} |\psi(x_b)| d\mu(0)(x_b)\Big| \leq \int_{[x_1-t,x_1)} |\psi(x_1)-\psi(x_b)| d\mu(0)(x_b) \\ &\quad + \int_{[x_1-t,x_1)} |\psi(x_b)| \left| e^{\int_{x_1-x_b}^{t}(p(s,x_1)-c_1(s))ds + \int_0^{x_1-x_b} p(s,x_i+(s-t))ds} - 1 \right| d\mu(0) \\ &\leq \left(\int_{[x_1-t,x_1)} d\mu(0)\right)\left(\sup|\psi'|t + \sup|\psi|\left|e^{(C_1(T)+C_p(T))t}-1\right|\right) \\ &\leq \left(\int_{(x_0,x_1)} d\mu(0)\right)\left(1 + (C_1(T)+C_p(T))e^{T(C_1(T)+C_p(T))}\right)t = L_{12}^D t. \end{aligned}$$

and

$$\begin{aligned} |D_{13}| &= \left|\int_{(x_1,x_1+t]} \psi(x) d\mu(t)\right| = \left|\int_{[0,t)} e^{\int_{t_b}^{t} p(s,x_1+(s-t_b))ds} f_1(t_b)\psi(x_1+(t-t_b))dt_b\right| \\ &\leq te^{TC_p(T)} C_1^c(T) M(T) e^{TK(T)} \int_{\mathbb{R}} d\mu(0) \leq L_{13}^D t. \end{aligned}$$

Similar calculations can be conducted for the further terms, what leads to the estimations for D_i and I_i in terms of $\int_{(x_{i-1},x_i)} d\mu(0)$ and $\sup_{[0,T]}\int_{\{x_{i-1}\}} d\mu(s)$, which is locally exponentially bounded by Lemma 4.4. Since these computations hold for every $\psi \in C^1(\mathbb{R})$ such that $\|\psi\|_{W^{1,\infty}} \leq 1$ and the resulting Lipschitz constant is independent of ψ, the assertion of the lemma follows. $\square$

5 Uniqueness for the linear problem

Lemma 5.1. *Assume that* $\mu_1, \mu_2 \in C([0,T], (\mathcal{M}, \rho_F))$ *are measure-transmission solutions of problem* (14)–(16). *Then* $\mu_1 = \mu_2$.

Proof. Step 1: We show the uniqueness of the solution $\mu(t)\lfloor_{(-\infty,x_0]}$, where for a given Borel set $B \subset \mathbb{R}$, $\mu\lfloor_B$ is a Borel measure defined by restriction of μ to the set B, i.e. $\mu\lfloor_B(A) := \mu(A \cap B)$.

Let μ be a solution of the problem. If $\mu(0)\lfloor_{(-\infty,x_0)} = 0$, then also $\mu(t)\lfloor_{(-\infty,x_0)} = 0$ for every time point t. Using its continuity with respect to time, see also Remark 4.8, we obtain that for every $T > t^* > 0$ and every $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi(x) d\mu(T) - \int_{\mathbb{R}} \varphi(x) d\mu(t^*) = \int_{t^*}^{T} \int_{\{x \neq x_i\}} \partial_x \varphi(x) d\mu(t) dt + \int_{t^*}^{T} \int_{\mathbb{R}} p(t,x) \varphi(x) d\mu(t) dt \quad (23)$$

We take the following test function

$$\varphi(x) = \varphi^\varepsilon(x) = \begin{cases} 1, & \text{if } x \leq x_0, \\ 1 - \frac{x - x_0}{\varepsilon}, & \text{if } x \in (x_0, x_0 + \varepsilon), \\ 0, & \text{if } x \geq x_0 + \varepsilon. \end{cases}$$

Although the function φ^ε is only $W^{1,\infty}$, it can be regularized.

Denoting $w(t) := \int_{\{x_0\}} d\mu(t)$ and using the property $\mu(t)\lfloor_{(-\infty,0)} = 0$ we calculate for $\varepsilon \to 0$

$$w(T) - w(t^*) = \int_{t^*}^{T} p(s, x_0) w(s) ds - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t^*}^{T} \left(\int_{(x_0, x_0 + \varepsilon)} d\mu(t) \right) dt,$$

The above limit is easy to evaluate by the transmission condition, which not only states that the Radon-Nikodym derivative of μ with respect to $\mathcal{L}^1$ exists but also has a limit at x_0^+. Taking this into account, after simple transformations we obtain a linear integral equation

$$w(T) - w(t^*) = \int_{t^*}^{T} p(s, x_0) w(s) ds - \int_{t^*}^{T} \frac{D\mu(t)}{D\mathcal{L}^1}(x_0^+) dt \quad (24)$$

$$w(T) - w(t^*) = \int_{t^*}^{T} p(s, x_0) w(s) ds - \int_{t^*}^{T} c_0(s) w(s) ds. \quad (25)$$

Continuity of w in 0 (see Definition 3.3 iii) allows us to conclude that this relation holds also for $t^* = 0$.

To prove uniqueness of $w(t)$, we take w_1, w_2 corresponding to the two solutions μ_1, μ_2 and subtract them. It leads to

$$|w_1(t) - w_2(t)| \leq \int_0^t |(p(s, x_0) - c_0(s))| \, |(w_1(s) - w_2(s))| \, ds.$$

Using the Gronwall's lemma for locally integrable functions (hence also for BV functions), see Lemma 2.2 in [25], we conclude that $w_1 = w_2$, and hence also the uniqueness of $\mu(t)\lfloor_{(-\infty,x_0]}$.

Step 2: We show the uniqueness of $\mu(t)\lfloor_{(x_0,x_1)}$.

This will be done by an adaptation of the dual equation method. Take a function φ compactly supported in $[0,\infty) \times (-\infty, x_1)$. Then, by continuity and proper time cut-off functions (compare Remark 4.8, which has to be extended to φ in non-product form, which is not difficult and follows by approximation in C^1 norm by finite sums of product functions), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(T,x)d(\mu_1(T)-\mu_2(T)) \\
&= \int_0^T \int_{\mathbb{R}} (p(t,x)\varphi(t,x) + \mathbf{1}_{x\neq x_i}\partial_x\varphi(t,x) + \partial_t\varphi(t,x))d(\mu_1(t)-\mu_2(t))dt. \qquad (26)
\end{aligned}$$

For every function φ with a support such that *supp* $\varphi(t,\cdot) \subset (-\infty, x_1)$, the above equality can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(T,x)d(\mu_1(T)-\mu_2(T)) \\
&= \int_0^T \int_{(x_0,x_1)} (p(t,x)\varphi(t,x) + \partial_x\varphi(t,x) + \partial_t\varphi(t,x))d(\mu_1(t)-\mu_2(t))dt,
\end{aligned}$$

since $\mu_1(t)\lfloor_{(-\infty,x_0]}= \mu_2(t)\lfloor_{(-\infty,x_0]}$ for a.e. $t \in [0,T]$ and

$$\int_0^T \int_{(-\infty,x_0]} (p(t,x)\varphi(t,x) + \partial_x\varphi(t,x) + \partial_t\varphi(t,x))d(\mu_1(t)-\mu_2(t))dt = 0.$$

Next, we choose as test function the solution of the backward equation,

$$\begin{cases} (\partial_t + \partial_x)\varphi(t,x) = -\tilde{p}(t,x)\varphi(t,x), \\ \varphi(T,x) = \varphi^T(x), \end{cases}$$

with a modified function p such that

$$\tilde{p}(t,x) = \begin{cases} p(t,x_0), & \text{if } x \le x_0 \\ p(t,x), & \text{if } x \in (x_0,x_1), \\ p(t,x_1), & \text{if } x \ge x_1. \end{cases}$$

The solution of the backward equation is given by

$$\varphi(t,x) = \varphi^T(x + (T-t))e^{\int_t^T \tilde{p}(s,x+(s-t))ds}.$$

To obtain the C^1 regularity demanded for a test function, we mollify $\varphi(t,x)$, $\varphi^\varepsilon := (\varphi * \rho^\varepsilon)(t,x)$, where ρ is a symmetric regularization kernel and we define φ for $t<0$ and $t>T$ by a continuous (for example constant in time) extension. Since φ^ε is smooth, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \varphi^\varepsilon(T,x)d(\mu_1(T)-\mu_2(T)) \\ = &\int_0^T \int_{(x_0,x_1)} (p(t,x)\varphi^\varepsilon(t,x) + (\partial_x+\partial_t)\varphi^\varepsilon(t,x))d(\mu_1(t)-\mu_2(t))dt. \end{aligned}$$

Passing with ε to zero, we obtain $\int_{\mathbb{R}} \varphi(T,x)d(\mu_1(T)-\mu_2(T)) = 0$.
Step 3 is devoted to the uniqueness of $\mu(t)\lfloor_{\{x_1\}}$.
We take the following test function:

$$\varphi^\varepsilon(x) = \begin{cases} 0 & \text{if } x \le x_1-\varepsilon \\ 1-\frac{x_1-x}{\varepsilon}, & \text{if } x\in(x_1-\varepsilon,x_1) \\ 1-\frac{x-x_1}{\varepsilon}, & \text{if } x\in(x_1,x_1+\varepsilon) \\ 0 & \text{if } x\ge x_1+\varepsilon \end{cases}$$

To be more precise, similarly as in *Step 1* we consider a smooth version of this test function. Denoting $w(t) := \int_{\{x_1\}} d\mu(t)$, we obtain

$$\begin{aligned} w(T)-w(0) &= \int_0^T p(s,x_1)w(s)ds \\ &+ \lim_{t^*\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{t^*}^T\left(\int_{(x_1,x_1+\varepsilon)} d\mu(t)\right)dt - \int_{t^*}^T\left(\int_{(x_1-\varepsilon,x_1)} d\mu(t)\right)dt\right). \end{aligned}$$

The term to the right of x_1 has a limit, which can be calculated similarly as in *Step 1.* We obtain

$$\begin{aligned} w(T)-w(0) &= \int_0^T p(s,x_1)w(s)ds \\ &- \int_0^T c_1(s)w(s)ds + \lim_{t^*\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^T\left(\int_{(x_1-\varepsilon,x_1)} d\mu(t)\right)dt. \end{aligned}$$

Since $\mu_1(t)\lfloor_{(-\infty,x_1)} = \mu_2(t)\lfloor_{(-\infty,x_1)}$ for a.e. $t\in[0,T]$, then

$$\lim_{t^*\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^T\left(\int_{(x_1-\varepsilon,x_1)} d(\mu_1-\mu_2)(t)\right)dt = \lim_{t^*\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^T 0 = 0.$$

Therefore, for every T it holds

$$w_1(T)-w_2(T) = \int_0^T p(s,x_1)(w_1(s)-w_2(s))ds - \int_0^T c_1(s)(w_1(s)-w_2(s))ds,$$

and

$$|w_1(T) - w_2(T)| \leq (C_p(T) + C_1^c(T)) \int_0^T |w_1(s) - w_2(s)| ds,$$

what by the Gronwall's lemma, [25], finishes the proof of the uniqueness at x_1.

Alternating further *Step 2* and *Step 3* we extend the uniqueness result to the solution $\mu(t)\lfloor_{(-\infty,x_i]}$ for any i, and finally obtain the uniqueness on the whole domain. $\square$

6 Time-dependent transport coefficient

In this section we extend our results to the case with the transport coefficient g_1, which depends on time, i.e., $g_1 \in BV([0,T])$ for every $T > 0$. We define a transformation of variables:

Definition 6.1.

$$\begin{aligned}
\tilde{t}(t) &:= \int_0^t g_1(s)ds, \qquad \textit{or equivalently} \qquad t(\tilde{t}) := \int_0^{\tilde{t}} \frac{ds}{g_1(s)}, \\
\partial_t &:= \frac{d\tilde{t}}{dt}\partial_{\tilde{t}}, \\
\tilde{p}(\tilde{t}, x) &:= \frac{p(t(\tilde{t}), x)}{g_1(t(\tilde{t}))} = \frac{p_1(t(\tilde{t}))p_2(x)}{g_1(t(\tilde{t}))}, \\
\tilde{c}_i(\tilde{t}) &:= \frac{c_i(t(\tilde{t}))}{g_1(t(\tilde{t}))}, \quad i = 0, \ldots, N.
\end{aligned}$$

Note that since g_1 is bounded away from zero on finite intervals, the transformation $t(\tilde{t})$ is a locally bi-Lipschitz homeomorphism and it preserves the total variation of the function. Consequently, $\tilde{p}$ and $\tilde{c}_i$ are in BV_{loc}.

Theorem 6.2. *If $\tilde{\mu}$ is a measure-transmission solution of*

$$\begin{aligned}
\partial_{\tilde{t}}\tilde{\mu}(\tilde{t}) + \partial_x \left(\mathbf{1}_{x\neq x_i}(x)\tilde{\mu}(\tilde{t})\right) &= \tilde{p}(\tilde{t}, x)\tilde{\mu}(\tilde{t}), \qquad (27)\\
\frac{D\tilde{\mu}(\tilde{t})}{D\mathcal{L}^1}(x_i^+) &= \tilde{c}_i(\tilde{t}) \int_{\{x_i\}} d\tilde{\mu}(\tilde{t}) \quad i = 0, \ldots, N, \\
\tilde{\mu}(0) &= \mu_0,
\end{aligned}$$

then $\mu(t) := \tilde{\mu}(\tilde{t}(t))$ solves

$$\begin{aligned}
\partial_t\mu(t) + g_1(t)\partial_x \left(\mathbf{1}_{x\neq x_i}\mu(t)\right) &= p(t, x)\mu(t), \qquad (28)\\
g_1(t)\frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) &= c_i(t) \int_{\{x_i\}} d\mu(t) \quad i = 0, \ldots, N, \\
\mu(0) &= \mu_0
\end{aligned}$$

in the sense of Definition 3.3. Moreover, $\mu(t) \in Lip([0,T], (\mathcal{M}, \rho_F))$ for every $T > 0$.

Proof. We take a test function $\varphi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R})$. By density of product functions in $C_c^\infty([0,\infty) \times \mathbb{R})$ with respect to C^1 norm, we assume that $\varphi(t,x) = \varphi_1(t)\varphi_2(x)$. For simplicity, we assume also that $\varphi_1(0) = 0$, since the initial condition is very simple to verify (compare Remark 4.8).

$$
\begin{aligned}
&\langle (\partial_t \mu(t) + g_1(t) (\partial_x \mathbf{1}_{x \neq x_i} \mu(t)) - p(t,x)\mu), \varphi \rangle := -\int_{\mathbb{R}^+} \partial_t \varphi_1(t) \int_{\mathbb{R}} \varphi_2(x) d\mu(t) dt \\
&- \int_{\mathbb{R}^+} g_1(t)\varphi_1(t) \int_{\mathbb{R}} \mathbf{1}_{x \neq x_i} \partial_x \varphi_2(x) d\mu(t) dt - \int_{\mathbb{R}^+} \varphi_1(t) \int_{\mathbb{R}} p(t,x)\varphi_2(x) d\mu(t) dt = \\
&- \int_{\mathbb{R}^+} \frac{dt}{d\tilde{t}} \partial_{\tilde{t}} \tilde{\varphi}_1(\tilde{t}) \int_{\mathbb{R}} \varphi_2(x) d\tilde{\mu}(\tilde{t}) \frac{d\tilde{t}}{dt} d\tilde{t} - \int_{\mathbb{R}^+} \tilde{g}_1(\tilde{t})\tilde{\varphi}_1(\tilde{t}) \left(\int_{\mathbb{R}} \mathbf{1}_{x \neq x_i} \partial_x \varphi_2(x) d\tilde{\mu}(\tilde{t}) \right) \frac{dt}{d\tilde{t}} d\tilde{t} \\
&- \int_{\mathbb{R}^+} \tilde{\varphi}_1(\tilde{t}) \left(\int_{\mathbb{R}} p(t(\tilde{t}),x)\varphi_2(x) d\tilde{\mu}(\tilde{t}) \right) \frac{dt}{d\tilde{t}} d\tilde{t} = - \int_{\mathbb{R}^+} \partial_{\tilde{t}} \tilde{\varphi}_1(\tilde{t}) \int_{\mathbb{R}} \varphi_2(x) d\tilde{\mu}(\tilde{t}) d\tilde{t} \\
&- \int_{\mathbb{R}^+} \tilde{\varphi}_1(\tilde{t}) \left(\int_{\mathbb{R}} \mathbf{1}_{x \neq x_i} \partial_x \varphi_2(x) d\tilde{\mu}(\tilde{t}) \right) d\tilde{t} - \int_{\mathbb{R}^+} \tilde{\varphi}_1(\tilde{t}) \left(\int_{\mathbb{R}} \tilde{p}(\tilde{t},x)\varphi_2(x) d\tilde{\mu}(\tilde{t}) \right) d\tilde{t} = 0.
\end{aligned}
$$

The last equality results from the observation that a function $\tilde{\varphi}_1 \in Lip([0,T])$ can be approximated by a sequence $\tilde{\varphi}_1^n \in C_c^\infty$, such that $\tilde{\varphi}_1^n \to \tilde{\varphi}_1$ locally uniformly and $(\tilde{\varphi}_1^n)' \to \tilde{\varphi}_1'$ in L^1 and almost everywhere. Therefore, $\tilde{\varphi}_1(\tilde{t})\varphi_2(x)$ is an admissible test function.

To prove the transmission condition one has to proceed exactly like in Lemma 4.6 and apply the chain rule for derivatives of locally Lipschitz functions. Lipschitz continuity results from the fact that the transformation of variables is Lipschitz and $\mu(t) = \tilde{\mu}(\tilde{t}(t))$. This also proves the initial condition. $\square$

Corollary 6.3. *The problem* (28) *admits a unique solution,* $\mu(t) \in Lip([0,T], (\mathcal{M}, \rho_F))$ *for every* $T > 0$.

Proof. Solutions of problem (27) are unique and there is a correspondence between the solutions of problems (27) and (28). $\square$

7 The nonlinear problem - existence of solutions

Theorem 7.1. *For every* $\mu_0 \in \mathcal{M}(\mathbb{R})$ *such that* $supp(\mu_0) \subset [x_0, x_N]$*, there exists a measure-transmission solution of problem* (7)–(9) *in sense of Definition 3.3.*

Proof. Step 1: To build a measure-transmission solution of the nonlinear problem, we first prove the local in time existence of v using Schauder's Fixed Point Theorem. We define a nonlinear operator S such that, for ε small enough, $S : L^1([t_0, t_0+\varepsilon]) \to L^1([t_0, t_0+\varepsilon])$ maps $w \mapsto v$ by the following formula

$$\begin{aligned} v(t) &= \left[w(t_0) + \int_{t_0}^{t} p(w(r), x_N) w(r) dr \right] + \int_{(t_0,t]} dh_N \\ &= \left[w(t_0) + \int_{t_0}^{t} p(w(r), x_N) w(r) dr \right] \\ &+ \int_{[x_N - \int_{t_0}^{t} g_1(w(r))dr, x_N)} e^{\int_{t_0}^{\tau(x)} p\left(w(s), x + \int_{t_0}^{s} g_1(w(r))dr\right) ds} d\mu(0), \end{aligned} \tag{29}$$

where $\tau(x)$ is the unique solution of $\int_{t_0}^{\tau(x)} g_1(w(t))dt = x_N - x$.

First we show that $S : K \to K$ is compact. Since v is right-continuous by definition (29), we can estimate total variation seminorm of v (see [9]) by

$$TV_{[t_0,t_0+\varepsilon]}(v) \le \int_{t_0}^{t_0+\varepsilon} |p(w(t), x_N| w(t) dt + \|h_N\|_{\mathcal{M}((t_0,t_0+\varepsilon])}.$$

Obviously,

$$\int_{t_0}^{t_0+\varepsilon} |p(w(t), x_N| w(t) dt \le \|p\|_\infty \|w\|_{L^1([t_0,t_0+\varepsilon])}.$$

On the other hand,

$$\begin{aligned} \|h_N\|_{\mathcal{M}((t_0,t_0+\varepsilon])} &= \int_{\left[x_N - \int_{t_0}^{t_0+\varepsilon} g_1(w(r)dr, x_N\right)} e^{\int_{t_0}^{\tau(x)} p\left(w(s), x + \int_{t_0}^{s} g_1(w(r))dr\right) ds} d\mu(t_0) \\ &\le e^{\varepsilon\|p\|_\infty} \|\mu(t_0)\|_{\mathcal{M}((x_{N-1},x_N])}. \end{aligned}$$

Altogether,

$$TV_{[t_0,t_0+\varepsilon]}(v) \le \|\mu(t_0)\| e^{\varepsilon\|p\|_\infty} + \|p\|_\infty \|w\|_{L^1([t_0,t_0+\varepsilon])}.$$

Consequently, we obtain for any t_0,

$$\begin{aligned} \|v\|_{L^1([t_0,t_0+\varepsilon])} &\le \varepsilon(|v(t_0)| + TV_{[t_0,t_0+\varepsilon]}(v)) \\ &\le \varepsilon\left(|w(t_0)| + \|\mu(t_0)\|_{\mathcal{M}((x_{N-1},x_N])} e^{\varepsilon\|p\|_\infty} + \|p\|_\infty \|w\|_{L^1([t_0,t_0+\varepsilon])}\right). \end{aligned}$$

Taking $R > 0$ and w such that $\|w\|_{L^1([t_0,t_0+\varepsilon])} \le R$, we obtain

$$\|v\|_{L^1([t_0,t_0+\varepsilon])} \le \varepsilon\left(w(t_0) + \|\mu(t_0)\|_{\mathcal{M}((x_{N-1},x_N])} e^{\varepsilon\|p\|_\infty} + R\|p\|_\infty\right).$$

We choose ε such that

$$\varepsilon < \min\left\{ \frac{R}{|w(t_0)| + \|\mu(t_0)\|_{\mathcal{M}((x_{N-1},x_N])} e^{\varepsilon\|p\|_\infty} + R\|p\|_\infty}, \frac{x_N - x_{N-1}}{\|g\|_{L^\infty}} \right\}. \tag{30}$$

With such choice of ε, we obtain that $\|v\|_{L^1([t_0,t_0+\varepsilon])} \leq R$ for every $t_0 \in [0,T]$. Moreover, the bound $\frac{x_N - x_{N-1}}{\|g\|_{L^\infty}}$ assures that the fixed point of operator S is indeed a solution of the nonlinear problem, not influenced by the transmission condition.

Consequently, $S(B(0,R)) \subset B(0,R)$, where $B(0,R)$ denotes a closed ball in $L^1([t_0,t_0+\varepsilon])$ with radius R. Moreover, since $\|v\|_{BV([t_0,t_0+\varepsilon])} = \|v\|_{L^\infty([t_0,t_0+\varepsilon])} + TV_{[t_0,t_0+\varepsilon]}(v)$,

$$\|v\|_{BV([t_0,t_0+\varepsilon])} \leq |v(t_0)| + 2TV_{[t_0,t_0+\varepsilon]}(v) \leq C(T)e^{\varepsilon\|p\|_\infty} + \|p\|_\infty R + C(T) = \tilde{R}.$$

To apply Schauder's Fixed Point Theorem we choose $K = B(0,R) \cap B_{BV([t_0,t_0+\varepsilon])}(0,\tilde{R})$, which is convex and such that $S(K) \subset K$. By Helly's theorem (see [7] Theorem 13.16) every bounded sequence in BV has a subsequence convergent strongly in L^1 to some BV function. By lower semicontinuity of the total variation with respect to L^1 convergence, the limit function also belongs to K. We can always modify the limit on zero measure sets so that it is right-continuous. In consequence, K is a compact subset of $L^1([t_0,t_0+\varepsilon])$.

Lemma 7.2. *$S : K \to K$ is continuous.*

Proof. Take $w^n \in K$ such that $w^n \to w$ in $L^1([t_0,t_0+\varepsilon])$. We have to prove that $S(w^n) \to S(w)$. It holds

$$\begin{aligned} v^n(t) = S(w^n(t)) &= w(t_0) + \int_{t_0}^{t} p(w^n(r), x_N) w^n(r) dr \qquad (31) \\ &+ \int_{\left[x_N - \int_{t_0}^{t} g_1(w^n(r))dr, x_N\right)} \exp\left(\int_{t_0}^{\tau^n(x)} p\left(w^n(s), x + \int_{t_0}^{s} g_1(w^n(r))dr \right) ds \right) d\mu(t_0), \end{aligned}$$

where $\tau^n(x)$ is defined implicitly by $x = x_N - \int_{t_0}^{\tau^n(x)} g_1(w^n(s))ds$. First, we show that $\tau^n(x)$ converges pointwise to $\tau(x)$ defined by $x = x_N - \int_{t_0}^{\tau(x)} g_1(w(s))ds$. Indeed, for a given x

$$\begin{aligned} 0 &= x - x = \int_{t_0}^{\tau^n(x)} g\left(w^n(s)\right) ds - \int_{t_0}^{\tau(x)} g\left(w(s)\right) ds \\ &= \int_{\tau(x)}^{\tau^n(x)} g_1(w^n(s)) ds + \int_{t_0}^{\tau(x)} \left(g_1(w^n(s)) - g_1(w(s))\right) ds \end{aligned}$$

Since g is Lipschitz and w^n converges in L^1, $\int_{t_0}^{\tau(x)} \left(g_1(w^n(s)) - g_1(w(s))\right) ds \to 0$. This implies also that

$$|\tau(x) - \tau^n(x)| \leq \frac{|\int_{\tau(x)}^{\tau_n(x)} g_1(w^n(s))ds|}{\inf_{n\in\mathbb{N}}(\inf_{[t_0,t_0+\varepsilon]} g_1(w^n(s)))} \to 0.$$

Thus, we conclude that $\tau^n(x) \to \tau(x)$ for every x.

Now, we consider the term $H^n(x) = \exp\left(\int_{t_0}^{\tau^n(x)} p\left(w^n(s), x + \int_{t_0}^{s} g_1(w^n(r))dr \right) ds \right)$. Since $p = p_1(v(t))p_2(x)$ is bounded and such that $p_2(x)$ is continuous and p_1 is Lipschitz continuous,

$\int_{t_0}^s g_1(w^n(r))dr \to \int_{t_0}^s g_1(w(r))dr$, and $\tau_n \to \tau$, we obtain convergence of H^n for every x,

$$H^n(x) \to H(x) = \exp\left(\int_{t_0}^{\tau(x)} p\left(w(s), x + \int_{t_0}^{s} g_1(w(r))dr\right) ds\right).$$

Similarly,

$$x_N - \int_{t_0}^t g_1(w^n(s))ds \to x_N - \int_{t_0}^t g_1(w(s))ds.$$

Next, since both $w^n(r)$ and $p(w^n(r), x_N)$ are bounded and converge strongly in $L^1([t_0, t_0+\varepsilon])$, we obtain

$$\int_{t_0}^t p(w^n(r), x_N)w^n(r)dr \to \int_{t_0}^t p(w(r), x_N)w(r)dr.$$

Additionally, $\sup_{n\in N} \|H^n\|_\infty < +\infty$ and $x_N - \int_0^t g_1(w(s))ds$ may be atom of $\mu(t_0)$ at most on a set of $\mathcal{L}^1$ measure 0. Applying Dominated Convergence Theorem, we conclude that $\int_{[x_N - \int_{t_0}^t g_1(w^n(r))dr, x_N)} H^n d\mu(t_0)$ converges almost everywhere. Finally, we obtain

$$\begin{aligned}\lim_{n\to\infty} v^n(t) = \lim_{n\to\infty} S(w^n(t)) &= \left[w(t_0) + \int_{t_0}^t p(w(r), x_N)w(r)dr\right] \\ &+ \int_{[x_N - \int_{t_0}^t g_1(w(r))dr, x_N)} H d\mu(t_0) = S(w(t))\end{aligned}$$

for $\mathcal{L}^1$ - a.e. $t \in [t_0, t_0+\varepsilon]$. This and uniform boundedness of v^n proves also that $\lim_{n\to\infty} S(w^n(t)) = S(w)$ in $L^1([t_0, t_0+\varepsilon])$. □

Having verified the assumptions, we apply Schauder's Fixed Point Theorem to conclude that there exists a function $v \in BV([t_0, t_0+\varepsilon])$ such that $S(v) = v$ and which satisfies the integral formula (29) with $w = v$.

Step 2: We show that the solution constructed on $[t_0, t_0+\varepsilon]$ can be prolonged to $\mathbb{R}^+$. Indeed, since the constants in the estimate in Lemma 4.4 are given by the supremum of some quantities and the transformation $\tilde{t}(t) = \int_0^t g_1(s)ds$ is Lipschitz continuous, we can extend the result from Lemma 4.4 to the following

Proposition 7.3. *For every $T > 0$ there exist constants $M(\tilde{t}(T))$ and $K(\tilde{t}(T))$ such that for any $t \in [0, T]$*

$$\int_{\mathbb{R}} d\mu(t) \le M(\tilde{t}(T))e^{K(\tilde{t}(T))\tilde{t}(t)} \int_{\mathbb{R}} d\mu(0).$$

The proposition can be proved similarly as Lemma 4.4 by replacing constants C_p, C_i, C_i^c by respective L^∞ norms; $C_p = \|p\|_\infty$, $C_i = \|p(\cdot, x_i) - c_i\|_\infty$, $i = 0, ..., N$, and $C_i^c = \|c_i\|_\infty$, and using the fact that there exists a Lipschitz constant L_1 such that

$$|\tilde{t}(t_1) - \tilde{t}(t_2)| \le L_1|t_1 - t_2|.$$

The constant L_1 does not depend on v, since g_1 is uniformly bounded (see Assumptions 3.2).

Since unique solvability of the linear problem is provided under the assumption that $\int_{\{x_i\}} d\mu(t) \to \int_{\{x_i\}} d\mu(t_k)$ for $t \to t_k$, see Definition 3.3 iii, for each next step we choose the starting time t_k in the interval $[t_{k-1} + \frac{3}{4}\varepsilon, t_{k-1} + \varepsilon]$ such that this assumption is fulfiled. For every initial data μ_0, we can construct the solution on $[0, T]$ for any T. Thus, by prolongation it exists in $[0, \infty)$.

This concludes also the proof of Theorem 7.1. $\square$

8 The nonlinear problem - uniqueness of solutions

Theorem 8.1. *For every $\mu_0 \in \mathcal{M}(\mathbb{R})$ such that supp$(\mu_0) \subset [x_0, x_N]$ the measure-transmission solution of problem (7)–(9) is unique.*

Proof. Suppose that $\mu_1 \neq \mu_2$, $\mu_1, \mu_2 \in C([0,\infty), (\mathcal{M}, \rho_F))$ are two different solutions. Consequently, there exists $t > 0$ such that $\mu_1(t) \neq \mu_2(t)$. We take $t_0 = \sup\{s > 0 : \mu_1(s) = \mu_2(s)\}$, what corresponds to the first difference point. Using the explicit formula

$$\begin{aligned} v(s) = v(0) \quad &+ \int_0^s p(v(r), x_N)v(r)dr \\ &+ \int_{[x_N - \int_0^s g_1(v(r))dr, x_N)} e^{\int_0^{\tau(x)} p\left(v(r), x + \int_0^{\xi} g_1(v(r))dr\right)d\xi} d\mu(0), \end{aligned} \tag{32}$$

we obtain that $v(t_0)$ is determined solely in terms of $v(t)$ for $t < t_0$ and, therefore, $v_1(t_0) = v_2(t_0)$. Using the uniqueness result for the linear problem we conclude that also $\mu_1(t_0) = \mu_2(t_0)$. It means that the first difference point is not of jump type and uniqueness holds on a closed interval.

To conclude about uniqueness on the whole $\mathbb{R}^+$, we need to prove additionally that the interval of uniqueness is relatively open in $\mathbb{R}^+$. To this aim we show that for each $t_0 \in \mathbb{R}^+$ with $v_1(t_0) = v_2(t_0)$, there exists $\varepsilon(t_0) > 0$ such that $v_1(t) = v_2(t)$, on the interval $t \in [0, t_0 + \varepsilon(t_0))$.

We proceed similarly as in Lemma 5.1 and, using (32), we estimate the difference of the two solutions in L^1,

$$\int_{t_0}^{t_0+\varepsilon} |v_1(t) - v_2(t)|dt \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 \quad &= \quad \int_{t_0}^{t_0+\varepsilon} \int_{t_0}^{t} |p(v_1(r), x_N)v_1(r) - p(v_2(r), x_N)v_2(r)|dr, \\ I_2 \quad &= \quad \int_{t_0}^{t_0+\varepsilon} [\int_{[x_N - \int_{t_0}^t g_1(v_1(r))dr, x_N)} e^{\int_{t_0}^{\tau_1(x)} p(v_1(s), x + \int_{t_0}^s g_1(v_1(r))dr)ds} d\mu(t_0) \\ &\quad - \int_{[x_N - \int_{t_0}^t g_1(v_2(r))dr, x_N)} e^{\int_{t_0}^{\tau_2(x)} p(v_2(s), x + \int_{t_0}^s g_1(v_2(r))dr)ds} d\mu(t_0) \,] \, dt. \end{aligned}$$

Estimates for the first term, I_1, are standard, because both v_i and $p(v_i(r),x_N)$ are bounded and p_1 is Lipschitz continuous. We obtain

$$|p(v_1(r),x_N)v_1(r)-p(v_2(r),x_N)v_2(r)|\le (L(p_1)\|p_2\|_\infty\|v_1\|_\infty+\|p\|_\infty)\,|v_1(r)-v_2(r)|,$$

where $L(p_1)=\|p_1'\|_\infty$ is a Lipschitz constant. Thus,

$$I_1\le \varepsilon\,(L(p_1)\|p_2\|_\infty\|v_1\|_\infty+\|p\|_\infty)\int_{t_0}^{t_0+\varepsilon}|v_1(r)-v_2(r)|dr.$$

More involved and interesting part is I_2. To simplify notation, we define

$$\begin{aligned} H_i(s,x) &:= p\left(v_i(s),x+\int_{t_0}^{s}g_1(v_i(r))dr\right), \quad i=1,2,\\ G_i(s) &:= \int_{t_0}^{s}g_1(v_i(r))dr, \quad i=1,2, \end{aligned}$$

and two intervals

$$\begin{aligned} J_{min} &:= [x_N-\min\{G_1(t_0+\varepsilon),G_2(t_0+\varepsilon)\},x_N),\\ J_{max} &:= [x_N-\max\{G_1(t_0+\varepsilon),G_2(t_0+\varepsilon)\},x_N), \end{aligned}$$

where $|J_{min}|$ and $|J_{max}|$ tend to 0 as $\varepsilon\to 0$.
I_2 can be rewritten as

$$I_2=\int_{t_0}^{t_0+\varepsilon}\left(\int_{[x_N-G_1(t),x_N)}e^{\int_{t_0}^{\tau_1(x)}H_1(s,x)ds}-\int_{[x_N-G_2(t),x_N)}e^{\int_{t_0}^{\tau_2(x)}H_2(s,x)ds}\right)d\mu(t_0)dt.$$

We estimate

$$\begin{aligned} |I_2| &\le \int_{t_0}^{t_0+\varepsilon}\int_{[x_N-\max\{G_1(t),G_2(t)\},x_N-\min\{G_1(t),G_2(t)\})}e^{\varepsilon\|p\|_\infty}d\mu(t_0)dt\\ &\quad+\int_{t_0}^{t_0+\varepsilon}\int_{[x_N-\min(G_1(t),G_2(t)),x_N)}(e^{\int_{t_0}^{\tau_1(x)}H_1(s,x)ds}-e^{\int_{t_0}^{\tau_2(x)}H_2(s,x)ds})d\mu(t_0)dt\\ &=:I_{21}+I_{22}. \end{aligned}$$

Now, we estimate the integrand in I_{22}. For all $x\in(x_{N-1},x_N)$, it holds

$$\left|e^{\int_{t_0}^{\tau_1(x)}H_1(s,x)ds}-e^{\int_{t_0}^{\tau_2(x)}H_2(s,x)ds}\right| \le e^{\varepsilon\|p\|_\infty}\left|1-e^{\int_{t_0}^{\tau_2(x)}H_2(s,x)ds-\int_{t_0}^{\tau_1(x)}H_1(s,x)ds}\right|$$

Furthermore,

$$\begin{aligned} &\left|\int_{t_0}^{\tau_2(x)}H_2(s,x)ds-\int_{t_0}^{\tau_1(x)}H_1(s,x)ds\right|\\ &\qquad\le|\tau_2(x)-\tau_1(x)|\|p\|_\infty+\int_{t_0}^{\min\{\tau_1,\tau_2\}}|H_2(s,x)-H_1(s,x)|ds, \end{aligned}$$

and

$$
\begin{aligned}
& \int_{t_0}^{\min\{\tau_1,\tau_2\}} |H_2(s,x)-H_1(s,x)|ds \\
= & \int_{t_0}^{\min\{\tau_1,\tau_2\}} |p\left(v_2(s), x+G_2(s)\right)-p\left(v_1(s), x+G_1(s)\right)| \, ds \\
\leq & \ \|p_2\|_\infty L(p_1) \int_{t_0}^{t_0+\varepsilon} |v_2(s)-v_1(s)|ds + \varepsilon \|p_1\|_\infty L(p_2) L(g) \int_{t_0}^{t_0+\varepsilon} |v_2(s)-v_1(s)|ds,
\end{aligned}
$$

where $L(g)$, $L(p_1)$ and $L(p_2)$ are Lipschitz constants for the functions g, p_1 and p_2 respectively.

Applying the inequality

$$
|1-e^x| \leq |x| e^{|x|}, \tag{33}
$$

we obtain

$$
\begin{aligned}
|I_{22}| \leq & \int_{t_0}^{t_0+\epsilon} \int_{[x_N-\min(G_1(t),G_2(t)),x_N)} e^{\varepsilon\|p\|_\infty} \left|1-e^{\int_{t_0}^{\tau_2(x)} H_2(s,x)ds - \int_{t_0}^{\tau_1(x)} H_1(s,x)ds}\right| \\
\leq & \int_{t_0}^{t_0+\varepsilon} \int_{[x_N-\min(G_1(t),G_2(t)),x_N)} e^{\varepsilon\|p\|_\infty} e^{2\varepsilon\|p\|_\infty} \Bigg(\sup_{x\in J_{min}} |\tau_2(x)-\tau_1(x)| \|p\|_\infty \\
& + \left(\|p_2\|_\infty L(p_1) + \varepsilon \|p_1\|_\infty L(p_2) L(g)\right) \int_{t_0}^{t_0+\varepsilon} |v_2(s)-v_1(s)|ds \Bigg) d\mu(t_0) dt \\
\leq & \ \varepsilon \|\mu(t_0)\|_{\mathcal{M}(\mathbb{R}^+)} e^{3\varepsilon\|p\|_\infty} \Bigg(\sup_{x\in J_{min}} |\tau_2(x)-\tau_1(x)| \\
& + \left(\|p_2\|_\infty L(p_1) + \varepsilon \|p_1\|_\infty L(p_2) L(g)\right) \int_{t_0}^{t_0+\varepsilon} |v_2(s)-v_1(s)|ds \Bigg),
\end{aligned}
$$

where the supremum $\sup|\tau_1(x)-\tau_2(x)|$ is estimated on the interval J_{min} due to the positivity of the function g (see Figure 5). Thus, combining the above expressions and using Fubini theorem for I_{21} (see Fig.5), we conclude

$$
\begin{aligned}
& \int_{t_0}^{t_0+\varepsilon} |v_1(t)-v_2(t)|dt \leq |I_1| + |I_{21}| + |I_{22}| \leq \\
\leq & \ \varepsilon \left(L(p_1)\|p_2\|_\infty \|v_1\|_\infty + \|p\|_\infty\right) \int_{t_0}^{t_0+\varepsilon} |v_1(r)-v_2(r)|dr \\
& + e^{\varepsilon\|p\|_\infty} \mu(t_0)(J_{max}) \sup_{x\in J_{min}} |\tau_1(x)-\tau_2(x)| + \varepsilon \|\mu(t_0)\|_{\mathcal{M}(\mathbb{R}^+)} e^{3\varepsilon\|p\|_\infty} \Big[\left((L(p_1)\|p_2\|_\infty \right. \\
& \left. + \varepsilon L(p_2)\|p_1\|_\infty L(g)\right) \int_{t_0}^{t_0+\varepsilon} |v_2(s)-v_1(s)|ds + \sup_{x\in J_{min}} |\tau_1(x)-\tau_2(x)| \Big] \\
\leq & \ \varepsilon K_1 \int_{t_0}^{t_0+\varepsilon} |v_1(r)-v_2(r)|dr + \left(e^{\varepsilon\|p\|_\infty} \mu(t_0)(J_{max}) + \varepsilon K_2\right) \sup_{x\in J_{min}} |\tau_1(x)-\tau_2(x)|.
\end{aligned}
$$

Next, we need to estimate $\sup_{x\in J_{min}} |\tau_1(x)-\tau_2(x)|$ in terms of $\int_{t_0}^{t_0+\varepsilon} |v_1(s)-v_2(s)|ds$.

$$\begin{aligned}\sup_{x\in J_{min}} |\tau_1(x)-\tau_2(x)| &= \sup_{y\in J_{min}} \left|\int_{x_N-y}^{x_N} \left(\frac{1}{g_1(v_1(\tau_1(z))}-\frac{1}{g_1(v_2(\tau_2(z))}\right) dz\right| \\ &\leq \frac{L(g)}{\min(g)^2}\int_{J_{min}} |v_1(\tau_1(y))-v_2(\tau_2(y))|\, dy\end{aligned}$$

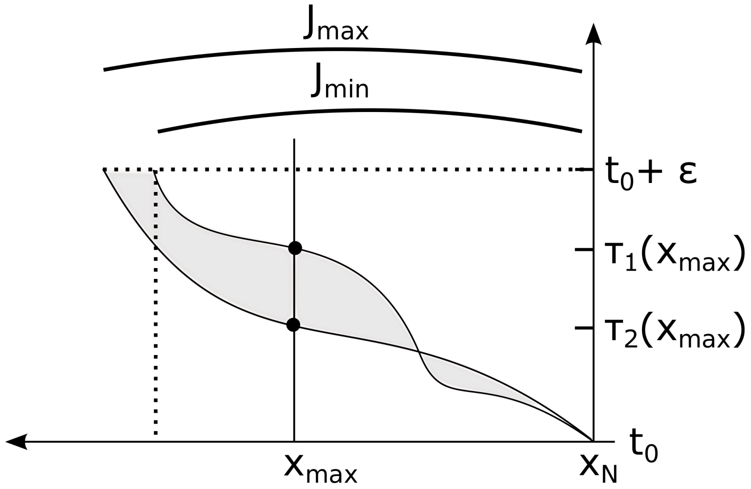

Figure 5: Schematic presentation of trajectories $x_N - G_1(t)$ and $x_N - G_2(t)$ or, equivalently in swapped variables, $\tau_1(y)$ and $\tau_2(y)$. Double integrals over the shaded region, whose maximum width with respect to time variable is assumed at some point $x_{\max}$ inside the *smaller* interval J_{min}, are transformed by use of Fubini theorem.

To estimate the integral on the right hand-side, we calculate

$$\begin{aligned}\int_{J_{min}} |v_1(\tau_1(y))-v_2(\tau_2(y))|dy &\leq \int_{J_{min}} |v_1(\tau_1(y))-v_2(\tau_1(y))|dy \\ &+\int_{J_{min}} |v_2(\tau_1(y))-v_2(\tau_2(y))|dy.\end{aligned}$$

The first integral satisfies

$$\begin{aligned}\int_{J_{min}} |v_1(\tau_1(y))-v_2(\tau_1(y))|dy &\leq \int_{t_0}^{t_0+\varepsilon} |v_1(s)-v_2(s)|\frac{dy_1}{ds}ds \\ &\leq \|g\|_\infty \int_{t_0}^{t_0+\varepsilon} |v_1(s)-v_2(s)|ds.\end{aligned}$$

On the other hand, using explicit formula (32) for v_2, we estimate

$$\int_{J_{min}} |v_2(\tau_1(y)) - v_2(\tau_2(y))|dy \leq \int_{J_{min}} \Bigg(\int_{[x_N - G_2(\tau_1(y)), x_N - G_2(\tau_2(y)))} e^{\varepsilon \|p\|_\infty} d\mu(t_0) + (e^{\sup_{x\in J_{min}} |\tau_2(x)-\tau_1(x)| \|p\|_\infty} - 1)\|v_2\|_\infty \Bigg) dy.$$

Application of the Fubini theorem and inequality (33) results in

$$\int_{J_{min}} |v_2(\tau_1(y)) - v_2(\tau_2(y))|dy \leq \left(e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max})\|g\|_\infty + \varepsilon\|g\|_\infty\|p\|_\infty e^{\varepsilon\|p\|_\infty}\|v_2\|_\infty\right) \sup_{x\in J_{min}} |\tau_2(x) - \tau_1(x)|.$$

We obtain

$$\sup_{x\in J_{min}} |\tau_1(x) - \tau_2(x)| \leq \frac{L(g)}{(\min g)^2}\|g\|_\infty \Bigg(\int_{t_0}^{t_0+\varepsilon} |v_1(s) - v_2(s)|ds + \left(e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max}) + \varepsilon\|p\|_\infty e^{\varepsilon\|p\|_\infty}\|v_2\|_\infty\right) \sup_{x\in J_{min}} |\tau_1(x) - \tau_2(x)| \Bigg).$$

It yields

$$\sup_{x\in J_{min}} |\tau_1(x) - \tau_2(x)| \leq \frac{1}{M_\varepsilon} \int_{t_0}^{t_0+\varepsilon} |v_1(s) - v_2(s)|ds$$

for ε small enough and $M_\varepsilon = \frac{(\min g)^2}{L(g)\|g\|_\infty} - e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max}) - \varepsilon\|p\|_\infty e^{\varepsilon\|p\|_\infty}\|v_2\|_\infty$. Since the interval J_{max} is given by $J_{max} = [x_N - \max\{G_1(t_0+\varepsilon), G_2(t_0+\varepsilon)\}, x_N)$, its length tends to 0 as $\varepsilon \to 0$, and also $\mu(t_0)(J_{max}) \to 0$. Therefore, we may choose ε small enough such that $M_\varepsilon > 0$. Then,

$$\begin{aligned} &\int_{t_0}^{t_0+\varepsilon} |v_1(s) - v_2(s)|ds \\ \leq\ & \varepsilon K_1 \int_{t_0}^{t_0+\varepsilon} |v_1(s) - v_2(s)|ds + \left(e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max}) + \varepsilon K_2\right) \sup_{x\in J_{min}} |\tau_1 - \tau_2| \\ \leq\ & \left(\varepsilon K_1 + \left(e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max}) + \varepsilon K_2\right)\frac{1}{M_\varepsilon}\right) \int_{t_0}^{t_0+\varepsilon} |v_1(s) - v_2(s)|ds \end{aligned}$$

Using again the properties of the interval J_{max}, we may conclude that there exists ε such that the term $\left(\varepsilon K_1 + \left(e^{\varepsilon\|p\|_\infty}\mu(t_0)(J_{max}) + \varepsilon K_2\right)\frac{1}{M_\varepsilon}\right) < 1$. Consequently, $v_1(t) = v_2(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$.

Finally, it follows from Lemma 5.1 that $\mu_1 = \mu_2$ on $[t_0, t_0+\varepsilon]$, what gives a contradiction and finishes the proof. □

Remark 8.2. It has to be pointed out that the method applied above is well-suited for proving uniqueness, but cannot be directly used in the proof of existence. Observe that ε in the last estimate depends strongly on the measure $\mu(t_0)$ and cannot be chosen uniformly in terms of the norm of measure $\mu(t_0)$, what would be the case if $\mu(0)$ were absolutely continuous with respect to $\mathcal{L}^1$ with bounded density. For example, taking $\mu^k(t_0) = \delta_{x_N - 1/k}$ leads to ε_k tending to 0 as $k \to \infty$. Therefore, Banach's Fixed Point Theorem cannot be directly applied to show the global existence of solutions. Nevertheless, using more involved calculations based on the metric structure and similar estimates as above, one could apply Banach's Fixed Point Theorem to show the existence and uniqueness. We omit such approach here for the clarity of presentation.

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