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Preprint no. 2011 - 003

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Structured Populations, Cell Growth and Measure Valued Balance Laws

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February 2, 2011

Abstract

A well-posedness theory of measure valued solutions to balance laws is presented. Nonlinear semigroups are constructed by means of the operator splitting algorithm. This approach allows to separate the differential terms from the integral ones, leading to a significant simplification of the proofs. Continuous dependence with respect to parameters is also shown. The whole framework allows a unified approach to a variety of structured population models, providing to each of them the basic well posedness and stability results.

1 Introduction

Structured population models have been widely studied in the mathematical biology literature for many years, see [5, 11, 13, 16, 17, 18, 20, 21, 22]. Most of these models can be written as evolutionary PDEs for the density of individuals [13, 21] with a specific structural variable x, for instance the age for the age-structured equation [22] or a phenotypic trait for the selection-mutation equations [5]. The typical functional space in which early achievements were obtained is the space of integrable functions or densities. For instance, global existence and continuity with respect to model ingredients with integrable initial data were established in [21, 22].

For many biological applications it is often necessary to consider initial data or coefficients which are not integrable functions, but measures. In fact, setting a model in the space of measures was suggested in [22, Section III.5] and, since then, several ways of dealing with this problem were proposed. The weak* semigroup approach was developed in [11], where global existence of solutions in the set of finite Radon measures was proved, together with their weak* continuous dependence from time and initial datum. A slightly different treatment of the problem, based on the theory of nonlinear semigroups in metric spaces, was presented in [16, 17]. An alternative construction of measure-valued solutions to these models can be obtained following ideas coming from kinetic theory [14, 7] by means of a Picard-type result for evolutions in the set of measures, see [6]. Here, we follow and extend the approach in [16, 17]

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by constructing the solutions using the operator splitting, or fractional step, method [9, 10] in metric spaces. This allows for a significant shortening of the proofs compared to [16, 17], while in the same time gaining more generality.

The main aim of this paper is the study of the following Cauchy problem

$$\begin{cases} \partial_t \mu + \partial_x \left(b(t,\mu)\,\mu\right) + c(t,\mu)\,\mu = \displaystyle\int_{\mathbb{R}^+} \left(\eta(t,\mu)\right)(y)\;\mathrm{d}\mu\,(y) \\ \mu(0) = \mu_o \end{cases} \quad \text{with} \quad \begin{cases} t \in [0,T] \\ x \in \mathbb{R}^+ \\ \mu \in \mathcal{M}^+(\mathbb{R}^+) \end{cases} \tag{1.1}$$

where $\mu(t)$ is the measure determining the distribution of the population with respect to the structural variable x.

The framework presented below applies to a variety of relevant biological models, allowing a unified analytical approach to several entirely different biological situations considered in the current literature. In § 3.1 we show that (1.1) includes the McKendrick age structured population model [18] as well as the nonlinear age structured model [21, 22]. Then, § 3.2 deals with the linear and nonlinear size structured models for cell division presented in [13] and in [20, Chapter 4], as well as with the size structured model for evolution of phytoplankton aggregates, see [3]. Also a simple cell cycle structured population model fits in the present setting, as shown in § 3.3. The body size structured model [11] is considered in § 3.4. Finally, the selection-mutation models in [1, 5, 8] are tackled in § 3.5.

This paper is organized as follows. Section 2 presents the analytical results, separately considering the linear autonomous case in § 2.1, the linear non autonomous case in § 2.2 and the general case in § 2.3. Section 3 shows that the present framework applies to a several models considered in the current literature, as already remarked. All proofs are deferred to Section 4.

2 Main Results

Throughout, $x \in \mathbb{R}^+ = [0, +\infty[$. We emphasize that $\mathbb{R}^+$ is used only for the ambient space, so it always refers to structural variables and not to time. Nevertheless, to avoid misunderstandings, we sometimes denote the ambient space by $\mathbb{R}^+_{\mathbf{x}}$ to underline that we refer to a structural variable. The choice of $\mathbb{R}^+$ as the structural variable ambient space does not play any specific role but it is adopted to include some typical nonlinear age-structured models.

As usual, given a metric space X, we denote by $\mathbf{C}([0,T];X)$ the space of continuous functions. When X is a normed space, $\mathbf{BC}([0,T];X)$ is the space of bounded continuous functions with the supremum norm and $\mathbf{W^{k,p}}([0,T];X)$ denotes the usual Sobolev space. The set of positive Radon measures on $\mathbb{R}^+$ with bounded total variation is denoted by $\mathcal{M}^+(\mathbb{R}^+)$, which is a complete metric space when equipped with the distance

$$d(\mu_1,\mu_2) = \sup\left\{ \int_{\mathbb{R}^+} \varphi\;\mathrm{d}(\,\mu_1 - \mu_2) \colon \varphi \in \mathbf{C^1}(\mathbb{R}^+;\mathbb{R}) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \leq 1 \right\} \tag{2.1}$$

and a subspace of $\left(\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right)^*$, the dual of $\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$ endowed with the usual norm $\|u\|_{\mathbf{W^{1,\infty}}} = \max\left\{\|u\|_{\mathbf{L^\infty}}, \|\partial_x u\|_{\mathbf{L^\infty}}\right\}$. Let us observe that the condition $\varphi \in \mathbf{C^1}(\mathbb{R}^+;\mathbb{R})$ in (2.1) can be substituted by $\varphi \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$ through a standard mollifying sequence argument applied to the test function φ, since its derivative is not involved in the value of the integral. Therefore, this metric is exactly the one induced by the dual norm of $\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$.

However, we use (2.1) since it simplifies some of the technical proofs below. The metric d is usually called the *Bounded Lipschitz* distance, see [23] or the *flat metric* distance, see [19]. The assumptions on the different ingredients of the model are:

$$\begin{aligned} b, c &\colon [0,T] \times \mathcal{M}^+(\mathbb{R}^+) \to \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R}) \text{ with } b(t,\mu)(0) \geq 0 \text{ for } (t,\mu) \in [0,T] \times \mathcal{M}^+(\mathbb{R}^+), \\ \eta &\colon [0,T] \times \mathcal{M}^+(\mathbb{R}^+) \to (\mathbf{BC} \cap \mathbf{Lip}) \left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right) . \end{aligned} \tag{2.2}$$

The space $\mathbf{Lip}\left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right)$ consists of all Lipschitz functions from $\mathbb{R}^+_{\mathbf{x}}$ with values in the metric space $(\mathcal{M}^+(\mathbb{R}^+), d)$. The space $\mathbf{BC}\left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right)$ denotes the set of functions that are bounded with respect to the norm $\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}$ and continuous with respect to d. A norm in the space $(\mathbf{BC} \cap \mathbf{Lip})\left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right)$ is defined as

$$\|\cdot\|_{\mathbf{BCL}} = \|\cdot\|_{\mathbf{BC_x}} + \mathbf{Lip}(\cdot), \quad \text{where} \quad \|f\|_{\mathbf{BC_x}} = \sup_{x \in \mathbb{R}^+} \left\|f(x)\right\|_{(\mathbf{W^{1,\infty}})^*} .$$

Note that $\left(\mathcal{M}^+(\mathbb{R}^+), d\right)$ is separable, hence strong measurability and weak measurability are equivalent, see [17] for more details. We also refer to [15, Appendix E.5] for the basic results about Banach space valued functions. In this framework, the integral

$$\int_{\mathbb{R}^+} \left(\eta(t,\mu)\right)(y) \, \mathrm{d}\mu\,(y)$$

appearing in (1.1) is a Bochner integral with values in $\mathcal{M}^+(\mathbb{R}^+_{\mathbf{x}})$.

Remark 2.1. *It is worth to note that the space* $(\mathbf{BC} \cap \mathbf{Lip})\left(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^+)\right)$ *is not a subspace of* $\mathbf{W^{1,\infty}}\left(\mathbb{R}^+; \left(\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right)^*\right)$*, although the set of Radon measures* $\mathcal{M}^+(\mathbb{R}^+)$ *is a nonnegative cone in* $\left(\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right)^*$*. As an example, consider* $f(x) = \delta(y = x)$ *where* δ *is Dirac delta. It is easy to check, that* f *is bounded with respect to the norm* $\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}$ *and Lipschitz continuous with respect to* d*, since*

$$\begin{aligned} \|f\|_{\mathbf{BC_x}} = \sup_{x \in \mathbb{R}^+} \left\|f(x)\right\|_{(\mathbf{W^{1,\infty}})^*} &= \sup_{x \in \mathbb{R}^+} \sup_{\{\psi \,:\, \|\psi\|_{\mathbf{W^{1,\infty}}} \leq 1\}} \int_{\mathbb{R}^+} \psi(x) \, \mathrm{d}\delta\,(x) \leq 1 \\ d(f(x_1), f(x_2)) &= \min\{2, |x_1 - x_2|\} \leq |x_1 - x_2| . \end{aligned}$$

However, $f'(x) = \delta'(x)$ *is not a well defined functional on* $\mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$*.*

We also need to assume some time regularity for our model functions, i.e.,

$$b, c \in \mathbf{BC^{\alpha,1}}\left([0,T] \times \mathcal{M}^+(\mathbb{R}^+);\ \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right) \tag{2.3}$$

$$\eta \in \mathbf{BC^{\alpha,1}}\left([0,T] \times \mathcal{M}^+(\mathbb{R}^+);\ (\mathbf{BC} \cap \mathbf{Lip})(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+))\right) . \tag{2.4}$$

Here, $\mathbf{BC^{\alpha,1}}([0,T] \times \mathcal{M}^+(\mathbb{R}^+); X)$ is the space of X valued functions which are bounded with respect to the $\|\cdot\|_X$ norm, Hölder continuous with exponent α with respect to time and Lipschitz continuous in d with respect to the measure variable. This space is equipped with the $\|\cdot\|_{\mathbf{BC^{\alpha,1}}}$ norm defined by

$$\|f\|_{\mathbf{BC^{\alpha,1}}} = \sup_{t \in [0,T], \mu \in \mathcal{M}^+(\mathbb{R}^+)} \left(\left\|f(t,\mu)\right\|_X + \mathbf{Lip}\left(f(t,\cdot)\right) + \mathrm{H}\left(f(\cdot,\mu)\right)\right) \tag{2.5}$$

where

$$\mathrm{H}(f(\cdot,\mu)) := \sup_{s_1,s_2\in[0,T]} \left(\left\| f(s_1,\mu) - f(s_2,\mu) \right\|_X / |s_1-s_2|^\alpha \right).$$

A relevant choice of functions b, c and η is the following:

$$b(t,\mu) = \tilde b\left(t, \int_{\mathbb{R}^+} \beta(y)\, \mathrm{d}\mu\,(y)\right), \quad c(t,\mu) = \tilde c\left(t, \int_{\mathbb{R}^+} \gamma(y)\, \mathrm{d}\mu\,(y)\right)$$

and

$$\eta(t,\mu) = \tilde\eta\left(t, \int_{\mathbb{R}^+} h(y)\, \mathrm{d}\mu\,(y)\right)$$

with $\beta,\gamma,h \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R}^+)$, $\tilde b, \tilde c \in \mathbf{BC}^{\alpha,\mathbf{1}}\left([0,T]\times\mathbb{R}^+_{\mathbf{x}};\ \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right)$ and the right-hand side $\tilde\eta \in \mathbf{BC}^{\alpha,\mathbf{1}}\left([0,T]\times\mathbb{R}^+_{\mathbf{x}};\ (\mathbf{BC}\cap\mathbf{Lip})(\mathbb{R}^+_{\mathbf{x}};\mathcal{M}^+(\mathbb{R}^+))\right)$. As a particular example, consider the following nonlinear functions b, c and η:

$$b(t,\mu)(x) = h_b(t)\, f_b(x)\, G_b\left(\int_{\mathbb{R}^+} \varphi_b\, \mathrm{d}\mu\right), \quad c(t,\mu)(x) = h_c(t)\, f_c(x)\, G_c\left(\int_{\mathbb{R}^+} \varphi_c\, \mathrm{d}\mu\right),$$

and

$$\left[\eta(t,\mu)(x)\right](A) = h_\eta(t)\, f_\eta(x)\, G_\eta\left(\int_{\mathbb{R}^+} \varphi_\eta\, \mathrm{d}\mu\right)\nu(A)$$

where $h_b, h_c, h_\eta \in \mathbf{C}^\alpha([0,T];\mathbb{R}^+)$, $f_b, f_c, f_\eta, \varphi_b, \varphi_c, \varphi_\eta, G_b, G_c, G_\eta \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R}^+)$ and $\nu \in \mathcal{M}^+(\mathbb{R}^+)$.

We begin our analytical study with the basic definition of solutions to (1.1).

Definition 2.2. *Given $T>0$, a function $\mu\colon [0,T] \to \mathcal{M}^+(\mathbb{R}^+)$ is a* weak solution *to* (1.1) *on the time interval $[0,T]$ if μ is narrowly continuous with respect to time and for all $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})\left([0,T]\times\mathbb{R};\mathbb{R}\right)$, the following equality holds:*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^+} \Big(\partial_t\varphi(t,x) + \big(b(t,\mu)\big)(x)\, \partial_x\varphi(t,x) - \big(c(t,\mu)\big)(x)\, \varphi(t,x)\Big)\, \mathrm{d}\mu_t\,(x)\, \mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \varphi(t,x)\, \mathrm{d}\big[\eta(t,\mu)(y)\big]\,(x)\right) \mathrm{d}\mu_t\,(y)\, \mathrm{d}t \qquad (2.6)\\
&= \int_{\mathbb{R}^+} \varphi(T,x)\ \mathrm{d}\mu_T\,(x) - \int_{\mathbb{R}^+} \varphi(0,x)\ \mathrm{d}\mu_o\,(x)\,.
\end{aligned}$$

Here, by *narrowly continuous functions* we refer to the narrow convergence introduced in [2, § 5.1]. The integral $\int_{\mathbb{R}^+} \varphi(t,x)\, \mathrm{d}\big[\eta(t,\mu)(y)\big]\,(x)$ denotes the integral of $\varphi(t,x)$ with respect to the measure $\eta(t,\mu)(y)$ in the variable x. Similarly, $\int_{\mathbb{R}^+} \varphi(T,x)\, \mathrm{d}\mu_T\,(x)$ is the integral of $\varphi(T,x)$ with respect to the measure $\mu(T)$ in the variable x.

2.1 The Linear Autonomous Case

The linear autonomous case of (1.1) reads

$$\begin{cases} \partial_t\mu + \partial_x\left(b(x)\,\mu\right) + c(x)\,\mu = \displaystyle\int_{\mathbb{R}^+} \eta(y)\ \mathrm{d}\mu\,(y) \\ \mu(0) = \mu_o \end{cases} \qquad \text{with} \qquad \begin{cases} t \in [0,T] \\ x \in \mathbb{R}^+ \end{cases} \qquad (2.7)$$

where the unknown $\mu = \mu(t)$ is in $\mathcal{M}^+(\mathbb{R}^+)$ for all times $t \in [0,T]$. In the present case, the assumption (2.2) reduces to

$$\begin{aligned} b, c &\in \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^+;\mathbb{R}) \text{ with } b(0) \geq 0\,, && (2.8)\\ \eta &\in (\mathbf{BC} \cap \mathbf{Lip})\left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right). && (2.9)\end{aligned}$$

A first justification of Definition 2.2 of weak solution and of assumptions (2.8)–(2.9) is provided by the following result.

Proposition 2.3. *With the notations introduced above:*

i) If $\eta(y)$ has density $g(y)$ for all $y \in \mathbb{R}^+$ with respect to the Lebesgue measure, with $g \in (\mathbf{BC} \cap \mathbf{Lip})\left(\mathbb{R}^+; \mathbf{L}^\mathbf{1}(\mathbb{R}^+;\mathbb{R}^+)\right)$, i.e.,

$$\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \varphi(x)\, \mathrm{d}\left[\eta(y)\right](x) \right) \mathrm{d}\mu_t\,(y) = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \varphi(x) g(x,y)\, \mathrm{d}x \right) \mathrm{d}\mu_t\,(y)$$

for all $\varphi \in \mathbf{C}(\mathbb{R}^+;\mathbb{R})$, with $g(y) \in \mathbf{L}^\mathbf{1}(\mathbb{R}^+;\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$ and $g(x,y) := g(y)(x) \in \mathbb{R}^+$ for all $x, y \in \mathbb{R}^+$, then η satisfies (2.9).

ii) If μ_o has density u_o with respect to the Lebesgue measure, with $u_o \in (\mathbf{L}^\mathbf{1} \cap \mathbf{C}^\mathbf{1})(\mathbb{R}^+;\mathbb{R}^+)$, then $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$.

iii) Let (2.8) hold together with i) and ii) above. If μ has density u with respect to the Lebesgue measure, with $u \in \mathbf{Lip}\left([0,T]; \mathbf{L}^\mathbf{1}(\mathbb{R}^+;\mathbb{R}^+)\right)$ and $u(t) \in \mathbf{C}^\mathbf{1}(\mathbb{R}^+;\mathbb{R}^+)$, then u is weak a solution to

$$\begin{cases} \partial_t u + \partial_x \left(b(x) u\right) + c(x) u = \displaystyle\int_{\mathbb{R}^+} g(x,y)\, u(t,y)\, \mathrm{d}y \\ u(0,x) = u_o(x) \\ u(t,0) = 0 \end{cases}$$

if and only if μ is a solution to the linear equation (2.7) *in the sense of Definition* 2.2*.*

The proof is immediate and, hence, omitted. To prove the well posedness of (2.7), we use the operator splitting algorithm, see [9] and [10, § 3.3]. To this aim, we consider separately the problems

$$\partial_t \mu + c(x)\, \mu = \int_{\mathbb{R}^+} \eta(y)\; \mathrm{d}\mu\,(y) \qquad \text{and} \qquad \partial_t \mu + \partial_x \left(b(x)\, \mu\right) = 0\,.$$

Remark that both problems are particular cases of (1.1), so that Definition 2.2 applies to both. Consider first the ODE part.

Lemma 2.4. *Let $c \in \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^+;\mathbb{R})$ and $\eta \in (\mathbf{BC} \cap \mathbf{Lip})\left(\mathbb{R}^+_{\mathbf{x}}; \mathcal{M}^+(\mathbb{R}^+)\right)$. Then, the Cauchy problem for*

$$\partial_t \mu + c(x)\, \mu = \int_{\mathbb{R}^+} \eta(y)\; \mathrm{d}\mu\,(y) \tag{2.10}$$

generates a local Lipschitz semigroup $\hat{S}\colon [0,T] \times \mathcal{M}^+(\mathbb{R}^+) \to \mathcal{M}^+(\mathbb{R}^+)$, in the sense that:

i) $\hat{S}_0 = \mathbf{Id}$ and for all $t_1, t_2 \in [0,T]$ with $t_1 + t_2 \in [0,T]$, we have $\hat{S}_{t_1} \circ \hat{S}_{t_2} = \hat{S}_{t_1+t_2}$.

ii) For all $t \in [0,T]$ *and for all* $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^+)$*, the following estimate holds:*

$$d(\hat{S}_t\mu_1, \hat{S}_t\mu_2) \leq \exp\Big(3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BCL}}\right) t\Big)\, d(\mu_1, \mu_2)\,.$$

iii) For all $t \in [0,T]$ *and for all* $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$ *define* $\mu(t) = \hat{S}_t\mu_o$*. Then, the solution to the Cauchy problem satisfies* $\mu \in \mathbf{Lip}([0,T], \mathcal{M}^+(\mathbb{R}^+))$ *and the following estimate holds:*

$$d(\hat{S}_t\mu_o, \mu_o) \leq \Big(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\Big)\, \exp\bigg(\Big(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\Big) t\bigg)\, \mu_o(\mathbb{R}^+)\, t\,.$$

iv) Let $c_1, c_2 \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$*,* η_1, η_2 *satisfy* (2.9)*, and denote by* $\hat{S}^1, \hat{S}^2$ *the corresponding semigroups. Then, for all* $t \in [0,T]$ *and* $\mu \in \mathcal{M}^+(\mathbb{R}^+)$

$$\begin{aligned} & d(\hat{S}^1_t\mu, \hat{S}^2_t\mu) \\ \leq\ & \Big(\|c_1 - c_2\|_{\mathbf{L}^\infty} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\Big)\, e^{\left(\|c_1\|_{\mathbf{L}^\infty} + \|c_2\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}} + \|\eta_2\|_{\mathbf{BC_x}}\right)t}\, \mu(\mathbb{R}^+)\, t \end{aligned}$$

v) For all $\mu \in \mathcal{M}^+(\mathbb{R}^+)$*, the orbit* $t \to \hat{S}_t\mu$ *of the semigroup is a weak solution to* (2.10) *in the sense of Definition* 2.2*.*

The proof is deferred to Paragraph 4.1, where we exploit the dual formulation of (2.10). The analogous result about the convective part is below.

Lemma 2.5. *Let* $b \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$ *with* $b(0) \geq 0$*. Then, the Cauchy problem*

$$\partial_t\mu + \partial_x\left(b(x)\,\mu\right) = 0 \tag{2.11}$$

generates a local Lipschitz semigroup $\check{S}\colon [0,T] \times \mathcal{M}^+(\mathbb{R}^+) \to \mathcal{M}^+(\mathbb{R}^+)$*, in the sense that*

i) $\check{S}_0 = \mathbf{Id}$ *and for all* $t_1, t_2 \in [0,T]$ *with* $t_1 + t_2 \in [0,T]$*, we have* $\check{S}_{t_1} \circ \check{S}_{t_2} = \check{S}_{t_1+t_2}$*.*

ii) For all $t \in [0,T]$ *and for all* $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^+)$*, the following estimate holds:*

$$d(\check{S}_t\mu_1, \check{S}_t\mu_2) \leq \exp\left(\|\partial_x b\|_{\mathbf{L}^\infty} t\right)\, d(\mu_1, \mu_2)\,.$$

iii) For all $t \in [0,T]$ *and for all* $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$ *define* $\mu(t) = \check{S}_t\mu_o$*. Then, the solution of the Cauchy problem satifies* $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}^+))$ *and the following estimate holds:*

$$d(\check{S}_t\mu_o, \mu_o) \leq \|b\|_{\mathbf{L}^\infty}\, \mu_o(\mathbb{R}^+)\, t\,.$$

iv) Let $b_1, b_2 \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$ *with* $b_1(0), b_2(0) \geq 0$ *and denote by* $\check{S}^1, \check{S}^2$ *the corresponding semigroups. Then, for all* $t \in [0,T]$ *and* $\mu \in \mathcal{M}^+(\mathbb{R}^+)$

$$d(\check{S}^1_t\mu, \check{S}^2_t\mu) \leq \|b_1 - b_2\|_{\mathbf{L}^\infty}\, \mu(\mathbb{R}^+)\, t\,.$$

v) For all $\mu \in \mathcal{M}^+(\mathbb{R}^+)$*, the orbit* $t \to \check{S}_t\mu$ *of the semigroup is a weak solution to* (2.11) *in the sense of Definition* 2.2*.*

The proof of the above Lemma is deferred to Paragraph 4.2. To apply [10, Corollary 3.3], see also [9, theorems 3.5 and 3.8], about the convergence of the operator splitting algorithm, we need to estimate the defect of commutativity of the two semigroups.

Proposition 2.6. *Let* (2.8) *and* (2.9) *hold. Let* $\hat{S}$ *be the semigroup defined in Lemma 2.4 and* $\check{S}$ *the one in Lemma 2.5. Then, for all* $\mu \in \mathcal{M}^+(\mathbb{R}^+)$ *and for all* $t \in [0,T]$*, the following estimate on the lack of commutativity of* $\check{S}$ *and* $\hat{S}$ *holds:*

$$d(\check{S}_t \hat{S}_t \mu, \hat{S}_t \check{S}_t \mu) \leq 3\ t^2\ \|b\|_{\mathbf{L}^\infty} \left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BCL}}\right)\ \exp\left[3\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BCL}}\right) t\right] \tag{2.12}$$

The above commutativity estimate allows us to apply the usual operator splitting technique, obtaining the following final result in the linear autonomous case.

Theorem 2.7. *Let* (2.8) *and* (2.9) *hold. The operator splitting procedure applied to the semigroups* $\check{S}$ *and* $\hat{S}$ *yields a local Lipschitz semigroup* S *that enjoys the following properties:*

i) $S_0 = \mathbf{Id}$ *and for all* $t_1, t_2 \in [0,T]$ *with* $t_1 + t_2 \in [0,T]$*, we have* $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$.

ii) For all $t \in [0,T]$ *and for all* $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^+)$,

$$d(S_t\mu_1, S_t\mu_2) \leq \exp\left[3\left(\|\partial_x b\|_{\mathbf{L}^\infty} + \|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BCL}}\right) t\right] d(\mu_1, \mu_2)\,.$$

iii) For all $t \in [0,T]$ *and for all* $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$,

$$d(S_t\mu_o, \mu_o) \leq \left(\|b\|_{\mathbf{L}^\infty} + \left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right) \exp\left[\left(\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right) t\right]\right) \mu_o(\mathbb{R}^+)\, t\,.$$

iv) For $i = 1, 2$*, let* b_i*,* c_i*,* η_i *satisfy assumptions* (2.8) *and* (2.9)*. Denote by* S^i *the corresponding semigroup. Then, for all* $\mu \in \mathcal{M}^+(\mathbb{R}^+)$ *and for all* $t \in [0,T]$,

$$\begin{array}{rcl} d(S^1_t\mu, S^2_t\mu) & \leq & \exp\left[5\left(\|b_1\|_{\mathbf{W^{1,\infty}}} + \|c_1\|_{\mathbf{W^{1,\infty}}} + \|\eta_1\|_{\mathbf{BCL}}\right) t\right] \\ & & \cdot\ t\ \mu(\mathbb{R}^+)\ \left(\|b_1 - b_2\|_{\mathbf{L}^\infty} + \|c_1 - c_2\|_{\mathbf{L}^\infty} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\right) \end{array}$$

v) For all $\mu \in \mathcal{M}^+(\mathbb{R}^+)$*, the orbit* $t \to S_t\mu$ *of the semigroup is a weak solution to linear autonomous problem* (2.7) *in the sense of Definition* 2.2.

vi) The following tangency condition holds: $\lim_{t\to 0+} \frac{1}{t}\, d(S_t\mu, \check{S}_t\hat{S}_t\mu) = 0$.

Above, *ii)* corresponds to the Lipschitz dependence from the initial datum, *iii)* to the time regularity of the solution, *iv)* shows the stability with respect to the defining equations and *vi)* allows for a characterization in terms of evolution equations in metric spaces, see [10].

2.2 The Linear Non Autonomous Case

We now assume that, for a fixed $\alpha \in \,]0,1]$,

$$b, c \;\in\; \mathbf{BC}^\alpha\left([0,T]; \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})\right) \text{ with } b(t)(0) \geq 0 \text{ for all } t \in [0,T] \tag{2.13}$$

$$\eta \;\in\; \mathbf{BC}^\alpha\left([0,T]; (\mathbf{Lip} \cap \mathbf{BC})(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^+))\right) \tag{2.14}$$

and consider the following linear non autonomous version of (1.1):

$$\left\{\begin{array}{l} \partial_t\mu + \partial_x\left(b(t,x)\,\mu\right) + c(t,x)\,\mu = \displaystyle\int_{\mathbb{R}^+} \eta(t,y)\ \mathrm{d}\mu\,(y) \\ \mu(0) = \mu_o \end{array}\right. \quad \text{with} \quad \left\{\begin{array}{rcl} t & \in & [0,T] \\ x & \in & \mathbb{R}^+\,. \end{array}\right. \tag{2.15}$$

The space $\mathbf{BC}^\alpha\left([0,T];X\right)$ consists of Hölder continuous, X valued functions with norm

$$\|f\|_{\mathbf{BC}^\alpha} = \|f\|_{\mathbf{BC_t}} + H(f) = \sup_{t\in[0,T]} \left\|f(t)\right\|_X + \sup_{s_1,s_2\in[0,T]} \frac{\left\|f(s_1)-f(s_2)\right\|_X}{|s_1-s_2|^\alpha}\,.$$

To simplify the statements of the next theorems, for any finite set of elements x_k in the normed space X, $k\in\{1,\ldots,n\}$, define

$$\left\|(x_1,\ldots,x_n)\right\|_X := \sum_{k=1}^n \|x_k\|_X\,.$$

Theorem 2.8. *Let* (2.13) *and* (2.14) *hold. Then, the linear nonautonomous problem* (2.15) *generates a global process* $P\colon [0,T]^2\times\mathcal{M}^+(\mathbb{R}^+)\to\mathcal{M}^+(\mathbb{R}^+)$*, in the sense that*

i) For all t_o,t_1,t_2,μ *satisfying* $0\le t_o\le t_1\le t_2\le T$ *and* $\mu\in\mathcal{M}^+(\mathbb{R}^+)$

$$\begin{aligned} &P(t_o,t_o)\mu=\mu\\ &P(t_1,t_o)\mu\in\mathcal{M}^+(\mathbb{R}^+)\\ &P(t_2,t_1)\circ P(t_1,t_o)\mu = P(t_2,t_o)\mu \end{aligned}$$

ii) For all t_o,t,μ_1,μ_2 *satisfying* $0\le t_o\le t\le T$ *and* $\mu_1,\mu_2\in\mathcal{M}^+(\mathbb{R}^+)$

$$d(P(t,t_o)\mu_1,P(t,t_o)\mu_2)\le \mathrm{e}^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\ d(\mu_1,\mu_2)$$

iii) For all t_o,t,μ *satisfying* $0\le t_o\le t\le T$ *and* $\mu\in\mathcal{M}^+(\mathbb{R}^+)$

$$d(P(t,t_o)\mu,P(t_o,t_o)\mu)\le \left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}\,\mathrm{e}^{2\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\,\mu(\mathbb{R}^+)\,(t-t_o)$$

iv) For $i=1,2$*, let* b_i*,* c_i*,* η_i *satisfy assumptions* (2.13) *and* (2.14)*. Call* P^i *the corresponding process. Then, for all* t_o,t,μ *satisfying* $0\le t_o\le t\le T$ *and* $\mu\in\mathcal{M}^+(\mathbb{R}^+)$*, there exists a constant* $C^*=C^*\left(t_o,T,\|b_1\|_{\mathbf{BC_t}},\|c_1\|_{\mathbf{BC_t}},\|\eta_1\|_{\mathbf{BC_t}}\right)$ *such that*

$$\begin{aligned} d\left(P^1(t,t_o)\mu,P^2(t,t_o)\mu\right) \;\le\;& C^*(t-t_o)\,\mathrm{e}^{5\left\|(b_1,b_2,c_1,c_2,\eta_1,\eta_2)\right\|_{\mathbf{BC_t}}(t-t_o)}\\ &\cdot\left\|(b_1,c_1,\eta_1)-(b_2,c_2,\eta_2)\right\|_{\mathbf{BC_t}}\mu(\mathbb{R}^+) \end{aligned}$$

v) For all $\mu\in\mathcal{M}^+(\mathbb{R}^+)$*, the trajectory* $t\to P(t,0)\mu$ *of the process is a weak solution to linear nonautonomous problem* (2.7) *in the sense of Definition* 2.2*.*

2.3 The Non Linear Case

This section is devoted to the general problem (1.1) and presents the main result of this paper.

Theorem 2.9. *Let* (2.2)*,* (2.3)*, and* (2.4) *hold. Then, there exists an unique solution* $\mu\in(\mathbf{BC}\cap\mathbf{Lip})\left([0,T];\mathcal{M}^+(\mathbb{R}^+)\right)$ *to the full nonlinear problem* (1.1)*. Moreover,*

i) for all $0\le t_1\le t_2\le T$ *there exist constants* K_1 *and* K_2*, such that*

$$d\left(\mu(t_1),\mu(t_2)\right)\le K_1\,\mathrm{e}^{K_2(t_2-t_1)}\,\mu_o(\mathbb{R}^+)(t_2-t_1)$$

ii) Let $\mu_1(0), \mu_2(0) \in \mathcal{M}^+(\mathbb{R}^+)$ *and* b_i, c_i, η_i *satisfy assumptions* (2.2), (2.3)*, and* (2.4) *for* $i = 1, 2$*. Let* μ_i *solve* (1.1) *with initial datum* $\mu_i(0)$ *and coefficients* (b_i, c_i, η_i)*,* $i = 1, 2$*. Then, there exist constants* C_1*,* C_2 *and* C_3 *such that for all* $t \in [0, T]$

$$d\left(\mu_1(t), \mu_2(t)\right) \leq \mathrm{e}^{C_1 t}\, d\left(\mu_1(0), \mu_2(0)\right) + C_2\, \mathrm{e}^{C_3 t}\, t \left\|(b_1, c_1, \eta_1) - (b_2, c_2, \eta_2)\right\|_{\mathbf{BC_t}} .$$

3 Biological Models

This section is devoted to show that (1.1) comprehends various relevant models in mathematical biology of current interest. We consider the models in below in their simplest version, often referring to their formulations in terms of $\mathbf{L^1}$ densities. However, they are currently studied in the framework of Radon measures, as highlighted in the given references.

3.1 The Age Structured Cell Population Model with Crowding

Consider the age structured population of cells evolving due to processes of mortality and equal mitosis. Here, mitosis is understood as the birth of two new cells and the death of a mother cell. Linear models are based on the assumption, that birth and death rates are linear functions of the population density, what excludes such phenomena as crowding effects or environment limitations. Hence, it is more reasonable to consider nonlinear models, as an example we recall [22, Ex. 5.1], where the death rate depends on the population density:

$$\begin{aligned} \partial_t p(t,x) + \partial_x p(t,x) &= -\left(\beta(x) + \mu(x) + \tau \int_{\mathbb{R}^+} p(t,y)\, \mathrm{d}y\right) p(t,x) \\ p(t,0) &= 2\int_{\mathbb{R}^+} \beta(y) p(t,y)\, \mathrm{d}y\ . \end{aligned} \tag{3.1}$$

Here, t is time; x is age; $p(t,x)$ is the density of cells having age x at time t; $\beta(x)$, $\mu(x)$ and τ are respectively the division rate, the natural mortality rate and the coefficient describing the influence of crowding effects on the evolution. Setting in (1.1)

$$\begin{aligned} \mu(t)(A) &= \textstyle\int_A p(t,x)\, \mathrm{d}x\ , & b(\mu)(x) &= 1, \\ c(\mu)(x) &= \beta(x) + \mu(x) + \tau\mu(t)(\mathbb{R}^+)\ , \quad \text{and} & \left(\eta(y)\right)(A) &= 2\beta(y)\delta(x=0) \quad \text{if } 0 \in A\ , \end{aligned}$$

we obtain (3.1). This model is at the basis of several studies. For instance, one may introduce a birth rate that depends on the population density

$$\beta(x) = \bar{b}(x)\bar{\beta}\left(\int_{\mathbb{R}^+} p(t,y)\, \mathrm{d}y\right) .$$

Otherwise, one may simplify (3.1) obtaining the well-known and widely studied McKendrick age structured model [18]. Refer to [21] and the references therein for further possibilities.

3.2 Nonlinear Size Structured Model for Asymmetric Cell Division

For unicellular organisms, structuring population by age does not apply well, mainly because age is not the most relevant parameter that determines mitosis. Therefore, it is often more reasonable to consider size structured models, see [13, Section I.4.3, Ex. 4.3.6], for which

$$\partial_t n + \partial_x\left(V(x) n\right) n = -\left(\mu(x) + b(x)\right) n + 2\int_{\mathbb{R}^+} b(y) d(x,y) n(t,y)\, \mathrm{d}y \tag{3.2}$$

where t is time, x is size, $n(t,x)$ is a density of cells having size x at time t, $b(x)$, $u(x)$ are respectively division rate and mortality rate. V describes the dynamics of evolution of the individual at the state x. If division occurs, a mother cell of size y divides into two daughter cells of sizes x and $y-x$, what is described by the kernel d. In general, the structural variable considered here does not need to be a size. It can be maturity (see Section 3.3), which is described by the cell diameter or by the level of a chemical substance significant for the cell division process. Another biological process which fits into (3.2) is the evolution of phytoplankton aggregates (without a coagulation term), see [3]. Setting in (1.1)

$$\begin{array}{rclcrcl} \mu(t)(A) & = & \int_A n(t,x)\,\mathrm{d}x\,, & & b(\mu)(x) & = & V(x)\,, \\ c(\mu)(x) & = & \mu(x)+b(x) & \text{and} & \eta(y)(A) & = & 2\int_A b(y)\,d(x,y)\,\mathrm{d}x\,, \end{array}$$

we obtain (3.2). In the linear case, with $d(x,y) = 2\delta(x = y/2)$, we obtain the model [20, Section 4.1] describing equal mitosis. If $d(x,y) = [\delta(x = \sigma y) + \delta(x = (1-\sigma)y)]$, we obtain the general mitosis model [20, Section 4.2]. Setting $d(x,y) = [\delta(x = y) + \delta(x = 0)]$, we return to the McKendrick model [18].

3.3 The Cell Cycle Structured Population Model

This model is a special case of the one mentioned in Section 3.2. It describes the structure of cells characterised by the position x in the cell cycle, where $0 < x_o \leq x \leq 1$. A new born cell has a maturity x_o and mitosis occurs only at a maturity $x = 1$. For simplicity, no mortality of cells is assumed, see [22, Ex. 2.3]. The model thus reads

$$\left\{ \begin{array}{rcl} \partial_t p(t,x) + \partial_x\left(xp(t,x)\right) & = & 0, \\ x_o p(t,x_0) & = & 2p(t,1), \\ p(0,x) & = & p_o(x)\,. \end{array} \right.$$

This model is a particular case of (1.1), obtained setting $b(\mu)(x) = x$, $c(\mu) = 0$, $\mu(t)(A) = \int_A p(t,x)\,\mathrm{d}x$ and $\eta(y)(A) = \left\{ \begin{array}{rl} 2\delta(x=0) \quad \text{if } 0 \in A, & \text{for } y = 1 \\ 0, & \text{for } y \neq 1 \end{array} \right.$.

3.4 Body Size Structured Model with Possible Cannibalistic Interactions

Let us now present a model slightly more general than the one in Section 3.2. This generalization is necessary for modeling those biological phenomenas, where the growth rate depends on the population density. As an example, we consider the following model, studied in [11, 16, 17], which describes the evolution of a body size structured population:

$$\left\{ \begin{array}{rcl} \partial_t n + \partial_x\left(g(x,n)\,n\right) + h(x,n)\,n & = & 0, \\ g(x_0,n)\,n(t,x_0) & = & \int_{x_0}^{x_m} \beta(y,n)\,n(t,y)\,\mathrm{d}y\,, \\ n(0,x) & = & n_o(x), \end{array} \right.$$

where t is time, x is the individual body size, x_o is the size of each new born individual, x_m is the maximum body size, $n(t,x)$ is the density of population having size x at time t (or rather concentration, if we allow $n(t,\cdot)$ to be a Radon measure), g describes the dynamics of individual's growth, h is a death rate and β is related to the influx of new individuals. It is worth mentioning that the dependence of the coefficients on n allows to model e. g. the evolution of a cannibalistic populations, see [12].

Again, this is a particular case of (1.1), obtained setting

$$\begin{array}{rclcrcl} \mu(t)(A) & = & \int_A n(t,x)\,\mathrm{d}x\ , & & b(\mu)(x) & = & g(x,n)\,, \\ c(\mu)(x) & = & h(x,n) & \text{and} & \eta(y)(A) & = & \delta(x=0)\otimes\beta\,. \end{array}$$

3.5 Selection-Mutation Models

Selection mutation models have been proposed in [1, 5, 8] to model species evolution. More precisely, one is interested in the evolution of a density of individuals $u(t,x)$ at time t with respect to a evolutionary variable $x \in \mathbb{R}^+$. For instance, one could think x as the maturation age of a species. These models typically include a selection part due to the environment that can be modelled by logistic growth and a mutation term in which offsprings are born with a slightly different trait than their parents. For instance, a typical model reads

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t}(t,x) & = & (1-\varepsilon)b(x)u(t,x) - m(x,P(t))u(t,x) + \varepsilon \displaystyle\int_{\mathbb{R}^+} b(y)\gamma(x,y)u(t,y)\,dy, \\ u(0,x) & = & u_o(x), \end{array} \right.$$

where $P(t) = \int_{\mathbb{R}^+} u(t,x)\,\mathrm{d}x$ is the total population, m is the death rate, b is fertility rate, and ε gives the probability of mutation of the offspring. Finally, the mutant population is modeled by an integral operator where $\gamma(x,y)$ is the density of probability that the trait of the mutant offspring of an individual with trait y is x. Also this model is a particular case of (1.1), obtained by setting

$$\begin{array}{rclcrcl} \mu(t)(A) & = & \int_A u(t,x)\,\mathrm{d}x\,, & & b(\mu)(x) & = & 0\,, \\ c(\mu)(x) & = & (1-\varepsilon)b(x) - m\left(x, \mu(t)(\mathbb{R}^+)\right) & \text{and} & \eta(y)(A) & = & \varepsilon \int_A b(y)\gamma(x,y)\,\mathrm{d}x\ . \end{array}$$

4 Proofs

4.1 The O.D.E. (2.10)

A convenient way to deal with the problem (2.10) relies on its dual formulation

$$\begin{array}{rcll} \partial_t\varphi - c(x)\varphi + \displaystyle\int_{\mathbb{R}^+} \varphi(t,y)\,\mathrm{d}[\eta(x)]\,(y) & = & 0 & \text{in } [0,T]\times\mathbb{R}^+ \\ \varphi(T) & = & \psi & \text{in } \mathbb{R}^+ \end{array} \tag{4.1}$$

with $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$ and c,η as in (2.8), (2.9). A function $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}^+;\mathbb{R})$ is a solution to the dual problem to (2.10), if it satisfies (4.1) in the classical strong sense. The relation between (2.10) and (4.1) is explained by the following Lemma.

Lemma 4.1. *Fix $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$. Then:*

i) Problem (2.10) admits a unique solution $\mu \in \mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}^+)\right)$. More precisely, for all $t,\tau \in [0,T]$,

$$d\left(\mu(t,\cdot),\mu(\tau,\cdot)\right) \le \left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{L}^\infty}\right)\exp\Big(\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{L}^\infty}\right)\,\max\{t,\tau\}\Big)\,\mu_o(\mathbb{R}^+)|t-\tau|.$$

ii) Fix $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. If $\mu \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}^+)\right)$ solves (2.10)*, then for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_1,t_2] \times \mathbb{R}^+; \mathbb{R})$ we have*

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{R}^+}\left(\partial_t\varphi(t,x) - c(x)\,\varphi(t,x) + \int_{\mathbb{R}^+}\varphi(t,y)\,\mathrm{d}\big[\eta(x)\big]\,(y)\right)\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t \qquad (4.2)\\ &= \int_{\mathbb{R}^+}\varphi(t_2,x)\;\mathrm{d}[\mu(t_2)]\,(x) - \int_{\mathbb{R}^+}\varphi(t_1,x)\;\mathrm{d}[\mu(t_1)]\,(x)\,.\end{aligned}$$

iii) If $\mu \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}^+)\right)$ solves (2.10)*, then for any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+; \mathbb{R})$, there exists a function $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^+; \mathbb{R})$ solving the dual problem* (4.1) *and such that*

$$\int_{\mathbb{R}^+}\psi(x)\,\mathrm{d}[\mu(t)]\,(x) = \int_{\mathbb{R}^+}\varphi_{T,\psi}(T-t,x)\,\mathrm{d}\mu_o\,(x)\,. \qquad (4.3)$$

iv) For any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+; \mathbb{R})$, let $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^+; \mathbb{R})$ solves the dual problem (4.1)*. Then the measure $\mu \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}^+)\right)$ defined by* (4.3) *solves* (2.10)*.*

v) If μ_o is positive, then also $\mu(t)$ is positive for all $t \in [0,T]$.

Preliminary to the proof of lemmas 2.4 and 4.1 is the study of (4.1).

Lemma 4.2. *For any $T > 0$ and $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+; \mathbb{R})$ there exists a unique solution $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^+; \mathbb{R})$ to* (4.1)*. If $\psi \geq 0$, then $\varphi_{T,\psi}(t,x) \geq 0$ for all $t \in [0,T]$ and $x \in \mathbb{R}$. Moreover, for $\tau \in [0,T]$ and $x \in \mathbb{R}^+$ the following estimates hold*

$$\left\|\varphi_{T,\psi}(\tau,\cdot)\right\|_{\mathbf{L}^\infty} \leq \|\psi\|_{\mathbf{L}^\infty}\,\mathrm{e}^{\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)(T-\tau)} \qquad (4.4)$$

$$\left\|\partial_x\varphi_{T,\psi}(\tau,\cdot)\right\|_{\mathbf{L}^\infty} \leq \|\psi\|_{\mathbf{W^{1,\infty}}}\,\mathrm{e}^{3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BCL}}\right)(T-\tau)} \qquad (4.5)$$

$$\sup_{\tau\in[T-t,T]}\left|\partial_\tau\varphi_{T,\psi}(\cdot,x)\right| \leq \|\psi\|_{\mathbf{L}^\infty}\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)t} \qquad (4.6)$$

If moreover φ_1, respectively φ_2, solves (4.1) *with terminal data ψ and parameters c_1, η_1, respectively c_2, η_2, then*

$$\begin{aligned}\left\|\varphi_1(\tau,\cdot) - \varphi_2(\tau,\cdot)\right\|_{\mathbf{L}^\infty} \leq\; & \|\psi\|_{\mathbf{W^{1,\infty}}}\left(\|c_1 - c_2\|_{\mathbf{L}^\infty} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\right)(T-\tau)\\ & \cdot\,\mathrm{e}^{\left(\|c_1\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}} + \|c_2\|_{\mathbf{L}^\infty} + \|\eta_2\|_{\mathbf{BC_x}}\right)(T-\tau)}\end{aligned} \qquad (4.7)$$

The proof of Lemma 4.2 is an immediate consequence of standard ODE estimates. The next proof slightly generalizes that of [16, Lemma 3.6], to which we refer for further details.

Proof of Lemma 4.1. The proof consists of several steps.

1. Regularization. Let $\rho \in \mathbf{C_c^\infty}(\mathbb{R}; \mathbb{R}^+)$ be such that $\int_{\mathbb{R}}\rho(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$ define the mollifiers $\rho^\varepsilon = \rho(x/\varepsilon)/\varepsilon$. We consider equation (2.10) with initial datum u_o^ε and coefficient η^ε, where

$$\begin{aligned} u_o^\varepsilon \cdot \mathcal{L}^1 &= \mu_o * \rho^\varepsilon \quad \text{and} \quad & u_o^\varepsilon &\in \mathbf{BC}(\mathbb{R}^+; \mathbb{R}^+)\\ \eta^\varepsilon(y) \cdot \mathcal{L}^1 &= \eta(y) * \rho^\varepsilon \quad \text{and} \quad & \eta^\varepsilon(y) &\in \mathbf{BC}(\mathbb{R}^+; \mathbb{R}^+)\,,\end{aligned}$$

with $\eta^\varepsilon \in (\mathbf{BC} \cap \mathbf{Lip})(\mathbb{R}^+; \mathbf{BC}(\mathbb{R}^+;\mathbb{R}^+))$. Here, the usual Lebesgue measure on $\mathbb{R}$ is denoted by $\mathcal{L}^1$. Above, the convolution on $\mathbb{R}^+$ is given by $(\nu * \rho^\varepsilon)(x) = \int_{\mathbb{R}^+} \rho^\varepsilon(x-\xi)\, \mathrm{d}\nu\,(\xi)$. Below, we denote $\eta^\varepsilon(y)(x) = \eta^\varepsilon(y,x)$. Note that

$$\|\eta^\varepsilon\|_{\mathbf{BC_x}} \leq \|\eta\|_{\mathbf{BC_x}}, \quad d(\mu_o, u_o^\varepsilon) \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \sup_{y\in\mathbb{R}^+} d\left(\eta(y), \eta^\varepsilon(y)\right) \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \tag{4.8}$$

where d is as in (2.1). Indeed, fix $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$ with $\|\psi\|_{\mathbf{W^{1,\infty}}} \leq 1$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^+} \psi(x)\, \mathrm{d}(\rho^\varepsilon * \mu_o - \mu_o)\,(x) &= \int_{\mathbb{R}^+} (\psi * \rho^\varepsilon - \psi)(x)\, \mathrm{d}\mu_o\,(x) \\
&\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \rho_\varepsilon(x-\xi)(\psi(\xi) - \psi(x))\, \mathrm{d}\xi \right) \mathrm{d}\mu_o\,(x) \\
&\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \rho_\varepsilon(x-\xi)|x-\xi|\, \mathrm{d}\xi \right) \mathrm{d}\mu_o\,(x) \leq \varepsilon \mu_o(\mathbb{R}^+)\,. \\
\int_{\mathbb{R}^+} \psi(x)\, \mathrm{d}\big(\rho^\varepsilon * \eta(y) - \eta(y)\big)\,(x) &= \int_{\mathbb{R}^+} (\psi * \rho^\varepsilon - \psi)(x)\, \mathrm{d}[\eta(y)]\,(x) \\
&\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \rho_\varepsilon(x-\xi)(\psi(\xi) - \psi(x))\, \mathrm{d}y \right) \mathrm{d}[\eta(y)]\,(x) \\
&\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \rho_\varepsilon(x-\xi)|x-\xi|\, \mathrm{d}y \right) \mathrm{d}[\eta(y)]\,(x) \\
&\leq \varepsilon[\eta(y)](\mathbb{R}^+) \leq \varepsilon \big\|\eta(y)\big\|_{(\mathbf{W^{1,\infty}})^*} \leq \varepsilon \|\eta\|_{\mathbf{BC_x}}\,.
\end{aligned}$$

2. Equality (4.3) **Holds in the Regular Case.** Note that

$$\left\{ \begin{array}{rcll}
\dfrac{\partial}{\partial t} u^\varepsilon(t,x) &=& -c(x) u^\varepsilon(t,x) + \displaystyle\int_{\mathbb{R}^+} \eta^\varepsilon(y,x) u^\varepsilon(t,y)\, \mathrm{d}y & (t,x) \in [0,T] \times \mathbb{R}^+ \\
u^\varepsilon(0,x) &=& u_o^\varepsilon(x) & x \in \mathbb{R}^+
\end{array} \right. \tag{4.9}$$

is a Cauchy problem for an ODE in $\mathbf{L^1}(\mathbb{R}^+;\mathbb{R})$ with a globally Lipschitz right hand side. Therefore, the existence and uniqueness of a classical solution u^ε is immediate, see [5]. Integrating (4.9) we obtain that for any $t \in [0,T]$ and for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}^+;\mathbb{R})$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^+} \left(\partial_\tau \varphi(\tau,x) - c(x)\, \varphi(\tau,x) + \int_{\mathbb{R}^+} \varphi(\tau,y)\, \eta^\varepsilon(x,y)\, \mathrm{d}y \right) u^\varepsilon(\tau,x)\, \mathrm{d}x\, \mathrm{d}\tau \\
&= \int_{\mathbb{R}^+} \varphi(t,x)\; u^\varepsilon(t,x)\, \mathrm{d}x - \int_{\mathbb{R}^+} \varphi(0,x)\; u_o^\varepsilon(x)\, \mathrm{d}x\;.
\end{aligned} \tag{4.10}$$

Choosing as φ a solution of the dual problem (4.1) with $T = t$, we obtain

$$\int_{\mathbb{R}^+} \psi(x)\;\, \mathrm{d}u_t^\varepsilon\,(x) = \int_{\mathbb{R}^+} \varphi_{t,\psi}^\varepsilon(0,x)\;\, \mathrm{d}u_o^\varepsilon\,(x)\,, \tag{4.11}$$

which is the smooth version of (4.3).

3. Convergence of the Regularizations. Let u^{ε_m}, respectively u^{ε_n}, solve problem (4.9) with ε replaced by ε_m, respectively ε_n. Moreover, let v be the solution to

$$\left\{ \begin{array}{rcll}
\dfrac{\partial}{\partial t} v(t,x) &=& -c(x) v(t,x) + \displaystyle\int_{\mathbb{R}^+} \eta^{\varepsilon_m}(y,x) v(t,y)\, \mathrm{d}y & (t,x) \in [0,T] \times \mathbb{R}^+ \\
v(0,x) &=& u_o^{\varepsilon_n}(x) & x \in \mathbb{R}^+
\end{array} \right.$$

By estimate (4.7) for dual problem and push-forward formula (4.11),

$$d\left(u^{\varepsilon_n}(t,\cdot), v(t,\cdot)\right) \leq \sup_{y\in\mathbb{R}^+} d\left(\eta(y), \eta^{\varepsilon}(y)\right) \, \mathrm{e}^{\left(2\|c\|_{\mathbf{L}^\infty}+\|\eta_{\varepsilon_n}\|_{\mathbf{BC_x}}+\|\eta_{\varepsilon_m}\|_{\mathbf{BC_x}}\right)T} u_o^{\varepsilon_n}(\mathbb{R}^+)\, T$$

while by estimates (4.4)–(4.5) for a dual problem and push-forward formula (4.11)

$$d\left(u^{\varepsilon_m}(t,\cdot), v(t,\cdot)\right) \leq \exp\left(3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta^{\varepsilon_m}\|_{\mathbf{BCL}}\right) T\right) \, d(u_o^{\varepsilon_m}, u_o^{\varepsilon_n})\,.$$

Therefore, by (4.8), $d\left(u^{\varepsilon_n}(t,\cdot), u^{\varepsilon_m}(t,\cdot)\right) \overset{n,m\to\infty}{\longrightarrow} 0$ uniformly with respect to time. By the completeness of $\mathcal{M}^+(\mathbb{R}^+)$, the sequence $u^{\varepsilon_n}(t,\cdot)\cdot\mathcal{L}^1$ converge uniformly with respect to time to a unique limit μ_t.

4. The Limit is Narrowly Lipschitz in Time. Using (4.6) and (4.8), we obtain the following uniform Lipschitz estimate for all $0 \leq \tau \leq t$

$$d\left(u^{\varepsilon}(t,\cdot), u^{\varepsilon}(\tau,\cdot)\right) \leq \left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right) \exp\left(\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)\, \tau\right) u_o^{\varepsilon}(\mathbb{R}^+)(t-\tau)\,.$$

Hence, μ_t is also narrowly Lipschitz in time.

5. The Limit Solves (2.10). We proved, that $u^{\varepsilon}(t,\cdot)$ converges narrowly and uniformly with respect to time to the unique limit μ_t. Notice, that $\partial_\tau\varphi(\tau,\cdot)$ and $c(x)\varphi(\tau,\cdot)$ are bounded continuous functions, while $\int_{\mathbb{R}^+}\varphi(\tau,y)\eta^{\varepsilon}(\cdot,y)\,\mathrm{d}y$ converges uniformly to $\int_{\mathbb{R}^+}\varphi(\tau,y)\eta(\cdot,y)\,\mathrm{d}y$. Thus, passage to the limit in the integral (4.10) completes the proof of *i)*.

6. *ii)* Holds. Assume that $\mu \in \mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}^+)\right)$ is a solution to (2.10) in the sense of Definition 2.2. Fix a $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}^+;\mathbb{R})$, then we prove that (4.2) holds for $t\in[t_1,t_2]$. Define $\varphi^{\varepsilon}(t,x) = \kappa_\varepsilon(t)\,\varphi(t,x)$, where

$$\kappa_\varepsilon \in \mathbf{C_c^\infty}([t_1,t_2[\,,[0,1])\,, \quad \kappa_\varepsilon(t_1) = 1\,, \quad \lim_{\varepsilon\to 0}\kappa_\varepsilon(\tau) = \chi_{[t_1,t_2[}(\tau)$$

and

$$\lim_{\varepsilon\to 0}\kappa'_\varepsilon = \delta(t=t_2) - \delta(t=t_1) \text{ in } \mathcal{M}^+([0,T])\,.$$

Use φ^{ε} as a test function in the definition of weak solution. Using the Lipschitz continuity of $t\to\mu(t)$ and the Lebesque Dominated Convergence Theorem we conclude

$$\begin{aligned}
&\int_{\mathbb{R}^+}\varphi(t_2,x)\,\mathrm{d}[\mu(t_2)]\,(x) - \int_{\mathbb{R}^+}\varphi(t_1,x)\,\mathrm{d}[\mu(t_1)]\,(x) = \lim_{\varepsilon\to 0}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\kappa^{\varepsilon}(t)\int_{\mathbb{R}^+}\varphi(t,x)\,\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t\\
&= \lim_{\varepsilon\to 0}\int_0^T \kappa^{\varepsilon}(t)\int_{\mathbb{R}^+}\left[\partial_t\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}^+}\varphi(t,y)\,\mathrm{d}[\eta(x)]\,(y)\right]\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^+}\left[\partial_t\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}^+}\varphi(t,y)\,\mathrm{d}[\eta(x)]\,(y)\right]\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t\ .
\end{aligned}$$

7. *iii)* Holds. Equality (4.3) follows setting $t_1 = 0$, $t_2 = t$ and $\varphi(s,x) = \varphi_{T,\psi}(s+(T-t_2),x)$.

8. *iv)* Holds. We proved that there exists a unique solution to (2.10) which also fulfills (4.3). This equation characterizes μ uniquely, hence each μ given by (4.3) is a solution to (2.10).

9. *v)* Holds. It immediately follows from the analogous property of the dual equation stated in Lemma 4.2 and push-forward formula (4.3). □

Proof of Lemma 2.4. Claims *i)* and *v)* follow from *iii)* in Lemma 4.1, since the dual problem to (2.10) is autonomous. Claim *iii)* is a consequence of *i)* in Lemma 4.1.

To prove *ii)*, choose $\psi \in \mathbf{C^1}(\mathbb{R}^+;\mathbb{R})$ with $\|\psi\|_{\mathbf{W^{1,\infty}}} \leq 1$ and $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^+)$. By the push-forward formula in *iii)* of Lemma 4.1 and by the estimates (4.4)–(4.5) for the dual problem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^+} \psi(x)\,\mathrm{d}(\,\hat S_t\mu_1 - \hat S\mu_2)(x) = \int_{\mathbb{R}^+} \varphi_{T,\psi}(T-t,x)\,\mathrm{d}(\,\mu_1-\mu_2)(x)\\
&\leq \sup\left\{\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\mu_1-\mu_2)(x) : \begin{array}{l}\varphi(x)\in \mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}} \leq \mathrm{e}^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)t}\end{array}\right\}\\
&\leq \sup\left\{\int_{\mathbb{R}^+}\psi(x)\,\mathrm{d}(\,\mu_1-\mu_2)(x) \colon \begin{array}{l}\psi(x)\in \mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\psi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\mathrm{e}^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)t}\\
&= \exp\Big(3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)t\Big)\; d(\mu_1,\mu_2)\,.
\end{aligned}
$$

Hence, *ii)* holds.

Finally, to prove *iv)*, let c_1, c_2 satisfy (2.8), η_1, η_2 satisfy (2.9) and call $\hat S^1, \hat S^2$ the corresponding semigroups. Then, using (4.7) and Lemma 4.1

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\hat S^1\mu - \hat S^2\mu) = \int_{\mathbb{R}^+}\Big(\varphi^1_{T,\psi}(T-t,\cdot)-\varphi^2_{T,\psi}(T-t,\cdot)\Big)\,\mathrm{d}\mu\,(x)\\
&\leq \Big(\|c_1-c_2\|_{\mathbf{L^\infty}}+\|\eta_1-\eta_2\|_{\mathbf{BC_x}}\Big)\,\mathrm{e}^{\left(\|c_1\|_{\mathbf{L^\infty}}+\|c_2\|_{\mathbf{L^\infty}}+\|\eta_1\|_{\mathbf{BC_x}}+\|\eta_2\|_{\mathbf{BC_x}}\right)t}\,\mu(\mathbb{R}^+)\;t
\end{aligned}
$$

completing the proof. □

4.2 The Transport Equation (2.11)

Proof of Lemma 2.5. Claims *i)* and *v)* are classical results, see for instance [2, Section 8.1]. Integrating along characteristics, we can explicitly write

$$
\check S_t\mu = X(t;0,\cdot)\#\mu \quad \text{where} \quad \left\{\begin{array}{l}\partial_\tau X(\tau;t,x) = b\left(X(\tau;t,x)\right)\\ X(t;t,x) = x\end{array}\right. \tag{4.12}
$$

Hence $(\check S_t\mu)(A) = \mu\left(X(0;t,A)\right)$ for any measurable subset A of $\mathbb{R}^+$. By the standard theory of ODEs, we have $X\left(t_o;t,X(t;t_o,x)\right) = x$. Using the definition (2.1) of the distance, we prove *ii)* as follows:

$$
\begin{aligned}
d(\check S_t\mu_1,\check S_t\mu_2) &= \sup\left\{\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\check S_t\mu_1-\check S_t\mu_2)(x) \colon \varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R}) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\right\}\\
&= \sup\left\{\int_{\mathbb{R}^+}\varphi\left(X(0;t,x)\right)\mathrm{d}(\,\mu_1-\mu_2)(x) \colon \begin{array}{l}\varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\\
&\leq \sup\left\{\int_{\mathbb{R}^+}\psi(x)\,\mathrm{d}(\,\mu_1-\mu_2)(x) \colon \begin{array}{l}\psi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\psi\|_{\mathbf{L^\infty}}\leq 1\\ \|\partial_x\psi\|_{\mathbf{L^\infty}}\leq \left\|\partial_x X(0;t,\cdot)\right\|_{\mathbf{L^\infty}}\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max\left\{1, \left\|\partial_x X(0;t,\cdot)\right\|_{\mathbf{L}^\infty}\right\} d(\mu_1,\mu_2)\\
&\leq \exp\left(\|\partial_x b\|_{\mathbf{L}^\infty} t\right) d(\mu_1,\mu_2)
\end{aligned}
$$

where we used [20, § 6.1.2]. Concerning *iii)*, i.e. the Lipschitz continuity with respect to time,

$$
\begin{aligned}
d(\check S_t\mu,\mu) &= \sup\left\{\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\check S_t\mu-\mu)(x)\colon \varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R}) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\right\}\\
&\leq \sup\left\{\int_{\mathbb{R}^+}\left|\varphi\left(X(0;t,x)\right)-\varphi(x)\right|\mathrm{d}\mu\,(x)\colon \begin{array}{l}\varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\\
&\leq \|b\|_{\mathbf{L}^\infty}\,\mu(\mathbb{R}^+)\,t\,.
\end{aligned}
$$

Finally, to prove *iv)*, let b_1, b_2 satisfy (2.8) and call $\check S^1, \check S^2$ the corresponding semigroups. Then, with obvious notation,

$$
\begin{aligned}
d(\check S^1_t\mu,\check S^2_t\mu) &= \sup\left\{\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\check S^1_t\mu-\check S^2_t\mu)(x)\colon \begin{array}{l}\varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\\
&= \sup\left\{\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\check S^1_t\mu)(x)-\int_{\mathbb{R}^+}\varphi(x)\,\mathrm{d}(\,\check S^2_t\mu)(x)\colon \begin{array}{l}\varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\\
&\leq \sup\left\{\int_{\mathbb{R}^+}\left|\varphi\left(X^1(0;t,x)\right)-\varphi\left(X^2(0;t,x)\right)\right|\mathrm{d}\mu\,(x)\colon \begin{array}{l}\varphi\in\mathbf{C^1}(\mathbb{R}^+;\mathbb{R})\\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\leq 1\end{array}\right\}\\
&\leq \|b_1-b_2\|_{\mathbf{L}^\infty}\,\mu(\mathbb{R}^+)\,t
\end{aligned}
$$

completing the proof. □

To prove Proposition 2.6, we need results on the dual formulation of (2.11), namely

$$
\begin{array}{rcll}
\partial_t\varphi + b(x)\,\partial_x\varphi &=& 0 & \in [0,T]\times\mathbb{R}^+\\
\varphi(T) &=& \psi & \in \mathbb{R}^+
\end{array}
\tag{4.13}
$$

with $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$ and b as in (2.8). We say that a map $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}^+)$ solves (2.11) if (4.13) is satisfied in the classical strong sense.

For completeness, we state the following results, whose proofs are found where referred.

Lemma 4.3. [16, Lemma 3.6] *Fix $\mu_o\in\mathcal{M}^+(\mathbb{R}^+)$. A map $\mu\colon[0,T]\to\mathcal{M}^+(\mathbb{R}^+)$ solves* (2.11) *with initial datum μ_o in the sense of Definition* 2.2 *if and only if for any $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$*

$$
\int_{\mathbb{R}^+}\psi(x)\,\mathrm{d}[\mu(t)]\,(x) = \int_{\mathbb{R}^+}\varphi_{T,\psi}(T-t,x)\,\mathrm{d}\mu_o\,(x)\,,
\tag{4.14}
$$

where $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}^+;\mathbb{R})$ is the solution of the dual problem (4.13) *for any $T>0$. Moreover, if μ_o is nonnegative, then so is μ.*

Lemma 4.4. [2, Lemma 8.1.2] *Fix $t_1,t_2\in[0,T]$ with $t_1<t_2$. If $\mu\in\mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}^+)\right)$ solves* (2.11)*, then for any $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([t_1,t_2]\times\mathbb{R}^+;\mathbb{R})$ we have*

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^+}\left(\partial_t\varphi(t,x)+\partial_x\varphi(t,x)\,b(x)\right)\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t\\
&= \int_{\mathbb{R}^+}\varphi(t_2,x)\,\mathrm{d}[\mu(t_2)]\,(x) - \int_{\mathbb{R}^+}\varphi(t_1,x)\,\mathrm{d}[\mu(t_1)]\,(x)\,.
\end{aligned}
\tag{4.15}
$$

Lemma 4.5. [16, Lemma 3.5] *For any $T > 0$ and $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$, there exists a unique solution $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}^+;\mathbb{R})$ of (4.13). Moreover, for $\tau \in [0,T]$ and $x \in \mathbb{R}^+$,*

$$\begin{aligned}
\left\|\varphi_{T,\psi}(\tau,\cdot)\right\|_{\mathbf{L}^\infty} &\leq \|\psi\|_{\mathbf{L}^\infty} \\
\left\|\partial_x\varphi_{T,\psi}(\tau,\cdot)\right\|_{\mathbf{L}^\infty} &\leq \left\|\psi'\right\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\partial_x b\|_{\mathbf{L}^\infty}(T-\tau)} \\
\left\|\partial_\tau\varphi_{T,\psi}(\cdot,x)\right\|_{\mathbf{L}^\infty} &\leq \left\|\psi'\right\|_{\mathbf{L}^\infty} \|b\|_{\mathbf{L}^\infty} .
\end{aligned}$$

4.3 The Operator Splitting Algorithm

Proof of Proposition 2.6. Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^+;\mathbb{R})$ with $\|\psi\|_{\mathbf{W^{1,\infty}}} \leq 1$ and $\mu \in \mathcal{M}^+(\mathbb{R}^+)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^+} \psi(x)\,\mathrm{d}(\hat{S}_t\check{S}_t\mu - \check{S}_t\hat{S}_t\mu) &= \int_{\mathbb{R}^+} \hat{\varphi}_{T,\psi}(T-t,x)\,\mathrm{d}(\check{S}_t\mu) - \int_{\mathbb{R}^+} \check{\varphi}_{T,\psi}(T-t,x)\,\mathrm{d}(\hat{S}_t\mu) \\
&= \int_{\mathbb{R}^+} \check{\varphi}_{T,(\hat{\varphi}_{T,\psi}(T-t,\cdot))}(T-t,x)\,\mathrm{d}\mu - \int_{\mathbb{R}^+} \hat{\varphi}_{T,(\check{\varphi}_{T,\psi}(T-t,\cdot))}(T-t,x)\,\mathrm{d}\mu \\
&= \int_{\mathbb{R}^+} \Big(\hat{\varphi}_{T,\psi}\left(T-t, X(T-t;T,x)\right) - \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}(T-t,x)\Big)\,\mathrm{d}\mu \\
&\leq \sup_{x\in\mathbb{R}^+} \left|\hat{\varphi}_{T,\psi}\left(T-t, X(T-t;T,x)\right) - \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}(T-t,x)\right| \mu(\mathbb{R}^+)
\end{aligned}$$

Set $\varphi_1 = \hat{\varphi}_{T,\psi}$ and $\varphi_2 = \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}$ and consider the term in the modulus. Use the estimates for the dual problem in Lemma 4.2 and (4.12) to obtain

$$\begin{aligned}
\varphi_1(T-t,&X(T-t;T,x)) - \varphi_2(T-t,x) \\
&= \psi(X(T-t;T,x)) - \int_{T-t}^T c(X(T-t;T,x))\varphi_1(s,X(T-t;T,x)) \\
&\quad + \int_{T-t}^T \int_{\mathbb{R}^+} \varphi_1(s,y)\,\mathrm{d}[\eta(X(T-t;T,x))](y)\,\mathrm{d}s - \psi(X(T-t;T,x)) \\
&\quad + \int_{T-t}^T c(x)\varphi_2(s,x)\,\mathrm{d}s - \int_{T-t}^T \int_{\mathbb{R}^+} \varphi_2(s,y)\,\mathrm{d}[\eta(x)](y)\,\mathrm{d}s \pm \int_{T-t}^T c(x)\varphi_1(s,x)\,\mathrm{d}s \\
&\quad \pm \int_{T-t}^T c(X(T-t;T,x))\varphi_1(s,x)\,\mathrm{d}s \pm \int_{T-t}^T \int_{\mathbb{R}^+} \varphi_1(s,y)\,\mathrm{d}[\eta(x)](y)\,\mathrm{d}s \\
&= \int_{T-t}^T c(x)\left(\varphi_2(s,x) - \varphi_1(s,x)\right)\mathrm{d}s + \int_{T-t}^T \varphi_1(s,x)\Big(c(x) - c\left(X(T-t;T,x)\right)\Big)\,\mathrm{d}s \\
&\quad + \int_{T-t}^T c\left(X(T-t;T,x)\right)\Big(\varphi_1(s,x) - \varphi_1\left(s,X(T-t;T,x)\right)\Big)\,\mathrm{d}s \\
&\quad + \int_{T-t}^T \int_{\mathbb{R}^+} \left(\varphi_1(s,x) - \varphi_2(s,x)\right)\mathrm{d}[\eta(x)](y)\,\mathrm{d}s \\
&\quad + \int_{T-t}^T \int_{\mathbb{R}^+} \varphi_1(s,y)\,\mathrm{d}[\eta\left(X(T-t;T,x)\right) - \eta(x)](y)\,\mathrm{d}s \\
&\quad \leq \|c\|_{\mathbf{L}^\infty} \int_{T-t}^T \sup_x \left|\varphi_1(s,x) - \varphi_2(s,x)\right|\mathrm{d}s + \|\partial_x c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t \int_{T-t}^T \sup_x \left|\varphi_1(s,x)\right|\mathrm{d}s \\
&\quad + \|c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t \int_{T-t}^T \sup_x \left|\partial_x\varphi_1(s,x)\right|\mathrm{d}s
\end{aligned}$$

$$+\|\eta\|_{\mathbf{BC_x}}\int_{T-t}^{T}\sup_x\left|\varphi_1(s,x)-\varphi_2(s,x)\right|\mathrm{d}s+\mathbf{Lip}(\eta)\|b\|_{\mathbf{L}^\infty}t\int_{T-t}^{T}\sup_x\left|\varphi_1(s,x)\right|\mathrm{d}s$$
$$:=I_1+I_2+I_3+I_4+I_5\,.$$

Using estimate (4.4) for the dual problem, we conclude that

$$\begin{aligned}
I_1 &\le \|c\|_{\mathbf{L}^\infty}\left\|\psi'\right\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty}t^2\ \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}\\
I_2 &\le \|\partial_x c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty}t^2\ \|\psi\|_{\mathbf{L}^\infty}\ \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}\\
I_4 &\le \|\eta\|_{\mathbf{BC_x}}\left\|\psi'\right\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty}t^2\ \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}\\
I_5 &\le \mathbf{Lip}(\eta)\ \|b\|_{\mathbf{L}^\infty}t^2\ \|\psi\|_{\mathbf{L}^\infty}\ \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}\ .
\end{aligned}$$

Directly from estimate (4.5) follows, that $I_3 \le \|c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty}t^2\ \|\psi\|_{\mathbf{W^{1,\infty}}}\ \mathrm{e}^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)t}$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^+}\psi(x)\,\mathrm{d}(\,\hat S_t\check S_t\mu-\check S_t\hat S_t\mu)\le\\
&\qquad\le 3t^2\ \|\psi\|_{\mathbf{W^{1,\infty}}}\|b\|_{\mathbf{L}^\infty}\exp\left[3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)t\right]\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BCL}}\right)\,.
\end{aligned}$$

Taking the supremum over all functions ψ we conclude the proof. □

Proof of Theorem 2.7. Points *i)*, *ii)*, *iii)* and *vi)* are consequences of the results obtained in [10, Corollary 3.3 and Lemma 3.4], see also [9, Proposition 3.2], combined with the estimates provided by Lemma 2.4 and Lemma 2.5.

Passing to *iv)*, we use [4, Theorem 2.9], to estimate the distance between $S^1_t\mu$ and $S^2_t\mu$:

$$d(S^1_t\mu,S^2_t\mu)\le\mathbf{Lip}(S^1_t)\int_0^t\liminf_{h\to0}\frac{1}{h}d(S^1_hS^1_\tau\mu,S^2_hS^1_\tau\mu)\ \mathrm{d}\tau\ . \tag{4.16}$$

Let $\nu=S^1_\tau\mu$. Using Lemma 2.4 and Lemma 2.5 compute

$$\begin{aligned}
d(S^1_h\nu,S^2_h\nu) &\le d(S^1_h\nu,\check S^1_h\hat S^1_h\nu)+d(\check S^1_h\hat S^1_h\nu,\check S^2_h\hat S^2_h\nu)+d(\check S^2_h\hat S^2_h\nu,S^2_h\nu)\\
&\le d(\check S^1_h\hat S^1_h\nu,\check S^1_h\hat S^2_h\nu)+d(\check S^1_h\hat S^2_h\nu,\check S^2_h\hat S^2_h\nu)+o(h)\\
&\le \exp(\|\partial_x b_1\|_{\mathbf{L}^\infty}h)\,d(\hat S^1_h\nu,\hat S^2_h\nu)+\|b_1-b_2\|_{\mathbf{L}^\infty}\,(\hat S^2_h\nu)(\mathbb{R}^+)\,h+o(h)\\
&\le\left(\|c_1-c_2\|_{\mathbf{L}^\infty}+\|\eta_1-\eta_2\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|\partial_x b_1\|_{\mathbf{L}^\infty}+\left\|(c_1,c_2)\right\|_{\mathbf{L}^\infty}+\left\|(\eta_1,\eta_2)\right\|_{\mathbf{BC_x}}\right)h}\,\nu(\mathbb{R}^+)\,h\\
&\quad+\|b_1-b_2\|_{\mathbf{L}^\infty}\ \mathrm{e}^{2\left(\|c_2\|_{\mathbf{L}^\infty}+\|\eta_2\|_{\mathbf{BC_x}}\right)h}\,\nu(\mathbb{R}^+)\,h+o(h)
\end{aligned}$$

therefore, by the lim inf formula (4.16),

$$d(S^1_t\mu,S^2_t\mu)\le\mathbf{Lip}(S^1)\left(\|b_1-b_2\|_{\mathbf{L}^\infty}+\|c_1-c_2\|_{\mathbf{L}^\infty}+\|\eta_1-\eta_2\|_{\mathbf{BC_x}}\right)\int_0^t(S^1_\tau\mu)(\mathbb{R}^+)\,\mathrm{d}\tau\ .$$

An estimate of $\mathbf{Lip}(S^1_t)$ is provided by *ii)*, while the latter term above is bounded using *iii)* and definition (2.1) of the metric:

$$\int_0^t(S^1_\tau\mu)(\mathbb{R}^+)\,\mathrm{d}\tau\ \le\ \int_0^t\left|(S^1_\tau\mu)(\mathbb{R}^+)-\mu(\mathbb{R}^+)\right|\mathrm{d}\tau+t\,\mu(\mathbb{R}^+)$$

$$\leq \int_0^t \left[\|b_1\|_{\mathbf{L}^\infty} + \left(\|c_1\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}}\right) e^{\left(\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)\tau}\right] \mu(\mathbb{R}^+)\, \tau \, d\tau$$
$$+t\, \mu(\mathbb{R}^+)$$

which proves point *iv)* in Theorem 2.7.

To complete the proof, we show that $t \mapsto S_t\mu$ solves the linear autonomous problem (2.7) in the sense of Definition 2.2. Fix $n \in \mathbb{N}$ and define $\varepsilon = T/n$. First, as in [9, Section 5.3], consider the following continuous operator splitting:

$$F^\varepsilon(t)\mu = \begin{cases} \check{S}_{2t-2i\varepsilon}\, (\hat{S}_\varepsilon \check{S}_\varepsilon)^i \mu & \text{for} \quad t \in \left[i\varepsilon, (i+1/2)\varepsilon\right[\\ \hat{S}_{2t-2(i+1)\varepsilon}\, \check{S}_\varepsilon\, (\hat{S}_\varepsilon \check{S}_\varepsilon)^i \mu & \text{for} \quad t \in \left[(i+1/2)\varepsilon, (i+1)\varepsilon\right[\end{cases}$$

where $i = 0, \ldots, n-1$. This formula is, in our case, equivalent to that given by [10, Corollary 3.3]. Define $\mu^\varepsilon(t) = F^\varepsilon(t)\mu_o$ for a $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$. For any $\varphi \in \mathbf{C_c^\infty}([0,T] \times \mathbb{R}^+; \mathbb{R})$,

$$\int_{\mathbb{R}^+} \varphi(T,x)\, d[\mu^\varepsilon(T)]\,(x) - \int_{\mathbb{R}^+} \varphi(0,x)\, d\mu_o\,(x) =$$
$$= \int_0^T \int_{\mathbb{R}^+} \left[\partial_t\varphi(t,x) + b(x)\partial_x\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}^+} \varphi(t,y)\, d[\eta(x)]\,(y)\right] d[\mu^\varepsilon(t)]\,(x)\, dt$$
$$+ R(\varepsilon) \tag{4.17}$$

where

$$\begin{aligned}
R(\varepsilon) \;=\; & \sum_{i=0}^{n-1} \int_{i\varepsilon}^{(i+1/2)\varepsilon} \left[\int_{\mathbb{R}^+} \big(\partial_t\varphi(t+\varepsilon/2,x) - 2c(x)\varphi(t+\varepsilon/2,x)\big)\, d[\mu^\varepsilon(t+\varepsilon/2)]\,(x)\right. \\
& + \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \varphi(t+\varepsilon/2,y)\, d[2\eta(x)]\,(y)\right) d[\mu^\varepsilon(t+\varepsilon/2)]\,(x) \\
& \left. - \int_{\mathbb{R}^+} \left(\partial_t\varphi(t,x) - 2c(x)\varphi(t,x) + \int_{\mathbb{R}^+} \varphi(t,y)\, d[2\eta(x)]\,(y)\right) d[\mu^\varepsilon(t)]\,(x)\right] dt \\
& + \sum_{i=0}^{n-1} \int_{(i+1/2)\varepsilon}^{(i+1)\varepsilon} \left[\int_{\mathbb{R}^+} \big(\partial_t\varphi(t-\varepsilon/2,x) + 2b(x)\partial_x\varphi(t-\varepsilon/2,x)\big)\, d[\mu^\varepsilon(t-\varepsilon/2)]\,(x)\right. \\
& \left. - \int_{\mathbb{R}^+} \big(\partial_t\varphi(t,x) + 2b(x)\partial_x\varphi(t,x)\big)\, d[\mu^\varepsilon(t)]\,(x)\right] dt
\end{aligned}$$

$$= \sum_{i=0}^{n-1} \int_{i\varepsilon}^{(i+1/2)\varepsilon} \left\{ \int_{\mathbb{R}^+} \big(\partial_t\varphi(t+\varepsilon/2,x) - \partial_t\varphi(t,x)\big)\, d[\mu^\varepsilon(t+\varepsilon/2)]\,(x) \right. \tag{4.18}$$
$$- \int_{\mathbb{R}^+} 2c(x)\, \big(\varphi(t+\varepsilon/2,x) - \varphi(t,x)\big)\, d[\mu^\varepsilon(t+\varepsilon/2)]\,(x) \tag{4.19}$$
$$+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \big(\varphi(t+\varepsilon/2,y) - \varphi(t,x)\big)\, d[2\eta(x)]\,(y)\, d[\mu^\varepsilon(t+\varepsilon/2)]\,(x) \tag{4.20}$$
$$+ \int_{\mathbb{R}^+} \left(\partial_t\varphi(t,x) - 2c(x)\varphi(t,x) + \int_{\mathbb{R}^+} \varphi(t,y)\, d[2\eta(x)]\,(y)\right) \tag{4.21}$$
$$\left. d[\mu^\varepsilon(t+\varepsilon/2) - \mu^\varepsilon(t)]\,(x) \right\} dt$$
$$+ \sum_{i=0}^{n-1} \int_{(i+1/2)\varepsilon}^{(i+1)\varepsilon} \left\{ \int_{\mathbb{R}^+} \big(\partial_t\varphi(t-\varepsilon/2,x) - \partial_t\varphi(t,x)\big)\, d[\mu^\varepsilon(t-\varepsilon/2)]\,(x) \right. \tag{4.22}$$

$$+\int_{\mathbb{R}^+} 2b(x)\left(\partial_x\varphi(t-\varepsilon/2,x)-\partial_x\varphi(t-\varepsilon/2,x)\right)\mathrm{d}[\mu^\varepsilon(t-\varepsilon/2)]\,(x) \tag{4.23}$$

$$+\int_{\mathbb{R}^+} \left(\partial_t\varphi(t,x)+2b(x)\partial_x\varphi(t,x)\right)\mathrm{d}[\mu^\varepsilon(t-\varepsilon/2)-\mu^\varepsilon(t)]\,(x)\Bigg\}\,\mathrm{d}t \tag{4.24}$$

Notice, that $t\to\mu^\varepsilon(t)$ is uniformly bounded in $\mathbf{BC}\left([0,T],(\mathcal{M}^+(\mathbb{R}^+),d)\right)$. Due to the regularity of φ, we have the following uniform convergences:

$$\begin{array}{rcccl}
\partial_t\varphi(t+\varepsilon/2,x)-\partial_t\varphi(t,x) & \rightrightarrows & 0 & \Rightarrow & (4.18)\to 0\\
2c(x)\left(\varphi(t+\varepsilon/2,x)-\varphi(t,x)\right) & \rightrightarrows & 0 & \Rightarrow & (4.19)\to 0\\
\left(\varphi(t+\varepsilon/2,y)-\varphi(t,y)\right) & \rightrightarrows & 0 & \Rightarrow & (4.20)\to 0\\
\partial_t\varphi(t-\varepsilon/2,x)-\partial_t\varphi(t,x) & \rightrightarrows & 0 & \Rightarrow & (4.22)\to 0\\
2b(x)\left(\partial_x\varphi(t-\varepsilon/2,x)-\partial_x\varphi(t,x)\right) & \rightrightarrows & 0 & \Rightarrow & (4.23)\to 0
\end{array}$$

as $\varepsilon\to 0$. To show the convergence of (4.21) and (4.24), it is sufficient to note that $\mu^\varepsilon(t)$ is uniformly Lipschitz continuous, i.e. $d\left(\mu^\varepsilon(t),\mu^\varepsilon(t-\varepsilon)\right)\le K\varepsilon$, where K is a Lipschitz constant independent from t, for instance the same as in Theorem 2.7, see [9, Proposition 3.2]. Moreover, according to [10, Corollary 4.4 and Proposition 4.6], $\mu^\varepsilon(t)$ converges uniformly with respect to time in d to $S_t\mu$. Hence, passing to the limit in (4.17) gives

$$\begin{aligned}
&\int_{\mathbb{R}^+}\varphi(T,x)\,\mathrm{d}[\mu(T)]\,(x)-\int_{\mathbb{R}^+}\varphi(0,x)\,\mathrm{d}\mu_o\,(x)\\
&\quad=\int_0^T\!\!\int_{\mathbb{R}^+}\left[\partial_t\varphi(t,x)+b(x)\partial_x\varphi(t,x)-c(x)\varphi(t,x)+\int_{\mathbb{R}^+}\varphi(t,y)\,\mathrm{d}[\eta(x)]\,(y)\right]\mathrm{d}[\mu(t)]\,(x)\,\mathrm{d}t\,.
\end{aligned}$$

We need now to show that the above equality holds for all test functions $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}^+;\mathbb{R})$. To this aim, fix $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}^+;\mathbb{R})$, choose a sequence $\varphi_n\in\mathbf{C_c^\infty}([0,T]\times\mathbb{R}^+;\mathbb{R}^+)$ such that $\varphi_n\to\varphi$ in $\mathbf{W^{1,\infty}_{loc}}$ as $n\to+\infty$ and $\sup_n\|\varphi_n\|_{\mathbf{W^{1,\infty}}}<C$. An application of a standard limiting procedure completes the proof of *v)*. □

Proof of Theorem 2.8. First, we prove that there exists a process P given by *i)*. Fix $n\in\mathbb{N}$, define $t_n^i=iT/2^n$ for $i=0,1,\ldots,2^n$ and approximate b, c and η as follows:

$$\begin{aligned}
b_n(t,x) &= \sum_{i=0}^{2^n-1} b(t_n^i,x)\,\chi_{[t_n^i,t_n^{i+1}[}(t)\\
c_n(t,x) &= \sum_{i=0}^{2^n-1} c(t_n^i,x)\,\chi_{[t_n^i,t_n^{i+1}[}(t)\\
\eta_n(t,x) &= \sum_{i=0}^{2^n-1} \eta(t_n^i,x)\,\chi_{[t_n^i,t_n^{i+1}[}(t)\,.
\end{aligned}$$

Call $S^{k,n}$ the semigroup constructed in Theorem 2.7 on the interval $\left[t_n^k,t_n^{k+1}\right[$. Assume $t_o\le t$, $t_o\in\left[t_n^{i_o},t_n^{i_o+1}\right[$, $t\in\left[t_n^i,t_n^{i+1}\right[$ and define the map $F^n_{t,t_o}\colon[0,T]\times\mathcal{M}^+(\mathbb{R}^+)\to\mathcal{M}^+(\mathbb{R}^+)$ by

$$F^n_{t,t_o}\mu=\begin{cases}
S^{i,n}_{t-t_o}\mu & \text{if } i=i_o\\
\left(S^{i,n}_{t-t_n^i}\circ S^{i-1,n}_{t_n^i-t_o}\right)\mu & \text{if } i=i_o+1\\
\left(S^{i,n}_{t-t_n^i}\circ\left(\bigcirc_{j=i_o+1}^{i-1}S^{j,n}_{T/2^n}\right)\circ S^{i_o,n}_{t_n^{i_o+1}-t_o}\right)\mu & \text{otherwise}
\end{cases} \tag{4.25}$$

We now prove that as $n \to +\infty$, F^n converges to a process P, see also [10, Definition 2.4], whose trajectories solve (2.7) in the sense of Definition 2.2. Assume first that $t_o = i_o T/2^n$ and $t = iT/2^n$ with $i > i_o$. Then, $F^n_{t,t_o}\mu = \left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\right)\mu$ and

$$
\begin{aligned}
d(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu) &= d\left(F^n_{iT/2^n,t_o}\mu,\ F^{n+1}_{2iT/2^{n+1},t_o}\mu\right) = d\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu,\ \bigcirc_{j=2i_o}^{2i-1} S^{j,n+1}_{T/2^{n+1}}\mu\right)\\
&\le d\left(S^{i-1,n}_{T/2^n}\bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu,\ S^{i-1,n}_{T/2^n}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right)\\
&\quad + d\left(S^{i-1,n}_{T/2^{n+1}}\left(S^{i-1,n}_{T/2^{n+1}}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right),\ S^{2i-1,n+1}_{T/2^{n+1}}\left(S^{2i-2,n+1}_{T/2^{n+1}}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right)\right)\\
&\le e^{3\left(\left\|(b_n,c_n)(t_n^{i-1})\right\|_{\mathbf{W^{1,\infty}}}+\left\|\eta_n(t_n^{i-1})\right\|_{\mathbf{BCL}}\right)T/2^n}\, d\left(F^n_{(i-1)T/2^n,t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1},t_o}\mu\right)\\
&\quad + e^{5\left(\left\|(b_n,c_n)(t_n^{i-1})\right\|_{\mathbf{W^{1,\infty}}}+\left\|\eta_n(t_n^{i-1})\right\|_{\mathbf{BCL}}\right)T/2^{n+1}}\, T/2^{n+1}\left(\left(\bigcirc_{j=2i_o}^{2i-2} S^{j,n+1}_{T/2^{n+1}}\mu\right)(\mathbb{R}^+)\right)\\
&\qquad \cdot\left(\left\|(b_n,c_n)(t_n^{i-1}) - (b_{n+1},c_{n+1})(t_{n+1}^{2i-1})\right\|_{\mathbf{L^\infty}} + \left\|\eta_n(t_n^{i-1}) - \eta_{n+1}(t_{n+1}^{2i-1})\right\|_{\mathbf{BC_x}}\right)\\
&\le e^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}T/2^n}\, d\left(F^n_{(i-1)T/2^n,t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1},t_o}\mu\right)\\
&\quad + e^{5\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\left(\left\|(b_n,c_n,\eta_n)-(b_{n+1},c_{n+1},\eta_{n+1})\right\|_{\mathbf{BC_t}}\right)T/2^{n+1}\,\mu(\mathbb{R}^+)\\
&= e^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}T/2^n}\, d\left(F^n_{(i-1)T/2^n,t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1},t_o}\mu\right)\\
&\quad + e^{5\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\left\|(b_n,c_n,\eta_n)-(b_{n+1},c_{n+1},\eta_{n+1})\right\|_{\mathbf{BC_t}}T/2^{n+1}\,\mu(\mathbb{R}^+)\,.
\end{aligned}
$$

where the last inequality holds due to the fact, that

$$
\left(\bigcirc_{j=2i_o}^{2i-2} S^{j,n+1}_{T/2^{n+1}}\mu\right)(\mathbb{R}^+) \le e^{2\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o-T/2^{n+1})}\,\mu(\mathbb{R}^+)
$$

Gronwall's inequality (see [17, Lemma 4.2]) allows us to obtain the estimate

$$
\begin{aligned}
d\left(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu\right) \;&\le\; \frac{1}{2}\left[\frac{e^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}-1}{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}}\right] e^{5\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\,\mu(\mathbb{R}^+)\\
&\quad\cdot\left\|(b_n,c_n,\eta_n)-(b_{n+1},c_{n+1},\eta_{n+1})\right\|_{\mathbf{BC_t}}\,.
\end{aligned}
$$

There exist a constant $C^* = C^*\left(T, \left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}\right)$, such that for all $t \in [t_o, T]$,

$$
\frac{1}{2}\left[\frac{e^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}-1}{3\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}}\right] \le C^*(t-t_o)
$$

hence,

$$
d\left(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu\right) \le C^*\, e^{5\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)}\left\|(b_n,c_n,\eta_n)-(b_{n+1},c_{n+1},\eta_{n+1})\right\|_{\mathbf{BC_t}}\,\mu(\mathbb{R}^+)(t-t_o).
$$

Due to the assumptions about Hölder regularity of functions b, c, η we conclude that there exist constants H_b, H_c, H_η such that

$$\begin{aligned}
&\sup_t \left\| b_n(t) - b_{n+1}(t) \right\|_{\mathbf{W^{1,\infty}}} \leq H_b 2^{-n\alpha} \\
&\sup_t \left\| c_n(t) - c_{n+1}(t) \right\|_{\mathbf{W^{1,\infty}}} \leq H_c 2^{-n\alpha} \\
&\sup_t \left\| \eta_n(t) - \eta_{n+1}(t) \right\|_{(\mathbf{W^{1,\infty}})^*} \leq H_\eta 2^{-n\alpha}
\end{aligned} \tag{4.26}$$

meaning that

$$\left\| b_n(t) - b_{n+1}(t) \right\|_{\mathbf{BC_t}} + \left\| c_n(t) - c_{n+1}(t) \right\|_{\mathbf{BC_t}} + \left\| \eta_n(t) - \eta_{n+1}(t) \right\|_{\mathbf{BC_t}} \leq (H_b + H_c + H_\eta) 2^{-n\alpha}$$

which implies $\sum_{n=1}^{\infty} \left\| (b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1}) \right\|_{\mathbf{BC_t}} < \infty$. Hence, for $m, k \to \infty$ series $\sum_{n=m}^{k} \left\| (b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1}) \right\|_{\mathbf{BC_t}}$ converges to 0. Thus, we conclude that for each $\mu \in \mathcal{M}^+(\mathbb{R}^+)$ the sequence $F^n_{t,t_o}\mu$ is a Cauchy sequence, which converge uniformly with respect to time to a measure $\nu \in \mathcal{M}^+(\mathbb{R}^+)$. By definition we set $P(t, t_o)\mu = \nu$. Claim *i)* follows then from the construction of F^n_{t,t_o}, since we are dealing with linear problems.

Now assume that $t_o \in \left] t_n^{i_o - 1}, t_n^{i_o} \right[$ and $t \in \left] t_n^i, t_n^{i+1} \right[$, meaning that t_o and t are not grid points. Then,

$$d\left(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu \right) \leq d\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu, \bigcirc_{j=2i_o}^{2i-1} S^{j,n+1}_{T/2^{n+1}}\mu \right) + o\left(\frac{1}{2^n} \right).$$

What holds due to the fact, that $F^n_{t,t_o}\mu$ is Lipschitz continuous and the length of time intervals $\left] t_n^{i_o - 1}, t_n^{i_o} \right[$, $\left] t_n^i, t_n^{i+1} \right[$ is equal to $T/2^n$.

To prove *iii)*, one can easily check, that for $t = iT/2^n$

$$\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu \right)(\mathbb{R}^+) \quad \leq \quad \mathrm{e}^{2\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)} \mu(\mathbb{R}^+)$$

Therefore,

$$\begin{aligned}
d\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu, \mu \right) &\leq d\left(S^{i-1,n}_{T/2^n}\left(\bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu \right), \bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu \right) + d\left(\bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu, \mu \right) \\
&\leq \left\|(b,c,\eta)\right\|_{\mathbf{BC_t}} \mathrm{e}^{\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}} T/2^n} \frac{T}{2^n} \left(\mathrm{e}^{2\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}((i-i_o)-1)T/2^n} \mu(\mathbb{R}^+) \right) + d\left(\bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu, \mu \right)
\end{aligned}$$

Hence, iterating the procedure we obtain

$$d\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu, \mu \right) \quad \leq \quad \left\|(b,c,\eta)\right\|_{\mathbf{BC_t}} \mathrm{e}^{2\left\|(b,c,\eta)\right\|_{\mathbf{BC_t}}(t-t_o)} \mu(\mathbb{R}^+)(t - t_o)$$

Passing to *ii*) and *iv)*, let b, $\tilde{b}$, c, $\tilde{c}$, η and $\tilde{\eta}$ satisfy assumptions (2.13) and (2.14). Call $S^{i,n}$ and $\tilde{S}^{i,n}$ corresponding semigroups constructed in Theorem 2.7 on the interval $\left[t_n^i, t_n^{i+1} \right[$.

Define maps $F^n_{t,t_o}\mu$ and $\tilde{F}^n_{t,t_o}\nu$ as in (4.25). Assume first that $t_o = i_oT/2^n$ and $t = iT/2^n$ with $i > i_o$. Then,

$$
\begin{aligned}
d(F^n_{t,t_o}\mu, \tilde{F}^n_{t,t_o}\nu) &= d(F^n_{iT/2^n,t_o}\mu, \tilde{F}^n_{iT/2^n,t_o}\nu) = d\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu, \bigcirc_{j=i_o}^{i-1} \tilde{S}^{j,n}_{T/2^n}\nu\right)\\
&= d\left(S^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu, \tilde{S}^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} \tilde{S}^{j,n}_{T/2^n}\nu\right)\\
&\le d\left(S^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu, S^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} \tilde{S}^{j,n}_{T/2^n}\nu\right) + d\left(S^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} \tilde{S}^{j,n}_{T/2^n}\nu, \tilde{S}^{i-1,n}_{T/2^n} \bigcirc_{j=i_o}^{i-2} \tilde{S}^{j,n}_{T/2^n}\nu\right)\\
&\le \mathrm{e}^{3\left(\left\|(b_n,c_n)(t_n^{i-1})\right\|_{\mathbf{W^{1,\infty}}}+\left\|\eta_n(t_n^{i-1})\right\|_{\mathbf{BCL}}\right)T/2^n} \; d(F^n_{(i-1)T/2^n,t_o}\mu, \tilde{F}^n_{(i-1)T/2^n,t_o}\nu)\\
&\quad + \mathrm{e}^{5\left(\left\|(b_n,c_n)(t_n^{i-1})\right\|_{\mathbf{W^{1,\infty}}}+\left\|\eta_n(t_n^{i-1})\right\|_{\mathbf{BCL}}\right)T/2^n} \; T/2^n \left(\left(\bigcirc_{j=i_o}^{i-2} \tilde{S}^{j,n}_{T/2^n}\nu\right)(\mathbb{R}^+)\right)\\
&\qquad \cdot \left(\left\|(b_n,c_n)(t_n^{i-1}) - (\tilde{b}_n,\tilde{c}_n)(t_n^{i-1})\right\|_{\mathbf{L}^\infty} + \left\|\eta_n(t_n^{i-1}) - \tilde{\eta}_n(t_n^{i-1})\right\|_{\mathbf{BC_x}}\right)\\
&\le \mathrm{e}^{3\left\|(b,c,t)\right\|_{\mathbf{BC_t}}T/2^n} \; d(F^n_{(i-1)T/2^n,t_o}\mu, \tilde{F}^n_{(i-1)T/2^n,t_o}\nu)\\
&\quad + \mathrm{e}^{5\left\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\right\|_{\mathbf{BC_t}}(t-t_o)} \left\|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\right\|_{\mathbf{BC_t}} T/2^n \; \nu(\mathbb{R}^+)
\end{aligned}
$$

Therefore, using Gronwall's inequality (see [17, Lemma 4.2]) we obtain

$$
\begin{aligned}
d(F^n_{t,t_o}\mu, \tilde{F}^n_{t,t_o}\nu) \le\; & \mathrm{e}^{3\left\|(b,c,t)\right\|_{\mathbf{BC_t}}(t-t_o)} \; d(\mu,\nu)\\
&+ \left[\frac{\mathrm{e}^{3\left\|(b,c,t)\right\|_{\mathbf{BC_t}}(t-t_o)} - 1}{3\left\|(b,c,t)\right\|_{\mathbf{BC_t}}}\right] \mathrm{e}^{5\left\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\right\|_{\mathbf{BC_t}}(t-t_o)} \; \nu(\mathbb{R}^+)\left\|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\right\|_{\mathbf{BC_t}}
\end{aligned}
$$

Using the arguments as in proof of *i)*, we conlude that there exists a constant C^*, such that

$$
\begin{aligned}
d(F^n_{t,t_o}\mu, \tilde{F}^n_{t,t_o}\nu) \le\; & \mathrm{e}^{3\left\|(b,c,\eta)\right\|_{\mathbf{BC}_t}(t-t_o)} \; d(\mu,\nu)\\
&+ C^*(t-t_o)\,\mathrm{e}^{5\left\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\right\|_{\mathbf{BC}_t}(t-t_o)} \; \nu(\mathbb{R}^+)\left\|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\right\|_{\mathbf{BC}_t}
\end{aligned}
$$

For t_o and t, which are not grid points, we prove this inequality using again the same arguments as in proof of *i)*. Therefore, passage to the limit with n ends the proof.

Passing to *v)*, let $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}^+;\mathbb{R})$ and $n \in \mathbb{N}$. From Theorem 2.7 we know that for each $i = 0, 1, \cdots, 2^{n-1}$, orbits of the semigroup $S^{i,n}$ are weak solutions of the linear non-autonomous problem (2.15) on $[t_n^{i-1}, t_n^i[$. Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^+} \varphi(T,x)\,\mathrm{d}\mu^n\,(T) - \int_{\mathbb{R}^+} \varphi(0,x)\,\mathrm{d}\mu_o\\
=\;& \int_0^T \int_{\mathbb{R}^+} \left[\partial_t\varphi(t,x) + b_n(t,x)\partial_x\varphi(t,x) - c_n(x)\varphi(t,x) + \int_{\mathbb{R}^+} \varphi(t,y)\,\mathrm{d}[\,\eta_n(t,x)](y)\right]\mathrm{d}\mu^n(t)\,\mathrm{d}t
\end{aligned}
$$

$$= \int_0^T \int_{\mathbb{R}^+} \Big[\partial_t \varphi(t,x) + b(t,x)\partial_x \varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}^+} \varphi(t,y)\, \mathrm{d}[\,\eta(t,x)](y) \Big] \mathrm{d}\mu^n(t)\ \mathrm{d}t$$
$$+R^n$$

where

$$R^n = \int_0^T \int_{\mathbb{R}^+} \Big(\big(b_n(t,x) - b(t,x)\big)\, \partial_x \varphi(t,x) + \big(c(t,x) - c_n(t,x)\big)\, \varphi(t,x) \Big)\, \mathrm{d}\mu^n\,(t)\ \mathrm{d}t + \int_0^T \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \varphi(t,y)\, \mathrm{d}[\,\eta_n(t,x) - \eta(t,x)](y)\, \mathrm{d}\mu^n\,(t)\ \mathrm{d}t$$

From the previous analysis in this proof we know, that $\mu^n(t)$ converges narrowly and uniformly with respect to time to the unique limit μ_t. Due to the assumptions (4.26) about Hölder regularity of functions b, c, η, we use the analogous arguments as in proof of claim *i)* in Lemma 4.1 and pass to the limit in the integral, what ends the proof. □

Proof of Theorem 2.9. Let b, c, η be functions given by (2.3)–(2.4) and $\mu_o \in \mathcal{M}^+(\mathbb{R}^+)$ be an initial measure in (1.1). Let us introduce a complete metric space $\mathbf{BC}(I; \bar{B}_R(\mu_o))$ where $I = [0, \varepsilon]$ with ε to be chosen later on and $\bar{B}_R(\mu_o) = \{\nu \in \mathcal{M}^+(\mathbb{R}^+)\ :\ d(\mu_o, \nu) \le R\}$. The space $\mathbf{BC}(I; \bar{B}_R(\mu_o))$ is equipped with the norm given by $\|\mu\|_{\mathbf{BC}} = \sup_{t\in[0,T]} \big\|\mu(t)\big\|_{(\mathbf{W^{1,\infty}})^*}$. This space is complete since $\bar{B}_R(\mu_o)$ is a closed subset of the complete metric space $\mathcal{M}^+(\mathbb{R}^+)$. We define the operator $\mathcal{T}$ on $\mathbf{BC}(I; \bar{B}_R(\mu_o))$ as follows

$$\begin{aligned} &\mathcal{T} : \mathbf{BC}(I; \bar{B}_R(\mu_o)) \longrightarrow \mathbf{BC}(I; \bar{B}_R(\mu_o)) \\ &\mathcal{T}(\mu) = \nu_{(b,c,\eta)(\mu)} \end{aligned}$$

Where $\nu_{(b,c,\eta)(\mu)}$ is the solution to (2.15) with coefficients $b(\cdot,\mu)$, $c(\cdot,\mu)$, $\eta(\cdot,\mu)$ and initial data μ_o. From assumptions on coefficients and the definition of norm $\|\cdot\|_{\mathbf{BC^{\alpha,1}}}$ (2.5) we observe

$$\begin{aligned} M_b &= \sup\nolimits_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}^+)} \big\|b(t,\nu)\big\|_{\mathbf{W^{1,\infty}}} < \infty \\ M_c &= \sup\nolimits_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}^+)} \big\|c(t,\nu)\big\|_{\mathbf{W^{1,\infty}}} < \infty \\ M_\eta &= \sup\nolimits_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}^+)} \big\|\eta(t,\nu)\big\|_{\mathbf{BCL}} < \infty \end{aligned}$$

For further simplicity we introduce a constant $M = M_b + M_c + M_\eta$. First, we need to prove, that the operator $\mathcal{T}$ is well defined, meaning that its image must be a bounded continuous function taking values in $\bar{B}_R(\mu_o)$. Continuity of $\nu_{(b,c,\eta)(\mu)}$ follows from *iii)* in Theorem 2.8. Moreover, for each $t \in [0, \varepsilon]$ we have

$$d(\mathcal{T}(\mu)(t), \mu_o) \le \big\|(b,c,\eta)\big\|_{\mathbf{BC_t}}\, \mathrm{e}^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}\, t}\, \mu_o(\mathbb{R}^+)\, t \le M\, \mathrm{e}^{2M\varepsilon}\, \mu_o(\mathbb{R}^+)\, \varepsilon \le R$$

We need to assume, that $\varepsilon < 1$. Then, $M\, \mathrm{e}^{2M}\, \mu_o(\mathbb{R}^+)\, \varepsilon \le R$, or equivalently

$$\varepsilon \ \le\ R \Big[M\, \mathrm{e}^{2M}\, \mu_o(\mathbb{R}^+) \Big]^{-1} =: \zeta_1 \tag{4.27}$$

Now, we prove that $\mathcal{T}$ is a contraction for ε small enough. To this end, we show, that $\mathcal{T}$ is Lipschitz operator with Lipschitz constant smaller than 1. Here, we use *iv)* in Theorem 2.8

$$||\mathcal{T}(\mu) - \mathcal{T}(\nu)\,||_{\mathbf{BC}} = \sup_{t\in[0,\varepsilon]} \big\|\mathcal{T}(\mu_1)(t) - \mathcal{T}(\mu_2)(t)\big\|_{(\mathbf{W^{1,\infty}})^*} = \sup_{t\in[0,\varepsilon]} d\big(\mathcal{T}(\mu_1)(t), \mathcal{T}(\mu_2)(t)\big)$$

$$
\begin{aligned}
&= \sup_{t\in[0,\varepsilon]} d\Big(\nu_{(b,c,\eta)(\mu_1)}(t), \nu_{(b,c,\eta)(\mu_2)}(t)\Big) \\
&\le \sup_{t\in[0,\varepsilon]} C^* \ \mathrm{e}^{5\Big(\big\|(b,c,\eta)(\mu_1)\big\|_{\mathbf{BC_t}}+\big\|(b,c,\eta)(\mu_2)\big\|_{\mathbf{BC_t}}\Big)t} \ \big\|(b,c,\eta)(\mu_1) - (b,c,\eta)(\mu_2)\big\|_{\mathbf{BC_t}} \mu_o(\mathbb{R}^+) \\
&\le \sup_{t\in[0,\varepsilon]} \left[\frac{\mathrm{e}^{3\big\|(b,c,\eta)(\mu_1)\big\|_{\mathbf{BC_t}}\varepsilon} - 1}{3\big\|(b,c,\eta)(\mu_1)\big\|_{\mathbf{BC_t}}}\right] \mathrm{e}^{10\big(M_b+M_c+M_\eta\big)\varepsilon} \ \mu_o(\mathbb{R}^+) \\
&\quad \cdot \ \big(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\big) \cdot d(\mu_1(t), \mu_2(t)) \\
&\le \left[\frac{\mathrm{e}^{3M\varepsilon} - 1}{M}\right] \mathrm{e}^{10M\varepsilon} \ \mu_o(\mathbb{R}^+) \cdot \ \big(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\big) \, \|\mu_1 - \mu_2\|_{\mathbf{BC}}
\end{aligned}
$$

where $\mathbf{Lip}(b) = \sup_{t\in[0,T]} \mathbf{Lip}(b(t,\cdot)) < \infty$, what holds due to assumptions on b (similarly for c and η). Lipschitz constant of $\mathcal{T}$ is smaller then 1, if the following inequality holds

$$
\mathbf{Lip}(\mathcal{T}) \ = \ \left[\frac{\mathrm{e}^{3M\varepsilon} - 1}{M}\right] \mathrm{e}^{10M\varepsilon} \ \mu_o(\mathbb{R}^+) \big(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\big) < 1
$$

We need to assume, that $\varepsilon < 1$. Hence,

$$
\begin{aligned}
\mathrm{e}^{3M\varepsilon} &< \Big[\mathrm{e}^{10M} \mu_o(\mathbb{R}^+) \big(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\big) / M\Big]^{-1} + 1 \\
\varepsilon &< \ln\left(\Big[\mathrm{e}^{10M} \mu_o(\mathbb{R}^+) \big(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\big) / M\Big]^{-1} + 1\right) / 3M =: \zeta_2
\end{aligned}
$$

We have just proved, that $\mathcal{T}$ is a contraction on a complete metric space $\mathbf{BC}(I, \bar{B}_R(\mu_o))$, where $\varepsilon = \min\{1, \zeta_1, \zeta_2\} > 0$. From the Banach fixed point theorem it follows, that there exists unique μ^*, such that $\mathcal{T}(\mu^*) = \mu^*$. Hence, an existence of the unique solution to (1.1) on the time interval $[0, \varepsilon]$ is proved. This solution can be extended on the whole $[0, T]$ interval, because ζ_1 and ζ_2 do not depend on time. Moreover, from *iii)* in Theorem 2.8, we conclude, that solution to (1.1) is Lipschitz continuous with respect to time.

Estimates in claims *i)* and *ii)* are consequences of estimates for the linear nonautonomous case (see Theorem 2.8). □

Acknowledgments. The first author acknowledges support from the project MTM2008-06349-C03-03 DGI-MCI (Spain) and 2009-SGR-345 from AGAUR-Generalitat de Catalunya. The second author acknowledges the support of GNAMPA. The third author is the coordinator and the fourth author is a Ph.D. student in the International Ph.D. Projects Programme of Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 (Ph.D. Programme: Mathematical Methods in Natural Sciences). The first, third and fourth authors also acknowledge the Isaac Newton Institute for the Mathematical Sciences, where part of this work was accomplished.

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