L^∞ a priori bounds for gradients of solutions to quasilinear inhomogenous fast-growing parabolic system

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Abstract

We prove boundedness of gradients of solutions to quasilinear parabolic system, which main part is a generalization to p-Laplacian and its right hand side's growth depending on gradient is not slower (and generally strictly faster) then $p-1$. Energy estimates and nonlinear iteration procedure of a Moser type are cornerstones of the used method.

Keywords:
quasilinear parabolic systems, p-Laplacian, regularity, boundedness of gradient, Moser-type iteration
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1. Introduction

1.1. General statement of the problem.

We are interested in obtaining a local boundedness of gradients of solutions to a following parabolic system in $\Omega \subset \mathbb{R}^n$:

$$u^i_t - (A^i_\alpha(\nabla u))_{x_\alpha} = f^i(x,t,\nabla u) \qquad i = 1,..,N$$

where the main part is a generalization of p-Laplacian and the right hand side grows as $1 + |\nabla u|^w$ or $|\nabla u|^w$ with w specified further. We say that a right-hand-side is a fast-growing one, when $w > p-1$ holds.
The existing literature on the regularity issue of parabolic equations and systems is impressive. Let us recall that for equations the existing results are quite strong: even for the right-hand-side growth of $1 + |\nabla u|^p$ one obtains $C^{1,\alpha}$ regularity of solutions: see classic monograph Ladyzhenskaya et al. [20] for the case $p = 2$ and DiBenedetto [9] for $p \in (1,\infty)$. Many further generalizations are possible: for instance in Bartier and Souplet [2] right-hand-side takes the form $e^u|\nabla u|^p$, which suffices for a boundedness of ∇u. Moreover, this growth condition seems to be optimal, because there are blowup results for gradients of solutions to equations, which right-hand-sides grow faster than p - compare Souplet [24]. In the case of systems, the regularity results are much weaker. One can construct irregular (i.e. unbounded or discontinous) functions, which solve homogenous parabolic systems. For $n > 2$ it suffices for irregularity that the coefficients $A(x,t)$ of the main part are discontinous (and still bounded) or that there is a relevant non-diagonality of the main part - for details, consult Arkhipova [1]. Nevertheless, there are many classes of main parts, which allow for higher regularity (even $C^{1,\alpha}$) in the homogenous case; these are: having structure close to Laplacian or p-Laplacian, like those studied in Ladyzhenskaya et al. [20] or DiBenedetto [9], respectively, or having main part depending solely on ∇u: see well-known paper by cas and Šverak [5] or more extensive research done by Choe and Bae [8]. As these papers consider homogenous systems, one may ask a natural question: what inhomogenous counterparts of such systems remain, in a certain sense, regular? The general answer is unknown, but there are several hints: on one hand, for right-hand-side growing like $1 + |\nabla u|^{p-1}$ the regularity of the homogenous case seems to be retained - see

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DiBenedetto [9]; on the other, unlike for equations, one cannot have right-hand-side growing as fast as $|\nabla u|^p$ without further assumptions, even in the case of the system with the simplest main part: i.e. an inhomogenous heat system. Recall the classical counterexample: for $n \geq 3$ bounded but discontinous function $u(x) = \frac{x}{|x|}$ with unbounded weak derivatives solves $u^i_t - \Delta u^i = u^i|\nabla u|^2 \quad [= (n-1)\frac{x^i}{|x|^3}]$, for details - see Struwe [25]. It turns out, that in the case of an inhomogenous system for $p = 2$ one has to assume additionally a certain smallness in order to obtain regularity - for details, refer to Tolksdorf [26], Pingen [23], Idone [16] or even the classical Ladyzhenskaya et al. [20]. The regularity issue for a general nonlinear inhomogenous parabolic system with right-hand-side growing at the rate $1 + |\nabla u|^w$ for w possibly close to p, which homogenous counterparts enjoy regularity, is not fully researched, especially for the case $p \neq 2$. There are several approaches to answering this question: some authors relax the notion of regularity, by resorting to partial regularity - see for example classical papers of Italian school: Campanato [4], Giaquinta and Struwe [14] and newer ones: Fanciullo [12], Frehse and Specovius-Neugebauer [13], Misawa [21], Duzaar and Mingione [11]; or by demanding a high integrability-type regularity[2], like in Naumann [22] Kinnunen and Lewis [19] or Bensoussan and Frehse [3] (in the last paper the growth of right-hand-side may be arbitrarily big!). Certain systems with peculiar structure or two-dimensional ones (or at least close to them in some sense) enjoy also high regularity, for results in this direction compare papers of Seregin, Arkhipova, Frehse, Kaplicky (and many others), for instance: Arkhipova [1], J. Naumann and Wolff [17], Kaplicky [18], Zajączkowski and Seregin [27].
In this note we concentrate on deriving a full regularity result, more precisely: the local boundedness of gradients, for a class of quasilinliear parabolic inhomogenous systems. Our aim is twofold: firstly to obtain results for a general inhomogenous parabolic system, which main part is analogous to the system considered in Choe and Bae [8], while retaining possibly general growth conditions for the right-hand-side. Secondly, to sharpen these results with respect to growth of the right-hand-side, restricting ourselves to less general systems, being close to $p-$Laplacian. For similar result on the level of solutions, compare Giorgi and O'Leary [15].
Let us emphasize, that we proceed in a manner typical for the regularity approach: we assume existence of solution u in a given class, which is often a deep problem itself, from which we derive higher regularity. Moreover, we concentrate on a priori estimates while conducting the proofs: the rigorous version of computations is commented on in the conclusion.

1.2. General definitions and assumptions.

Consider parabolic problem in $\Omega \subset \mathbb{R}^n$:

$$u^i_t - (A^i_\alpha(\nabla u))_{x_\alpha} = f^i(x,t,\nabla u) \qquad i = 1,..,N \tag{1}$$

As all our results have a local character, any further specification of Ω is irrelevant.
We say that a vector valued function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,p}(\Omega))$ is a solution to (1) iff

$$\int_{\Omega_T} -u^i\phi^i_t + A^i_\alpha(\nabla u)\phi^i_{x_\alpha} = \int_{\Omega_T} f^i(x,t,\nabla u)\phi^i \qquad \forall_{\phi \in C^\infty_0(\Omega_T)} \tag{2}$$

Globally, following notions will be used:

- δ^α_β denotes the Kronecker delta,
- $Q_R(x_0,t_0)$ denotes parabolic cylinder, i.e. $B_R(x_0) \times (t_0 - R^p, t_0)$; when possible, cut short to Q_R,
- $\eta_{\rho,R} \in C^\infty_0(Q_R)$ denotes standard parabolic cutoff function for $Q_\rho \subset Q_R$, when possible, cut short to η, which satisfies: $\eta = 1$ in Q_ρ, $\eta = 0$ outside Q_R, $|\nabla\eta| \leq c(R-\rho)^{-1}$, $|\eta_{,t}| \leq c(R-\rho)^{-p}$.

Throughout the article, summation convention is in use.

[2] such results are especially interesting, as our result may be easily strenghtened via higher regularity

1.3. The structure of results.

We show our results in a following order:

1. First we derive a general result for inhomogenous version of system analyzed in Choe and Bae [8], where ellipticity assumptions for the main part are generalized by introducing exponent q (theorem 2). Here, loosely speaking, admissible growth for the right-hand-side is $1 + |\nabla u|^{p-1}$, so this result may be seen as parallel to DiBenedetto [9].
2. Next we allow for for faster growths of for the right-hand-side, at the cost of assuming that the main part is closer to p- Laplacian, in the sense that it is not enriched with terms involving $q > p$ (theorem 3).
3. Finally, we state the result for the least general case, i.e. for 3D p-Laplacian with right-hand-side growing as $|\nabla u|^w$ (theorem 1).

As the last result seems to be the most traceable one, let us give incentive to studying the technical remainder of this paper by stating theorem 1 now:

Consider $u = (u^i) \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$ solving the 3D p-Laplace system:

$$u_t^i - \operatorname{div}(|\nabla u|^{p-2}\nabla u^i) = f^i(t,x,u,\nabla u) \quad i = 1,2,3 \tag{3}$$

Theorem 1. *Let Ω be an arbitrary domain. Assume a growth condition: $|f^i(x,t,\nabla u)| \le |\nabla u|^w,\ w \le p$ and initial integrability[3] $|\nabla u|_{L^{\tilde p}_{loc}} < \infty$. If **one** of the following conditions is fullfilled:*

1. *$w \le p-1,\ \tilde p = p$;*
2. *$p \ge 2,\ w < \frac{\tilde p + 4p - 3}{5},\ \nabla u \in L^{\tilde p}(\Omega_T)$;*
3. *$p \le 2,\ w \le \frac{p}{2},\ \tilde p = p$*

then ∇u is bounded:

$$|\nabla u|_{L^\infty_{loc}} < C|\nabla u|_{L^{\tilde p}_{loc}} \tag{4}$$

For the proof, see the end of the next section.

Observe, that by point 2. merely from existence, i.e for $\tilde p = p$, one has $w < p - \frac{3}{5}$. This corresponds for $p = 2$ with Campanato's notion of controllable growth (see for example Campanato [4]). Therefore this result may be seen as a generalization of classical results over gradients of p-Laplacian-like systems. Utilizing results on high integrability of certain systems, one may relax growth condition further. For example for the system analysed in Naumann [22] we have $w < \varepsilon + \frac{9}{5}$, because $\nabla u \in L^{\varepsilon+4}(\Omega_T)$, which can be taken as the initial integrability $L^{\tilde p}$.

Let $|f^i(x,t,\nabla u)| \le |\nabla u|^w$ hold. In all our theorems there is no explicit assumption, that $w < p$. In fact the inequality $w \le p$ is enforced by the rigorous treatment of energy estimates. Simultaneously we know from the counterexample recalled in the introduction, that $w = p$ is generally not admissible. Therefore our results can be viewed as a way to quantify the possible boundedness of gradients by means of higher integrability. As in the just mentioned case of theorem 1, for $p = 2$ one has $w < p - \frac{3}{5}$, which can be boosted in some cases to $w < \varepsilon + \frac{9}{5}$, because $\nabla u \in L^{\varepsilon+4}$. For $w = p$ one would need $\nabla u \in L^{\varepsilon+5}$.

2. Boundedness of gradient of solution

As outlined in the introduction, first we prove the general theorem. As the cornerstone of the analysis is energy method, we derive formal estimates for the sake of transparency. For a rigorous justification of the formal estimates please consult the conclusion of this note. We analyze solutions of:

$$u_t^i - (A_\alpha^i(\nabla u))_{x_\alpha} = f^i(x,t,\nabla u) \qquad i = 1,..,N \tag{5}$$

where the main part comes from Choe and Bae [8] and the right-hand-side grows as $1 + |\nabla u|^w,\ w \le \delta$ for a certain $\delta \le p$, obtaining the boundedness of ∇u. More precisely, one has a following:

[3] From existence one has $\tilde p = p$, so this assumption may be void. It only helps to quantify the results when we have some additional knowledge on integrability.

Theorem 2. *Under the following assumptions:*

(A0) ellipticity-type: A^i_α *is given by potential* $F \in C^2(\mathbb{R}),\ F'(0) \geq 0,$ *as follows:*

$$A^i_\alpha(Q) = (F(|Q|)_{Q^i_\alpha} \tag{6}$$

and F enjoys ellipticity:

$$(F(|Q|)_{Q^i_\alpha Q^j_\beta}\zeta^i_\alpha\zeta^j_\beta \in [\lambda|Q|^{p-2}, \frac{1}{\lambda}|Q|^{p-2} + |Q|^{q-2}]|\zeta|^2 \tag{7}$$

where:

$$1 < p \leq q < p+1 < \infty \tag{8}$$

(A1) growth-type:

$$|f^i| \leq +c_1|\nabla u|^w + c_2 \tag{9}$$

where

$$w \geq 0, \quad c_i \in L^\infty(\Omega_T) \tag{10}$$

(A2) initial integrability: Let

$$M := max(2, p, 2q-p, w+1, 2w-p+2) \tag{11}$$

$$\nabla u \in L^{s_0+M}_{loc}(\Omega_T) \tag{12}$$

with s_0 *satisfying:*

$$s_0 \geq 0, \qquad s_0 + 2 + \frac{np}{2} - \frac{Mn}{2} > 0 \tag{13}$$

and

$$\begin{cases} s_0 > p-2 \ for\ c_2 \neq 0 \\ s_0 > p - 2w - 2 \ for\ c_2 = 0 \end{cases} \tag{14}$$

Gradient of solution to (5) *is locally bounded; moreover, for any* $Q_{R_0} \subset \Omega$ *with* $R_0 < 1$ *following inequality holds:*

$$|\nabla u|_{L^\infty(Q_{\frac{R_0}{2}})} \leq C(\fint_{Q_{R_0}} |\nabla u|^{s_0+M})^{\frac{1}{s_0+2+\frac{np}{2}-\frac{Mn}{2}}} + C \tag{15}$$

Proof. First we derive formal energy inequalities, then implement iteration scheme.

Differentiate formally system (5) to obtain:

$$u^i{,}_{tx_\gamma} - (A^i_{\alpha, u^j_{x_\beta}}(\nabla u)u^j_{x_\beta x_\gamma})_{x_\alpha} = (f^i(x,t,\nabla u))_{x_\gamma} \qquad i = 1,..,N \tag{16}$$

testing (16) by $u^i_{x_\gamma}|\nabla u|^s\eta^2$ one gets:

$$\begin{aligned} I + II := \Big[\frac{1}{s+2}\sup_t \int_{B_R}(|\nabla u|^{s+2}\eta^2)\Big] + \Big[\int_{Q_R} A^i_{\alpha\,, u^j_{x_\beta}}(\nabla u)u^j_{x_\beta x_\gamma}(|\nabla u|^s u^i_{x_\gamma x_\alpha}\eta^2 + s|\nabla u|^{s-2}u^i_{x_\gamma}u^k_{x_\alpha x_\delta}u^k_{x_\delta}\eta^2)\Big] \\ = -\int_{Q_R} f^i(x,t,\nabla u)[u^i_{x_\gamma x_\gamma}|\nabla u|^s\eta^2 + su^i_{x_\gamma}|\nabla u|^{s-2}u^k_{x_\delta x_\gamma}u^k_{x_\delta}\eta^2 + 2u^i_{x_\gamma}|\nabla u|^s\eta\eta_{x_\gamma}] \\ + \frac{2}{s+2}\int_{Q_R}|\nabla u|^{s+2}\eta\eta_t - \int_{Q_R} A^i_{\alpha, u^j_{x_\beta}}(\nabla u)u^k_{x_\beta x_\gamma}u^k_{x_\gamma}|\nabla u|^s\eta\eta_{x_\alpha} \end{aligned} \tag{17}$$

Consider II. Utilizing ellipticity assumption $(A0)$ with $\zeta^l_\rho := u^l_{x_\beta x_\rho}$ one estimates the first summand of II as follows:

$$\int_{Q_R} A^i_{\alpha\,'(\nabla u)^j_\beta}(\nabla u)u^j_{x_\beta x_\gamma}|\nabla u|^s u^i_{x_\gamma x_\alpha}\eta^2 \geq \lambda\int_{Q_R}|\nabla u|^{p-2}|\nabla^2 u|^2|\nabla u|^s\eta^2 \tag{18}$$

and because A^i_α is given by potential F, from differentiation we estimate the second summand of II:

$$\int_{Q_R} A^i_{\alpha\,'(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma} s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\alpha x_\delta} u^k_{x_\delta}\eta^2 = \int_{Q_R}\Big[F''(|\nabla u|)\frac{u^i_{x_\alpha}u^j_{x_\beta}}{|\nabla u|^2} + F'(|\nabla u|)\Big(\frac{\delta^i_{x_\alpha}\delta^j_{x_\beta}}{|\nabla u|} - \frac{u^i_{x_\alpha}u^j_{x_\beta}}{|\nabla u|^3}\Big)\Big] u^j_{x_\beta x_\gamma} s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\alpha x_\delta} u^k_{x_\delta}\eta^2$$
$$= s\int_{Q_R}\eta^2 F''(|\nabla u|)|\nabla u|^{s-4}\underbrace{(u^i_{x_\gamma}u^j_{x_\beta}u^j_{x_\beta x_\gamma})}_{c^i}\underbrace{(u^i_{x_\alpha}u^k_{x_\delta}u^k_{x_\delta x_\alpha})}_{c^i} + s\int_{Q_R}\eta^2 F'(|\nabla u|)|\nabla u|^{s-5}\Big(\underbrace{u^k_{x_\delta}u^k_{x_\delta x_\alpha}u^i_{x_\gamma}u^i_{x_\gamma x_\alpha}|\nabla u|^2}_{=\frac14(|\nabla u|^2)_{x_\alpha}(|\nabla u|^2)_{x_\alpha}|\nabla u|^2} - \underbrace{u^i_{x_\alpha}u^i_{x_\gamma}u^j_{x_\beta}u^j_{x_\beta x_\gamma}u^k_{x_\delta}u^k_{x_\delta x_\alpha}}_{=\frac14 u^i_{x_\alpha}(|\nabla u|^2)_{x_\alpha}u^i_{x_\gamma}(|\nabla u|^2)_{x_\gamma}}\Big) \tag{19}$$

From the ellipticity assumption $(A0)$ one has $F''(|s|) \geq 0$, $F'(0) \geq 0$ therefore it holds: $F'(|s|) \geq 0$. This, in conjunction with a following computation: $u^i_{x_\alpha}(|\nabla u|^2)_{x_\alpha} u^i_{x_\gamma}(|\nabla u|^2)_{x_\gamma} \leq |\nabla u|^2|\nabla|\nabla u|^2|^2 = |\nabla u|^2(|\nabla u|^2)_{x_\beta}(|\nabla u|^2)_{x_\beta}$, implies that equation (19) takes the form:

$$\int_{Q_R} A^i_{\alpha\,'(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma} s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\alpha x_\delta} u^k_{x_\delta}\eta^2 \geq 0 \tag{20}$$

Summing up (18) (20) we conclude, that II satisfies:

$$\int_{Q_R} A^i_{\alpha\,'(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma}(|\nabla u|^s u^i_{x_\gamma x_\alpha}\eta^2 + s|\nabla u|^{s-2}u^i_{x_\gamma}u^k_{x_\gamma x_\delta}u^k_{x_\delta}\eta^2) \geq \lambda\int_{Q_R}|\nabla u|^{p-2}|\nabla^2 u|^2|\nabla u|^s\eta^2 \tag{21}$$

Inputting inequality (21) into (17) we arrive at:

$$\frac{1}{s+2}\sup_t\int_{Q_R}|\nabla u|^{s+2}\eta^2 + \lambda\int_{Q_R}|\nabla u|^{p-2}|\nabla^2 u|^2|\nabla u|^s\eta^2 \leq$$
$$\int_{Q_R} f^i(x,t,\nabla u)[u^i_{x_\gamma x_\gamma}|\nabla u|^s\eta^2 + su^i_{x_\gamma}|\nabla u|^{s-2}u^k_{x_\delta x_\gamma}u^k_{x_\delta}\eta^2 + 2u^i_{x_\gamma}|\nabla u|^s\eta\eta_{x_\gamma}] + \frac{2}{s+2}\int_{Q_R}|\nabla u|^{s+2}\eta\eta_t - \int_{Q_R}A^i_{\alpha(\nabla u)^j_\beta}(\nabla u)u^k_{x_\beta x_\gamma}u^k_{x_\gamma}|\nabla u|^s\eta\eta_{x_\alpha}$$
$$\leq c\int_{Q_R}[1+|\nabla u|^w][(1+s)|\nabla^2 u||\nabla u|^s\eta^2 + 2|\nabla u|^{s+1}\eta|\nabla\eta|] + \frac{c}{s+2}\int_{Q_R}|\nabla u|^{s+2}\eta|\eta_t| + \frac{c}{\lambda}\int_{Q_R}[|\nabla u|^{p-2}+|\nabla u|^{q-2}]|\nabla^2 u||\nabla u|^{s+1}\eta|\nabla\eta| \tag{22}$$

where the last inequality is valid in view of growth $(A1)$ and ellipticity $(A0)$ assumptions.
Absorb $|\nabla^2 u|$ from right-hand-side of (22) using Young's inequality:

$$\frac{1}{s+2}\sup_t\int_{Q_R}|\nabla u|^{s+2}\eta^2 + (\lambda-\varepsilon)\int_{Q_R}|\nabla u|^{p-2}|\nabla^2 u|^2|\nabla u|^s\eta^2 \leq$$
$$c\int_{Q_R}(1+|\nabla u|^w)|\nabla u|^{s+1}\eta|\nabla\eta| + \frac{c}{s+2}\int_{Q_R}|\nabla u|^{s+2}\eta|\eta_t| + c\int_{Q_R}|\nabla u|^s[|\nabla u|^p+|\nabla u|^{2q-p}]\eta^2|\nabla\eta|^2 + c(1+s)\int_{Q_R}\eta^2|\nabla u|^s[|\nabla u|^{2w-p+2}+|\nabla u|^{2-p}] \tag{23}$$

By estimates for derivatives of cutoff function η we obtain:

$$\frac{1}{s+2}\sup_t\int_{Q_R}|\nabla u|^{s+2}\eta^2 + (\lambda-\varepsilon)\int_{Q_R}|\nabla u|^{p-2}|\nabla^2 u|^2|\nabla u|^s\eta^2 \leq$$
$$\frac{c}{(R-\rho)^{\max(2,p)}}\int_{Q_R}|\nabla u|^s\Big[|\nabla u|^p + |\nabla u| + |\nabla u|^{w+1} + \frac{1}{(2+s)}|\nabla u|^2 + |\nabla u|^p + |\nabla u|^{2q-p} + (1+s)(|\nabla u|^{2w-p+2} + |\nabla u|^{2-p})\Big] \tag{24}$$

for $0 < \rho < R < 1$. As for some $w,\ p$ the exponents $2w-p+2, 2-p$ may be nonpositive, we estimate respective powers of $|\nabla u|$ using $|\nabla u|^s$ as follows:

$$\int_{Q_R}|\nabla u|^s[(1+s)(|\nabla u|^{2w-p+2} + |\nabla u|^{2-p})] \leq (1+s)\int_{Q_R}[1 + |\nabla u|^{\max(s+2-p,s+2w+2-p)}] \tag{25}$$

for the last inequality to hold, we must assume

$$s > max(p - 2w - 2,\ p - 2) \tag{26}$$

Because summand $|\nabla u|^{2-p}$ occurs only if $c_2 \neq 0$ in the growth condition $(A1)$: $|f^i| \leq c_1|\nabla u|^w + c_2$, the above assumption (26) can be written as:

$$\begin{cases} s > p - 2\ for\ c_2 \neq 0 \\ s > p - 2w - 2\ for\ c_2 = 0 \end{cases} \tag{27}$$

In the forthcoming iteration scheme we construct growing sequence of s_i, therefore it is sufficient to assume

$$\begin{cases} s_0 > p - 2\ for\ c_2 \neq 0 \\ s_0 > p - 2w - 2\ for\ c_2 = 0 \end{cases} \tag{28}$$

which coincides with our initial integrability assumption $(A2)$.
By computation, a following inequality holds:

$$\int_{Q_R} |\nabla(|\nabla u|^{\frac{p+s}{2}}\eta)|^2 \leq \int_{Q_R} |\nabla u|^{p+s}|\nabla\eta|^2 + (p+s)^2\int_{Q_R}|\nabla u|^{p+s-2}|\nabla^2 u|^2\eta^2 \tag{29}$$

Adding to both sides $\frac{\lambda-\varepsilon}{(p+s)^2}\int_{Q_R}|\nabla u|^{p+s}|\nabla\eta|^2$ and considering properties of η, as $(s+p)^2 \leq c(1+s^2)$, we arrive from (24), by virtue of (29), at:

$$\sup_t \int_{Q_R} |\nabla u|^{s+2}\eta^2 + \int_{Q_R}|\nabla(|\nabla u|^{\frac{p+s}{2}}\eta)|^2 \leq \frac{c(1+s^3)}{(R-\rho)^M}\int_{Q_R}[1+|\nabla u|^{s+M}] \tag{30}$$

taking into account (25) if neccessary. Recall, that $M := max(2, p, 2q-p, w+1, 2w-p+2)$.
By Hölder and critical-Sobolev inequalities (respectively), one gets:

$$\int_{Q_\rho}|\nabla u|^{p+s+(s+2)\frac{2}{n}} \leq \int_{t_0-R^2}^{t_0}\Big[\int_{B_R}|\nabla u|^{s+2}\eta^2\Big]^{\frac{2}{n}}\Big[\int_{B_R}|\nabla u|^{(s+p)\frac{n}{n-2}}\eta^{\frac{2n}{n-2}}\Big]^{\frac{n-2}{n}}$$
$$\leq \Big[\sup_t\int_{B_R}|\nabla u|^{s+2}\eta^2\Big]^{\frac{2}{n}}\int_{t_0-R^2}^{t_0}\Big[\int_{B_R}(|\nabla u|^{\frac{p+s}{2}}\eta)^{\frac{2n}{n-2}}\Big]^{\frac{n-2}{n}} \leq \Big[\sup_t\int_{B_R}|\nabla u|^{s+2}\eta^2\Big]^{\frac{2}{n}}\int_{Q_R}|\nabla(|\nabla u|^{\frac{p+s}{2}}\eta)|^2 \overset{(30)}{\leq} \Big[\frac{c(1+s^3)}{(R-\rho)^M}\int_{Q_R}1+|\nabla u|^{s+M}\Big]^{1+\frac{2}{n}} \tag{31}$$

Inequality (31) is our desired energy estimate, which we now iterate. Define recursively numbers s_i: $s_{i+1} + M = p + s_i + (s_i+2)\frac{2}{n}$, then:

$$s_i = (1+\frac{2}{n})^i[s_0 + n + 2 - \frac{n(p-M)}{2}] - [2 - \frac{n(p-M)}{2}] \tag{32}$$

Utilizing the initial-integrability assumption, i.e. $s_0 + 2 + n - \frac{n(p-M)}{2} > 0$, we have:

$$s_i \overset{i\to\infty}{\to} \infty; \quad \frac{s_i}{(1+\frac{2}{n})^i} \overset{i\to\infty}{\to} s_0 + 2 + n - \frac{n(p-M)}{2} \tag{33}$$

Let:

$$\psi_i = ⨍_{S_{R_i}} |\nabla u|^{s_i+M} \tag{34}$$

then (31) with η_{R_{i+1},R_i} can be written as:

$$|S_{R_{i+1}}|\psi_{i+1} \leq [C(1+s_i^p)(\frac{2^{i+2}}{R_0})^M|S_{R_i}|(1+\psi_i)]^{1+\frac{p}{n}} \implies R_{i+1}^{n+2}\psi_{i+1} \leq [C(1+s_i^p)(\frac{2^{i+2}}{R_0})^M R_{i+1}^{n+2}(1+\psi_i)]^\beta \implies$$
$$\psi_{i+1} \leq C(1+s_i^p)^\beta 2^i(1+\psi_i)]^\beta \tag{35}$$

with $\beta := 1 + \frac{2}{n}$; the last inequality given by $R_i := \frac{R_0}{2}(1 + 2^{-i})$. As we know from (33) that asymptotically s_i behaves like β^i, finally (35) folds to:

$$\psi_{i+1} \leq C^i \psi_i^{\beta} + C^i \tag{36}$$

which, by a standard computation (see Choe and Bae [8] for details), gives

$$\psi_{i+1} \leq C^{\beta^{i+1}} \psi_0^{\beta^{i+1}} + (i+1) C^{\beta^{i+1}} \tag{37}$$

From the above considerations one gets, using the definition of ψ:

$$R_0^{-\frac{n+2}{s_{i+1}+M}} |\nabla u|_{L^{s_{i+1}+M}(Q_{\frac{R_0}{2}})} \leq (\fint_{S_{R_{i+1}}} |u|^{s_{i+1}+M})^{\frac{1}{s_{i+1}+M}} = \psi_{i+1}^{\frac{1}{s_{i+1}+M}} \overset{(37)}{\leq} (C^{\beta^{i+1}} \psi_0^{\beta^{i+1}})^{\frac{1}{s_{i+1}+M}} + ((i+1) C^{\beta^{i+1}})^{\frac{1}{s_{i+1}+M}} \overset{i\to\infty}{\to} C \psi_0^{\frac{1}{s_0+2+n-\frac{Mn}{p}}} + C \tag{38}$$

in view of (33).

As $s_i + M \overset{i\to\infty}{\to} \infty$, (38) in tandem with initial integrability assumption gives the following uniform bound:

$$|\nabla u|_{L^\infty(Q_{\frac{R_0}{2}})} \leq C(\fint_{Q_{R_0}} |\nabla u|^{s_0+M})^{\frac{1}{s_0+2+n-\frac{Mn}{p}}} + C \tag{39}$$

□

In the next theorem we resign from the term possessing $q > p$ in the ellipticity assumption. This allows us, in turn, to obtain bigger growths of right-hand-side, as now it is possible to derive estimates for negative $s > -\frac{\lambda}{\Lambda}$.

Theorem 3. *Gradient of solution to* (5) *is locally bounded, under the following assumptions:*

(A0) ellipticity-type: A^i_α is given by potential $F \in C^2(\mathbb{R})$ as follows:

$$A^i_\alpha(Q) = (F(|Q|)_{Q^i_\alpha} \tag{40}$$

and F enjoys ellipticity:

$$(F(|Q|)_{Q^i_\alpha Q^j_\beta} \zeta^i_\alpha \zeta^j_\beta \in [\lambda |Q|^{p-2}, \Lambda |Q|^{p-2}] |\zeta|^2 \tag{41}$$

(A1) growth-type:

$$|f^i| \leq c |\nabla u|^w, \quad w \geq 0, \quad c \in L^\infty(\Omega_T) \tag{42}$$

(A2) initial integrability Let

$$M := max(2, p, w+1, 2w - p + 2) \tag{43}$$

$$\nabla u \in L^{s_0+M}_{loc}(\Omega_T) \tag{44}$$

with s_0 satisfying

$$\begin{cases} s_0 > \max(-\frac{\lambda}{\Lambda}, p - 2w - 2) \\ s_0 + 2 + \frac{np}{2} - \frac{Mn}{2} > 0 \end{cases} \tag{45}$$

moreover, for any $Q_{R_0} \subset \Omega$ with $R_0 < 1$ following inequality holds:

$$|\nabla u|_{L^\infty(Q_{\frac{R_0}{2}})} \leq C(\fint_{Q_{R_0}} |\nabla u|^{s_0+M})^{\frac{1}{s_0+2+\frac{np}{2}-\frac{Mn}{2}}} + C \tag{46}$$

Proof. For $s \geq 0$ theorem 3 is a special case of theorem 2, therefore it suffices to show it in the case of negative s. The only difference in the energy estimates is the lack of positivity of the left-hand-side term of (17), where sign of s plays a role:

$$\int_{Q_R} A^i_{\alpha\, '(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma} s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\gamma x_\delta} u^k_{x_\delta} \eta^2 \tag{47}$$

it can be however estimated as follows:

$$\int_{Q_R} A^i_{\alpha\, '(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma} s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\gamma x_\delta} u^k_{x_\delta} \eta^2 = s \int_{Q_R} \eta^2 F''(|\nabla u|)|\nabla u|^{s-4} u^i_{x_\gamma} u^j_{x_\beta} u^j_{x_\beta x_\gamma} u^i_{x_\alpha} u^k_{x_\delta} u^k_{x_\delta x_\alpha} \geq s\Lambda \int_{Q_R} \eta^2 |\nabla u|^{p+s-2} |\nabla^2 u|^2 \tag{48}$$

which allows for a following counterparty of (21)

$$\int_{Q_R} A^i_{\alpha\, '(\nabla u)^j_\beta}(\nabla u) u^j_{x_\beta x_\gamma} (|\nabla u|^s u^i_{x_\gamma x_\alpha} \eta^2 + s|\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\gamma x_\delta} u^k_{x_\delta} \eta^2) \geq (\lambda + s\Lambda) \int_{Q_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 \tag{49}$$

From this inequality on one proceeds identically as in proof of theorem 2. □

Finally, let us comment on the theorem 1.

Proof of theorem 1. Observe that for p-Laplace system $\lambda = \Lambda$. Therefore, the points 2. and 3. of the above theorem are a direct consequence of theorem 3; point 1., which is not entirely covered by theorem 3, stems from theory in DiBenedetto [9]. □

Please recall, that by point 2. merely from existence, i.e for $\tilde{p} = p$, one has $w < p - \frac{3}{5}$.

3. Conclusion.

3.1. Note on the rigorous estimates.

The above computations are formal. To perform them rigorously, transform the considered problem using Steklov averages with respect to time and use finite differences instead of differentiating it with respect to space. This procedure has been presented in DiBenedetto [9], DiBenedetto and Friedman [10] for homogenous systems. In our case we need to deal additionaly with quasilinear right-hand-side, for which the testing function $u^i_{x_\gamma}|\nabla u|^s \eta^2$ may not be admissible. In order to begin iteration, we need to have: $s_0 + w + 1 \leq p$ and to perform it at the i-th step: $s_i + w + 1 \leq s_i + M$. However the second inequality holds from the definition of M, the first may prove sometimes troublesome. In such cases one can resort to testing with $F_n(u^i_{x_\gamma}|\nabla u|^s \eta^2)$, where $F_n(x)$ is a Lipschitz truncation at the level n. As the estimates are valid for every n we can proceed as before. Observe however, that we do not encounter these difficulties during computations for theorem 1. For additional rigorous treatment (especially for s nonpositive, consult Choe and Bae [8] and reference therein[4] as well as Choe [7, 6]).

3.2. Further research.

There are many possible generalizations to the result. The most obvious one is to allow for bigger growths of right-hand-side using some natural extra assumptions, especially as some of them appear naturally in the existence theory, like boundedness of the solution (for this, consult the forthcoming paper). It would be interesting to obtain a general result for critically growing right-hand-side (i.e. like $1 + |\nabla u|^p$), with some smallness assumption, which would generalize the classical results mentioned in the introduction for the heat system.

[4] Observe, however, that in Choe and Bae [8] there are allowed $s_0 > -2$, which seems to be an error as far as $s \in (-2, -1]$ are concerned.

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