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Fundamental problems to equations of compressible chemically reacting flows

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The following thesis is dedicated to the mathematical analysis of a model governing the flow of chemically reacting compressible mixtures. We investigate the existence of weak solutions to the Navier-Stokes system supplemented by the reaction-diffusion equations for the species. We put particular emphasis on the reversible reactions and the state equation which depends on the chemical composition of the mixture. In the first approach, we consider the isothermal steady flow with diagonal, linear diffusion. However, a direct attempt to generalize this result for the case of heat-conducting fluids leads to inconsistency with the second law of thermodynamics. To avoid this discrepancy, the more complex form of the species diffusion flux, the so called multicomponent diffusion, has to be considered. As a result, a new type of degeneration in the species mass balance equations arises, which cannot be handled using the standard renormalization techniques. We solve this problem by introducing the entropy variables which make it possible to derive the basic a-priori estimate. The next important aspect of the studied model is vanishing viscosity at the vacuum states. Postulating particular form of the viscosity coefficients enables to obtain higher regularity of the density, necessary to define the notion of the weak solution. Under this assumption we show sequential stability of weak solutions to the flow of two-component mixture with the multicomponent diffusion. Further, for an additional modification of the barotropic pressure in the neighborhood of small densities, we present the complete existence result and prove the sequential stability of weak variational entropy solutions to the flow of heat-conducting mixture of arbitrary large number of reacting species.

Keywords

Navier-Stokes-Fourier system, compressible flow, thermal conduction, chemically reacting mixtures, multicomponent diffusion, sequential stability of solutions, large data, weak solutions, weak variational entropy solutions

AMS Mathematics Subject Classification

35B45, 35D05, 35Q30, 76N10, 80A32

Streszczenie

Niniejsza rozprawa poświęcona jest analizie matematycznej przepływów chemicznie reagujących mieszanin ściśliwych. Badamy istnienie słabych rozwiązań dla układu równań Naviera-Stokesa uzupełnionego równaniami reakcji-dyfuzji poszczególnych składników. Skupiamy się na opisie reakcji odwracalnych i równaniu stanu uwzględniającym skład chemiczny mieszaniny. W pierwszej kolejności rozważamy izotermiczny przepływ stacjonarny z diagonalną, liniową dyfuzją. Jednak bezpośrednia próba przeniesienia tego wyniku na przypadek płynów przewodzących ciepło prowadzi do sprzeczności z drugą zasadą termodynamiki. Aby tego uniknąć, konieczne jest wprowadzenie bardziej ogólnej postaci strumienia dyfuzji składników, tzw. dyfuzji wieloskładnikowej. W rezultacie, mamy do czynienia z nowym rodzajem degeneracji w równaniach bilansu masowego składników, który nie pozwala na wykorzystanie standardowych technik renormalizacyjnych. Rozwiązujemy ten problem poprzez zastosowanie zmiennych entropijnych, dzięki którym udaje się uzyskać podstawowe oszacowanie a-priori. Kolejnym ważnym aspektem analizowanego modelu jest zerowanie się lepkości w obszarach próżni. Przyjęcie, że współczynniki lepkości są funkcjami gęstości spełniającymi odpowiednią relację pozwala na uzyskanie wyższej regularności gęstości, koniecznej do zdefiniowania pojęcia słabego rozwiązania. Przy tym założeniu pokazujemy ciągową stabilność słabych rozwiązań dla równań przepływu dwuskładnikowej mieszaniny z ogólną dyfuzją. Dla dodatkowej modyfikacji ciśnienia barotropowego w okolicach próżni przedstawiamy kompletny dowód istnienia słabych rozwiązań oraz dowodzimy ciągowej stabilności słabych wariacyjnych rozwiązań entropijnych dla przewodzącej ciepło mieszaniny dowolnie wielu chemicznie reagujących gazów.

Słowa kluczowe

równania Naviera-Stokesa-Fouriera, przepływ ściśliwy, przewodnictwo ciepła, mieszaniny chemicznie reagujące, wieloskładnikowa dyfuzja, ciągowa stabilność rozwiązań, duże dane, słabe rozwiązania, słabe wariacyjne rozwiązania entropijne

Klasyfikacja tematyczna według AMS

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Contents

Chapter 1

Introduction

This dissertation develops the existence theory for the flows of chemically reacting gaseous mixtures. We analyze the model formulated by the full Navier-Stokes system supplemented by the reaction-diffusion equations for the species. Our principal intention is to handle the reversible reactions and the pressure which depends on concentration of the chemical species. As a consequence, one has to deal with more complex balance law for the entropy. This is the main source of the a-priori estimates, provided the model is consistent with the principles of continuum mechanics and thermodynamics. To ensure that, more general than usual forms of the transport fluxes must be considered. The mathematical investigation of such systems encounters various problems arising mostly due to a strong cross-diffusion in the species transport and the insufficient information about the density satisfying only the hyperbolic equation. Because of these difficulties, for a long time, the effort of mathematicians focused on systems simplifying either the reactive or dynamical aspects of the flow, or minimizing the coupling between them. They still can be used to model particular phenomena, but most of nowadays applications require more detailed description. For example, in modeling hydrogen-oxygen system included in a number of chemical mechanisms, as many as 20 different reactions can be taken into account involving the eight species:

$$H_2,\ \ O_2,\ \ H_2O,\ \ OH,\ \ O,\ \ H,\ \ HO_2\ \text{ and }\ H_2O_2.$$

Moreover, depending upon the temperature, pressure, and extent of reaction, all the reverse reactions can become important [104]. The main motivation of this thesis is to undertake a first step in solving some fundamental mathematical problems for such type of systems. Our approach charges from several fragmentary results that have been obtained for compressible Navier-Stokes equations so far. Below we present a brief survey on current state of the art in this area.

1.1 Overview of the theory

The existence theory for the compressible Navier-Stokes type of systems started to develop at the end of 70's when the first local existence theorems for solutions with arbitrary norm (Tani [100], Solonnikov [98]) and global existence theorems for initial data sufficiently close to an equilibrium [65] were established. In [65] Matsumura and Nishida considered the full Navier-Stokes system (coupled with heat equation) and proved the existence of global solution around the *constant* steady state, for the Cauchy problem with no external force. It was then extended in [66,67] to the case of bounded domains and a half-space for small external forces. Under similar assumptions, the issue of asymptotic stability was addressed by Valli [106]. It was a starting point to further studies devoted to existence of almost-periodic, periodic and stationary solutions for barotropic flow in a bounded domain with zero Dirichlet boundary condition [106–108]. Later on, these results were generalized by Valli and Zajączkowski [109] for the heat-conducting fluids with the inflow and outflow through the boundary.

The earliest result for the case of large external potential forces is due to Matsumura and Padula [68] who adopted the energy method to prove global in time existence of strong solutions which tend to a nontrivial steady state. For the relevant existence result in the L^p approach we refer to the papers of Mucha and Zajączkowski [70, 75, 76].

Concerning compressible *steady* flow, until early 90's, existence of strong solutions near the equilibrium corresponding to small perturbations of zero external forces became well understood due to results of Padula [87], Beirao da Veiga [6] and Farwig [31]. They applied the energy method in the spirit of [66], so the existence was obtained in the Hilbert and the Sobolev spaces. The same approach was used by Novotný and Padula in [80] to investigate the small perturbations of large potential forces, however the proof was relatively long and could not be easily generalized for more complicated boundary-value problems. An advance in this subject was a method of decomposition of velocity introduced by the same authors in [81] and then adapted by Novotný and Pileckas to treat the large potential forces [82].

At the same time, in parallel to the theory of strong solutions, there appeared the first results concerning discontinuous (weak) global solutions (Hoff [46–49], Serre [93, 94], Shelukhin [96]). In case of large data some new ideas were exposed already by Padula [86] for the 2-dimensional case and by Kazhikhov [52] when convection in the momentum equation is neglected. It is here also worth to mention the result of Vaigant and Kazhikhov [105], who established the existence of global classical and weak solution to the initial-value problem on a square for the density dependent viscosity coefficients satisfying additional growth conditions. But the real breakthrough came with the pioneering works of Lions [60, 61] and with his subsequent book [62] from 1998, where he gave the global existence results for the steady as well as the non-steady barotropic Navier-Stokes system and also for some compressible models with temperature. The proofs were essentially using properties of quantity called *the effective viscous flux* or *the effective pressure*. A compactness of this quantity was studied already by Novotný [79] using the method of decomposition from [81]. Later on, this approach was improved by Feireisl [32] to handle the case when density is not a-priori bounded in L^2, by introducing the tool for studying density oscillations, which was then adopted by Novo and Novotný [77] to treat the steady case. The comparison of these methods together with complete approximation scheme can be found in the book of Novotný and Straškraba [85], mostly for the Dirichlet boundary conditions. For the steady problem with slip boundary conditions we refer to the papers of Mucha and Pokorný [71, 89], where also a new idea of construction of approximate solution has been introduced.

For the full Navier-Stokes system, the question of existence of *the weak variational solutions* was addressed by Feireisl [33]. Later on, the concept of *the weak variational entropy solutions* (i.e. a solution satisfying the balance of mass, momentum, the entropy inequality and the global balance of total energy) was introduced by Feireisl and Novotný [35], who proved the sequential stability of these solutions. Then, in the monograph [36], also the existence of the weak variational entropy solutions was proven and several asymptotic limits were investigated. These remarkable results and their later generalizations are the only ones for the temperature-dependent viscosity coefficients with physically acceptable growth conditions. For the steady flows, the existence of weak solutions was shown first by Mucha and Pokorný [72] for the adiabatic exponent $\gamma > 3$. Then, it has been observed by Novotný and Pokorný that considering temperature-dependent viscosity coefficients helps to attain more realistic values of γ, see [83, 84]. Moreover, in [84], the authors proved that if $\gamma > \frac{4}{3}$ then not only the entropy inequality but the weak formulation of the total energy balance is fulfilled. More recently, a generalization of this result to the case of slip boundary conditions has been proven in [50].

Another important achievement in this theory is due to Feireisl and Novotný [37] who showed the *weak-strong uniqueness* of weak variational entropy solutions, meaning that they coincide with the strong ones, emanating from the same initial data, as long as the latter exists. Proving this property provides a strong argument in favor to call the weak formulation based on the entropy inequality the suitable one. It is, in a sense, a generalization of the classical result of Prodi [90] and Serrin [95] to the case of heat-conducting compressible fluid flows.

Recently, many studies have been focused on the compressible flows with density dependent viscosity. There is a sequel of papers devoted to sequential stability of weak solutions to the barotropic Navier-Stokes system [14, 69] and its generalization to the heat-conducting case [13]. These studies were originally developed for the Korteweg and shallow water models [10, 15] and all they relay on a new mathematical concept of entropy, that has been discovered by Bresch, Desjardins and Lin [15]. Although the stability result is widely regarded as a most difficult step in the complete proof of existence, the construction of sufficiently smooth approximate solutions is still an open problem; some ideas can be found in [12].

Much less is known about the physically reasonable models that include the chemical reactions. In fact, we are aware of only one global-in time result applicable to a large number of reversibly reacting gaseous species. The existence of solutions to the system based on the Navier-Stokes equations equipped with physically relevant constitutive relations was established by Giovangigli [45]. He assumed, however, that the initial conditions are sufficiently close to an equilibrium state.

The serious difficulty in analysis of such models is the mixed hyperbolic-parabolic type of associated PDEs and a strong coupling between the species mass balance equations and the rest of the system. But the most troublesome is the form of diffusion flux which does not allow to apply the standard tools for parabolic systems. For this reason, in majority of studies, the diffusion fluxes are described by *the Fick law*. This approximation does not take into account the cross-effects that are well-known to play an important role in many phenomena. Furthermore, such an assumption leads to inconsistency with the second law of thermodynamics, when the pressure depends on the chemical composition of the mixture. In that case, the sign of the entropy production may fail to be non-negative, which contradicts physical admissibility of the process. This in turn interferes with obtaining the fundamental a-priori estimates causing that the methods from [13, 36, 62] break down.

A similar problem appeared in the works of Frehse, Goj and Málek [40, 41] who proved existence and uniqueness of solutions to the steady Stokes-like system for a mixture of two (non-reactive) fluids (see also [42] for relevant result on quasi-stationary model). The authors considered the model assigning densities and velocity fields to each species of the fluid (cf. [91]). In their case, neglecting some nonlinear interaction terms in the source of momentum caused that the basic energy equality was no longer preserved.

Regarding models with the Fick approximation, the literature is more exhaustive, especially for one dimensional case [18, 28, 114] and for the multidimensional combustion models. As far as the latter are concerned, the global existence of weak solutions was recently proven by Donatelli and Trivisa [26] and then extended in [25] to treat the pressure dependence on the mass fraction of fuel. Concerning multicomponent models, the issue of global-in-time existence of weak variational entropy solutions was investigated by Feireisl, Petzeltová and Trivisa [38]. They generalized the proof from [36] to the case of chemically reacting flows, when there is no interaction among the species diffusion fluxes and the pressure does not depend on the chemical composition of the mixture. Since the proof is based on the energy-entropy method, it is not clear if it can be extended to cover more realistic state equation without including a general form of diffusion. Nevertheless, it should be emphasized that it provides an exact mathematical complementation of the scale analysis and numerical experiments presented in the work of Klein et al. [54].

1.2 Main results

The thesis consists of four main parts. Each of them, as a separate chapter, is a distinct result based on the series of articles [73, 74, 111–113].

The Fick approximation – adaptation of Lions' approach, [111]. In this chapter we examine the system of equations governing the steady flow of a polyatomic isothermal reactive gaseous mixture. The model covers situation when the pressure depends on species concentrations and when the diffusion fluxes are approximated by the Fick law with density-dependent coefficients. It is shown that this problem admits a weak solution provided the adiabatic exponent for the mixture γ is greater than $\frac{7}{3}$. In the proof we follow the idea of Lions [62] based on the DiPerna-Lions transport theory [24]. This approach requires the L^2 integrability of the density in order to get the renormalized solution of the continuity equation which is later on used to exploit the *effective viscous flux* equality. In the barotropic case it is possible to improve the value of γ be applying the oscillations defect measure introduced by Feireisl [32]. It seems, however, that here this technique could not be used due to a presence of the molecular part of the pressure.

The case of two species – vanishing viscosity coefficients, [73, 113]. Here, we study the Cauchy problem for the system of equations governing flow of isothermal reactive mixture of compressible gases. The first part of this chapter is devoted to the proof of sequential stability of weak solutions when the state equation essentially depends on the species concentration and the viscosity vanish on vacuum. The main difficulty in comparison to the systems with viscosity coefficients bounded from below by a constant [36, 62], is lack of information about the velocity vector field. On the other hand, a special relation among the viscosity coefficients

$$\nu(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho) \tag{1.1}$$

proposed by Bresch and Desjardins in [11] allows to show some better properties of the density. But still, it is not enough to deduce compactness of the most restrictive convective term when no additional term, like capillary or a drag force, is present. In the second part of the chapter, under further assumption on the barotropic pressure, we give a detailed description of the approximate scheme and prove the existence of weak solutions for arbitrary large initial data.

System of reaction-diffusion equations – the entropy variables, [74]. In this part of the thesis we analyze the system of reaction-diffusion equations for the compressible mixture of gases, taking rigorously into account the thermodynamical regime. It implies, in particular, that the diffusion terms are non-symmetric, not positively defined and cross-diffusion effects must be taken into account. In consequence one has to deal with a system of parabolic equations with hyperbolic deviation. Our main achievement is the proof of existence of weak solutions for arbitrary number of reacting species under the assumption that the total density is as regular as it follows from the previous chapter. Here, in contrary to [40], the nonlinear diffusion terms appear in all the species diffusion forces, while the matrix of diffusion coefficients C satisfies the assumptions from [45], Chapter 7. We present our technique only for a particular form of C, however, the entropy-like estimate we obtain in the course of proof seems to be of independent interest as it indicates possible renormalization of the system.

Heat-conducting mixtures – a key role of the entropy estimate, [112]. The purpose of the last chapter is to bring together and complement the results from [74, 113] in order to treat the complete mathematical model governing the motion of n-component, heat-conducting and chemically reacting gaseous mixture. We prove sequential stability of weak variational entropy solutions when the state equation essentially depends on the species concentration and the species diffusion fluxes depend on the gradients of partial pressures, analogously as in [74]. The complete existence result, however important and interesting, seems to be very complex and technically complicated task and is left for future study. Of crucial importance for our analysis is the fact that viscosity coefficients are related by (1.1) and the source terms enjoy the admissibility condition dictated by the second law of thermodynamics. Thanks to this, we are able to combine the entropy inequality with the Bresch-Desjardins type of estimate [11]. The latter, similarly as in [113], compensates the missing information about the species densities, which is due to a more general form of diffusion fluxes. However, the presence of the temperature causes that the method scratched in [113] and [74] is now more intricate.

It is to be noticed that, unlike to [13], we perform the limit passage in the framework of weak variational entropy formulation, similar in spirit to [38]. What is still lacking to end up with the usual weak solution, is better regularity of the temperature required to pass to the limit in the internal energy balance.

1.3 Presentation of the model

In the following section we first present the set of equations governing multicomponent reactive flow. The derivation of such model from the kinetic theory of gases can be found in numerous textbooks [17, 45, 110]. Next, we introduce some equivalent formulations that can be used interchangeably depending on the context and purpose. Then, in Section 1.3.3, we supplement the system of field equations by the set of constitutive relations characteristic for compressible, viscous, heat-conductive and chemically-reacting mixtures.

1.3.1 Equations of motion

To describe the flow of n-component chemically reacting compressible mixture, we will use the full Navier-Stokes-Fourier (NSF) system suplemented by the set of n reaction-diffusion equations for the species:

$$\left.\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \\ \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla\pi = \varrho\mathbf{f} \\ \partial_t(\varrho E) + \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(\pi\mathbf{u}) + \operatorname{div}\mathbf{Q} - \operatorname{div}(\mathbf{S}\mathbf{u}) = \varrho\mathbf{f}\cdot\mathbf{u} \\ \partial_t\varrho_k + \operatorname{div}(\varrho_k\mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho\vartheta\omega_k, \quad k\in\{1,...,n\} \end{aligned}\right\} \text{ in } (0,T)\times\Omega. \tag{1.2}$$

These equations express the physical laws of conservation of mass, momentum, total energy and the balances of species mass, respectively.

Here, $\mathbf{u}:\mathbb{R}^3\to\mathbb{R}^3$ is the velocity field, $\varrho:\mathbb{R}^3\to\mathbb{R}$ denotes the total mass density being a sum of species densities ϱ_k, $k\in\{1,\dots,n\}$. The last unknown quantity is the temperature $\vartheta:\mathbb{R}^3\to\mathbb{R}$ which appears implicitly in all the equations of (1.2) except for the continuity equation. Next, $\mathbf{S}$ denotes the viscous stress tensor, the internal pressure is denoted by π, $\mathbf{f}$ is a given external force, E is the total energy per unit mass, $\mathbf{Q}$ stands for the heat flux, $\mathbf{F}_k$, $k\in\{1,\dots,n\}$ denote the species diffusion fluxes and ω_k, $k\in\{1,\dots,n\}$ are the chemical source terms, also termed the species production rates.

In (1.2), t denotes time $t\in(0,T)$ and the length of the time interval T is usually assumed to be arbitrary large, but finite. The space domain Ω is a bounded subset of $\mathbb{R}^3$ that will be different in different chapters. The vectors belonging to the physical space $\mathbb{R}^3$ as well as the tensors are printed in the boldface style.

1.3.2 Alternative formulations

In what follows we give different formulations for the species mass balances and the energy equation. The first of them is very convenient in case when the diffusion fluxes for species are approximated by the Fick law; the second one allows to show several properties of function ϑ as a solution to quasilinear parabolic equation.

The species mass conservation equations. They can be equivalently written in terms of species mass fractions:

$$\partial_t(\varrho Y_k) + \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho\vartheta\omega_k,$$

where Y_k, $k \in \{1,\dots,n\}$ are defined by $Y_k = \frac{\varrho_k}{\varrho}$ and they satisfy:

$$\sum_{k=1}^{n} Y_k = 1. \tag{1.3}$$

We remark that we will freely switch from one notation to the other using the species unknowns $(\varrho, \varrho_1, \dots, \varrho_n)$ or equivalently $(\varrho, Y_1, \dots, Y_n)$. Note, however, that due to the law of mass conservation, system (1.2) is a priori linearly dependent. Indeed, assuming the following constraints for diffusion fluxes and the production rates for species

$$\sum_{k=1}^{n} \mathbf{F}_k = 0, \quad \sum_{k=1}^{n} \omega_k = 0,$$

we sum the n species mass conservation equations and derive the continuity equation. Therefore, to solve the system, one should eliminate one equation or constraint (1.3). In the subsequent chapters we will follow one of two strategies: either we solve the n species equations with the species unknowns $(\varrho_1, \dots, \varrho_n)$ or we will look for a solution to the n species equations and the continuity equation in parallel and then investigate compatibility with (1.3).

The internal energy equation. The total energy E per unit mass may be written in the form

$$E = \frac{1}{2}|\mathbf{u}|^2 + e,$$

where the first component is the kinetic energy whereas $e = e(\varrho, \vartheta, Y)$, $Y = (Y_1, \dots Y_n)^T$ stands for the specific internal energy. As mentioned above, it is sometimes more convenient to replace the total energy by the internal energy balance. It can be derived by subtracting from the total energy balance the balance of kinetic energy:

$$\partial_t(\varrho e) + \operatorname{div}(\varrho e \mathbf{u}) + \operatorname{div} \mathbf{Q} = -\pi \operatorname{div} \mathbf{u} + \mathbf{S} : \nabla \mathbf{u}. \tag{1.4}$$

It should be, however, underlined that these formulations have different physical meaning and system (1.2) is equivalent to the one with the total energy equation replaced by the internal energy equation, provided the motion is sufficiently smooth. In the next section we will transform equation (1.4) into another form, based on the thermodynamical concept of entropy.

1.3.3 Constitutive relations

The purpose of this section is to specify the constitutive relations of the gaseous mixture, i.e. to supplement system (1.2) with a set of expressions determining the form of thermodynamical functions and transport fluxes in terms of macroscopic variable gradients and transport coefficients in the spirit of [45], Chapter 2. Precise evaluation of the form of transport coefficients is a very difficult task in the modeling, in fact, only some approximate expressions are available. We will specify the structural properties to be imposed on them, separately in the subsequent chapters.

Thermal equation of state. We consider the pressure $\pi = \pi(\varrho, \vartheta, Y)$, which can be decomposed into

$$\pi = \pi_m + \pi_c, \tag{1.5}$$

where the latter component depends solely on the density and it corresponds to the barotropic process of viscous gas. It is the only non-vanishing component of the pressure when temperature goes to zero, thus will be termed a "cold pressure" or a barotropic correction. The first component $\pi_m = \pi_m(\varrho, \vartheta, Y)$ is the classical molecular pressure of the mixture which is determined through the *Boyle law* as a sum of partial pressures p_k:

$$\pi_m(\varrho, \vartheta, Y) = \sum_{k=1}^{n} p_k(\varrho, \vartheta, Y_k) = R \sum_{k=1}^{n} \frac{\varrho \vartheta Y_k}{m_k},$$

with m_k the molar mass of the k-th species and R the perfect gas constant.
Likewise the pressure, the internal energy $e = e(\varrho, \vartheta, Y)$ can be decomposed into

$$e = e^{st} + e_m + e_c, \quad e^{st}(Y) = \sum_{k=1}^{n} Y_k e_k^{st}, \quad e_m(\vartheta, Y) = \vartheta \sum_{k=1}^{n} c_{vk} Y_k, \tag{1.6}$$

where $e_k^{st} = const.$ is the formation energy of the k-th species, at the standard temperature ϑ^{st}, while c_{vk} is the constant-volume specific heat of the k-th species. The "cold" components of the internal energy $e_c = e_c(\varrho)$ and pressure π_c are related through the following equation of state:

$$\varrho^2 \frac{\mathrm{d}e_c(\varrho)}{\mathrm{d}\varrho} = \pi_c(\varrho). \tag{1.7}$$

The last relation is a consequence of the second law of thermodynamics which postulates the existence of a state function called the entropy.

The entropy equation. The entropy of a thermodynamical system is defined (up to an additive constant) by the differentials of energy, total density, and species mass fractions via the Gibbs relation:

$$\vartheta \mathrm{D}s = \mathrm{D}e + \pi \mathrm{D}\left(\frac{1}{\varrho}\right) - \sum_{k=1}^{n} g_k \mathrm{D}Y_k, \tag{1.8}$$

where D denotes the total derivative with respect to the state variables $\{\varrho, \vartheta, Y\}$; whereas g_k are the Gibbs functions

$$g_k = h_k - \vartheta s_k. \tag{1.9}$$

Here, $h_k = h_k(\vartheta)$ denotes the specific enthalpy and $s_k = s_k(\vartheta, \varrho_k)$ is the specific entropy of the k-th species

$$h_k(\vartheta) = e_k^{st} + c_{pk}\vartheta,$$

$$s_k(\vartheta, \varrho_k) = s_k^{st} + c_{vk} \log \frac{\vartheta}{\vartheta^{st}} + \frac{R}{m_k} \log \frac{\Gamma^{st} m_k}{\varrho_k}, \tag{1.10}$$

where $s_k^{st} = const.$ denotes the formation entropy of k-th species and $\Gamma^{st} = \frac{p^{st}}{R\vartheta^{st}}$ is the standard concentration (at the standard temperature ϑ^{st} and the standard pressure p^{st}).
The constant-volume (c_{vk}) and constant-pressure (c_{pk}) specific heats for the k-th species are constants related by the following formula

$$c_{pk} = c_{vk} + \frac{R}{m_k}. \tag{1.11}$$

In accordance with (1.4), (1.6), (1.7) and (1.8), the specific entropy of the mixture, expressed as a weighted sum of the species specific entropies

$$s = \sum_{k=1}^{n} Y_k s_k, \tag{1.12}$$

is governed by the following equation

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta}\mathbf{F}_k\right) = \sigma, \tag{1.13}$$

where σ is the entropy production rate

$$\sigma = \frac{\mathbf{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \frac{\sum_{k=1}^{n} g_k \varrho \vartheta \omega_k}{\vartheta}. \tag{1.14}$$

By virtue of the second law of thermodynamics, the entropy production rate must be non-negative for any admissible process.

The stress tensor. The viscous part of the stress tensor obeys the *Newton rheological law*, namely:

$$\mathbf{S} = 2\mu \mathbf{D}(\mathbf{u}) + \nu \operatorname{div} \mathbf{u}\mathbb{I}, \tag{1.15}$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ and μ, ν are the Lamé viscosity coefficients (dependent on temperature and pressure) satisfying the following conditions

$$\mu > 0, \quad 2\mu + 3\nu \geq 0. \tag{1.16}$$

These inequalities are the consequence of the second law of thermodynamics. It implies, in particular, that the production of entropy associated with the flow of viscous fluid must not be negative, i.e. that $\frac{\mathbf{S}(\mathbf{u}):\mathbf{u}}{\vartheta} \geq 0$.
Rewriting relation (1.15) as a sum of two orthogonal components, we get the following expression

$$\mathbf{S}(\mathbf{u}) = 2\mu\left(\mathbf{D}(\mathbf{u}) - \frac{1}{3} \operatorname{div} \mathbf{u}\mathbb{I}\right) + \xi \operatorname{div} \mathbf{u}\mathbb{I}, \tag{1.17}$$

where μ is the shear viscosity coefficient and $\xi = \frac{2}{3}\mu + \nu$ is the bulk viscosity coefficient, thus the dissipation is equal to

$$\mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} = \frac{1}{2}\mu \sum_{i,j=1}^{3} \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3} \operatorname{div} \mathbf{u} \, \delta_{ij} \right)^2 + \xi (\operatorname{div} \mathbf{u})^2.$$

So, it is nonnegative provided $\mu, \xi \geq 0$.

The species diffusion fluxes. Following [45], Chapter 2, Section 2.5.1, we postulate that the diffusion flux of the k-th species is given by

$$\mathbf{F}_k = -C_0 \sum_{l=1}^{n} C_{kl} \mathbf{d}_l, \quad k = 1, ...n, \tag{1.18}$$

where C_0, C_{kl} are multicomponent flux diffusion coefficients and $\mathbf{d}_k$ is the species k diffusion force specified, in the absence of external forces, by the following relation

$$\mathbf{d}_k = \nabla \left(\frac{p_k}{\pi_m} \right) + \left(\frac{p_k}{\pi_m} - \frac{\varrho_k}{\varrho} \right) \nabla \log \pi_m. \tag{1.19}$$

The main properties of the flux diffusion matrix $C = (C_{k,l})_{k,l=1}^{n}$ are the following (see [45]):

$$C\mathcal{Y} = \mathcal{Y} C^T, \quad N(C) = \mathbb{R}Y, \quad R(C) = U^{\perp}, \tag{1.20}$$

where $\mathcal{Y} = \operatorname{diag}(Y_1, \ldots, Y_n)$ is the diagonal matrix of species mass fractions Y_k, $N(C)$ is the nullspace of matrix C and by $R(C)$ we denote its range; $U = (1, \ldots, 1)^T$ and $U^{\perp}$ is the orthogonal complement of $\mathbb{R}U$.
Another important condition on C, postulated for example by Waldmann [110], is that wherever it can be defined, the matrix

$$D_{kl} = \frac{C_{kl}}{\varrho_k}, \quad k, l \in \{1, \ldots, n\}, \tag{1.21}$$

is symmetric and positive definite over the physical hyperplane $U^{\perp}$, which corresponds to the positivity of entropy production rate associated with the diffusive process. For more details on evaluation of C from the kinetic theory of gases as well as its mathematical properties, we refer the reader to [45], Chapters 4, 7 and the references therein.
For the purposes of Chapter 2 we recall here some additional properties of diagonal diffusion matrix C for $n-1$ species, formulated as Lemma 7.5.10 in [45].

Lemma 1.1. *Assume that there exist scalars $\alpha_k, k \in \{1, \ldots, n-1\}$, $n \geq 2$ such that*

$$\forall x \in U^{\perp}, \quad \forall k \in \{1, \ldots, n-1\}, \quad (Cx)_k = \sum_{l=1}^{n} C_{kl} x_l = \alpha_k x_k.$$

Then the matrix C is in the form

$$C = \begin{pmatrix} \alpha_1(1-y_1) & -\alpha_1 y_1 & \ldots & -\alpha_1 y_1 \\ & \ddots & & \vdots \\ -\alpha_1 y_{n-1} & \ldots & \alpha_{n-1}(1-y_{n-1}) & -\alpha_{n-1} y_{n-1} \\ -\alpha_1 + \sigma & \ldots & -\alpha_{n-1} + \sigma & \sigma \end{pmatrix}, \tag{1.22}$$

where $y_k = Y_k / \langle Y, U \rangle$, $k \in \{1, \ldots, n\}$, and $\sigma = \sum_{k=1}^{n-1} \alpha_k y_k$. In addition, we have $\alpha_i > 0$, for $i \in \{1, \ldots, n-1\}$, and whenever $\alpha_i \neq \alpha_j$, we must have $Y_i Y_j = 0$.

The heat flux. It is the sum of two components

$$\mathbf{Q} = \sum_{k=1}^{n} h_k \mathbf{F}_k - \kappa \nabla \vartheta, \tag{1.23}$$

where the second term represents the *Fourier law* with the thermal conductivity coefficient κ. The first term describes transfer of energy due to the species molecular diffusion.

The species production rates. We already included in model (1.2) the linear dependence with respect to ϑ and ϱ in the right hand side of the species mass balance equations. In accordance with the second law of thermodynamics, we additionally assume that ω_k, $k = 1, \ldots, n$ enjoy the following condition

$$-\int_{\Omega} \sum_{k=1}^{n} g_k \varrho \omega_k \ \mathrm{d}x \geq 0, \tag{1.24}$$

for any admissible thermodynamical process [45].

Chapter 2

The Fick approximation

The Fick law relates the diffusive flux to the gradients of concentrations

$$\mathbf{F}_k = -\varrho D_k^{ap} \nabla Y_k,$$

where D_k^{ap} denotes the empirical diffusion coefficient of the k-th species of the mixture. This is a very common approximation often used implicitly in the classical textbooks as well as in various studies devoted to scale analysis and numerical experiments. From the point of view of mathematical analysis such a simplification leads, on one hand, to the the system of elliptic (in a steady case) or parabolic (in case of time-dependent flow) equations of species mass conservation, on the other hand, it is an obstacle to obtain series of a-priori estimates resulting from the entropy balance.

2.1 Introduction and main result

We investigate the model of motion of four-component gaseous mixture undergoing an isothermal, reversible chemical reaction constituted by

$$A + B \rightleftharpoons C \tag{2.1}$$

and we assume that the reaction takes place in the presence of a dilutant D. For reversible reactions, the postulate that the process occurs in a constant temperature is met when the reaction is slow enough to enable the surroundings to continually compensate the difference in heats between the substrates and products.

We will focus on the stationary flow that can be characterised by the state variables: the total mass density $\varrho = \varrho(x)$, the velocity vector field $\mathbf{u} = \mathbf{u}(x)$ and the species mass fractions $Y_k = Y_k(x)$ for $k \in \{A, B, C, D\}$, throughout the set of balance equations:

$$\left.\begin{aligned} \operatorname{div}(\varrho\mathbf{u}) &= 0 \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla\pi &= \varrho\mathbf{f} \\ \operatorname{div}(\varrho Y_A\mathbf{u}) + \operatorname{div}(\mathbf{F}_A) &= \varrho\omega_A \\ \operatorname{div}(\varrho Y_B\mathbf{u}) + \operatorname{div}(\mathbf{F}_B) &= \varrho\omega_B \\ \operatorname{div}(\varrho Y_C\mathbf{u}) + \operatorname{div}(\mathbf{F}_C) &= \varrho\omega_C \end{aligned}\right\} \text{ in } \Omega, \tag{2.2}$$

expressing the conservation of mass, momentum and the conservation of species mass, respectively.

In this chapter Ω is a bounded subset of $\mathbb{R}^3$ and we supplement system (2.2) by the impermeability conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{F}_k\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.3}$$

together with the no-slip boundary condition

$$\mathbf{u}\times\mathbf{n}|_{\partial\Omega} = \mathbf{0}. \tag{2.4}$$

We assume that the total mass is given,

$$\int_\Omega \varrho \,\mathrm{d}x = M > 0.$$

The model is consistent with the principle of mass conservation, thus necessarily

$$\sum_{k\in S} Y_k = 1, \tag{2.5}$$

and

$$\sum_{k\in S} \omega_k = 0, \tag{2.6}$$

where by S we denote the set of all species $\{A, B, C, D\}$. Note that we consider only the first three mass fractions as unknowns and use (2.5) to evaluate the mass fraction of the remaining species Y_D.

The internal pressure π is a function of density ϱ and mass fractions $Y = \{Y_A, Y_B, Y_C, Y_D\}$ and it obeys the following equation of state

$$\pi(\varrho, Y) = \varrho^\gamma + R\varrho\left(\sum_{k\in S}\frac{Y_k}{m_k}\right), \quad \gamma > 1. \tag{2.7}$$

The viscous part of the stress tensor $\mathbf{S}$ is given by (1.15), where μ, ν are constant viscosity coefficients satisfying

$$\mu > 0, \quad 2\mu + 3\nu \geq 0. \tag{2.8}$$

The species mass fluxes $\mathbf{F}_k$, $k \in \{A, B, C\}$, are given by the Fick empirical law

$$\mathbf{F}_k = -D_k(\varrho)\nabla Y_k, \quad k\in\{A,B,C\}, \quad \mathbf{F}_D = -\mathbf{F}_A - \mathbf{F}_B - \mathbf{F}_C, \tag{2.9}$$

where $D_k(\varrho)$ stand for the diffusion coefficients and $D_k(\varrho) = c_k D(\varrho)$, $c_k \in \mathbb{R}$. This approximation corresponds to diffusion matrix C which is diagonal only for the species A, B, C. For the moment we assume, however, that it is diagonal with respect to all four species, so

$$D_k(\varrho) = D(\varrho) \quad \text{for} \quad k\in\{A,B,C\}, \quad \text{and} \quad \mathbf{F}_D = -D(\varrho)\nabla Y_D.$$

We shall come back to the general case (2.9) at the end of this chapter.

Furthermore, the common diffusion coefficient satisfies

$$D(\cdot)\in C([0,\infty)), \quad \underline{D}(1+\varrho^{\frac{\gamma}{2}}) \leq D(\varrho) \leq \overline{D}(1+\varrho^{\frac{\gamma}{2}}), \tag{2.10}$$

for some positive constants $\underline{D}$, $\overline{D}$.

The production rates ω_k are continuous function and for $k \in \{A, B, C\}$ we have

$$-\underline{\omega} \leq \omega_k(Y) \leq \overline{\omega}, \quad \text{for all} \quad 0 \leq Y_k \leq 1, \tag{2.11}$$

$\omega_D = 0$, moreover we suppose

$$\omega_k(Y) \geq 0 \quad \text{whenever} \quad Y_k = 0. \tag{2.12}$$

The main result of this chapter concerns the existence of weak solutions in the sense specified by the following definition.

Definition 2.1. *We say that a set of functions $(\varrho, \mathbf{u}, Y)$ is a weak solution to the problem (2.2–2.4) provided $\varrho \in L^\gamma(\Omega)$, $\mathbf{u} \in W_0^{1,2}(\Omega)$, $Y \in W^{1,2}(\Omega)$, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$, $k \in S$, $Y_k, \varrho \geq 0$, $\sum_{k\in S} Y_k = 1$ a.e. in Ω, and the following integral equalities are satisfied*

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\xi \, \mathrm{d}x = 0 \quad \forall \xi \in C^\infty(\overline{\Omega}), \tag{2.13}$$

$$\int_\Omega (-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla\varphi + \mathbf{S}(\mathbf{u}) : \nabla\varphi) \, \mathrm{d}x - \int_\Omega \pi(\varrho, Y) \operatorname{div}\varphi \, \mathrm{d}x = \int_\Omega \varrho\mathbf{f} \cdot \varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega),$$

$$-\int_\Omega \varrho\mathbf{u}Y_k \cdot \nabla\phi \, \mathrm{d}x + \int_\Omega D(\varrho)\nabla Y_k \cdot \nabla\phi \, \mathrm{d}x = \int_\Omega \varrho\omega_k\phi \, \mathrm{d}x \quad \forall \phi \in C^\infty(\overline{\Omega}),$$

for all $k \in \{A, B, C\}$.

We will also use the notion of the renormalized solution to the continuity equation

Definition 2.2. *Let $\mathbf{u} \in W_{loc}^{1,2}(\mathbb{R}^3)$ and $\varrho \in L_{loc}^{6/5}(\mathbb{R}^3)$ solve*

$$\operatorname{div}(\varrho\mathbf{u}) = 0$$

in the sense of distributions on $\mathbb{R}^3$, then the pair $(\varrho, \mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$\operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho)) \operatorname{div}\mathbf{u} \, \mathrm{d}x = 0, \tag{2.14}$$

in the sense of distributions on $\mathbb{R}^3$, for all $b \in W^{1,\infty}(0,\infty) \cap C^1([0,\infty))$, such that $sb'(s) \in L^\infty(0,\infty)$.

The main result of this chapter is

Theorem 2.3. *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, let $\mu > 0$, $2\mu + 3\nu \geq 0$, $\gamma > \frac{7}{3}$, $M>0$ and let $\mathbf{f} \in L^\infty(\Omega)$. Then there exists a weak solution to the problem (2.2–2.4) in the sense of Definition 2.1. Moreover $\int_\Omega \varrho \, \mathrm{d}x = M$, and if $\gamma \geq 3$, then $\varrho \in L^{2\gamma}(\Omega)$, otherwise if $\frac{7}{3} < \gamma < 3$ then $\varrho \in L^{3\gamma-3}(\Omega)$. Additionally, the pair $(\varrho, \mathbf{u})$, extended by 0 outside Ω, is a renormalized solution to the continuity equation in the sense of Definition 2.2.*

The proof of this theorem is based on several ideas from the theory of weak solutions to the steady Navier-Stokes equations, which originates from the pioneering work of Lions [62]. He proved the existence of weak solutions for the steady as well as the non-steady Navier-Stokes system for barotropic flow. This means that the internal pressure has a particular form $\pi(\varrho) \approx \varrho^\gamma$, where $\gamma > 1$ is the adiabatic constant, which is determined by the number of degree of freedom of a single gas molecule. From the mathematical point of view, the value of γ indicates the quality of a-priori estimates of ϱ. The proof proposed by Lions relies on the DiPerna-Lions transport theory [24] requiring L^2 integrability of the density, which entails $\gamma \geq \frac{5}{3}$. The most difficult part of the proof was to show the strong convergence of the density in order to pass to the limit in the nonlinear term ϱ^γ. Here, one essentially uses the compactness of quantity called *effective viscous flux* or *effective pressure*. The compactness of this quantity was studied already in [79] using the method of decomposition proposed by Novotný and Padula in [81]. Later, these ideas were extended by Feireisl [32] by introducing a tool for studying density oscilations in the case of nonsteady equations. His method was then adapted to the steady system by Novo and Novotný [77] and led to the proof existence of weak solutions for an optimal value of adiabatic constant, $\gamma > \frac{3}{2}$. A modification of this approach in the case of steady flows with slip boundary conditions has been introduced by Mucha and Pokorný in a two dimensional case in [71] and in 3D in [89]. More recently, these ideas were also extended to treat the heat-conducting case [72,83,84], as it was mentioned in the introduction.

From the point of view of existence theory for steady flows of chemically reacting mixtures, the investigations devoted to stability and asymptotic analysis presented in [56, 58] are of particular interest, since, to the best of our knowledge, the time-independent problems were not considered so far. They seem to be worse than the nonsteady cases in the sense that the energy inequality by itself does not give any information about the sequence of weak solutions. This is precisely the reason why the value of γ obtained in the course of our proof is far from being physically relevant.

2.2 Existence of solutions

This section is devoted to the proof of Theorem 2.3. First we introduce the approximative system following the approach from [38, 62, 85]. At this stage the classical theory for elliptic equations together with the fixed point argument are sufficient to show the existence of regular solutions. In this section we also show the basic energy estimate that is used, with some modifications, throughout all the paper. Next, we let $\varepsilon \to 0$ in order to get rid of artificial diffusion in the continuity equation. The requirement imposed on the adiabatic constant, $\gamma > \frac{7}{3}$ is necessary to get the boundedness of velocity gradient in $L^2(\Omega)$ and enables to apply Lions technique of showing the strong convergence of density.

2.2.1 Approximative system

Combining the ideas from [85] and [38] we introduce the following approximative system. For the constant parameters $h, \varepsilon, \eta > 0$ we will look for a triple $(\varrho_{\eta,\varepsilon}, \mathbf{u}_{\eta,\varepsilon}, Y_{\eta,\varepsilon})$ (we will skip the subscripts when no confusion can arise) satisfying:

• the approximate continuity equation

$$\begin{aligned} \varepsilon\varrho + \operatorname{div}(\varrho\mathbf{u}) &= \varepsilon\Delta\varrho + \varepsilon h, \\ \nabla\varrho\cdot\mathbf{n}|_{\partial\Omega} &= 0, \end{aligned} \tag{2.15}$$

where $h = \frac{M}{|\Omega|}$,

• the approximate momentum equation

$$\begin{aligned} \tfrac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \tfrac{1}{2}\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S}(\mathbf{u}) + \nabla\pi(\varrho, Y) &= \varrho\mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} &= 0, \end{aligned} \tag{2.16}$$

• the approximate species mass balance equations

$$\begin{aligned} \varepsilon\varrho Y_k + \operatorname{div}(\varrho\mathbf{u}Y_k) - \operatorname{div}(D_\eta\nabla Y_k) - \varepsilon\Delta(\varrho Y_k) &= \varrho\omega_k + \varepsilon h K(Y_k), \quad k\in\{A,B,C\} \\ \nabla Y_k\cdot\mathbf{n}|_{\partial\Omega} &= 0, \end{aligned} \tag{2.17}$$

where we denoted

$$K(Y_k) = \begin{cases} \frac{\int_\Omega Y_k \, \mathrm{d}x}{\sum_{i\in S}\int_\Omega Y_i \, \mathrm{d}x} & \text{for } \varrho \neq 0, \\ 0 & \text{for } \varrho = 0. \end{cases}$$

Observe, that due to restriction (2.5) the approximate equation for species D writes as

$$\varepsilon\varrho Y_D + \mathrm{div}\,(\varrho \mathbf{u} Y_D) - \mathrm{div}(D_\eta \nabla Y_D) - \varepsilon\Delta(\varrho Y_D) = \varepsilon h(1 - K(Y_A) - K(Y_B) - K(Y_C))$$

and its r.h.s. is equal to $\varepsilon h K(Y_D)$ for $\varrho > 0$ otherwise it is εh.
Further, $D(\varrho)_\eta$ is a standard regularization of function $D(\varrho)$ (extended by constant $D(0)$ to the negative real line) by means of mollifiers, i.e.

$$D(\varrho)_\eta = D * \psi_\eta(\varrho) = \int_{\mathbb{R}} \psi_\eta(\varrho - \xi) D(\xi) \, \mathrm{d}\xi,$$

and the regularizing kernel satisfies the following properties

$$\psi \in C^\infty(\mathbb{R}), \quad \mathrm{supp}\, \psi \subset (-1,1), \quad \psi(t) = \psi(-t) \geq 0, \quad \int_{\mathbb{R}} \psi(t)\mathrm{d}t = 1, \quad \psi_\eta = \frac{1}{\eta}\psi\left(\frac{t}{\eta}\right).$$

For more properties of such convolutions we refer the reader to [30], Appendix C.4, in particular, regularized diffusion coefficients conserve the property (2.10) uniformly with respect to η.
The aim of this section is to prove the following theorem.

Theorem 2.4. *Let $\epsilon > 0$, $k \in S$, $h = \frac{M}{|\Omega|}$. Under assumptions of Theorem 2.3, there exists a triple $(\varrho, \mathbf{u}, Y)$ being a regular solution to (2.15-2.17), such that $\varrho \in W^{2,p}(\Omega)$, $\mathbf{u} \in W^{2,p}(\Omega)$, $Y_k \in W^{2,p}(\Omega)$, $k \in S$, for all $p < \infty$. Moreover, $\varrho \geq 0$ in Ω, $\int_\Omega \varrho \, \mathrm{d}x = M$, $Y_k \geq 0$ and $\sum_{k\in S} Y_k = 1$.*

The proof of this theorem is based on several auxiliary lemmas and it is presented in the next section.

2.2.2 Existence for fixed parameters

Step 1: We denote for $p \in [1, \infty]$:

$$M^p = \left\{\mathbf{w} \in W^{1,p}(\Omega); \mathbf{w}|_{\partial\Omega} = \mathbf{0}\right\},$$

and define the operator

$$\mathcal{S} \, : \, M^\infty \to W^{2,p}(\Omega),$$

$1 \leq p < \infty$, $\mathcal{S}(\mathbf{u}) = \varrho$, where ϱ solves the approximate continuity equation (2.15) with the Neumann boundary condition. We then claim that the following result holds true

Lemma 2.5. *Let assumptions of Theorem 2.4 be satisfied. Then the operator $\mathcal{S}$ is well defined for all $p < \infty$. Moreover, if $\mathcal{S}(\mathbf{u}) = \varrho$, then $\varrho \geq 0$ in Ω and $\int_\Omega \varrho \, \mathrm{d}x = \int_\Omega h \, \mathrm{d}x$. Additionally, if $\|\mathbf{u}\|_{W^{1,\infty}(\Omega)} \leq L$, $L > 0$, then*

$$\|\varrho\|_{2,p} \leq C(\varepsilon, p, \Omega)(1 + L)h, \quad 1 < p < \infty. \tag{2.18}$$

The above lemma is an analogue of Proposition 4.29 from [85], so we omit the proof.

Step 2: Our next aim is to show the non-negativity of the species concentrations under assumption that the solution to (2.15–2.17) is sufficiently smooth, i.e. $\varrho, \mathbf{u}$ and $Y_k \in W^{2,p}(\Omega)$, for any $p < \infty$, $k \in S$, $\sum_{k \in S} Y_k = 1$ and $\varrho \geq 0$.

It will follow directly from the features of the species production terms provided we justify that the set $\Omega_k^- = \{x \in \Omega : Y_k(x) < 0\}$ is sufficiently regular to integrate by parts. Assuming for a moment that this is the case, we integrate each of equations of (2.17) over Ω_k^-, and by the smoothness of Y_k we obtain

$$\varepsilon \int_{\Omega_k^-} \varrho Y_k \, \mathrm{d}x + \int_{\partial\Omega_k^-} \varrho Y_k \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\sigma - \int_{\partial\Omega_k^-} (D_\eta \nabla Y_k + \varepsilon \nabla(\varrho Y_k)) \cdot \mathbf{n} \, \mathrm{d}\sigma$$
$$= \int_{\Omega_k^-} \varrho \omega_k \, \mathrm{d}x + \varepsilon \int_{\Omega_k^-} hK(Y_k^+) \, \mathrm{d}x. \quad (2.19)$$

The same can be done for the equation of species D, for which $\omega_D = 0$ and we obtain "$\geq$" instead of "$=$".
The second integral on the l.h.s. vanishes due to the boundary conditions on $\partial\Omega$ and on $\partial\Omega_k^-$, the third one is non-positive since $\partial_{\mathbf{n}} Y_k \geq 0$ on $\partial\Omega_k^-$, $\partial_{\mathbf{n}} \varrho = 0$ on $\partial\Omega$ and $\mathbf{F}_k \cdot \mathbf{n} = 0$ on $\partial\Omega$. Finally, the first integral on the r.h.s. is non-negative due to assumptions imposed on ω_k (2.12), thus we end up with

$$\int_{\Omega_k^-} \varrho Y_k \, \mathrm{d}x \geq \int_{\Omega_k^-} hK(Y_k^+) \, \mathrm{d}x.$$

Since the r.h.s. is non-negative and the l.h.s is non-positive, the inequality may hold only in the case when $\int_{\Omega_k^-} \varrho Y_k \, \mathrm{d}x = 0$, thus $\varrho Y_k \geq 0$, in particular $Y_k \geq 0$, $k \in S$ for $\varrho > 0$.

Let us now focus on the case when $\varrho = 0$, we denote

$$\Omega_k^0 = \{x \in \Omega : (\varrho(x) = 0) \cap (Y_k(x) < 0)\}.$$

Observe that by the fact that $\int_\Omega \varrho \, \mathrm{d}x = M > 0$, provided Ω_k^0 has sufficiently smooth boundary, the set $\partial\Omega_k^0 \setminus \partial\Omega$ is nonempty, moreover, it has a non-zero measure.
Next, multiplying each of equations of (2.17) by Y_k and integrating by parts over Ω_k^0 we obtain

$$\|\sqrt{D_\eta(0)} \nabla Y_k\|_{L^2(\Omega_k^0)} = 0, \quad (2.20)$$

which suggests that Y_k is constant on Ω_k^0, but due to the zero boundary condition $Y_k = 0$ on Ω_k^0, $k \in \{A, B, C\}$. For the last species, similar reasoning leads to

$$\|\sqrt{D_\eta(0)} \nabla Y_D\|_{L^2(\Omega_D^0)} = \varepsilon h \int_{\Omega_D^0} Y_D \, \mathrm{d}x \leq 0, \quad (2.21)$$

thus also $Y_D \geq 0$ a.e. in Ω.

Let us now come back to the issue of regularity of $\partial\Omega_k^-$; the proof for $\partial\Omega_k^0$ will follow the same way. What we actually claim is that the set Ω_k^- may be approximated by sufficiently smooth sets $\Omega_{k,\delta_n}^- = \{x \in \Omega : Y_k(x) < \delta_n\}$ for $\delta_n > 0$, such that all the integrations by parts from the preceding step can be executed for Ω_{k,δ_n}^- and we obtain the final result by letting $\delta_n \to 0^+$.

The existence of such sets can be deduced from the following variant of the Sard theorem for the maps which are differentiable in the Sobolev sense, whose prove can be found e.g. in [22], see also [39].

Theorem 2.6. *Let Ω be an open set in $\mathbb{R}^n$ and let $u \in W^{k,p}(\Omega)$ with values in $\mathbb{R}^m$, where n, m are two integers such that $n > m$. If $k = n - m + 1$ and $p > n$, then the set of critical values of u, namely the image according to u of the critical set*

$$S = \{x \in \mathbb{R}^n : \ \mathrm{rank}(\nabla u(x)) < m\},$$

has zero m-measure.

The rough idea of the proof is first to choose sufficiently smooth representative of u that coincides with u (together with all derivatives) on the set F_ε, such that the measure of $\Omega \setminus F_\varepsilon$ is less then ε, then to apply the classical Sard theorem [92], and to let ε to zero. Therefore, similarly to the case of continuously differentiable functions, we may use the Implicit Function Theorem to infer that for a.e. $y \in \mathbb{R}^m$, the connected components of the level set $E_y = u^{-1}(y)$ are at most $n - m$ dimensional and can be locally parametrized by a $W^{k,p}$ function.

In our case, the structure of system (2.15)-(2.17) implies that, provided $\mathbf{u} \in W^{2,p}(\Omega)$, the regularity of ϱ can be improved up to $W^{3,p}(\Omega)$ and the same holds for Y_k. Hence, Theorem 2.6 can be used to prove that there exists a sequence $\delta_n \in \mathbb{R}$ convergent to zero, such that these parts of $\partial\Omega^-_{k,\delta_n}$ which do not adhere to $\partial\Omega$ belong to the regularity class $W^{2,p}$. In particular, Ω^-_{k,δ_n} is a Lipschitz domain.

Step 3: We now prove the existence of solutions to the species mass balance equations for $\mathbf{u}$, ϱ given. The main idea consists on applying the Leray-Schauder fixed point theorem to the mapping

$$\mathcal{T} : W^{2,p} \to W^{2,p}, \quad \mathcal{T}(\tilde{Y}_k) \to Y_k,$$

where Y_k is a solution to the boundary-value problem

$$\begin{gathered} -\,\mathrm{div}((D_\eta + \varepsilon\varrho)\nabla Y_k) = \varrho\omega_k(\tilde{Y}) + \varepsilon h K(\tilde{Y}^+) - \varepsilon\varrho\tilde{Y}_k - \mathrm{div}\left(\varrho\mathbf{u}\tilde{Y}_k\right) + \varepsilon\,\mathrm{div}(\nabla\varrho\tilde{Y}_k), \\ \nabla Y_k \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{gathered} \tag{2.22}$$

for $k \in \{A, B, C\}$, where we denoted

$$K(\tilde{Y}^+) = \begin{cases} \dfrac{\int_\Omega \tilde{Y}_k^+ \,\mathrm{d}x}{\sum_{i\in S}\int_\Omega \tilde{Y}_i^+ \,\mathrm{d}x} & \text{for } \varrho \neq 0 \\ 0 & \text{for } \varrho = 0 \end{cases} \quad \text{and} \quad \tilde{Y}_k^+ = \begin{cases} 0 & \text{if } \quad \tilde{Y}_k < 0 \\ \tilde{Y}_k & \text{if } 0 \le \tilde{Y}_k < 1 \\ 1 & \text{if } \quad 1 \le \tilde{Y}_k \end{cases}.$$

We have the following lemma.

Lemma 2.7. *Let assumptions of Theorem 2.4 be fulfilled and let $\mathbf{u} \in M^\infty$ and ϱ be given by Lemma 2.5. Then, the operator $\mathcal{T}$ is continuous and compact from $W^{2,p}(\Omega)$ into itself, such that the set*

$$\{Y_k \in W^{2,p}(\Omega) : Y_k = t\mathcal{T}Y_k \ \textit{for some}\ t \in [0,1]\}$$

is bounded.

Proof. The existence and uniqueness of solution to the system (2.22) is a consequence of Lax-Milgram theorem. Evidently, the mapping $\mathcal{T}$ is compact, since the r.h.s of (2.22) is sufficiently smooth and of lower order, it is also continuous.
To conclude, we should show boundedness of possible fixed points to

$$t\mathcal{T}(Y_k) = Y_k, \quad t \in [0,1].$$

The above equality rewrites as

$$
\begin{aligned}
-\operatorname{div}((D_\eta+\varepsilon\varrho)\nabla Y_k) = t\left(\varrho\omega_k+\varepsilon hK(Y_k^+)-\varepsilon\varrho Y_k-\operatorname{div}\left(\varrho\mathbf{u}Y_k\right)+\varepsilon\operatorname{div}(\nabla\varrho Y_k)\right),\\
\nabla Y_k\cdot\mathbf{n}|_{\partial\Omega}=0.
\end{aligned}
\tag{2.23}
$$

Multiplying the first equation by Y_k integrating by parts and using the boundary conditions we get

$$
\begin{aligned}
&\int_\Omega (D_\eta+\varepsilon\varrho)|\nabla Y_k|^2\,\mathrm{d}x\\
&=t\varepsilon\int_\Omega\left(hK(Y_k^+)Y_k-\varrho Y_k^2\right)\,\mathrm{d}x-\frac{t\varepsilon}{2}\int_\Omega\nabla\varrho\cdot\nabla Y_k^2\,\mathrm{d}x-\frac{t}{2}\int_\Omega\operatorname{div}(\varrho\mathbf{u})Y_k^2\,\mathrm{d}x+t\int_\Omega\varrho\omega_kY_k\,\mathrm{d}x.
\end{aligned}
$$

By the approximate continuity equation we obtain

$$
\int_\Omega (D_\eta+\varepsilon\varrho)|\nabla Y_k|^2\,\mathrm{d}x+t\varepsilon\int_\Omega\left(h\frac{Y_k^2}{2}+\varrho\frac{Y_k^2}{2}\right)\,\mathrm{d}x=t\int_\Omega\varrho\omega_kY_k\,\mathrm{d}x+t\varepsilon\int_\Omega hK(Y_k^+)Y_k\,\mathrm{d}x.
$$

Thus, using the Cauchy inequality and boundedness of ω_k, we show

$$
\|Y_k\|_{W^{1,2}(\Omega)}\leq c, \tag{2.24}
$$

with a costant c independent of t. Finally, we may estimate the norm of second gradient of Y_k directly from (2.23), we have

$$
\begin{aligned}
&-(D_\eta+\varepsilon\varrho)\Delta Y_k\\
&\qquad=-t\left(\varrho\omega_k+\varepsilon hK(Y_k^+)-\varepsilon\varrho Y_k-\operatorname{div}\left(\varrho\mathbf{u}Y_k\right)+\varepsilon\operatorname{div}(\nabla\varrho Y_k\right)-\nabla(D_\eta+\varepsilon\varrho)\cdot\nabla Y_k.
\end{aligned}
$$

Due to regularity of $\varrho,\mathbf{u}$ and estimate (2.24), we first justify that $Y_k\in W^{2,2}(\Omega)\hookrightarrow W^{1,6}(\Omega)\hookrightarrow L^\infty(\Omega)$. Then, by the bootstrap procedure, we arrive at $\|Y_k\|_{W^{2,p}(\Omega)}\leq c$. $\Box$

Step 4: Having prepared the necessary information we are ready to proceed with the proof of existence of regular solutions to the approximate momentum equation. We will use the Leray-Schauder fixed point theorem for the operator

$$
\mathcal{T}:M^\infty\to M^\infty, \tag{2.25}
$$

such that $\mathbf{v}=\mathcal{T}(\mathbf{u})$ is a solution of the problem

$$
\begin{aligned}
-\operatorname{div}\mathbf{S}(\mathbf{v})=-\tfrac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u}-\tfrac{1}{2}\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\nabla\pi(\varrho,Y)+\varrho\mathbf{f},\\
\varrho=S(\mathbf{u}),\\
\mathbf{v}|_{\partial\Omega}=\mathbf{0}.
\end{aligned}
\tag{2.26}
$$

Again, the existence of unique solution to this system can be shown by the direct application of the Lax-Milgram theorem. Regarding compactness and continuity of operator $\mathcal{T}$, the only difference with respect to the situation studied in [85], Chapter 4.7, is the presence of additional term in the pressure. But, by Lemma 2.7 one can see that the right hand side is still sufficiently smooth and bounded in $L^p(\Omega)$ for $1<p<\infty$ in order to estimate the norm of solution in $W^{2,p}(\Omega)$. The last information needed to verify the hypothesis of the Leray-Schauder fixed point theorem is the boundedness of possible fixed points to

$$
t\mathcal{T}(\mathbf{u})=\mathbf{u},\quad t\in[0,1] \tag{2.27}
$$

which will be derived from the first *a priori* estimate.

Lemma 2.8. *Let assumptions of Theorem 2.4 be fulfilled. Let $t \in [0,1]$, $\mathbf{u} \in M^\infty$ be a fixed point $\mathbf{u} = t\mathcal{T}(\mathbf{u})$. Then there exists a constant $c > 0$ independent of $t \in [0,1]$, such that*

$$\|\mathbf{u}\|_{1,2} \le c. \tag{2.28}$$

Proof. Taking as a test function in (2.27) the solution $\mathbf{u}$ which satisfies $\mathcal{S}(\mathbf{u}) = \varrho$ one gets

$$\int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x = -t \int_\Omega \nabla \pi(\varrho, Y) \cdot \mathbf{u} \, \mathrm{d}x + t \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x.$$

Next, we use (2.15) to get

$$\int_\Omega \nabla \varrho^\gamma \cdot \mathbf{u} \, \mathrm{d}x = \varepsilon\gamma \int_\Omega \varrho^{\gamma-2} |\nabla \varrho|^2 \, \mathrm{d}x + \varepsilon \frac{\gamma}{\gamma - 1} \int_\Omega \varrho^\gamma \, \mathrm{d}x - \varepsilon \frac{\gamma}{\gamma-1} \int_\Omega h \varrho^{\gamma-1} \, \mathrm{d}x,$$

thus we have

$$\begin{aligned} \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x + t\varepsilon\gamma \int_\Omega \varrho^{\gamma-2} |\nabla \varrho|^2 \, \mathrm{d}x + t\varepsilon \frac{\gamma}{\gamma-1} \int_\Omega \varrho^\gamma \, \mathrm{d}x \\ = t\varepsilon \frac{\gamma}{\gamma-1} \int_\Omega h\varrho^{\gamma-1} \, \mathrm{d}x + tR \int_\Omega \left(\sum_{k \in S} \frac{Y_k}{m_k} \right) \varrho \operatorname{div} \mathbf{u} \, \mathrm{d}x + t \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x. \end{aligned} \tag{2.29}$$

We now want to show that the first term on the l.h.s. can be used to control the norm of $\mathbf{u}$ in $W_0^{1,2}(\Omega)$, to this purpose we prove a simple generalization of the Korn inequality, which can be of independent interest.

Lemma 2.9. *For $\mathbf{u} \in W_0^{1,2}(\Omega)$ and $\mathbf{S}$ satisfying* (1.15) *and* (2.8)*, there exists a constant c depending on Ω and μ such that*

$$c\|\mathbf{u}\|^2_{W^{1,2}(\Omega)} \le \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x.$$

Proof. Rewriting the viscous part of the stress tensor in the form (1.17), we can estimate

$$\int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \ge \mu \int_\Omega \left(|\nabla \mathbf{u}|^2 + (\nabla \mathbf{u})^T : \nabla \mathbf{u} - \frac{2}{3} (\operatorname{div} \mathbf{u})^2 \right) \mathrm{d}x = \mu \int_\Omega \left(|\nabla \mathbf{u}|^2 + \frac{1}{3} (\operatorname{div} \mathbf{u})^2 \right) \mathrm{d}x,$$

and we conclude by application of the Poincaré inequality. □

Comming back to (2.29), we infer

$$\begin{aligned} c\|\mathbf{u}\|^2_{1,2} + t\varepsilon\gamma \int_\Omega \varrho^{\gamma-2} |\nabla \varrho|^2 \, \mathrm{d}x + t \frac{\varepsilon\gamma}{\gamma-1} \int_\Omega \varrho^\gamma \, \mathrm{d}x \\ \le t \frac{\gamma h \varepsilon}{\gamma - 1} \int_\Omega \varrho^{\gamma-1} \, \mathrm{d}x + tR \int_\Omega \left(\sum_{k \in S} \frac{Y_k}{m_k} \right) \varrho \operatorname{div} \mathbf{u} \, \mathrm{d}x + t \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x. \end{aligned} \tag{2.30}$$

The first term on the r.h.s. can be absorbed by the corresponding one on the l.h.s. Next, the Hölder and Young inequalities yield

$$t \int_\Omega \frac{Y_k}{m_k} \varrho \operatorname{div} \mathbf{u} \, \mathrm{d}x + t \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \le \frac{t}{m_k} \|\varrho\|_2 \|\mathbf{u}\|_{1,2} + t\|\mathbf{f}\|_\infty \|\varrho\|_{6/5} \|\mathbf{u}\|_{1,2}. \tag{2.31}$$

In order to control the norm of ϱ in $L^2(\Omega)$ we test the approximate momentum equation (2.27) by the function

$$\Phi = \mathcal{B} \left(\pi^\beta - \frac{1}{|\Omega|} \int_\Omega \pi^\beta \, \mathrm{d}x \right),$$

where $\beta \in (0,1]$ and $\mathcal{B}$ is the Bogovskii operator. Its definition and main properties are recalled in Lemma 6.4, we know in particular that

$$\|\nabla\Phi\|_p \le c(p,\Omega)\|\pi^\beta\|_p$$

and due to the Sobolev imbedding

$$\|\Phi\|_{\bar p} \le c(p,\Omega)\|\pi^\beta\|_p, \quad 1<p<\infty, \qquad \bar p = \begin{cases} \frac{3p}{3-p} & \text{if } p<3, \\ \in [1,\infty) & \text{if } p=3, \\ \infty & \text{if } p>3. \end{cases}$$

This testing results in the following identity

$$\begin{aligned} t\int_\Omega \pi^{1+\beta}\,\mathrm{d}x = &-t\frac{1}{2}\int_\Omega \varrho\,(\mathbf{u}\otimes\mathbf{u}) : \nabla\Phi\,\mathrm{d}x + t\frac{1}{2}\int_\Omega \varrho\,(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Phi\,\mathrm{d}x \\ &+ \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla\Phi\,\mathrm{d}x - t\int_\Omega \varrho\mathbf{f}\cdot\Phi\,\mathrm{d}x + t\frac{1}{|\Omega|}\int_\Omega \pi\,\mathrm{d}x\int_\Omega \pi^\beta\,\mathrm{d}x = \sum_{i=1}^5 I_i. \end{aligned} \tag{2.32}$$

The "worst" estimate here is due to the convective term:

$$I_1+I_2 \le t\|\varrho\|_{(1+\beta)\gamma}\|\mathbf{u}\|_{1,2}^2\|\pi\|^\beta_{\frac{3\gamma\beta(1+\beta)}{2\gamma(1+\beta)-3}} \le ct^3\|\varrho\|_{(1+\beta)\gamma}\|\varrho\|_2^2\|\pi\|^\beta_{\frac{3\gamma\beta(1+\beta)}{2\gamma(1+\beta)-3}} \tag{2.33}$$

where in the last inequality we used (2.30) and (2.31) to estimate the $W^{1,2}$ norm of $\mathbf{u}$ by the L^2 norm of ϱ. Next, provided $\frac{3\gamma\beta(1+\beta)}{2\gamma(1+\beta)-3} \le \beta+1$ meaning that

$$\beta = \begin{cases} \frac{2\gamma-3}{\gamma} & \text{if } \gamma<3 \\ 1 & \text{if } \gamma\ge 3, \end{cases} \tag{2.34}$$

we can transform (2.33) ino the following form:

$$I_1+I_2 \le ct^3\|\varrho\|^{1+2a}_{(1+\beta)\gamma}\|\pi\|^\beta_{\beta+1}, \tag{2.35}$$

where we additionally used the interpolation inequality $\|\varrho\|_2 \le \|\varrho\|_1^{1-a}\|\varrho\|^a_{(\beta+1)\gamma}$, with $a = \frac{(\beta+1)\gamma}{2(\beta+1)\gamma-2}$. Hence, from (2.32) we deduce in particular that independently of $t\in[0,1]$ and ε we have

$$\|\varrho\|_{(1+\beta)\gamma} \le c, \tag{2.36}$$

if only $1+2a+\beta\gamma < \gamma(1+\beta)$. Therefore, by virtue of (2.34), one can see that the relevant condition on γ is $\gamma > \frac{7}{3}$.
Estimate (2.36), along with (2.30), leads to the following conclusion

$$\|\mathbf{u}\|^2_{1,2} + t\varepsilon\left(\|\varrho\|^\gamma_\gamma + \|\nabla\varrho^{\frac{\gamma}{2}}\|_2^2\right) \le ct, \tag{2.37}$$

that finishes the proof of Lemma 2.8. □
This information allows us to repeat the procedure described in Chapter 4.3 of [85] which together with the Lemmas 2.5, 2.7 yields the existence of regular solutions, and hence completes the proof of Theorem 2.4. □

2.2.3 Limit passage $\eta, \varepsilon \to 0$

Although the construction from previous section corresponds only to regularized system, it is clear that the final estimates are completely independent of ε. In particular, taking $t = 1$ in (2.37) we get

$$\|\mathbf{u}\|_{1,2}^2 + \varepsilon \left(\|\varrho\|_\gamma^\gamma + \|\nabla \varrho^{\frac{\gamma}{2}}\|_2^2 \right) \leq c,$$

in addition, the Bogovskii type of estimates gives

$$\|\pi(\varrho, Y)\|_{1+\beta} \leq c,$$

so, we can repeat the first a-priori estimate for ∇Y_k and since $0 \leq Y_k \leq 1$, $k \in S$, we have

$$\|Y_k\|_{W^{1,2}(\Omega)} \leq c$$

independently of ε. Moreover, taking ϱ as a test function in the approximate continuity equation one gets

$$\sqrt{\varepsilon}\|\nabla \varrho\|_2 \leq c.$$

These estimates can be used to deduce that, at least for a suitable subsequence, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } W^{1,2}(\Omega), \tag{2.38}$$

$$\varrho_\varepsilon \to \varrho \quad \text{weakly in } L^{(1+\beta)\gamma}(\Omega), \tag{2.39}$$

$$\varepsilon \nabla \varrho_\varepsilon \to 0 \quad \text{strongly in } L^2(\Omega), \tag{2.40}$$

$$(Y_k)_\varepsilon \to Y_k \quad \text{weakly in } W^{1,2}(\Omega), \tag{2.41}$$

$$(Y_k)_\varepsilon \to Y_k \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{2.42}$$

We are hence in a position to conclude that there exists a triple of functions $(\varrho, \mathbf{u}, Y)$ which satisfy the integral equalities:

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \xi \, \mathrm{d}x = 0 \quad \forall \xi \in C^\infty(\overline{\Omega}), \tag{2.43}$$

$$\int_\Omega \left(-\varrho \left(\mathbf{u} \otimes \mathbf{u} \right) : \nabla \varphi + \mathbf{S}(\mathbf{u}) : \nabla \varphi \right) \, \mathrm{d}x - \int_\Omega \overline{\pi(\varrho, Y)} \operatorname{div} \varphi \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega),$$

$$\int_\Omega \varrho \mathbf{u} Y_k \nabla \cdot \phi \, \mathrm{d}x = \int_\Omega \overline{D(\varrho) \nabla Y_k} \cdot \nabla \phi \, \mathrm{d}x - \int_\Omega \varrho \omega_k \phi \, \mathrm{d}x \quad \forall \phi \in C^\infty(\overline{\Omega}),$$

for $k \in \{A, B, C\}$. Here and in the sequel $\overline{g(\varrho, \mathbf{u}, Y)}$ denotes the weak limit of a sequence $g(\varrho_\varepsilon, \mathbf{u}_\varepsilon, Y_\varepsilon)$.
Accordingly, there left two problems that need to be solved, namely, is it true that $\overline{\pi(\varrho, Y)} = \pi(\varrho, Y)$ and is $\overline{D(\varrho)\nabla Y_k} = D(\varrho)\nabla Y_k$? As we already have an information about strong convergence of Y_k for $k \in S$, the positive answer for the first question is, as will be seen in the sequel, in fact equivalent to the strong convergence of the density. Moreover, having proved this, it will be straightforward to see that the second hypothesis holds true as well. It is an easy consequence of boundedness of $D(\varrho)$ in $L^2(\Omega)$. The aim of the following reasoning is to derive the weak compactness identity for the effective pressure, which is the key point of proving the strong convergence of the density.

Since $\varrho_\varepsilon \mathbf{u}_\varepsilon$ and $\nabla \varrho_\varepsilon$ possess zero normal traces, it is possible to extend the approximate continuity equation to the whole $\mathbb{R}^3$

$$\varepsilon 1_\Omega \varrho_\varepsilon + \operatorname{div}(1_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \operatorname{div}(1_\Omega \nabla \varrho_\varepsilon) + \varepsilon 1_\Omega h. \tag{2.44}$$

Next, we introduce the operators $\mathcal{A} = \nabla\Delta^{-1}$, $\mathcal{R} = \nabla \otimes \nabla\Delta^{-1}$ specified by (6.2) and (6.3) in the Appendix. We will use some general results on such operators as continuity but also some facts about commutators, see Lemmas 6.5, 6.6 and 6.7.

First, we test the approximate momentum equation by the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla\Delta^{-1})[1_\Omega\varrho_\varepsilon],\ \zeta \in C_0^\infty(\Omega),$$

observe that this operation "gains" one derivative thus using only the $L^{(1+\beta)\gamma}(\Omega)$ integrability of ϱ_ε we justify that this is an admissible test function. Evidently $\sum_{i=1}^3 \mathcal{R}_{i,i}[v] = v$, thus integrating by parts we obtain from this testing the following equivalence:

$$\begin{aligned}
&\int_\Omega \zeta\left(\pi(\varrho_\varepsilon, Y_\varepsilon)\varrho_\varepsilon - \mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega\varrho_\varepsilon]\right)\,\mathrm{d}x \\
&= -\frac{1}{2}\int_\Omega \zeta\partial_j(\varrho u_j)u_i\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \zeta\varrho_\varepsilon u_i u_j \mathcal{R}_{i,j}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon u_i u_j \partial_j\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad + \int_\Omega S_{i,j}\partial_j\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \pi(\varrho_\varepsilon, Y_\varepsilon)\partial_i\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon\zeta f^i\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x,
\end{aligned} \tag{2.45}$$

where we used the Einstein summation convention. Now, adding and substracting the term $\int_\Omega \zeta\varrho_\varepsilon u_j\mathcal{R}_{i,j}[1_\Omega\varrho_\varepsilon u_i]\,\mathrm{d}x$ we may rewrite the r.h.s. in the form which lets the commutator appear

$$\begin{aligned}
\text{r.h.s} &= \int_\Omega \zeta\varrho_\varepsilon u_j\mathcal{R}_{i,j}[1_\Omega\varrho_\varepsilon u_i]\,\mathrm{d}x - \int_\Omega \zeta\varrho_\varepsilon u_i u_j\mathcal{R}_{i,j}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad - \int_\Omega \zeta\varrho_\varepsilon u_j\mathcal{R}_{i,j}[1_\Omega\varrho_\varepsilon u_i]\,\mathrm{d}x - \frac{1}{2}\int_\Omega \zeta\partial_j(\varrho u_j)u_i\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon u_i u_j\partial_j\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad + \int_\Omega S_{i,j}\partial_j\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \pi(\varrho_\varepsilon, Y_\varepsilon)\partial_i\zeta\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon\zeta f^i\mathcal{A}_i[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x.
\end{aligned}$$

Using the fact that $R_{i,j}[v] = \partial_i\mathcal{A}_j[v]$, the basic properties of the Riesz operator:

$$\mathcal{R}_{i,j}[v] = \mathcal{R}_{j,i}[v], \quad \int_{\mathbb{R}^3} \mathcal{R}_{i,j}[v]u\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathcal{R}_{j,i}[v]u\,\mathrm{d}x, \quad v \in L^p(\mathbb{R}^3),\ u \in L^{p'}(\mathbb{R}^3)$$

and the approximate continuity equaion, we transform (2.45) into

$$\begin{aligned}
&\int_\Omega \zeta\left(\pi(\varrho_\varepsilon, Y_\varepsilon)\varrho_\varepsilon - \mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega\varrho_\varepsilon]\right)\,\mathrm{d}x \\
&= \int_\Omega \zeta\left(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}[1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon] - \varrho_\varepsilon(\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega\varrho_\varepsilon]\right)\,\mathrm{d}x \\
&\quad - \int_\Omega \zeta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\Delta^{-1}[\operatorname{div} 1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon]\,\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \zeta(\nabla\varrho_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad + \frac{\varepsilon}{2}\int_\Omega (\nabla\varrho_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\left(\zeta\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\right)\,\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \zeta(\varrho_\varepsilon - h)\mathbf{u}_\varepsilon\cdot\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad - \int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}_\varepsilon) : \nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x \\
&\quad - \int_\Omega \pi(\varrho_\varepsilon, Y_\varepsilon)\nabla\zeta\otimes\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\varrho_\varepsilon\zeta\nabla\Delta^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x = \sum_{i=1}^9 I_i.
\end{aligned} \tag{2.46}$$

Finally, I_2 may be expressed by means of approximate (extended) continuity equation (2.44) in the following way

$$I_2 = -\varepsilon \int_\Omega \zeta \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta^{-1}[\operatorname{div} 1_\Omega \nabla \varrho_\varepsilon] \, \mathrm{d}x + \varepsilon \int_\Omega \zeta \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta^{-1}[1_\Omega(\varrho_\varepsilon - h)] \, \mathrm{d}x. \tag{2.47}$$

We will compare (2.46) with a similar expression obtained by testing the limit momentum equation with the function

$$\varphi(x) = \zeta(x)\phi, \quad \phi = (\nabla \Delta^{-1})[1_\Omega \varrho], \ \zeta \in C_0^\infty(\Omega),$$

more precisely

$$\begin{aligned} \int_\Omega \zeta \left(\overline{\pi(\varrho, Y)} \varrho - \mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \, \mathrm{d}x &= \int_\Omega \zeta \left(\varrho \mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \varrho(\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \, \mathrm{d}x \\ &- \int_\Omega \varrho_\varepsilon (\mathbf{u} \otimes \mathbf{u}) : \nabla \zeta \otimes \nabla \Delta^{-1}[1_\Omega \varrho] \, \mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla \zeta \nabla \Delta^{-1}[1_\Omega \varrho] \, \mathrm{d}x \\ &- \int_\Omega \overline{\pi(\varrho, Y)} \nabla \zeta \otimes \nabla \Delta^{-1}[1_\Omega \varrho] \, \mathrm{d}x - \int_\Omega \mathbf{f} \varrho \zeta \nabla \Delta^{-1}[1_\Omega \varrho] \, \mathrm{d}x = \sum_{i=1}^5 I_i. \end{aligned} \tag{2.48}$$

Now, observe that

$$(\nabla \Delta^{-1})[1_\Omega \varrho_\varepsilon] \to (\nabla \Delta^{-1})[1_\Omega \varrho] \quad \text{in } C(\overline{\Omega}), \tag{2.49}$$

which is the consequence of Lemma 6.5. Recalling (2.38-2.41) we show that the ε-dependent integrals on the right hand side of (2.46) vanish, whence I_6, I_7, I_8, I_9 converge to their counterparts in (2.48).

In what follows we give some more details of these limit passages. Firstly, due to the compact imbeddung $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \le p < 6$, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{strongly in } L^p(\Omega), \quad 1 \le p < 6, \tag{2.50}$$

taking into account also (2.39) we therefore get

$$\mathbf{u}_\varepsilon \varrho_\varepsilon \to \mathbf{u} \varrho \quad \text{weakly in } L^p(\Omega), \quad 1 \le p < \frac{6(1+\beta)\gamma}{6 + (1+\beta)\gamma}. \tag{2.51}$$

Since $\frac{6(1+\beta)\gamma}{6+(1+\beta)\gamma} > 2$ for $\gamma > \frac{7}{3}$ and by virtue of (2.40) and Lemma 6.5, we check that the most restrictive term of I_2 (2.47) goes to 0, i.e.

$$\varepsilon \int_\Omega \zeta \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta^{-1}[\operatorname{div} 1_\Omega \nabla \varrho_\varepsilon] \, \mathrm{d}x \to 0.$$

The second integral in (2.47) is evidently convergent to 0 on account of (2.51) and (2.49), the same argument works also for I_5.

Next, due to (2.38) and (2.40)

$$\varepsilon (\nabla \varrho_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \to 0 \quad \text{weakly in } L^1(\Omega),$$

which, when coupled with (2.49), implies the zero limit of I_3.

Similarly, observe, that (2.40) together with (2.50) yields

$$\varepsilon \nabla \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon \to 0 \quad \text{strongly in } L^p(\Omega), \quad 1 \le p < \frac{3}{2}.$$

Moreover, using the second inequality from Lemma 6.5 and the fact that ζ is smooth, we show weak convergence of the second component of the integrant in I_4 in the same space as (2.40), thus $I_4 \to 0$.
In order to show convergence of I_6, I_7, I_8 to the corresponding integrals in the formula (2.48) it suffices to justify the weak convergence of $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$, $\mathbf{S}(\mathbf{u}_\varepsilon)$, $\pi(\varrho_\varepsilon, Y_\varepsilon)$ to $\varrho \mathbf{u} \otimes \mathbf{u}$, $\mathbf{S}(\mathbf{u})$, $\overline{\pi(\varrho, Y)}$ respectively, in $L^p(\Omega)$ for any $p \geq 1$; the passage to the limit in I_9 is trivial. Observe, that due to (2.38) and (2.51) we have

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^p(\Omega), \quad 1 \leq p < \frac{6(1+\beta)\gamma}{6 + 2(1+\beta)\gamma},$$

and $p > 1$ for $\gamma > \frac{7}{3}$. Next, in view of (2.38) we deduce that

$$\mathbf{D}(\mathbf{u}_\varepsilon) \to \mathbf{D}(\mathbf{u}), \quad \operatorname{div} \mathbf{u}_\varepsilon \to \operatorname{div} \mathbf{u} \quad \text{weakly in } L^2(\Omega),$$

thus also $\mathbf{S}(\mathbf{u}_\varepsilon) \to \mathbf{S}(\mathbf{u})$ weakly in $L^2(\Omega)$. Finally, using (2.39), (2.41), or directly from the Bogovskii estimate, we are able to extract the subsequence such that

$$\pi(\varrho_\varepsilon, Y) \to \overline{\pi(\varrho, Y)} \quad \text{weakly in } L^{1+\beta}(\Omega).$$

Summarizing, by letting ε to 0 in (2.46) and substracting from it (2.48) we deduce

$$\begin{aligned} &\int_\Omega \zeta \left(\pi(\varrho_\varepsilon, Y_\varepsilon)\varrho_\varepsilon - \mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x - \int_\Omega \zeta \left(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon] - \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x \\ \to &\int_\Omega \zeta \left(\overline{\pi(\varrho, Y)}\varrho - \mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \, \mathrm{d}x - \int_\Omega \zeta \left(\varrho \mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \varrho (\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \, \mathrm{d}x. \end{aligned} \tag{2.52}$$

Our next aim is to show that the last terms from both sides cancels. For this purpose we will apply Lemma 6.6.

We take $\mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$, $r_\varepsilon = \varrho_\varepsilon$ and check that they fulfill assumptions of Lemma 6.6 with $p = \frac{6(1+\beta)\gamma}{(1+\beta)\gamma+6}$ and $q = (1+\beta)\gamma$, where by $\varrho_\varepsilon, \mathbf{u}_\varepsilon, \varrho, \mathbf{u}$ we mean the functions extended by 0 outside Ω to the whole space. Thus there is enough room to choose $s > 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and so Lemma 6.6 yields

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{R}[1_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon] - \varrho_\varepsilon (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \to \varrho \mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \varrho (\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \varrho],$$

weakly in $L^s(\Omega)$. Substituting this result into (2.52) we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega \zeta \left(\pi(\varrho_\varepsilon, Y_\varepsilon)\varrho_\varepsilon - \mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \right) \, \mathrm{d}x = \int_\Omega \zeta \left(\overline{\pi(\varrho, Y)}\varrho - \mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \right) \, \mathrm{d}x. \tag{2.53}$$

We would now like to express $\mathbf{S}(\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon]$ and $\mathbf{S}(\mathbf{u}) : \mathcal{R}[1_\Omega \varrho]$ in terms of divergence of $\mathbf{u}_\varepsilon$ and $\mathbf{u}$, respectively. There is no problem with the second part of (1.15) as we have

$$\nu \operatorname{div} \mathbf{u}_\varepsilon \mathbb{I} : \mathcal{R}[1_\Omega \varrho_\varepsilon] = \nu \sum_{i=1}^{3} \operatorname{div} \mathbf{u}_\varepsilon \mathcal{R}_{i,i}[1_\Omega \varrho_\varepsilon] = \nu 1_\Omega \operatorname{div} \mathbf{u}_\varepsilon \varrho_\varepsilon.$$

To handle the first part, we integrate by parts and we check that

$$\int_\Omega \zeta \mu \left(\nabla \mathbf{u}_\varepsilon + (\nabla \mathbf{u}_\varepsilon)^T \right) : \mathcal{R}[1_\Omega \varrho_\varepsilon] \, \mathrm{d}x = \int_\Omega \mathcal{R} : [\zeta \mu \left(\nabla \mathbf{u}_\varepsilon + (\nabla \mathbf{u}_\varepsilon)^T \right)] \varrho_\varepsilon \, \mathrm{d}x. \tag{2.54}$$

Observe that $\mathcal{R} : \left[\nabla \mathbf{u}_\varepsilon + (\nabla \mathbf{u}_\varepsilon)^T\right] = 2\sum_{i,j=1}^3 \partial_i \mathcal{A}_j \partial_j u_{\varepsilon,i} = 2\sum_i^3 \partial_i \sum_{j=1}^3 \mathcal{R}_{j,j} u_i = 2\,\mathrm{div}\,\mathbf{u}_\varepsilon$, thus, the r.h.s. of (2.54) can be rewritten as

$$\int_\Omega \mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)]\varrho_\varepsilon \,\mathrm{d}x = \int_\Omega \zeta 2\mu \,\mathrm{div}\,\mathbf{u}_\varepsilon \varrho_\varepsilon \,\mathrm{d}x + \int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u}_\varepsilon)]\right) \varrho_\varepsilon \,\mathrm{d}x. \quad (2.55)$$

Repeating the same procedure for the limit stress tensor $\mathbf{S}(\mathbf{u})$ we obtain from (2.53) the following expression

$$\int_\Omega \zeta \left(\overline{\pi(\varrho, Y)\varrho} - (2\mu+\nu)\,\overline{\mathrm{div}\,\mathbf{u}\varrho}\right) \,\mathrm{d}x = \int_\Omega \zeta \left(\overline{\pi(\varrho, Y)}\varrho - (2\mu+\nu)\,\mathrm{div}\,\mathbf{u}\varrho\right) \,\mathrm{d}x$$
$$+ \lim_{\varepsilon\to 0} \int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u}_\varepsilon)]\right) \varrho_\varepsilon \,\mathrm{d}x - \int_\Omega \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u})] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u})]\right) \varrho \,\mathrm{d}x. \quad (2.56)$$

In order to show that the two last integrals cancel we will apply Lemma 6.7 to each row of the matrix $\mathbf{D}(\mathbf{u}_\varepsilon)$, i.e. we take

$$w = \zeta, \quad V_i = \partial_i u_{\varepsilon,j} + \partial_j u_{\varepsilon,i},\ j = 1,2,3$$

and due to (2.37) $\mathbf{V} \in L^2(\mathbb{R}^3)$. Since ζ extended by 0 outside Ω belongs in particular to $W^{1,\infty}(\mathbb{R}^3)$ we can take any $s \in (1,6)$ and $\alpha = \frac{6-s}{2s}$ for which

$$\|\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u}_\varepsilon)]\|_{W^{\alpha,s}(\mathbb{R}^3)} \le c.$$

Next, we may use the fact that $W^{\alpha,s}(\mathbb{R}^3)$ is continuously embedded into $L^a(\mathbb{R}^3)$ for any $1 \le a \le 6$ and the embedding is compact for $a < 6$. Moreover, since $\frac{1}{q} = \frac{1}{a} + \frac{1}{(1+\beta)\gamma} < 1$, thus

$$\left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u}_\varepsilon)] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u}_\varepsilon)]\right) \varrho_\varepsilon \to \left(\mathcal{R} : [\zeta 2\mu \mathbf{D}(\mathbf{u})] - \zeta \mathcal{R} : [2\mu \mathbf{D}(\mathbf{u})]\right) \varrho \quad (2.57)$$

weakly in $L^q(\mathbb{R}^3)$. Therefore, we may now reduce (2.56) to the following remarkable identity:

$$\int_\Omega \zeta \left(\overline{\pi(\varrho, Y)\varrho} - (2\mu+\nu)\,\overline{\mathrm{div}\,\mathbf{u}\varrho}\right) \,\mathrm{d}x = \int_\Omega \zeta \left(\overline{\pi(\varrho, Y)}\varrho - (2\mu+\nu)\,\mathrm{div}\,\mathbf{u}\varrho\right) \,\mathrm{d}x. \quad (2.58)$$

In what follows, we will exploit (2.58) by use of the renormalized continuity equation.
Applying Lemma 6.8 to the limit continuity equation we can verify that the pair of functions $(\varrho, \mathbf{u})$ extended by zero outside of Ω is a solution to the renormalized continuity equation, as specified in Definition 2.2. Moreover, taking $b(\varrho) = \varrho \log \varrho$ and $\xi = 1$ it can be deduced from (2.43) that

$$\int_\Omega \mathrm{div}\,\mathbf{u}\varrho \,\mathrm{d}x = 0.$$

Alternatively, one can derive the same relation applying Theorem 1.1 from [78], see also [72].

We now test the approximate continuity equation (2.15) with $\log(\varrho_\varepsilon + \eta)$, $\eta > 0$, note that due to Lemma 2.5 this is an admissible test function

$$\varepsilon \int_\Omega (\varrho_\varepsilon - h)\log(\varrho_\varepsilon + \eta) \,\mathrm{d}x - \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \frac{\nabla \varrho_\varepsilon}{\varrho_\varepsilon + \eta} \,\mathrm{d}x - \varepsilon \int_\Omega \Delta \varrho_\varepsilon \log(\varrho_\varepsilon + \eta) \,\mathrm{d}x = 0.$$

Integrating by parts in the last integral, and using the boundary conditions we obtain the following expression

$$\varepsilon \int_\Omega (\varrho_\varepsilon - h)\log(\varrho_\varepsilon + \eta)\ \mathrm{d}x - \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \frac{\nabla \varrho_\varepsilon}{\varrho_\varepsilon + \eta}\ \mathrm{d}x + \varepsilon \int_\Omega \frac{|\nabla \varrho_\varepsilon|^2}{\varrho_\varepsilon + \eta}\ \mathrm{d}x = 0,$$

but the last term is non-negative, so we have

$$\varepsilon \int_\Omega (\varrho_\varepsilon - h)\log(\varrho_\varepsilon + \eta)\ \mathrm{d}x - \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \frac{\nabla \varrho_\varepsilon}{\varrho_\varepsilon + \eta}\ \mathrm{d}x \le 0.$$

Next, let $\eta \to 0$. Due to regularity of $\varrho_\varepsilon, \mathbf{u}_\varepsilon$, the only problematic term is $-\int_\Omega h \log(\varrho_\varepsilon + \eta)\ \mathrm{d}x$ for $\varrho_\varepsilon < 1 - \eta$, but as the above inequality has the right sign, we can handle it by use of the Lebesgue monotone convergence theorem. Integrating by parts once more, we end up with

$$\varepsilon \int_\Omega (\varrho_\varepsilon - h)\log \varrho_\varepsilon\ \mathrm{d}x + \int_\Omega \operatorname{div} \mathbf{u}_\varepsilon \varrho_\varepsilon\ \mathrm{d}x \le 0. \tag{2.59}$$

Since $x \log x$ is a convex function it satisfies the following inequality

$$(1 + \log \varrho_\varepsilon)(\varrho_\varepsilon - h) \ge \varrho_\varepsilon \log \varrho_\varepsilon - h \log h$$

and by the fact that $\int_\Omega \varrho_\varepsilon\ \mathrm{d}x = \int_\Omega h\ \mathrm{d}x$ we derive from (2.59) that

$$\varepsilon \int_\Omega \varrho_\varepsilon \log \varrho_\varepsilon\ \mathrm{d}x - \varepsilon \int_\Omega h \log h\ \mathrm{d}x + \int_\Omega \operatorname{div} \mathbf{u}_\varepsilon \varrho_\varepsilon\ \mathrm{d}x \le 0,$$

so, after letting $\varepsilon \to 0$ we finally arrive at

$$\int_\Omega \overline{\operatorname{div} \mathbf{u} \varrho}\ \mathrm{d}x \le 0.$$

Because of this, identity (2.58) may be transformed into:

$$\int_\Omega \overline{\pi(\varrho, Y)}\varrho\ \mathrm{d}x \ge \int_\Omega \overline{\pi(\varrho, Y)\varrho}\ \mathrm{d}x,$$

and by the definition of π we thus have

$$\int_\Omega \left(\overline{\varrho^\gamma}\varrho + R\varrho \sum_{k\in S} \overline{\frac{Y_k}{m_k}\varrho} \right)\ \mathrm{d}x \ge \int_\Omega \left(\overline{\varrho^{\gamma+1}} + \overline{R\varrho \sum_{k\in S} \frac{Y_k}{m_k}\varrho} \right)\ \mathrm{d}x. \tag{2.60}$$

This inequality can be used to show strong convergence of density as soon as one justifies

$$\overline{\varrho^\gamma}\varrho \le \overline{\varrho^{\gamma+1}}, \qquad \varrho \sum_{k\in S} \overline{\frac{Y_k}{m_k}\varrho} \le \overline{\varrho \sum_{k\in S} \frac{Y_k}{m_k}\varrho}. \tag{2.61}$$

To do this we will use a well known result about weak convergence of monotone functions composed with weakly converging sequences, whose proof can be found e.g. in [36], Theorem 10.19.

Lemma 2.10 (Weak convergence of monotone functions). *Let Ω be a domain in $\mathbb{R}^N$ and $(P,G) \in C(\mathbb{R}) \times C(\mathbb{R})$ be a couple of non-decreasing function. Assume that u_n is a sequence of functions from $L^1(\Omega)$ with values in $\mathbb{R}$ such that*

$$\left.\begin{array}{r} P(u_n) \to \overline{P(u)}, \\ G(u_n) \to \overline{G(u)}, \\ P(u_n)G(u_n) \to \overline{P(u)G(u)} \end{array}\right\} \text{weakly in } L^1(\Omega).$$

(i) Then

$$\overline{P(u)}\ \overline{G(u)} \leq \overline{P(u)G(u)} \quad \text{a.e. in } \Omega.$$

(ii) If, in addition

$$G(z) = z, \quad P \in C(\mathbb{R}), \quad P \text{ is non-decreasing}$$

and

$$\overline{P(u)G(u)} = \overline{P(u)}\ \overline{G(u)},$$

then

$$\overline{P(u)} = P(u).$$

Applying this lemma to (2.61), we see that the first inequality is evidently true since $P(\varrho_\varepsilon) = \varrho_\varepsilon^\gamma$ and $G(\varrho_\varepsilon) = \varrho_\varepsilon$ are increasing. Regarding the second inequality, by the strong convergence of $(Y_k)_\varepsilon$ in $L^p(\Omega)$ for $p < 6$, we check that $\overline{Y_k\varrho} = Y_k\varrho$, thus $\varrho\overline{Y_k\varrho} = Y_k\varrho^2$, while the r.h.s. satisfies $\overline{\varrho Y_k\varrho} = Y_k\overline{\varrho^2} \geq Y_k\varrho^2$, where, we applied Lemma 2.10 with $P(\varrho_\varepsilon) = G(\varrho_\varepsilon) = \varrho_\varepsilon$. Hence, by comparison of (2.60) with (2.61) we obtain, using the statement (ii) of Lemma 2.10, that

$$\overline{\varrho^\gamma} = \varrho^\gamma \quad \text{a.e. in } \Omega.$$

This in turn implies the strong convergence of the density as $L^\gamma(\Omega)$ is a uniformly convex Banach space. The proof of Theorem 2.3 is now complete. □

2.3 A note on the non-diagonal diffusion

The Fick law is a relevant approximation for the diffusion flux of species, provided the molar masses of species do not differ much from the average mass, i.e. $\frac{p_k}{\pi_m} \approx \frac{\varrho_k}{\varrho}$. Thus, the species diffusion forces defined by (1.19) may be approximated by

$$\mathbf{d}_k \approx \nabla \frac{p_k}{\pi_m} \approx \nabla Y_k \quad \text{for } k \in \{A, B, C\}.$$

Recall that we used this approximation to express the diffusion forces only for species taking an active part in reaction (2.1), but since $\sum_{k\in S} \mathbf{d}_k = 0$ we get the same for the dilutant, i.e. $\mathbf{d}_D = \nabla Y_D$. Such a case corresponds to the diffusion matrix C that is diagonal for 3 species only, see (1.18), and in general, there is no reason to assume that the diffusion coefficients D_A, D_B, D_C coincide.

However, in agreement with Lemma 1.1, page 21, the species diffusion coefficients for substrates D_A and D_B must be the same, otherwise restriction $Y_AY_B = 0$ would exclude reaction (2.1). This in turn implies that we may formally reduce system (2.2) to the following one

$$\left.\begin{array}{r} \operatorname{div}(\varrho\mathbf{u}) = 0 \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla\pi = \varrho\mathbf{f} \\ \operatorname{div}(\varrho Y_{AB}\mathbf{u}) + \operatorname{div}(\mathbf{F}_{AB}) = \varrho\omega_{AB} \\ \operatorname{div}(\varrho Y_C\mathbf{u}) + \operatorname{div}(\mathbf{F}_C) = \varrho\omega_C \end{array}\right\} \quad \text{in } \Omega, \tag{2.62}$$

where we denoted $Y_{AB} = Y_A + Y_B$, $\omega_{AB} = \omega_A + \omega_B$, $\mathbf{F}_{AB} = -D\nabla Y_{AB}$ and $D = D_A = D_B$. If in addition $D \neq D_C$, then

$$Y_{AB}Y_C = 0,$$

and so, in the absence of dilutant $Y_D = 0$ we would have either $Y_{AB} = 0,\ Y_C = 1$ or $Y_{AB} = 1,\ Y_C = 0$. By this observation, the whole reasoning performed for a system with diagonal diffusion matrix may be now repeated with merely several minor changes. In particular, a kind of maximum principle, derived for the approximate system (2.17), does still hold for the last species. More precisely, we may again start from the following equality

$$\varepsilon \int_{\Omega_D^-} \varrho Y_D \,\mathrm{d}x + \int_{\partial\Omega_k^-} \varrho Y_D \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_{\partial\Omega_D^-} (D_{AB,\eta}\nabla Y_{AB} + D_{C,\eta}\nabla Y_C - \varepsilon\nabla(\varrho Y_D))\cdot\mathbf{n}\,\mathrm{d}\sigma$$
$$= \varepsilon \int_{\Omega_D^-} h(1 - K(Y_{AB}^+) - K(Y_C^+))\,\mathrm{d}x.$$

Recall that Y_{AB} and Y_C are smooth and at the boundary of Ω_D^- they are allowed to take only one of two values 1 or 0. Because the boundary of Ω_D^- is regular and $Y_{AB} = 1 - Y_C$ at this part of $\partial\Omega_D^-$ which does not adhere to $\partial\Omega$, we may assume, without loss of generality, that $Y_{AB} = 1$ and $Y_C = 0$ at $\partial\Omega_D^- \setminus \partial\Omega$. Since the smooth function does not permit the jumps and $Y_{AB}Y_C = 0$ in Ω, Y_C must remain equal to 0 in the small neighbourhood of $\partial\Omega_D^- \setminus \partial\Omega$. In consequence

$$D_{C,\eta}\nabla Y_C \cdot \mathbf{n}|_{\partial\Omega_D^-\setminus\partial\Omega} = 0,$$

moreover

$$D_\eta \nabla Y_{AB}\cdot\mathbf{n}|_{\partial\Omega_D^-\setminus\partial\Omega} = -D_\eta\nabla Y_D\cdot\mathbf{n}|_{\partial\Omega_D^-\setminus\partial\Omega} \leq 0,$$

which is sufficient to repeat the argument from the case of diagonal diffusion with equal diffusion coefficients, and therefore $Y_D \geq 0$.

Chapter 3

Two species kinetics

We investigate the system of equations describing flow of two-component compressible gaseous mixture in the periodic domain $\Omega = \mathbb{T}^3$. The species A and B undergo an isothermal, reversible chemical reaction

$$A \leftrightarrows B. \tag{3.1}$$

The above reaction can be formally obtained from (2.1) by assuming that the amounts of one of reactants from the l.h.s. and of dilutant are much bigger than of the other species. Then, their concentration may be assumed to be constant.
At first sight this situation seems to be simpler, than the one considered in the previous chapter, however, now our main aim is to investigate the issue of the strong cross-diffusion phenomenon occurring in majority of multicomponent fluids. A careful mathematical analysis of the simplest model for binary mixture will be later on explored to describe more complex flows.

3.1 Introduction

To describe the dynamic of isothermally reacting binary mixture may we use system (1.2) with decoupled energy equation. Thus, the set of state variables consists only of the total mass density $\varrho = \varrho(t,x)$, $\varrho = \varrho_A + \varrho_B$, the velocity vector field $\mathbf{u} = \mathbf{u}(t,x)$ and the species A mass fraction $Y_A = Y_A(t,x)$:

$$\left.\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S} + \nabla \pi &= 0 \\ \partial_t(\varrho Y_A) + \operatorname{div}(\varrho Y_A \mathbf{u}) + \operatorname{div}(\mathbf{F}_A) &= \varrho \omega \end{aligned}\right\} \text{ in } (0,T) \times \Omega. \tag{3.2}$$

Here, $\pi = \pi(\varrho, Y)$ is the internal pressure, $\omega = \omega(\varrho, Y)$ is the species A production rate, $\mathbf{F}_A = \mathbf{F}_A(\varrho, Y)$ denotes the diffusion flux of the species A and $\mathbf{S} = \mathbf{S}(\varrho, \mathbf{u})$ is the viscous part of stress tensor.

We assume that the pressure $\pi = \pi(\varrho, Y)$ obeys the following state equation

$$\pi(\varrho, Y) = \pi_c(\varrho) + \pi_m(\varrho, Y), \tag{3.3}$$

where $\pi_c(\varrho) = \varrho^\gamma$, $\gamma > 1$ is the barotropic part of the pressure also referred to as a "cold pressure". By π_m we denote the classical molecular pressure given by the constitutive equation

$$\pi_m(\varrho, Y) = \sum_{k \in S} p_k = \varrho \left(\sum_{k=1}^{n} \frac{Y_k}{m_k} \right), \tag{3.4}$$

where $S = \{A, B\}$, m_k is the molar mass of the k-th species and we assume that $m_A \neq m_B$. Note that we take the perfect gas constant $R = 1$.
The species mass flux $\mathbf{F}_A$ yields diffusion effects due to the mole fraction gradients and pressure gradients and is given in a general form

$$\mathbf{F}_k = -\sum_{l\in S} C_{kl}\mathbf{d}_l, \quad k \in S, \tag{3.5}$$

with $\mathbf{d}_k$ specified by (1.19). Supposing the following form of the matrix C :

$$C = C_0(\varrho, Y_A, Y_B)\begin{pmatrix} Y_B & -Y_A \\ -Y_B & Y_A \end{pmatrix}, \tag{3.6}$$

we verify, by use of (3.5), that

$$\mathbf{F}_A = -C_0\mathbf{d}_A = -\frac{C_0}{\pi_m}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_A - \frac{\varrho_A}{\varrho m_B}\nabla\varrho\right),$$

$$\mathbf{F}_B = -C_0\mathbf{d}_B = -\frac{C_0}{\pi_m}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_B - \frac{\varrho_B}{\varrho m_A}\nabla\varrho\right).$$

Note that such form of matrix C corresponds to the process in which both species play the symmetric role and it satisfies all general properties listed in (1.20) and taken over from [45], Chapter 7.
In addition, we assume that the diffusion coefficient C_0 is proportional to the Boyle pressure $C_0 \approx \pi_m$ (we take $\frac{C_0}{\pi_m} = 1$).
An important consequence of (3.6) is that $\mathbf{F}_B + \mathbf{F}_A = 0$, therefore we can consider only the first mass fraction as unknown and use the relation

$$Y_A + Y_B = 1, \tag{3.7}$$

to evaluate the mass fraction of the remaining species.
The molar production rate ω is a Lipschitz continuous function. We will additionally postulate existence of constants $\underline{\omega}$ and $\overline{\omega}$ such that

$$-\underline{\omega} \leq \omega(Y_A, Y_B) \leq \overline{\omega}, \qquad \text{for all } 0 \leq Y_A, Y_B \leq 1, \tag{3.8}$$

and we suppose

$$\omega(Y_A, Y_B) \geq 0 \quad \text{whenever} \quad Y_A = 0. \tag{3.9}$$

The viscous stress tensor $\mathbf{S}$ is given by (1.15), where the viscosity coefficients $\mu(\varrho)$, $\nu(\varrho)$ are $C^2(0,\infty)$ functions satisfying restriction (2.8) together with

$$\nu(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho). \tag{3.10}$$

Remark 3.1. *The above condition is a necessary mathematical assumption, by which regularity of the density can be improved. It was proposed by Bresch and Desjardins in [11] as an extension of the particular case considered e.g. in [15], where* $\mu(\varrho) = \varrho$, $\nu(\varrho) = 0$.

Following Mellet and Vasseur [69], we stipulate that there exists positive constant $\underline{\mu'} \in (0, 1]$ such that

$$\begin{gathered} \mu'(\varrho) \geq \underline{\mu'}, \quad \mu(0) \geq 0, \\ |\nu'(\varrho)| \leq \frac{1}{\underline{\mu'}}\mu'(\varrho), \\ \underline{\mu'}\mu(\varrho) \leq 2\mu(\varrho) + 3\nu(\varrho) \leq \frac{1}{\underline{\mu'}}\mu(\varrho). \end{gathered} \tag{3.11}$$

In addition, for arbitrary small $\varepsilon > 0$ and $\gamma \geq 3$ we suppose that

$$\liminf_{\varrho \to \infty} \frac{\mu(\varrho)}{\varrho^{\frac{\gamma}{3}+\varepsilon}} > 0. \tag{3.12}$$

The main difficulty concerning systems with viscosity coefficients vanishing when density equals 0 is lack of information about the velocity vector field. It is no longer in $L^2((0,T)\times\Omega)$ as in the case for constant viscosity coefficients. In fact, it cannot even be defined on vacuum. Although this degeneracy causes additional difficulties, it also contributes some benefits, if only relation (3.10) is satisfied. It provides particular mathematical structure that yields global in time integrability of $\nabla\sqrt{\varrho}$. This property was observed for the first time by Bresch, Desjardins and Lin [15] for the Korteweg equations and for the 2-dimensional viscous shallow water model [10]. Later on, Mellet and Vasseur coupled these ideas with the additional estimate for the norm of $\varrho\mathbf{u}^2$ in $L^\infty(0,T;L\log L(\Omega))$ and proved the sequential stability of weak solutions to the barotropic compressible Navier-Stokes system with the viscosity coefficients satisfying conditions (3.10-3.12). Concerning the stability result, it is possible to extend this approach to treat the case of selfgravitating [29] gases, however existence of regular approximate solutions in this framework is still elusive. The main difficulty is to preserve the logarithmic estimate for the velocity at the level of construction of solution. To the best of our knowledge, when no additional drag terms are present, this is still an open problem.

Nevertheless, some progress has been achieved in the case when further assumption on the zero Kelvin isothermal curve of the equation of state in the neighbourhood of small densities is enforced. This strategy was proposed in the work of Bresch and Desjardins [13] for the heat conducting fluids as a way to get close to a solid state in tension. Their condition was designed to recover the standard cold component of the pressure ϱ^γ far from vacuum and to encompass plasticity and elasticity effects of solid materials, for which low densities may lead to negative pressures. By this modification the compactness of velocity can be obtained without requiring more a priori regularity than expected from the usual energy approach. In this framework the globally well posed system can be constructed by parabolic regularization of the total and partial masses conservation equations and by adding to the momentum equation the capillarity force regularizing the density together with the hyperdiffusive term providing integrability of higher derivatives of velocity. Then, the existence of solutions follows from the fixed point argument applied to the momentum equation combined with the standard theory for the semi-linear parabolic equations of species production.

This is, in a sense, opposite with respect to systems with constant viscosity coefficients, for which there is not enough information about density [62]. As it was mentioned in the introduction, the main difficulty for such systems is to prove the strong convergence of the density, necessary to pass to the limit in the nonlinear term ϱ^γ. However, this cannot be done using only the a-priori estimates. In fact, one needs to replace the missing part of information by compactness of quantity called the effective viscous flux. This in turn requires some knowledge on the velocity field assured by the uniform ellipticity of the differential operator associated with the viscous part of the stress tensor.

The objective of this chapter is to investigate the issue of large data existence of solutions to system (3.2). Let us emphasize that the model we consider is consistent with principles of continuum mechanics and does not violate the second law of thermodynamics when the heat conductivity is taken into account. In contrast, the presence of the species concentration in the state equation and approximation of the diffusion flux by the Fick law would result in the entropy production rate which may fail to be non-negative, see (1.14). This, in turn, would contradict thermodynamic admissibility of the process. In consequence, to be phisically consistent, one has to deal with more general form of diffusion (3.5) leading to a new type of degeneration in the system (3.2) which involves the second space derivatives of ϱ. Therefore, more regularity for the density, than we can prove for the Navier-Stokes-type systems with constant viscosity coefficients, is needed. Here, the theory developed in [15], [69] is applied as a possible way to overcome this difficulty.

In the first part of this chapter we establish the sequential stability of weak solutions to system (3.2) i.e. the closedness of the family of solutions bounded by a priori bounds in the framework of weak formulation. Then, we complement this result by constructing regular enough approximate solutions which preserve the mathematical structure of the system, but only when further restriction on the pressure is postulated.

3.2 Sequential stability of solutions

In what follows we will prove that provided the sequence of sufficiently smooth solutions to system (3.2) has been constructed, it converges to weak solution specified in the definition below. This procedure consist of deriving several a-priori estimates from which a sufficient compactness can be deduced in order to pass to the limit in the weak formulation of the problem. This procedure is usually the last step of the proof of existence of solutions, regarded as a most difficult one and it is a strong motivation to look for a suitable approximation of the original system.

3.2.1 Weak formulation and main result

We consider system (3.2) with the initial conditions

$$\begin{gathered}\varrho(0,\cdot)=\varrho^0, \quad \varrho\mathbf{u}(0,\cdot)=(\varrho\mathbf{u})^0=\mathbf{m}^0, \quad \varrho Y_A(0,\cdot)=(\varrho Y_A)^0=\varrho_A^0,\\ \varrho^0(x)\geq 0, \quad \text{and} \quad 0\leq \varrho_A^0(x)\leq \varrho^0(x) \text{ a.e. on } \Omega.\end{gathered} \tag{3.13}$$

The weak solutions to (3.2-3.12) and (3.13) are specified by the following definition.

Definition 3.2. *A triple $(\varrho,\mathbf{u},Y_A)$ is said to be a weak solution of (3.2-3.12) supplemented with the initial data (3.13) if:*

$$\begin{gathered}\varrho\in L^\infty(0,T;L^1\cap L^\gamma(\Omega)), \quad \sqrt{\varrho}\in L^\infty(0,T;H^1(\Omega)),\\ \sqrt{\varrho}\mathbf{u}\in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\mu(\varrho)}\nabla\mathbf{u}\in L^2(0,T;L^2(\Omega)),\\ \varrho\geq 0, \quad 0\leq Y_A\leq 1, \quad \textit{a.e. in } (0,T)\times\Omega,\\ \sqrt{\varrho}\nabla Y_A\in L^2(0,T;L^2(\Omega)),\end{gathered}$$

and equations of system (3.2) *hold in the following sense:*
1. The continuity equation

$$\begin{cases}\partial_t\varrho+\operatorname{div}(\sqrt{\varrho}\sqrt{\varrho}\mathbf{u})=0\\ \varrho(0,x)=\varrho^0(x)\end{cases}$$

is satisfied in the sense of distributions.
2. The weak formulation of the momentum equation

$$\begin{aligned}&\int_\Omega \mathbf{m}^0\cdot\phi(0,x)\ \mathrm{d}x+\int_0^T\int_\Omega\left(\sqrt{\varrho}(\sqrt{\varrho}\mathbf{u})\cdot\partial_t\phi+\sqrt{\varrho}\mathbf{u}\otimes\sqrt{\varrho}\mathbf{u}:\nabla\phi\right)\ \mathrm{d}x\ \mathrm{d}t\\ &+\int_0^T\int_\Omega p(\varrho,Y_A,Y_B)\operatorname{div}\phi\ \mathrm{d}x\ \mathrm{d}t-\int_0^T\langle 2\mu(\varrho)\mathbf{D}(\mathbf{u}),\nabla\phi\rangle\ \mathrm{d}t-\int_0^T\langle\nu(\varrho)\operatorname{div}\mathbf{u},\operatorname{div}\phi\rangle\ \mathrm{d}t=0\end{aligned}$$

holds for any smooth, compactly supported test function $\phi(t,x)$ such that $\phi(T,\cdot)=0$. In this formula, the last two terms should be understood as

$$\langle 2\mu(\varrho)\mathbf{D}(\mathbf{u}),\nabla\phi\rangle = -\int_\Omega \frac{\mu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_j\partial_{ii}\phi_j \, \mathrm{d}x - 2\int_\Omega \mu'(\varrho)\sqrt{\varrho}\mathbf{u}_j\partial_i\sqrt{\varrho}\partial_i\phi_j \, \mathrm{d}x$$
$$- \int_\Omega \frac{\mu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_i\partial_{ji}\phi_j \, \mathrm{d}x - 2\int_\Omega \mu'(\varrho)\sqrt{\varrho}\mathbf{u}_i\partial_j\sqrt{\varrho}\partial_i\phi_j \, \mathrm{d}x$$

and

$$\langle \nu(\varrho)\operatorname{div}\mathbf{u},\operatorname{div}\phi\rangle = -\int_\Omega \frac{\nu(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\mathbf{u}_i\partial_{ij}\phi_j \, \mathrm{d}x - 2\int_\Omega \nu'(\varrho)\sqrt{\varrho}\mathbf{u}_i\partial_i\sqrt{\varrho}\partial_j\phi_j \, \mathrm{d}x.$$

3. The weak formulation of the mass balance equation for species A

$$\int_\Omega \varrho_A^0\cdot\psi(0,x)\, \mathrm{d}x + \int_0^T\int_\Omega \left(\sqrt{\varrho}Y_A\sqrt{\varrho}\mathbf{u}\cdot\partial_t\psi + \sqrt{\varrho}Y_A\sqrt{\varrho}\mathbf{u}\cdot\nabla\psi\right)\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T \langle\mathbf{F}_A,\nabla\psi\rangle \, \mathrm{d}t = \int_0^T\int_\Omega \varrho\omega\psi \, \mathrm{d}x\, \mathrm{d}t$$

is satisfied for any smooth, compactly supported test function $\psi(t,x)$ such that $\psi(T,\cdot)=0$, where the last term on the left hand side (l.h.s.) denotes

$$\langle\mathbf{F}_A,\nabla\psi\rangle = \frac{1}{m_A}\int_\Omega \varrho Y_A\Delta\psi \, \mathrm{d}x + \frac{2}{m_A}\int \sqrt{\varrho}Y_A\nabla\sqrt{\varrho}\cdot\nabla\psi$$
$$+\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \sqrt{\varrho}Y_A^2\nabla\sqrt{\varrho}\cdot\nabla\psi \, \mathrm{d}x - \frac{1}{2}\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \varrho Y_A^2\Delta\psi \, \mathrm{d}x.$$

We can now formulate our main result.

Theorem 3.3. *Let $\gamma>1$ and let $\mu(\varrho)$, $\nu(\varrho)$ be two $C^2(0,\infty)$ functions satisfying (3.10-3.12). Assume that $\{\varrho_n,\mathbf{u}_n,Y_{A,n}\}_{n\in\mathbb{N}}$ is a sequence of smooth solutions to (3.2-3.12) satisfying weak formulation in the sense of Definition 3.2 and the energy-entropy inequalities (3.18), (3.20) and (3.26), with the initial data*

$$\varrho_n(0,\cdot)=\varrho_n^0,\quad \varrho_n\mathbf{u}_n(0,\cdot)=\varrho_n^0\mathbf{u}_n^0=\mathbf{m}_n^0,\quad \varrho_nY_{A,n}(0,\cdot)=\varrho_n^0Y_{A,n}^0=\varrho_{A,n}^0,$$

satisfying

$$\inf_{x\in\Omega}\varrho_n^0(x)>0,\quad \varrho_n^0\to\varrho^0 \quad \text{in } L^1(\Omega),\quad \varrho_n^0\mathbf{u}_n^0\to(\varrho\mathbf{u})^0 \quad \text{in } L^1(\Omega),$$
$$0\le Y_{A,n}^0\le 1,\quad \varrho_n^0Y_{A,n}^0\to(\varrho Y_A)^0 \quad \text{in } L^1(\Omega),$$

together with the following bounds

$$\begin{gathered}\int_\Omega \left(\tfrac{1}{2}\varrho_n^0\left|\mathbf{u}_n^0\right|^2+\tfrac{1}{\gamma-1}\left(\varrho_n^0\right)^\gamma-\tfrac{1}{m_B}\varrho_n^0\log\varrho_n^0\right)\, \mathrm{d}x<\infty,\quad \int_\Omega \tfrac{1}{\varrho_n^0}\left|\nabla\mu\left(\varrho_n^0\right)\right|^2\, \mathrm{d}x<\infty,\\ \int_\Omega \varrho_n^0\left(Y_{A,n}^0\right)^2\, \mathrm{d}x<\infty,\quad \int_\Omega \varrho_n^0\left(1+\left|\mathbf{u}_n^0\right|^2\right)\ln\left(1+\left|\mathbf{u}_n^0\right|^2\right)\, \mathrm{d}x<\infty.\end{gathered} \tag{3.14}$$

Then, up to a subsequence, $\{\varrho_n,\sqrt{\varrho_n}\mathbf{u}_n,Y_{A,n}\}$ converges strongly to the weak solution of the problem (3.2-3.12) in the sense of the above definition. More precisely, we have

$$\begin{aligned}\varrho_n &\to \varrho && \text{strongly in } C^0(0,T;L^{\frac{3}{2}}(\Omega)),\\ \sqrt{\varrho_n}\mathbf{u}_n &\to \sqrt{\varrho}\mathbf{u} && \text{strongly in } L^2(0,T;L^2(\Omega)),\\ \varrho_n\mathbf{u}_n &\to \varrho\mathbf{u} && \text{strongly in } L^2(0,T;L^1(\Omega)),\\ Y_{A,n} &\to Y_A && \text{strongly in } L^p(0,T;L^p(\Omega)),\end{aligned}$$

for any p finite and any $T > 0$.

We have divided the proof into two parts that are presented separately in the following subsections.

3.2.2 A priori estimates

In this section we present the a priori estimates, being derived for a sequence of smooth solutions $(\varrho_n, \mathbf{u}_n, Y_{A,n})$ to (3.2-3.12); we skip the subindex n when no confusion can arise.
We start with the conservation of mass. Integrating the continuity equation over Ω we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho \; \mathrm{d}x = 0,$$

i.e. knowing that $\int_\Omega \varrho^0(x)\; \mathrm{d}x = M$, we deduce that $\int_\Omega \varrho(t,x)\; \mathrm{d}x = M$ for any $t \in [0,T]$.
Moreover, since ϱ is smooth and $\varrho^0 > 0$, we have the following estimate

$$\varrho(\tau, x) \geq \inf_{x\in\Omega} \varrho^0(x) \exp\left(-\int_0^\tau \|\operatorname{div} \mathbf{u}\|_{L^\infty(\Omega)} \mathrm{d}t\right), \tag{3.15}$$

in particular $\varrho > 0$.

Correspondingly, the sum of masses of both species must be conserved, in particular we have the following lemma (a kind of weak maximal principle).

Lemma 3.4. *For any smooth solution of (3.2) we have*

$$Y_A, Y_B \geq 0 \quad \text{on } \Omega \times (0,T), \tag{3.16}$$

and

$$Y_A + Y_B = 1. \tag{3.17}$$

Proof. Let ϕ_ε be a sequence of smooth functions such that

$$\operatorname{supp} \phi_\varepsilon \subset \Omega_T^-, \quad 0 \leq \phi_\varepsilon \leq 1, \quad \phi_\varepsilon(x) = 1 \quad \text{for} \quad \operatorname{dist}((t,x), \partial\Omega_T^-) \geq \varepsilon,$$

where $\Omega_T^- = \{(t,x) \in ((0,T)\times\Omega) : Y_A(t,x) < 0\}$.[1]
Multiplying the species mass balance equation by ϕ_ε and integrating over $(0,T)\times\Omega$ we obtain

$$\begin{aligned} -&\int_{\Omega_T^-} \varrho Y_A \partial_t \phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t - \int_{\Omega_T^-} \varrho Y_A \mathbf{u}\cdot\nabla\phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t + \int_{\Omega_T^-} \frac{1}{m_A} Y_A \nabla\varrho\cdot\nabla\phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t \\ +&\int_{\Omega_T^-} \frac{1}{m_A}\varrho\nabla Y_A\cdot\nabla\phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t - \int_{\Omega_T^-} Y_A \nabla\pi_m(\varrho, Y)\cdot\nabla\phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t = \int_{\Omega_T^-} \varrho\omega(Y)\phi_\varepsilon \; \mathrm{d}x \; \mathrm{d}t. \end{aligned}$$

Observe that when $\varepsilon \to 0^+$ then the four-component vector $(\partial_t\phi_\varepsilon, \nabla\phi_\varepsilon)$ approximates $-\mathbf{n} = -(n_t, \mathbf{n}_x)$, which is the inter normal vector to the boundary of Ω_T^-, so we get

$$\begin{aligned} &\int_{\partial\Omega_T^-} \varrho Y_A n_t \; \mathrm{d}S_{t,x} + \int_{\partial\Omega_T^-} \varrho Y_A \mathbf{u}\cdot\mathbf{n}_x \; \mathrm{d}S_{t,x} - \int_{\partial\Omega_T^-} \frac{1}{m_A} Y_A\nabla\varrho\cdot\mathbf{n}_x \; \mathrm{d}S_{t,x} \\ -&\int_{\partial\Omega_T^-} \frac{1}{m_A}\varrho\nabla Y_A\cdot\mathbf{n}_x \; \mathrm{d}S_{t,x} + \int_{\partial\Omega_T^-} Y_A\nabla\pi_m(\varrho, Y)\cdot\mathbf{n}_x \; \mathrm{d}S_{t,x} = \int_{\Omega_T^-} \varrho\omega(Y) \; \mathrm{d}x \; \mathrm{d}t. \end{aligned}$$

[1] If Ω_T^- is not a regular domain, we may use the Sard theorem [92] and the Implicit Function Theorem to find a sequence of sets $\Omega_{T,\delta_n}^- = \{(t,x) \in ((0,T)\times\Omega) : Y_A(t,x) < \delta_n\}$ for $\delta_n > 0$, such that $\partial\Omega_{T,\delta_n}^-$ is as smooth as Y_A, and pass with $\delta_n \to 0^+$.

Now, due to the fact that $Y_A|_{\partial\Omega_T^-}=0$ all but the penultimate integral from the l.h.s. vanish and we are left only with

$$-\int_{\partial\Omega_T^-}\frac{1}{m_A}\varrho\nabla Y_A\cdot\mathbf{n}_x\ \mathrm{d}S_{t,x}=\int_{\Omega_T^-}\varrho\omega(Y)\ \mathrm{d}x\ \mathrm{d}t.$$

Due to assumption (3.9), the r.h.s. of the above equality is nonnegative. On the other hand, we know that $\frac{\partial Y_A}{\partial\mathbf{n}}\Big|_{\partial\Omega_T^-}$ is positive, hence the l.h.s. must be nonpositive. Therefore, the only possibility is that the Lebesgue measure of the set Ω_T^- is equal 0. In particular, in view of smoothness of Y_A we have (3.16) and then, the similar token applied to the continuity equation enables to verify (3.17).□

In the next step we present the usual energy approach to the second equation of system (3.2) which leads to the following equality.

Lemma 3.5. *The following equality holds for any smooth solution of (3.2)*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac12\varrho|\mathbf{u}|^2+\frac{1}{\gamma-1}\varrho^\gamma-\frac{1}{m_B}\varrho\log\varrho\right)\ \mathrm{d}x+\int_\Omega 2\mu(\varrho)|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x$$
$$+\int_\Omega\nu(\varrho)|\operatorname{div}\mathbf{u}|^2\ \mathrm{d}x-\int_\Omega\varrho Y_A\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\operatorname{div}\mathbf{u}\ \mathrm{d}x=0.\quad(3.18)$$

Proof. We test the momentum equation by $\mathbf{u}$ and integrate by parts. □

Transforming the last term from the l.h.s. of (3.18), we can derive some useful bounds. First observe that due to Lemma 3.2.2 we may apply the Cauchy inequality (with ε) to estimate

$$\int_\Omega\varrho Y_A\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\operatorname{div}\mathbf{u}\ \mathrm{d}x$$
$$\le\int_\Omega\frac{\varrho^{\frac12}}{\mu(\varrho)^{\frac12}}\mu(\varrho)^{\frac12}|\operatorname{div}\mathbf{u}|\varrho^{\frac12}\ \mathrm{d}x\le\varepsilon\int_\Omega\frac{\varrho}{\mu(\varrho)}\mu(\varrho)|\operatorname{div}\mathbf{u}|^2\ \mathrm{d}x+c(\varepsilon)\int_\Omega\varrho\ \mathrm{d}x.$$

The last term is controlled since $\varrho\in L_\infty(0,T;L^1(\Omega))$, while the first one is absorbed by the l.h.s. of (3.18) provided that

$$\mu(\varrho)\ge c\varrho^m\quad\text{for}\quad\varrho>1,\ m\ge1,$$
$$\mu(\varrho)\ge c\varrho^n\quad\text{for}\quad\varrho\le1,\ n\le1$$

and that ϵ is sufficiently small.
Indeed, since $2\mu(\varrho)+3\nu(\varrho)\ge\underline{\mu}'\mu(\varrho)$ and $(\operatorname{div}\mathbf{u})^2\le3|\mathbf{D}(\mathbf{u})|^2$ thus, taking ε sufficiently small we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac12\varrho|\mathbf{u}|^2+\frac{1}{\gamma-1}\varrho^\gamma-\frac{1}{m_B}\varrho\log\varrho\right)\ \mathrm{d}x+\int_\Omega\mu(\varrho)|\mathbf{D}(\mathbf{u})|^2+\nu(\varrho)(\operatorname{div}\mathbf{u})^2\ \mathrm{d}x\le c.$$

Therefore, due to (3.19) and (3.11), we have the following estimate

$$\|\sqrt{\varrho}\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\varrho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))}+\|\sqrt{\mu(\varrho)}\mathbf{D}(\mathbf{u})\|^2_{L^2(0,T;L^2(\Omega))}\le c.\quad(3.19)$$

In order to proceed we need to find some better estimate of the norm of density than in $L^\infty(0,T;L^\gamma(\Omega))$. It will be a consequence of integrability of gradient of $\sqrt{\varrho}$ obtained by a modification of entropy inequality proved for the first time by Bresch and Desjardins [10]. We will roughly recall the most important steps from the original proof and focus on the new features of the system. More details can be found in the last section, in the proof of Lemma 3.17.

Lemma 3.6. *Let $\mu(\varrho)$, $\nu(\varrho)$ be two $C^2(0,\infty)$ functions satisfying (3.10) and (3.11). Then, any smooth solution of (3.2) satisfies*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2+\frac{1}{\gamma-1}\varrho^\gamma-\frac{1}{m_B}\varrho\log\varrho\right)\,\mathrm{d}x+\int_\Omega\nabla\phi(\varrho)\cdot\nabla\pi(\varrho,Y)\,\mathrm{d}x\\&\qquad+\frac{1}{2}\int_\Omega\mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,\mathrm{d}x-\int_\Omega\varrho Y_A\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x=0\end{aligned}\tag{3.20}$$

for ϕ such that $\nabla\phi(\varrho)=2\frac{\mu'(\varrho)\nabla\varrho}{\varrho}$.

Proof. We start with the following observation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\varrho\mathbf{u}\cdot\nabla\phi(\varrho)\,\mathrm{d}x=\int_\Omega\nabla\phi(\varrho)\partial_t(\varrho\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x,\tag{3.21}$$

where the first term on the r.h.s. may be evaluated by multiplying the momentum equation by $\nabla\phi(\varrho)$ and integrating by parts

$$\begin{aligned}&\int_\Omega\partial_t(\varrho\mathbf{u})\nabla\phi(\varrho)\,\mathrm{d}x\\&=-2\int_\Omega\nabla\phi(\varrho)\cdot\nabla\mu(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x-\int_\Omega\nabla\phi(\varrho)\cdot\nabla\pi(\varrho,Y)\,\mathrm{d}x-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x\\&\quad-\int_\Omega(2\mu(\varrho)+\nu(\varrho))\Delta\phi(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x+2\int_\Omega\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\mu(\varrho)\,\mathrm{d}x.\end{aligned}\tag{3.22}$$

Next, multiplying continuity equation by $|\nabla\phi(\varrho)|^2$ we get the following "renormalized" version

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\,\mathrm{d}x\\&=-\int_\Omega\varrho\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\,\mathrm{d}x+\int_\Omega\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u}\,\mathrm{d}x+\int_\Omega\varrho\left(\nabla\phi(\varrho)\right)^2\operatorname{div}\mathbf{u}\,\mathrm{d}x.\end{aligned}\tag{3.23}$$

From (3.21), (3.22) and (3.23) we therefore deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\varrho\mathbf{u}\cdot\nabla\phi(\varrho)+\frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\right)\,\mathrm{d}x+\int_\Omega\nabla\phi(\varrho)\cdot\nabla p(\varrho,Y)\,\mathrm{d}x\\&\qquad=-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x.\end{aligned}\tag{3.24}$$

Now, the r.h.s. may be transformed into the form

$$\begin{aligned}&-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x\\&\qquad=\int_\Omega\nu(\varrho)(\operatorname{div}\mathbf{u})^2\,\mathrm{d}x+\int_\Omega2\mu(\varrho)|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega\mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,\mathrm{d}x\end{aligned}$$

and thus (3.18) summed up with (3.24) implies (3.20). □

To make use of this lemma we should verify that all the negative contributions from the l.h.s. and the whole r.h.s. are bounded. Note that, for instance, the pressure term is equal to

$$\begin{aligned}&\nabla\phi(\varrho)\cdot\nabla\pi(\varrho,Y)\\&=\gamma\mu'(\varrho)\varrho^{\gamma-2}|\nabla\varrho|^2+\mu'(\varrho)\left(\frac{Y_A}{m_A}+\frac{Y_B}{m_B}\right)\varrho^{-1}|\nabla\varrho|^2+\mu'(\varrho)\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\nabla\varrho\cdot\nabla Y_A\end{aligned}\tag{3.25}$$

where the first two parts have a positive sign on the l.h.s. of (3.20), while to control the last term we need the following result.

Lemma 3.7. *For any smooth solution of (3.2) we have*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho Y_A^2\ \mathrm{d}x + \frac{1}{\max\{m_A,m_B\}}\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x \\ &\quad\le \int_\Omega \varrho|\omega(Y)|Y_A\ \mathrm{d}x + \frac{1}{4}\left(\frac{1}{\min\{m_A,m_B\}} - \frac{1}{\max\{m_A,m_B\}}\right)\int_\Omega |\nabla\varrho\cdot\nabla Y_A|\ \mathrm{d}x. \end{aligned} \tag{3.26}$$

Proof. Multiplying the species mass balance equation by Y_A and integrating over Ω we deduce

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho Y_A^2\ \mathrm{d}x + \int_\Omega\left(\frac{1-Y_A}{m_A}+\frac{Y_A}{m_B}\right)\varrho|\nabla Y_A|^2\ \mathrm{d}x \\ &\qquad = \left(\frac{1}{m_B}-\frac{1}{m_A}\right)\int_\Omega Y_A(1-Y_A)\nabla\varrho\cdot\nabla Y_A\ \mathrm{d}x + \int_\Omega \varrho\omega(Y)Y_A\ \mathrm{d}x. \end{aligned}$$

Now, since $0\le Y_A\le 1$ and we have $\frac{1-Y_A}{m_A}+\frac{Y_A}{m_B}\ge\frac{1}{\max\{m_A,m_B\}}$ and $Y_A(1-Y_A)\le\frac{1}{4}$. □
To estimate the r.h.s. of (3.26) we use the Cauchy inequality

$$\int_\Omega |\nabla\varrho\cdot\nabla Y_A|\ \mathrm{d}x \le c(\varepsilon)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x + \varepsilon\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x$$

with $\varepsilon < \frac{4\min\{m_A,m_B\}}{\max\{m_A,m_B\}-\min\{m_A,m_B\}}$. And thus, we can integrate (3.26) with respect to time to get

$$\begin{aligned} &\|\sqrt{\varrho}Y_A\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{\varrho}\nabla Y_A\|^2_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le c\|Y_A\|_{L^\infty((0,T)\times\Omega)}\|\varrho\|_{L^\infty(0,T;L^1(\Omega))} + c(m_A,m_B)\,\|\nabla\sqrt{\varrho}\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{3.27}$$

Hence, the assertion of Lemma 3.6 gives rise to the following inequality

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{1}{\gamma-1}\varrho^\gamma - \frac{1}{m_B}\varrho\log\varrho\right)\ \mathrm{d}x + \int_\Omega \gamma\mu'(\varrho)\varrho^{\gamma-2}|\nabla\varrho|^2\ \mathrm{d}x \\ &\quad + \int_\Omega \mu'(\varrho)\left(\frac{Y_A}{m_A}+\frac{Y_B}{m_B}\right)\varrho^{-1}|\nabla\varrho|^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega \mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\ \mathrm{d}x \\ &\quad \le \left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \varrho|\operatorname{div}\mathbf{u}|\ \mathrm{d}x + \left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \mu'(\varrho)|\nabla\varrho||\nabla Y_A|\ \mathrm{d}x. \end{aligned} \tag{3.28}$$

The first term from the r.h.s is bounded on account of Lemma 3.5. In order to estimate last term we use the Cauchy inequality (with ε) to show

$$\int_\Omega \mu'(\varrho)\nabla\varrho\cdot\nabla Y_A\ \mathrm{d}x \le c(\varepsilon)\int_\Omega \frac{(\mu'(\varrho))^2}{\varrho}|\nabla\varrho|^2\ \mathrm{d}x + \varepsilon\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x.$$

So, the Gronwall-type argument applied to the first integral coupled with (3.27) applied to the second one yields boundedness of the l.h.s. of (3.28). In particular, since the initial data satisfy (3.19), we can integrate (3.28) with respect to time to obtain

$$\begin{aligned} &\|\sqrt{\varrho}\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mu'(\varrho)\nabla\sqrt{\varrho}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\varrho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} \\ &\qquad + \|\sqrt{\mu'(\varrho)\varrho^{\gamma-2}}\nabla\varrho\|^2_{L^2(0,T;L^2(\Omega))} + \|\sqrt{\mu(\varrho)}\mathbf{A}(\mathbf{u})\|^2_{L^2(0,T;L^2(\Omega))} \le c, \end{aligned} \tag{3.29}$$

where we denoted $\mathbf{A}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} - \nabla^T\mathbf{u}\right)$.
Now, one can check that via the Sobolev imbedding theorem we have

$$\frac{1}{c_s^2}\|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^6(\Omega))} \le \|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;H^1(\Omega))} \le \|\nabla\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^2)} + \|\varrho\|^{\gamma}_{L^\infty(0,T;L^\gamma(\Omega))} \tag{3.30}$$

where c_s is the constant from the Sobolev inequality. Moreover, applying the interpolation inequality we obtain

$$\|\varrho_n^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \le \|\varrho_n^\gamma\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))}\|\varrho_n^\gamma\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))} \le c. \tag{3.31}$$

Our ultimate goal before the limit passage is dedicated to better integrability of velocity.

Lemma 3.8. *Let assumptions (3.10), (3.11) be valid. Then for any $\delta \in (0,2)$ the smooth solution of (3.2) satisfies*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varrho}{2}(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x + \frac{\underline{\mu}'}{2}\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x \le c\int_\Omega \mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x \\ + c\left(\int_\Omega\left(\frac{\pi(\varrho,Y)^2\varrho^{-\frac{\delta}{2}}}{\mu(\varrho)}\right)^{\frac{2}{2-\delta}}\ \mathrm{d}x\right)^{\frac{2-\delta}{2}}\left(\int_\Omega \varrho(2+\ln(1+|\mathbf{u}|^2))^{\frac{2}{\delta}}\ \mathrm{d}x\right)^{\frac{\delta}{2}}.\end{aligned} \tag{3.32}$$

Proof. We follow the same strategy as in the work of Mellet and Vasseur [69] (Lemma 3.2). Multiplying the momentum equation by $(1+\ln(1+|\mathbf{u}|^2))\mathbf{u}$ and employing (3.11) we verify

$$\begin{aligned}\int_\Omega \frac{1}{2}\varrho\partial_t((1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2))\ \mathrm{d}x + \int_\Omega \frac{1}{2}\varrho\mathbf{u}\cdot\nabla(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x \\ + \underline{\mu}'\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x \\ \le -\int_\Omega (1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla\pi(\varrho,Y)\ \mathrm{d}x + c\int_\Omega \mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x.\end{aligned} \tag{3.33}$$

Multiplying continuity equation by $\frac{1}{2}(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)$ and integrating by parts

$$\int_\Omega \frac{1}{2}\partial_t\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x = \int_\Omega \frac{1}{2}\varrho\mathbf{u}\cdot\nabla(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x,$$

so the first two terms from the l.h.s. of (3.33) give $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\ \mathrm{d}x$. To control the r.h.s. of (3.33) we first integrate by parts

$$\begin{aligned}\left|\int_\Omega (1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla\pi(\varrho,Y)\ \mathrm{d}x\right| \\ \le \left|\int_\Omega \frac{2u_iu_k}{1+|\mathbf{u}|^2}\partial_iu_k\pi(\varrho,Y)\ \mathrm{d}x\right| + \left|\int_\Omega (1+\ln(1+|\mathbf{u}|^2))\operatorname{div}\mathbf{u}\ \pi(\varrho,Y)\ \mathrm{d}x\right|,\end{aligned}$$

then using the Hölder and Cauchy inequalities we show the following estimate

$$\begin{aligned}\int_\Omega (1+\ln(1+|\mathbf{u}|^2))\mathbf{u}\cdot\nabla\pi(\varrho,Y)\ \mathrm{d}x \\ \le \int_\Omega \mu(\varrho)|\nabla\mathbf{u}|^2\ \mathrm{d}x + \frac{1}{2}\underline{\mu}'\int_\Omega \mu(\varrho)(1+\ln(1+|\mathbf{u}|^2))|\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x \\ + c\int_\Omega (2+\ln(1+|\mathbf{u}|^2))\frac{(\pi(\varrho,Y))^2}{\mu(\varrho)}\ \mathrm{d}x.\end{aligned}$$

Hence (3.32) is obtained by applying to the last term from above the Hölder inequality with $p = \frac{2}{2-\delta}$, $q = \frac{2}{\delta}$ $(\frac{1}{p} + \frac{1}{q} = 1)$ and $\delta \in (0, 2)$. □
Observe that due to (3.29) the r.h.s. of (3.32) may be partially controlled, we know in particular that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho(1 + |\mathbf{u}|^2) \ln(1 + |\mathbf{u}|^2)\, \mathrm{d}x \leq c \left(\int_\Omega \left(\frac{(\pi(\varrho, Y))^2 \varrho^{-\frac{\delta}{2}}}{\mu(\varrho)} \right)^{\frac{2}{2-\delta}} \mathrm{d}x \right)^{\frac{2-\delta}{2}} + c. \tag{3.34}$$

Next, since $\mu(\varrho) > \underline{\mu}' \varrho$, thus $\varrho(1 + |\mathbf{u}|^2) \ln(1 + |\mathbf{u}|^2)$ is bounded in $L^\infty(0, T; L^1(\Omega))$ if only $(\pi(\varrho, Y))^2 \varrho^{-1-\frac{\delta}{2}}$ belongs to $L^1((0, T) \times \Omega)$. By virtue of Lemma 3.2.2 and estimate (3.31) this is true for $\gamma < 3$, otherwise the boundedness of the r.h.s. of (3.34) follows from the additional assumption (3.12).

3.2.3 Passage to the limit

In the previous section we showed uniform estimates for the sequence of of smooth solutions $\{\varrho_n, \mathbf{u}_n, Y_n\}_{n\in\mathbb{N}}$ under assumption that the initial data satisfy (3.19). For convenience of the reader we list all of them once more

$$\|\varrho_n\|_{L^\infty(0,T;L^1(\Omega)\cup L^\gamma(\Omega))} \leq c, \tag{3.35}$$
$$\|\varrho_n^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq c, \tag{3.36}$$
$$\|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{3.37}$$
$$\|\varrho_n|\mathbf{u}_n|^2 \ln(1 + |\mathbf{u}_n|^2)\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{3.38}$$
$$\|Y_n\|_{L^\infty((0,T)\times\Omega)} \leq c, \tag{3.39}$$
$$\|\nabla\sqrt{\varrho_n}\|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{3.40}$$
$$\|\sqrt{\varrho_n}\nabla\mathbf{u}_n\|_{L^2(0,T;L^2(\Omega))} \leq c, \tag{3.41}$$
$$\|\sqrt{\varrho_n}\nabla Y_n\|_{L^2(0,T;L^2(\Omega))} \leq c. \tag{3.42}$$

In this section we present the proof of Theorem 3.3. It will be split into several steps.

1. Convergence of $\sqrt{\varrho_n}$

Lemma 3.9. *If $\mu(\varrho)$ satisfies (3.11), then for a subsequence we have*

$$\sqrt{\varrho_n} \to \sqrt{\varrho} \quad \text{a.e. and } L^2((0,T)\times\Omega) \text{ strongly.}$$

Moreover $\varrho_n \to \varrho$ strongly in $C(0, T; L^{\frac{3}{2}}(\Omega))$.

Proof. By (3.35) and (3.40) we see that $\sqrt{\varrho_n} \in L^\infty(0, T; H^1(\Omega))$. Next, from the renormalized continuity equation coupled with (3.37) and (3.41) we also get that $\partial_t\sqrt{\varrho_n}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. Hence, the Aubin-Lions lemma implies strong convergence on every compact subset in $L^2((0, T) \times \Omega)$.
In order to proceed we observe that by the Sobolev imbedding theorem $\sqrt{\varrho_n} \in L^\infty(0, T; L^6(\Omega))$. Therefore, from the continuity equation $\partial_t\varrho_n \in L^\infty(0, T; W^{-1,\frac{3}{2}}(\Omega))$ which together with boundedness of $\nabla\varrho_n$ in $L^\infty(0, T; L^{\frac{3}{2}}(\Omega))$ establishes compactness of $\{\varrho_n\}$ in $C^0(0, T; L^{\frac{3}{2}}(\Omega))$.

2. Convergence of the pressure
In view of (3.36) and by the fact that ϱ_n^γ converges almost everywhere to ϱ^γ, we deduce that ϱ_n^γ

converges strongly to ϱ^γ in $L^1((0,T)\times\Omega)$.
Concerning the molecular pressure, since $\varrho_n \in L^\infty(0,T;L^3(\Omega))$, thus (3.39) implies

$$\varrho_n Y_{A,n} \quad \text{is bounded in } L^\infty(0,T;L^p(\Omega))$$

for any $p\in[1,3]$. Additionally, note that the space gradient of $\varrho_n Y_{A,n}$ equals

$$\nabla(\varrho_n Y_{A,n}) = Y_{A,n}\nabla\varrho_n + \sqrt{\varrho_n}\sqrt{\varrho_n}\nabla Y_{A,n}$$

and is bounded in $L^2(0,T;L^q(\Omega))$ for $q\in[1,\frac{3}{2}]$, therefore $\varrho_n Y_{A,n}\in L^2(0,T;W^{1,\frac{3}{2}}(\Omega))$.
Now, let us verify that the time derivative

$$\partial_t(\varrho_n Y_{A,n}) = -\operatorname{div}(\varrho_n Y_{A,n}\mathbf{u}_n) - \operatorname{div}(\mathbf{F}_{A,n}) + \varrho_n\omega_n \quad \text{is bounded in } L^2(0,T;W^{-1,\frac{3}{2}}(\Omega)).$$

Indeed, as $\varrho_n\mathbf{u}_n Y_{A,n} = \sqrt{\varrho_n}\mathbf{u}_n\sqrt{\varrho_n}Y_{A,n}$ belongs to $L^\infty(0,T;L^q(\Omega))$ and

$$\mathbf{F}_{A,n} = \frac{1}{m_A}\nabla(\varrho_n Y_{A,n}) - \frac{Y_{A,n}}{m_A}\nabla(\varrho_n Y_{A,n}) - \frac{Y_{A,n}}{m_B}\nabla(\varrho_n(1-Y_{A,n}))$$

is bounded in $L^2(0,T;L^q(\Omega))$ for $q\in[1,\frac{3}{2}]$ we have, by the Aubin-Lions lemma, compactness of $\{\varrho_n Y_{A,n}\}$ in $L^2(0,T;L^p(\Omega))$ for $p\in(1,3)$.

3. Strong convergence of $Y_{A,n}$
As a consequence of the last result we have (up to a subsequence) that $\varrho_n Y_{A,n}$ converges a.e. to some ϱ_A and we define $Y_A = \frac{\varrho_A}{\varrho}$. Moreover, since ϱ_n converges a.e. to ϱ it can be easily deduced that $Y_{A,n} = \frac{\varrho_n Y_{A,n}}{\varrho_n}$ converges a.e. to Y_A whenever $\{\varrho(t,x)\neq 0\}$. As a matter of fact this is also true in the set $\{\varrho(t,x)=0\}$ on account of (3.39) and the Fatou lemma. In particular, we have a strong convergence of $Y_{A,n}$ in $L^p(0,T;L^p(\Omega))$ for any p finite.

4. Convergence of the convective term
Having proved strong convergence of density and the additional estimate for velocity (3.38), convergence in the convective and the viscosity terms can be shown identically as in the work of Mellet & Vasseur [69]. Below we recall their final result.

Lemma 3.10. *Let $p\in[1,\frac{3}{2})$, then up to a subsequence we have*

$$\begin{aligned}&\varrho_n\mathbf{u}_n \to \mathbf{m} \quad \textit{a.e. in } (0,T)\times\Omega \textit{ and strongly in } L^2(0,T;L^p(\Omega)),\\ &\sqrt{\varrho_n}\mathbf{u}_n \to \frac{\mathbf{m}}{\sqrt{\varrho}} \quad \textit{strongly in } L^2((0,T)\times\Omega).\end{aligned}$$

In particular, we have $\mathbf{m}(t,x)=0$ a.e. on $\{\varrho(t,x)=0\}$ and there exists a function $\mathbf{u}(t,x)$ such that $\mathbf{m}(t,x)=\varrho(t,x)\mathbf{u}(t,x)$ and

$$\begin{aligned}\varrho_n\mathbf{u}_n &\to \varrho\mathbf{u} \quad \textit{strongly in } L^2(0,T;L^p(\Omega)),\\ \sqrt{\varrho_n}\mathbf{u}_n &\to \sqrt{\varrho}\mathbf{u} \quad \textit{strongly in } L^2((0,T)\times\Omega).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\mu(\varrho_n)\mathbf{D}(\mathbf{u}_n) &\to \mu(\varrho)\mathbf{D}(\mathbf{u}) \quad \textit{in } \mathcal{D}'((0,T)\times\Omega),\\ \nu(\varrho_n)\operatorname{div}\mathbf{u}_n &\to \nu(\varrho)\operatorname{div}\mathbf{u} \quad \textit{in } \mathcal{D}'((0,T)\times\Omega).\end{aligned}$$

3.3 First level of approximation-construction of solution

In this section we present a possible approach to the issue of solvability of system (3.2). As it was already announced the strategy requires either to consider additional friction of the form $\varrho|\mathbf{u}|\mathbf{u}$ or to modify the cold component of the pressure in the regime of small densities.

The second way seems more natural as ultimately we want to investigate the full system describing the motion of chemically reacting and heat conducting fluids for which it is not so evident that in the degenerated regimes (of low temperatures and densities) the medium behaves as a fluid. For further discussion on this topic we refer to [13] and references therein.

From now on π_c denotes a continuous function such that

$$\pi_c'(\varrho) = \begin{cases} c\varrho^{-4k-1} & \text{for } \varrho \leq 1, \quad k > 1, \\ \varrho^{\gamma-1} & \text{for } \varrho > 1, \quad \gamma > 1 \end{cases} \tag{3.43}$$

for some constant $c > 0$. By this modification, the compactness of velocity can be obtained without Lemma 3.8, so to construct the approximate solution one should only care about preserving the structure (3.20). The basic idea is contained already in the work [12] and consists of introducing the smoothing operator $\delta\varrho\nabla\left(\mu'(\varrho)\Delta^{2s+1}\mu(\varrho)\right)$ with s sufficiently large, inspired from the capillarity forces [15]. In the next step we improve regularity of velocity using the biharmonic operator $\eta\Delta^2\mathbf{u}$. Finally, to get the estimate for the norm of $\Delta^{s+1}\varrho$ in $L^2((0,T)\times\Omega)$ at the level of Faedo-Galerkin approximation, we also need to regularize the the continuity equation by adding $\varepsilon\Delta\varrho$.

At the points when construction of approximate solution does not differ much from the case of single-component barotropic flow we present only main arguments and give the reference where all the details can be found. For the sake of simplicity we assume that $\mu(\varrho) = \varrho$, $\nu(\varrho) = 0$.

For the constant parameters $\varepsilon, \eta, \kappa_1, \kappa_2, \delta > 0$ (we skip all the indexes when no confusion can arise) we will be looking for a set of four functions $(\varrho, \mathbf{u}, \varrho_A, \varrho_B)$ satisfying the following regularization of the original system.

1. Approximate continuity equation:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0, \tag{3.44}$$

with the initial condition

$$\varrho(0,x) = \varrho_\delta^0(x), \tag{3.45}$$

where

$$\varrho_\delta^0 \in C^{2+\nu}(\Omega), \quad \inf_{x\in\Omega}\varrho_\delta^0(x) > 0. \tag{3.46}$$

2. The Faedo-Galerkin approximation for the weak formulation of the momentum balance:

$$\begin{aligned} &\int_\Omega \varrho\mathbf{u}(T)\boldsymbol{\phi}\,\mathrm{d}x - \int_\Omega \mathbf{m}^0\boldsymbol{\phi}\,\mathrm{d}x + \eta\int_0^T\int_\Omega \Delta\mathbf{u}\cdot\Delta\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega 2\varrho\mathbf{D}(\mathbf{u}) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \pi_{\kappa_2}(\varrho,\varrho_A,\varrho_B)\operatorname{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \delta\int_0^T\int_\Omega \varrho\nabla\Delta^{2s+1}\varrho\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_0^T\int_\Omega (\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t = 0 \end{aligned} \tag{3.47}$$

satisfied for any test function $\boldsymbol{\phi} \in X_n$, where $X_n = \operatorname{span}\{\boldsymbol{\phi}_i\}_{i=1}^n$ where $\{\boldsymbol{\phi}_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$, such that $\boldsymbol{\phi}_i \in W^{2,2}(\Omega)$ for all $i \in \mathbb{N}$. The regularized internal pressure is equal to

$$\pi_{\kappa_2}(\varrho,\varrho_A,\varrho_B)=\pi_c(\varrho)+\left(\frac{\varrho_A}{\sqrt{\varrho}m_A}+\frac{\varrho_B}{\sqrt{\varrho}m_B}\right)_{\kappa_2}\sqrt{\varrho}.$$

3. The species mass balance equations with truncated and regularized coefficients:

$$\begin{aligned}\partial_t\varrho_A-\varepsilon\Delta\varrho_A+\operatorname{div}(\varrho_A\mathbf{u})-\operatorname{div}\left(\left(\frac{\varrho_B^+}{\varrho m_A}+\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\nabla\varrho_A-\left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\nabla\varrho\right)&=\varrho\omega_{\kappa_1},\\ \partial_t\varrho_B-\varepsilon\Delta\varrho_B+\operatorname{div}(\varrho_B\mathbf{u})-\operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_B}+\frac{\varrho_B^+}{\varrho m_A}\right)_{\kappa_1}\nabla\varrho_B-\left(\frac{\varrho_B^+}{\varrho m_A}\right)_{\kappa_1}\nabla\varrho\right)&=-\varrho\omega_{\kappa_1},\end{aligned}\tag{3.48}$$

we set

$$\varrho_i^+=\begin{cases}0 & \text{if } \varrho_i<0,\\ \varrho_i & \text{if } 0\le\varrho_i<\varrho,\\ \varrho & \text{if } \varrho\le\varrho_i,\end{cases}\qquad \text{for } i\in S.\tag{3.49}$$

The initial conditions are

$$\begin{aligned}&\varrho_A(0,x)=\varrho^0_{A,\delta}(x),\quad \varrho_B(0,x)=\varrho^0_{B,\delta}(x),\\ &\varrho^0_{A,\delta},\varrho^0_{B,\delta}\in C^{2+\nu}(\Omega),\quad \varrho^0_{A,\delta}+\varrho^0_{B,\delta}=\varrho^0_\delta.\end{aligned}\tag{3.50}$$

Moreover, the restriction $\varrho_A(t,x)+\varrho_B(t,x)=\varrho(t,x)$ is satisfied for $(t,x)\in[0,T]\times\Omega$.

The operators $f\to f_{\kappa_i}$, $\kappa_i=(\kappa_t^i,\kappa_x^i)$, $i=1,2$ are the standard smoothing operators that apply to the variables x and t in the case of functions $\varrho,\varrho_A,\varrho_B$. However, the regularization over time in (3.48) means that instead of $\varrho,\varrho_A,\varrho_B$ we consider their continuous extensions respectively in the class $V_{\mathbb{R}}$ that will be specified later on. We also assume that the supports of these extensions are contained in the time-space cylinder $(-2T,2T)\times\Omega$. Hence we define

$$f_\kappa(s,y)=(f*\zeta_{\kappa_x})*\psi_{\kappa_t}=\int_{\mathbb{R}}\psi_{\kappa_t}(s-\tau)\int_{\mathbb{T}^3}\zeta_{\kappa_x}(y-z)f(\tau,z)\;\mathrm{d}z\;\mathrm{d}\tau,$$

where

$$\zeta_{\kappa_x}(y)=\frac{1}{\kappa_x^3}\zeta\left(\frac{y}{\kappa_x}\right)$$

and $\zeta(y)$ is a regularizing kernel

$$\zeta\in C_c^\infty(\mathbb{T}^3),\quad \operatorname{supp}\zeta\subset(-1,1)^3,\quad \zeta(y)=\zeta(-y)\ge0,\quad \int_{\mathbb{T}^3}\zeta(y)\;\mathrm{d}y=1.$$

Similarly, we define a regularizing kernel for the time coordinate

$$\psi\in C_c^\infty(\mathbb{R}),\;\; \operatorname{supp}\psi\subset(-1,1),\;\; \psi(s)=\psi(-s)\ge0,\;\; \int_{\mathbb{R}}\psi(s)\;\mathrm{d}s=1,\;\; \psi_{\kappa_t}(s)=\frac{1}{\kappa_t}\psi\left(\frac{s}{\kappa_t}\right).$$

We start with the proof of well posedness of our approximate system.

Theorem 3.11. *Let $\varepsilon,\kappa_1,\kappa_2,\eta,\delta$ be fixed positive parameters. Approximate problem (3.44-3.50) admits a strong solution $\{\varrho,\mathbf{u},\varrho_A,\varrho_B\}$ belonging to the regularity class*

$$\varrho\in C([0,T];C^{2+\nu}(\Omega)),\quad \partial_t\varrho,\in C([0,T];C^{0,\nu}(\Omega)),\quad \inf_{[0,T]\times\Omega}\varrho>0,\quad \mathbf{u}\in C^1([0,T],X_n),$$

$$\varrho_i\in L^\infty(0,T;W^{1,2}(\Omega)),\quad \partial_t\varrho_i,\Delta\varrho_i\in L^2((0,T)\times\Omega),\quad i\in\{A,B\},\quad \varrho_A+\varrho_B=\varrho.$$

Proof. The strategy of the proof is following:

1. We linearize system (3.48).
2. We set $\mathbf{u} \in C([0,T];X_n)$ for which we find the mappings $\mathbf{u} \mapsto \varrho(\mathbf{u})$ and $\mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ determining the unique solution to the continuity equation and the species mass balance equations.
3. For sufficiently small time interval $[0,\tau^0]$ we find the unique solution to the momentum equation applying the Banach fixed point theorem. Then we extend the existence result for the maximal time interval.
4. We recover the semi-linear system (3.48) using a version of Leray-Schauder fixed point theorem.

The proof will be given in the following subsections.

3.3.1 Continuity equation

Here we present the argument for existence of smooth, unique solution to problem (3.44-3.46) in the situation when the vector field $\mathbf{u}(x,t)$ is given and belongs to $C([0,T];X_n)$.
The following result can be proven by the Galerkin approximation and the well known statements about the regularity of linear parabolic systems (for the details of the proof see [36], Lemma 3.1).

Lemma 3.12. *Let $\mathbf{u} \in C([0,T];X_n)$ for n fixed and let $\varrho_\delta^0 \in C^{2+\nu}(\Omega)$, $\nu \in (0,1)$ be such that*

$$0 < \underline{\varrho^0} \le \varrho^0 \le \overline{\varrho^0} < \infty.$$

Then there exists the unique classical solution to (3.44-3.46), i.e. $\varrho \in V_{[0,T]}$, where

$$V_{[0,T]} = \left\{ \begin{array}{l} \varrho \in C\left([0,T];C^{2+\nu}(\Omega)\right), \\ \partial_t \varrho \in C\left([0,T];C^{0,\nu}(\Omega)\right). \end{array} \right\} \tag{3.51}$$

Moreover, the mapping $\mathbf{u} \mapsto \varrho(\mathbf{u})$ maps bounded sets in $C([0,T];X_n)$ into bounded sets in $V_{[0,T]}$ and is continuous with values in $C\left([0,T];C^{2+\nu'}(\Omega)\right)$, $0<\nu'<\nu<1$.
Finally,

$$\underline{\varrho^0} e^{-\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \le \varrho(\tau,x) \le \overline{\varrho^0} e^{\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \quad \text{for all } \tau \in [0,T],\ x \in \Omega. \tag{3.52}$$

3.3.2 Linearized species mass balance equations

In this subsection we shall prove the existence of solutions to the linearization of system (3.48). For $\widetilde{\varrho_A}, \widetilde{\varrho_B} \in L^\infty\left(0,T;W^{1,2}(\Omega)\right)$ fixed, $\mathbf{u}$ and $\varrho(\mathbf{u})$ satisfying the assumptions and assertion of Lemma 3.12, we investigate the following system of linear parabolic equations with smooth coefficients

$$\begin{aligned} &\partial_t \varrho_A - \varepsilon\Delta\varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A} + \frac{\widetilde{\varrho_A^+}}{\varrho m_B}\right)_{\kappa_1} \nabla\varrho_A - \left(\frac{\widetilde{\varrho_A^+}}{\varrho m_B}\right)_{\kappa_1} \nabla\varrho\right) = \varrho\left(\omega\left(\widetilde{\varrho_A}\right)\right)_{\kappa_1}, \\ &\partial_t \varrho_B - \varepsilon\Delta\varrho_B + \operatorname{div}(\varrho_B \mathbf{u}) - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_A^+}}{\varrho m_B} + \frac{\widetilde{\varrho_B^+}}{\varrho m_A}\right)_{\kappa_1} \nabla\varrho_B - \left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A}\right)_{\kappa_1} \nabla\varrho\right) = -\varrho\left(\omega\left(\widetilde{\varrho_A}\right)\right)_{\kappa_1}. \end{aligned} \tag{3.53}$$

The existence of unique solution to system (3.53) with the initial conditions (3.50) is stated in the following lemma.

Lemma 3.13. *Let $\kappa_1 > 0$ and assumptions of Lemma 3.12 be satisfied. Suppose that $\varrho^0_{A,\delta}, \varrho^0_{B,\delta} \in C^{2+\nu}(\Omega)$, then the problem (3.53) with the initial data (3.50) possesses the unique strong solution (ϱ_A, ϱ_B) belonging to the regularity class $\left(V_{[0,T]}\right)^2$.*
Moreover, the mapping $\mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ maps bounded sets in $C([0,T];X_n)$ into bounded sets in $\left(V_{[0,T]}\right)^2$ and is continuous with values in $\Big(C([0,T];C^{2+\nu'}(\Omega))\Big)^2$.
In addition

$$\varrho_A + \varrho_B = \varrho. \tag{3.54}$$

Proof. Existence of unique classical solution can be shown using classical results about solvability of the linear parabolic Cauchy problem with variable coefficients:

$$\begin{gathered}\mathcal{L}(t,x,\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial x})u = \\ \partial_t u - \textstyle\sum_{i,j=1}^n a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(t,x)\frac{\partial u}{\partial x_i} + a(t,x)u = f(t,x) \quad \text{in } (0,T)\times\mathbb{R}^3, \\ u(0,\cdot) = u^0.\end{gathered}$$

A relevant existence theory for such systems, not only within the framework of continuously differentiable functions but also for the Sobolev spaces can be found in the book of Ladyženskaja, Solonnikov and Uralceva [57]. Here, however, it is more convenient to apply the result from the analytic semigroup theory taken over from the book of Lunardi [63], which requires merely continuity of coefficients with respect to time.

Theorem 3.14 (Theorem 5.1.9 in [63])**.** *Let all the coefficients of operator $\mathcal{L}$ and f be uniformly continuous functions belonging to $C^{0,\nu}([0,T]\times\mathbb{R}^3)$, with $0<\nu<1$, and let $u^0 \in C^{2+\nu}(\mathbb{R}^3)$. Then the above problem has a unique solution from the class $u \in C^{1,2+\nu}([0,T]\times\mathbb{R}^3)$ which satisfies the inequality*

$$\|u\|_{C^{1,2+\nu}([0,T]\times\mathbb{R}^3)} \le c\left(\|f\|_{C^{0,\nu}([0,T]\times\mathbb{R}^3)} + \|u^0\|_{C^{2+\nu}(\mathbb{R}^3)}\right). \tag{3.55}$$

Note, in particular, that the assertion of Lemma 3.12 guaranties uniform continuity in the time interval [0,T] of the "worst" term proportional to $\Delta\varrho$ which plays the role of force in system (3.53). Thus, the existence of regular, unique solution belonging to the class $\left(V_{[0,T]}\right)^2$ is straightforward. The continuity of the mapping $\mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ follows from uniqueness of solution in the class $\left(V_{[0,T]}\right)^2$, compact embeddings in the spaces of Hölder continuous functions and the Arzelà-Ascoli theorem.

The proof of (3.54) follows by subtracting both equations of (3.48) from the approximate continuity equation, we obtain

$$\begin{gathered}\partial_t\xi - \varepsilon\Delta\xi + \operatorname{div}(\xi\mathbf{u}) - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A} + \frac{\widetilde{\varrho_A^+}}{\varrho m_B}\right)_{\kappa_1}\xi\right) = 0, \\ \xi(0,x) = 0,\end{gathered} \tag{3.56}$$

where we denoted $\xi = \varrho - \varrho_A - \varrho_B$. The unique solution of the resulting system must be, due to the initial condition, equal to 0 for (t,x) in $[0,T]\times\Omega$.
By this remark, the proof of Lemma 3.13 is complete. □

3.3.3 Momentum equation

Now we prove that there exists $T = T(n)$ and $\mathbf{u} \in C([0,T]; X_n)$ satisfying (3.47). To this purpose we apply the fixed point argument to the mapping

$$\mathcal{T} : C([0,T]; X_n) \to C([0,T]; X_n), \quad \mathcal{T}[\mathbf{u}](t) = \mathcal{M}_{\varrho(t)} \left[P_n \mathbf{m}^0 + \int_0^t P_n \mathcal{N}(\mathbf{u})(s) \mathrm{d}s \right], \tag{3.57}$$

where P_n is the orthogonal projection of $L^2(\Omega)$ onto X_n,

$$\mathcal{N}(\mathbf{u}) = -\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \operatorname{div}(2\varrho \mathbf{D}(\mathbf{u})) + \nabla \pi_{\kappa_2} - \delta \varrho \nabla \Delta^{2s+1} \varrho + \eta \Delta^2 \mathbf{u} + \varepsilon (\nabla \varrho \cdot \nabla) \mathbf{u}$$

and

$$\mathcal{M}_\varrho [\cdot] : X_n \to X_n, \int_\Omega \varrho \mathcal{M}_\varrho [\mathbf{w}] \phi \, \mathrm{d}x =< \mathbf{w}, \phi >, \quad \mathbf{w}, \phi \in X_n.$$

First, observe that $P_n \mathcal{N}(\mathbf{u})(t)$ is bounded in X_n for $t \in [0,T]$. Using the equivalence of norms on the finite dimensional space X_n we can easily check that

$$\begin{aligned} \|P_n \mathcal{N}(\mathbf{u})\|_{X_n} \le c \big[\|\mathbf{u}\|_{X_n} + \|\varrho\|_{L^\infty(\Omega)} \left(\|\mathbf{u}\|^2_{X_n} + \|\mathbf{u}\|_{X_n} \right) \\ + \|\varrho\|^\gamma_{L^\infty(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} \|\varrho\|_{W^{4s+3,\infty}(\Omega)} \big]. \end{aligned} \tag{3.58}$$

To justify that the last term on the r.h.s. is bounded, one needs to know that the unique solution ϱ to the approximate continuity equation (3.44) is more regular than it was indicated in Lemma 3.12. More precisely, using the fact that $\mathbf{u}$ is actually smooth with respect to space, we can put the term $\operatorname{div}(\varrho \mathbf{u})$ to the r.h.s. of (3.44) and then bootstrap the procedure leading to regularity (3.51), see e.g. [57], Chapter IV. By this argument, the term $P_n \varrho \nabla \Delta^{2s+1} \varrho$ in the approximate momentum equation makes sense, i.e. it is bounded in $L^1(0,T; X_n)$.
Concerning the operator $\mathcal{M}_\varrho$, it is easy to see that provided $\varrho(t,x) \ge \underline{\varrho} > 0$, one has

$$\|\mathcal{M}_\varrho\|_{\mathcal{L}(X_n, X_n)} \le \underline{\varrho}^{-1}.$$

Moreover, since $\mathcal{M}_\varrho - \mathcal{M}_{\varrho'} = \mathcal{M}_{\varrho'} \left(\mathcal{M}^{-1}_{\varrho'} - \mathcal{M}^{-1}_\varrho \right) \mathcal{M}_\varrho$ we verify that

$$\|\mathcal{M}_{\varrho(t)} - \mathcal{M}_{\varrho'(t)}\|_{\mathcal{L}(X_n, X_n)} \le c \underline{\varrho}^{-2} \|(\varrho - \varrho')(t)\|_{L^1(\Omega)}$$

for $t \in [0,T]$. Thus, by virtue of continuity of mappings $\mathbf{u} \to \varrho(\mathbf{u})$ and $\mathbf{u} \to (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ and the estimates established in Lemmas 3.12 and 3.13 one can check that $\mathcal{T}[\mathbf{u}]$ maps the ball

$$B_{R,\tau^0} = \left\{ \mathbf{u} \in C([0,\tau^0], X_n) : \|\mathbf{u}\|_{C([0,\tau^0], X_n)} \le R, \mathbf{u}(0,x) = P_n \left(\frac{\mathbf{m}^0}{\varrho^0_\delta} \right) \right\}$$

into itself and it is a contraction, for sufficiently small $\tau^0 > 0$. It therefore possesses the unique fixed point satisfying (3.47) on the time interval $[0, \tau^0]$. In view of previous remarks, the proof of this step can be done by a minor modification of the procedure described in [85], Section 7.7, so we skip this part.
Additionally, the time regularity of $\mathbf{u}$ may be improved by differentiating (3.57) with respect to time and estimating the norm of the resulting r.h.s. in X_n, so we get

$$\mathbf{u} \in C^1([0,\tau^0], X_n).$$

This is the crucial information that enables to extend this solution to the maximal time interval $[0,T]$. Indeed, provided the system enjoys the estimates independent of τ^0, we can iterate the local construction of solution described above to get the solution for any $T > 0$. The existence of such a bound is based on the energy estimate and a bound from below for the density (3.52). Both of them can be derived analogously to [85], so for the sake of consistency, we recall here only the idea of the proof.

We first differentiate (3.47) with respect to t, then we observe that it is possible to use $\mathbf{u}$ as a test function, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho|^2 + \varrho e_c(\varrho)\right)\,\mathrm{d}x + \int_\Omega \left(2\varrho|\mathbf{D}(\mathbf{u})|^2 + \eta|\Delta\mathbf{u}|^2 + \delta\varepsilon|\Delta^{s+1}\varrho|^2\right)\,\mathrm{d}x$$
$$\leq \int_\Omega \left(\frac{\varrho_A}{\sqrt{\varrho}m_A} + \frac{\varrho_B}{\sqrt{\varrho}m_B}\right)_{\kappa_2}\sqrt{\varrho}\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x, \quad (3.59)$$

where $\varrho^2\frac{\mathrm{d}e_c(\varrho)}{\mathrm{d}\varrho} = \pi_c(\varrho)$.
Applying the Cauchy inequality (with λ) we see that the r.h.s. may be bounded as follows

$$\left|\int_\Omega \left(\frac{\varrho_A}{\sqrt{\varrho}m_A} + \frac{\varrho_B}{\sqrt{\varrho}m_B}\right)_{\kappa_2}\sqrt{\varrho}\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x\right| \leq \lambda\int_\Omega \varrho|\,\mathrm{div}\,\mathbf{u}|^2\,\mathrm{d}x + c(\lambda,\kappa_2)$$
$$\leq 3\lambda\int_\Omega \varrho|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x + c(\lambda,\kappa_2), \quad (3.60)$$

where the last inequality in (3.60) follows by the following observation

$$(\mathrm{div}\,\mathbf{u})^2 = \sum_{i,j=1}^3 \partial_i u_i \partial_j u_j \leq \sum_{i,j=1}^3 \frac{1}{2}\left((\partial_i u_i)^2 + (\partial_j u_j)^2\right) \leq 3|\mathbf{D}(\mathbf{u})|^2.$$

Hence, for λ sufficiently small, the r.h.s. of (3.59) can be absorbed by the l.h.s. and we get several, uniform in time estimates, in particular

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\eta}\Delta\mathbf{u} \in L^2(0,T;L^2(\Omega)). \quad (3.61)$$

From these bounds, using the estimate for $\varrho\pi(\varrho)$ and the Korn-Poincaré inequality we deduce boundedness of the $L^2(0,T;W^{2,2}(\Omega))$ norm of $\mathbf{u}$. Next, by the equivalence of norms of $\mathbf{u}$ we actually have that $\mathbf{u} \in L^2(0,T;X_n)$ also $\mathbf{u} \in L^2(0,T;W^{1,\infty}(\Omega))$. Therefore the bounds from below and above for ϱ can be derived exactly as in the proof of estimate (3.52) from Lemma 3.12. This in turn allows us to explore the uniform estimate on $\varrho|\mathbf{u}|^2$ following from (3.59) to show the boundedness of $\mathbf{u}$ in $C([0,T];L^2(\Omega))$. Having this, we can again take advantage of equivalence of norms, to deduce that we have uniformly in time

$$\|\mathbf{u}\|_{C([0,T];X_n)} \leq c.$$

At this point we can return to the procedure of construction of local in time solution and repeat it until we reach an approximate solution defined on $[0,T]$ for arbitrary large, but finite $T > 0$, exactly as in [85], Section 7.7.

From this we can deduce boundedness of the $L^2(0,T;W^{2,2}(\Omega))$ norm of $\mathbf{u}$.

3.3.4 Nonlinear equations of species mass conservation

Completing the proof of Theorem 3.11 requires to check that the original system (3.48) can be recovered. To this purpose we will need the following version of the fixed point theorem (for the proof see e.g. [44], Theorem 11.3)

Theorem 3.15. *Let $\mathcal{T}: X \to X$ be a continuous, compact mapping, X a Banach space. Let for any $\lambda \in [0,1]$ the fixed points $\lambda \mathcal{T} u = u$, $u \in X$ be bounded. Then $\mathcal{T}$ possesses at least one fixed point in X.*

We will apply it to the mapping

$$\mathcal{T}: W_{[0,T]} \times W_{[0,T]} \to W_{[0,T]} \times W_{[0,T]}, \quad \mathcal{T}(\widetilde{\varrho_A}, \widetilde{\varrho_B}) = (\varrho_A, \varrho_B),$$

where (ϱ_A, ϱ_B) is a unique, global in time solution to system (3.53) and $W_{[0,T]}$ denotes the following class of functions

$$W_{[0,T]} = \{L^2(0,T; W^{2,2}(\Omega)) \cap L^\infty(0,T; W^{1,2}(\Omega))\}. \tag{3.62}$$

For κ_1 fixed we show the boundedness of $\mathcal{T}$ in the class $\left(V_{[0,T]}\right)^2$ using Theorem 3.14, moreover, the obtained solution is unique. Therefore, proving compactness and continuity of this mapping in $C([0,T]; C^{2+\nu'}(\Omega))$ follows exactly as in the proof of Lemma 3.13.
The only assumption of the theorem above that needs to be checked is that any fixed point to $\lambda \mathcal{T}(\varrho_A, \varrho_B) = (\varrho_A, \varrho_B)$ is bounded for $\lambda \in [0,1]$. This identity rewrites as

$$\begin{aligned}\partial_t \varrho_A - \left(\varepsilon + \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) \Delta \varrho_A + \left(\mathbf{u} - \nabla \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) \nabla \varrho_A + \operatorname{div} \mathbf{u} \varrho_A \\ = \lambda \varrho \left(\omega\left(\varrho_A\right)\right)_{\kappa_1} - \lambda \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho\right),\end{aligned} \tag{3.63}$$

and similarly for the species B. So, we first multiply the above equation by ϱ_A and we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\varrho_A^2}{2} \,\mathrm{d}x + \int_\Omega \left(\varepsilon + \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) |\nabla \varrho_A|^2 \,\mathrm{d}x \\ = \int_\Omega \varrho_A \mathbf{u} \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \varrho_A \,\mathrm{d}x.\end{aligned} \tag{3.64}$$

The right hand side can be estimated due to assumed regularity of $\varrho, \mathbf{u}$ and by the definition of $\omega\left(\varrho_A\right)$, we obtain

$$\begin{aligned}\left| \int_\Omega \varrho_A \mathbf{u} \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \varrho_A \,\mathrm{d}x \right| \\ \le \|\mathbf{u}\|_{L^\infty(\Omega)} \|\varrho_A\|_{L^2(\Omega)} \|\nabla \varrho_A\|_{L^2(\Omega)} + c \|\nabla \varrho_A\|_{L^2(\Omega)} \|\nabla \varrho\|_{L^2(\Omega)} + c\overline{\omega} \|\varrho\|_{L^\infty(\Omega)} \|\varrho_A\|_{L^1(\Omega)},\end{aligned} \tag{3.65}$$

where the r.h.s. is absorbed by the l.h.s. after application of the Cauchy inequality. The same holds for ϱ_B. Next, multiplying (3.63) by $\partial_t \varrho_A$ we get

$$\begin{aligned}\int_\Omega |\partial_t \varrho_A|^2 \,\mathrm{d}x + \frac{\varepsilon}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \varrho_A|^2 \,\mathrm{d}x + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} |\nabla \varrho_A|^2 \,\mathrm{d}x \\ = -\int_\Omega \left(\nabla \varrho_A \cdot \mathbf{u}\, \partial_t \varrho_A + \varrho_A \operatorname{div} \mathbf{u}\, \partial_t \varrho_A\right) \,\mathrm{d}x - \lambda \int_\Omega \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_A}\right)_{\kappa_1} \nabla \varrho\right) \partial_t \varrho_A \,\mathrm{d}x \\ + \frac{1}{2} \int_\Omega \partial_t \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} |\nabla \varrho_A|^2 \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \partial_t \varrho_A \,\mathrm{d}x.\end{aligned}$$

By the properties of mollifiers, regularity of ϱ and $\mathbf{u}$ we can estimate the r.h.s., note, however, that this cannot be done independently of κ.
Reassuming, we have shown that

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_A\|^2_{W^{1,2}(\Omega)} + \int_0^T \|\partial_t \varrho_A\|_{L^2(\Omega)}\,\mathrm{d}t \le c(\kappa_1) \tag{3.66}$$

and from this we may deduce that also

$$\|\nabla^2 \varrho_A\|_{L^2((0,T)\times\Omega)} \le c(\kappa_1). \tag{3.67}$$

Moreover, the fixed point satisfies $\varrho_A + \varrho_B = \varrho$, so the proof of Theorem 3.11 is now complete.
□

3.4 Second level of approximation

In this section we first derive the estimates uniform with respect to κ_1 and then subtract subsequences in order to let $\kappa_1 \to 0$ in the approximate system. Having this, we prove that the species densities ϱ_A, ϱ_B are nonnegative, which is necessary to remove truncations from the coefficients of system (3.48). The last part of this section is devoted to the limit passage with the dimension of the Faedo-Galerkin approximation. Observe that the final regularity of solutions does not allow to test the momentum equation by $\mathbf{u}$, it is, however, sufficient to use $\nabla \log \varrho$ instead and hence we end up with the Bresch-Desjardins estimate, as it was announced in the introduction.

3.4.1 Passage to the limit $\kappa_1, \kappa_2 \to 0$

From what was written in the previous section, we deduce that the first energy estimate holds independently of κ_1, thus we have

$$\begin{aligned}&\|\sqrt{\varrho_{\kappa_1}}\mathbf{u}_{\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{\varrho_{\kappa_1}}\nabla \mathbf{u}_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))} + \|\sqrt{\eta}\Delta \mathbf{u}_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))}\\ &+ \|\sqrt{\varepsilon\delta}\Delta^{s+1}\varrho_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))} + \|\sqrt{\delta}\nabla\Delta^s \varrho_{\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|\pi_c(\varrho_{\kappa_1})\|_{L^\infty(0,T;L^1(\Omega))} \le c. \end{aligned}\tag{3.68}$$

By this we see that the construction of $\varrho_{\kappa_1}(\mathbf{u}_{\kappa_1})$ performed in Lemma 3.12 can be repeated. In particular, the sequence ϱ_{κ_1} is uniformly separated from 0 as long as n is fixed.
In addition, repeating estimate (3.64) we verify that also

$$\|\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}\|_{L^2(0,T;W^{1,2}(\Omega))} \le c. \tag{3.69}$$

Thus, the time derivatives of $\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}$ can be estimated in $L^2(0,T;W^{-1,2}(\Omega))$ directly from (3.53).

Having n fixed, all the norms of $\mathbf{u}_{\kappa_1}$ are equivalent and the limit function $\mathbf{u} \in C([0,T];X_n)$, thus passage to the limit in the continuity equation is trivial and the limit $\varrho \in V_{[0,T]}$ on account of Lemma 3.12. Concerning the species mass balance equations, the Aubin-Lions argument can be applied and we get compactness of ϱ_{A,κ_1} in $L^2(0,T;L^q(\Omega))$ for $q < 6$, in particular $\varrho_{A,\kappa_1} \to \varrho_A$ a.e. on $(0,T)\times\Omega$. By this and the bounds from (3.68) we easily check that the limit equation of mass conservation of species A

$$\partial_t \varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) - \varepsilon\Delta\varrho_A - \operatorname{div}\left(\left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)\nabla\varrho_A - \left(\frac{\varrho_A^+}{\varrho m_B}\right)\nabla\varrho\right) = \varrho\omega, \tag{3.70}$$

is satisfied in the sense of distributions on $(0,T)\times\Omega$, but the standard density argument enables to extend the class of test functions to $L^2(0,T;W^{1,2}(\Omega))$. Moreover, due to (3.69) the initial condition is satisfied in the sense of distributions on Ω. Similarly for ϱ_B.
The passage to the limit in the momentum equation is straightforward.

3.4.2 Non-negativity of partial densities

Our next goal is to deduce from the form of system (3.44-3.50) that for $\kappa = 0$ the limit functions ϱ_A, ϱ_B satisfy not only the mass constraint (3.54) but also they are nonnegative a.e. in $(0,T)\times\Omega$.

We have

Lemma 3.16. *Let $\delta, \varepsilon, \eta > 0$, n be fixed natural number and let $(\varrho, \mathbf{u}, \varrho_A, \varrho_B)$ be a solution to (3.44-3.50) with $\kappa = 0$ as specified above. Then*

$$\varrho_A = \varrho_A^+, \quad \varrho_B = \varrho_B^+ \quad \textit{a.e. in } (0,T)\times\Omega.$$

Proof. In what follows, we focus only on the proof of nonnegativity of ϱ_A, the case of ϱ_B can be shown analogously. By virtue of (3.69), we are allowed to test (3.70) with a function $(\varrho_{A-}+l)^{q-1}$, $l > 0$, $q \in (1,2]$, where

$$\varrho_{A-} = \begin{cases} -\varrho_A & \text{if } \varrho_A < 0, \\ 0 & \text{if } 0 \le \varrho_A, \end{cases}$$

and then pass to the limit $l \to 0^+$. Observe that $\varrho_A^+\varrho_{A-} = 0$ and $\varrho_B^+\varrho_{A-} = \varrho\varrho_{A-}$ in case when $\varrho_A < 0$ or $\varrho_B^+\varrho_{A-} = 0$ for $\varrho_A \ge 0$, thus

$$-\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_{A-}^q \,\mathrm{d}x - \frac{4\varepsilon(q-1)}{q^2}\int_\Omega \left|\nabla\varrho_{A-}^{q/2}\right|^2 \mathrm{d}x - \frac{4(q-1)}{m_A q^2}\int_\Omega \left|\nabla\varrho_{A-}^{q/2}\right|^2 \mathrm{d}x$$
$$= (1-q)\int_\Omega \mathbf{u}\cdot\nabla\varrho_{A-}\varrho_{A-}^{q-1}\,\mathrm{d}x + \int_\Omega \varrho\omega\left(\frac{\varrho_A}{\varrho}\right)\varrho_{A-}^{q-1}\,\mathrm{d}x. \quad (3.71)$$

Since $\varrho_{A-} \ge 0$ enforces $\omega\left(\frac{\varrho_A}{\varrho}\right) \ge 0$, we can put the last term from the r.h.s. to the l.h.s., so multiplying the above expression by -1 we get

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_{A-}^q \,\mathrm{d}x + \frac{4\varepsilon(q-1)}{q^2}\int_\Omega \left|\nabla\varrho_{A-}^{q/2}\right|^2 \mathrm{d}x + \frac{4(q-1)}{m_A q^2}\int_\Omega \left|\nabla\varrho_{A-}^{q/2}\right|^2 \mathrm{d}x + \int_\Omega \varrho\omega\left(\frac{\varrho_A}{\varrho}\right)\varrho_{A-}^{q-1}\,\mathrm{d}x$$
$$= \frac{2(q-1)}{q}\int_\Omega \mathbf{u}\cdot\nabla\varrho_{A-}^{q/2}\varrho_{A-}^{q/2}\,\mathrm{d}x. \quad (3.72)$$

Now, the r.h.s. may be bounded by use of the Cauchy inequality

$$\left|\int_\Omega \mathbf{u}\cdot\nabla\varrho_{A-}^{q/2}\varrho_{A-}^{q/2}\,\mathrm{d}x\right| \le \|\mathbf{u}\|_{L^\infty(\Omega)}\left(\epsilon\|\nabla\varrho_{A-}^{q/2}\|_{L^2(\Omega)}^2 + c(\epsilon)\|\varrho_{A-}^{q/2}\|_{L^2(\Omega)}^2\right)$$

and the first of the resulting terms is absorbed by the l.h.s. of (3.72) provided $\frac{4}{m_A q^2} > \frac{2\epsilon\|\mathbf{u}\|_\infty}{q}$, while the other is bounded since $\varrho_A \in L^\infty(0,T;L^2(\Omega))$.

Further, as the three last terms from the l.h.s. of (3.72) are nonnegative we get that $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_{A-}^q\,\mathrm{d}x \le c(q-1)$, thus, passing to the limit $q \to 1^+$ and integrating by time we conclude that

$$\int_\Omega \varrho_{A-}(t)\,\mathrm{d}x \le \int_\Omega \varrho_{A-}(0)\,\mathrm{d}x.$$

Since the integrant from the r.h.s. is equal to 0 a.e. in Ω, there must be $\varrho_{A-}(t,x) = 0$ a.e. in $(0,T)\times\Omega$. □

Obviously, positiveness of species masses coupled with (3.54) leads to the following inequality

$$0 \le \varrho_A, \varrho_B \le \varrho, \quad \text{a.e. in } (0,T)\times\Omega.$$

This fact allows us to verify that the estimates uniform with respect to κ_1 are in fact uniform with respect to κ_2. Therefore passage to the limit $\kappa_2 \to 0$ can be performed identically as the previous one.

3.4.3 Passage to the limit with dimension of the Galerkin approximation

Observe that the estimates derived in the previous section are independent of n. In particular, due to bounds from (3.68) we deduce that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{2,2}(\Omega)), \tag{3.73}$$

and

$$\varrho_n \to \varrho \quad \text{weakly in } L^2(0,T;W^{2s+2,2}(\Omega)), \tag{3.74}$$

at least for a suitable subsequence. In addition the r.h.s. of the linear parabolic problem

$$\begin{aligned} \partial_t\varrho_n - \varepsilon\Delta\varrho_n &= \operatorname{div}(\varrho_n\mathbf{u}_n), \\ \varrho_n(0,x) &= \varrho^0_\delta, \end{aligned}$$

is uniformly bounded in $L^2(0,T;L^6(\Omega))$ and the initial condition is sufficiently smooth, thus, applying the $L^p - L^q$ theory to this problem we conclude that $\{\partial_t\varrho_n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;L^6(\Omega))$. Thus, the standard compact embeddings imply $\varrho_n \to \varrho$ a.e. in $(0,T)\times\Omega$ and therefore passage to the limit in the approximate continuity equation is straightforward. Having that, we can also identify the limit for $n\to\infty$ in all terms of the momentum equation, except for the convective term, the additional capillarity force and the pressure.

To handle the first one observe that

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)),$$

due to the uniform estimates (3.68) and the strong convergence of the density. Next, one can show that for any $\phi \in \cup_{n=1}^\infty X_n$ the family of functions $\int_\Omega \varrho_n\mathbf{u}_n(t)\phi \,\mathrm{d}x$ is bounded and equi-continuous in $C([0,T])$, thus via the Arzelà-Ascoli theorem and density of smooth functions in $L^2(\Omega)$ we get that

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{in } C([0,T];L^2_{\text{weak}}(\Omega)). \tag{3.75}$$

Finally, by the compact embedding $L^2(\Omega)\subset W^{-1,2}(\Omega)$ and the weak convergence of $\mathbf{u}_n$ (3.73) we verify that

$$\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega).$$

Concerning the capillarity term, we first rewrite it in the form

$$\int_\Omega \varrho_n\nabla\Delta^{2s+1}\varrho_n\cdot\phi\,\mathrm{d}x = \int_\Omega \Delta^s\operatorname{div}(\varrho_n\phi)\,\Delta^{s+1}\varrho_n\,\mathrm{d}x.$$

Due to (3.74) and boundedness of the time derivative of ϱ_n, we infer that

$$\varrho_n \to \varrho \quad \text{strongly in } L^2(0,T;W^{2s+1,2}(\Omega)), \tag{3.76}$$

thus

$$\int_\Omega \Delta^s\operatorname{div}(\varrho_n\phi)\,\Delta^{s+1}\varrho_n\,\mathrm{d}x \to \int_\Omega \Delta^s\operatorname{div}(\varrho\phi)\,\Delta^{s+1}\varrho\,\mathrm{d}x,$$

for every $\phi\in\cup_{n=1}^\infty X_n$. Moreover, by the penultimate estimate of (3.68) and since the set $\cup_{n=1}^\infty X_n$ is dense in $W^{2s+1}(\Omega)$, this convergence holds for all $\phi\in L^2(0,T;W^{2s+1}(\Omega))$.

Passage to the limit in the molecular part of the pressure is an easy task, since due to (3.69) there exist the subsequences such that

$$\varrho_{k,n}\to\varrho_k, \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\ k\in S.$$

So the only uncertain part is the nonlinear barotropic pressure. Its strong convergence is a consequence of pointwise convergence of the density, and the bounds from (3.68). Taking s sufficiently large we can show that the density is separated from 0 uniformly with respect to all approximation parameters except for δ. Indeed, since by the Sobolev embedding $\|\varrho^{-1}\|_{L^\infty(\Omega)} \leq c\|\varrho^{-1}\|_{W^{3,k}(\Omega)}$ for $k > 1$ and

$$\|\nabla^3 \varrho^{-1}\|_{L^k(\Omega)} \leq (1 + \|\nabla^3 \varrho\|_{L^{2k}(\Omega)})^3 (1 + \|\varrho^{-1}\|_{L^{4k}(\Omega)})^4,$$

is bounded on account of (3.68), provided that $2s + 1 \geq 4$, we have

$$\|\varrho^{-1}\|_{L^\infty((0,T)\times\Omega)} \leq c(\delta) \quad \text{a.e. in } (0,T) \times \Omega. \tag{3.77}$$

By this observation, passage to the limit $n \to \infty$ in the species mass balance equations may be performed identically as the passage $\kappa_1 \to 0$ from the previous subsection.

Note that, due to the weak lower semicontinuity of convex functions we can pass to the limit in (3.59). Indeed, by the strong convergence of density and velocity we check that

$$\int_\Omega \left(\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + \frac{\delta}{2} |\nabla^{2s+1} \varrho_n|^2 + \varrho_n e_c(\varrho_n) \right) \, \mathrm{d}x \to \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{\delta}{2} |\nabla^{2s+1} \varrho|^2 + \varrho e_c(\varrho) \right) \, \mathrm{d}x \tag{3.78}$$

in the sense of distributions on $(0,T)$ and for any smooth, nonnegative function $\psi \in C^\infty([0,T])$ we have

$$\begin{aligned}
&\int_0^T \psi \int_\Omega 2\varrho |\mathbf{D}(\mathbf{u})|^2 \, \mathrm{d}x \, \mathrm{d}t + \eta \int_0^T \psi \int_\Omega |\Delta \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t + \delta\varepsilon \int_0^T \psi \int_\Omega |\Delta^{s+1} \varrho|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \liminf_{n\to\infty} \int_0^T \psi \int_\Omega 2\varrho_n |\mathbf{D}(\mathbf{u}_n)|^2 \, \mathrm{d}x \, \mathrm{d}t + \eta \int_0^T \psi \int_\Omega |\Delta \mathbf{u}_n|^2 \, \mathrm{d}x \, \mathrm{d}t + \delta\varepsilon \int_0^T \psi \int_\Omega |\Delta^{s+1} \varrho|^2 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{3.79}$$

Similar argument can be applied to the integral form of the energy inequality which i now satisfied for a.a. τ in $(0,T)$

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{\delta}{2} |\nabla^{2s+1} \varrho|^2 + \varrho e_c(\varrho) \right) (\tau) \, \mathrm{d}x + \int_0^\tau \int_\Omega 2\varrho |\mathbf{D}(\mathbf{u})|^2 + \eta |\Delta \mathbf{u}|^2 + \delta\varepsilon |\Delta^{s+1} \varrho|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq \int_\Omega \left(\frac{1}{2} \varrho^0 |\mathbf{u}^0|^2 + \frac{\delta}{2} |\nabla^{2s+1} \varrho^0|^2 + \varrho^0 e_c(\varrho^0) \right) \, \mathrm{d}x + \int_0^\tau \int_\Omega \left(\frac{\varrho_A}{m_A} + \frac{\varrho_B}{m_B} \right) \operatorname{div} \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{3.80}$$

and the r.h.s. is bounded.

3.5 Third level of approximation

As an outcome of Section 3.4 we obtain the weak solution to the following system

$$\begin{gathered}
\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0,\\
\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(2\varrho\mathbf{D}(\mathbf{u})) + \nabla\pi - \delta\varrho\nabla\Delta^{2s+1}\varrho + \varepsilon(\nabla\varrho\cdot\nabla)\mathbf{u} + \eta\Delta^2\mathbf{u} = \mathbf{0},\\
\partial_t\varrho_A - \varepsilon\Delta\varrho_A + \operatorname{div}(\varrho_A\mathbf{u}) - \operatorname{div}\left(\left(\tfrac{\varrho_B}{\varrho m_A} + \tfrac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_A - \left(\tfrac{\varrho_A}{\varrho m_B}\right)\nabla\varrho\right) = \varrho\omega,\\
\partial_t\varrho_B - \varepsilon\Delta\varrho_B + \operatorname{div}(\varrho_B\mathbf{u}) - \operatorname{div}\left(\left(\tfrac{\varrho_A}{\varrho m_B} + \tfrac{\varrho_B}{\varrho m_A}\right)\nabla\varrho_B - \left(\tfrac{\varrho_B}{\varrho m_A}\right)\nabla\varrho\right) = -\varrho\omega,
\end{gathered} \tag{3.81}$$

where the first equation holds a.e. on $(0,T)\times\Omega$, and the remaining ones are satisfied in the sense of distributions. Moreover, it results from the construction that

$$0 \le \varrho_A, \varrho_B \le \varrho, \quad \text{and} \quad \varrho_A + \varrho_B = \varrho \quad \text{a.e. in } (0,T)\times\Omega$$

and the energy inequality (3.59) is satisfied in the sense of distributions on $(0,T)$.

3.5.1 Estimates independent of ε, η and δ

At this level of approximation, it is relatively easy to derive the Bresch-Desjardins inequality. Indeed, as we know now that $\varrho \in L^2(0,T;W^{2s+2,2}(\Omega)) \cap L^\infty((0,T)\times\Omega)$ and $\mathbf{u} \in L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ we can differentiate the approximate continuity equation with respect to x to observe that $\nabla\varrho$ satisfies the following system

$$\begin{aligned}
\partial_t(\nabla\varrho) - \varepsilon\Delta(\nabla\varrho) &= -\nabla\operatorname{div}(\varrho\mathbf{u}),\\
\nabla\varrho(0,x) &= \nabla\varrho^0_\delta.
\end{aligned} \tag{3.82}$$

Since the r.h.s. of (3.82) is bounded in $L^2((0,T)\times\Omega)$, we can again apply the maximal $L^p - L^q$ theory for such problems to deduce that $\partial_t\nabla\varrho \in L^2((0,T)\times\Omega)$. Hence, the function $\nabla\phi = 2\frac{\nabla\varrho}{\varrho}$ belongs to $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2s+1,2}(\Omega))$, thus it is an admissible test function for the approximate momentum equation $(3.81)_2$.
Our next aim is to prove the following inequality:

Lemma 3.17. *We have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho e_c(\varrho)\right)\,\mathrm{d}x + \int_\Omega \nabla\phi(\varrho)\cdot\nabla\pi(\varrho,\varrho_A)\,\mathrm{d}x\\
&\qquad + \frac{1}{2}\int_\Omega \varrho|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,\mathrm{d}x + \int_\Omega\left(2\delta|\Delta^{s+1}\varrho|^2 + \delta\varepsilon|\Delta^{s+1}\varrho|^2\right)\,\mathrm{d}x + \eta\int_\Omega|\Delta\mathbf{u}|^2\,\mathrm{d}x\\
&\quad \le -\varepsilon\int_\Omega(\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x + \varepsilon\int_\Omega\Delta\varrho\frac{|\nabla\phi|^2}{2}\,\mathrm{d}x + \varepsilon\int_\Omega\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)\,\mathrm{d}x\\
&\qquad - \varepsilon\int_\Omega\operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\,\mathrm{d}x - \eta\int_\Omega\Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\,\mathrm{d}x + \int_\Omega\left(\frac{\varrho_A}{m_A}+\frac{\varrho_B}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x
\end{aligned} \tag{3.83}$$

in $\mathcal{D}'(0,T)$, *where* $e_c(\varrho) = \int_0^\varrho y^{-2}\pi_c(y)\;dy \ge 0$.

Proof. The basic idea of the proof is to find the explicit form of the term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\mathbf{u}\cdot\nabla\phi(\varrho) + \varrho|\nabla\phi(\varrho)|^2\right)\,\mathrm{d}x. \tag{3.84}$$

For this purpose we first multiply the approximate continuity equation by $\frac{|\nabla\phi(\varrho)|^2}{2}$ and we obtain the following sequence of equalities

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\,\mathrm{d}x \\
&=\int_\Omega\left(\varrho\partial_t\frac{|\nabla\phi(\varrho)|^2}{2}-\frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})+\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x \\
&=\int_\Omega\left(\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\partial_t\varrho\right)-\frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})+\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x \\
&=\int_\Omega\left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)-\varrho\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\phi(\varrho)+\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x \\
&\quad-\int_\Omega\left(\frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})+\varrho\mathbf{u}\otimes\nabla\phi(\varrho):\nabla^2\phi(\varrho)+\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\varrho\operatorname{div}\mathbf{u}\right)\right)\,\mathrm{d}x \\
&=\int_\Omega\left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)-\varrho\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\phi(\varrho)+\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x \\
&\quad+\int_\Omega\left(\varrho\mathbf{u}\Delta\phi(\varrho)\cdot\nabla\phi(\varrho)+\frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})-\operatorname{div}(\varrho\mathbf{u}\otimes\nabla\phi(\varrho)\nabla\phi(\varrho))\right)\,\mathrm{d}x \\
&\quad+\int_\Omega\left(\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u}+\varrho\,|\nabla\phi(\varrho)|^2\operatorname{div}\mathbf{u}\right)\,\mathrm{d}x \\
&=\int_\Omega\left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)-\varrho\nabla\mathbf{u}:\nabla\phi(\varrho)\otimes\nabla\phi(\varrho)+\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\,\mathrm{d}x \\
&\quad+\int_\Omega\left(\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u}+\varrho\,|\nabla\phi(\varrho)|^2\operatorname{div}\mathbf{u}\right)\,\mathrm{d}x.
\end{aligned}
\tag{3.85}
$$

In the above series of equalities, each one holds pointwisely with respect to time due to the regularity of ϱ and $\nabla\phi$. This is not the case of the middle integrant of (3.84), for which one should really think of weak in time formulation. Denote

$$V=W^{2s+1,2}(\Omega),\quad H=L^2(\Omega)\quad\text{and}\quad\mathbf{v}=\varrho\mathbf{u},\quad\mathbf{h}=\nabla\phi.$$

We know that $\mathbf{v}\in L^2(0,T;H)$ and its weak derivative with respect to time variable $\mathbf{v}'\in L^2(0,T;V^*)$, where V^* denotes the dual space to V. Moreover, $\mathbf{h}\in L^2(0,T;V)$, $\mathbf{h}'\in L^2(0,T;H^*)$. Now, let $\mathbf{v}_m$, $\mathbf{h}_m$ denote the standard mollifications in time of $\mathbf{v}$ and $\mathbf{h}$ respectively. By the properties of mollifiers we know that

$$\mathbf{v}_m,\mathbf{v}_m'\in C^\infty(0,T;H),\quad\mathbf{h}_m,\mathbf{h}_m'\in C^\infty(0,T;V),$$

and

$$
\begin{aligned}
\mathbf{v}_m&\to\mathbf{v}\quad\text{in } L^2(0,T;H), &\mathbf{h}_m&\to\mathbf{h}\quad\text{in } L^2(0,T;V),\\
\mathbf{v}_m'&\to\mathbf{v}'\quad\text{in } L^2(0,T;V^*), &\mathbf{h}_m'&\to\mathbf{h}'\quad\text{in } L^2(0,T;H^*).
\end{aligned}
\tag{3.86}
$$

For these regularized sequences we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mathbf{v}_m\cdot\mathbf{h}_m\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}_m,\mathbf{h}_m)_H=(\mathbf{v}_m',\mathbf{h}_m)_H+(\mathbf{v}_m,\mathbf{h}_m')_H.\tag{3.87}$$

Using the Riesz representation theorem we verify that $\mathbf{v}_m'\in C^\infty(0,T;H)$ uniquely determines the functional $\Phi_{\mathbf{v}_m'}\in H^*$ such that

$$(\mathbf{v}_m',\psi)_H=\langle\Phi_{\mathbf{v}_m'},\psi\rangle_{H^*,H}\quad\forall\psi\in H.$$

Since $H^* \subset V^*$ densely, this functional belongs to V^* in the sense

$$\langle \Phi_{\mathbf{v}'_m}, \psi \rangle_{V^*,V} = \int_\Omega \mathbf{v}'_m \cdot \psi \,\mathrm{d}x \quad \forall \psi \in V.$$

Therefore, by identification of H and H^*, the first term on the r.h.s. of (3.87) can be understood as

$$(\mathbf{v}'_m, \mathbf{h}_m)_H = \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{H^*,H} = \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{V^*,V}.$$

For the second term from the r.h.s. of (3.87), we use the Riesz representation theorem to write

$$(\mathbf{v}_m, \mathbf{h}'_m)_H = \langle \mathbf{v}_m, \mathbf{h}'_m \rangle_{H,H^*}$$

and thus we obtain

$$-\int_0^T (\mathbf{v}_m, \mathbf{h}_m)_H \psi' \,\mathrm{d}t = \int_0^T \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{V^*,V} \psi \,\mathrm{d}t + \int_0^T \langle \mathbf{v}_m, \mathbf{h}'_m \rangle_{H,H^*} \psi \,\mathrm{d}t \quad \forall \psi \in \mathcal{D}(0,T).$$

Observe, that both integrants from the r.h.s. are uniformly bounded in $L^1(0,T)$, thus, using (3.86), we let $m \to \infty$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}, \mathbf{h})_H = \langle \mathbf{v}', \mathbf{h} \rangle_{V^*,V} + \langle \mathbf{v} \cdot \mathbf{h}' \rangle_{H,H^*} \quad \text{in } \mathcal{D}'(0,T).$$

Coming back to our original notation, this means that the operation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \mathbf{u} \cdot \nabla \phi(\varrho) \,\mathrm{d}x = \int_\Omega \partial_t(\varrho \mathbf{u}) \cdot \nabla \phi \,\mathrm{d}x + \int_\Omega \varrho \mathbf{u} \cdot \partial_t \nabla \phi \,\mathrm{d}x \tag{3.88}$$

is well defined and is nothing but equality between two scalar distributions. By the fact that $\partial_t \nabla \phi$ exists a.e. in $(0,T) \times \Omega$ we may use approximate continuity equation to write

$$\int_\Omega \varrho \mathbf{u} \cdot \partial_t \nabla \phi \,\mathrm{d}x = \int_\Omega (\operatorname{div}(\varrho \mathbf{u}))^2 \phi'(\varrho) \,\mathrm{d}x - \varepsilon \int_\Omega \operatorname{div}(\varrho \mathbf{u}) \phi'(\varrho) \Delta \varrho \,\mathrm{d}x, \tag{3.89}$$

whence the first term on the r.h.s. of (3.88) may be evaluated by testing the approximate momentum equation by $\nabla \phi(\varrho)$

$$\begin{aligned} &\langle \partial_t(\varrho \mathbf{u}), \nabla \phi \rangle_{V^*,V} \\ &\quad = -\int_\Omega 2\varrho \Delta \phi(\varrho) \operatorname{div} \mathbf{u} \,\mathrm{d}x + 2\int_\Omega \nabla \mathbf{u} : \nabla \phi(\varrho) \otimes \nabla \varrho \,\mathrm{d}x - 2\int_\Omega \nabla \phi(\varrho) \cdot \nabla \varrho \operatorname{div} \mathbf{u} \,\mathrm{d}x \\ &\qquad - \int_\Omega \nabla \phi(\varrho) \cdot \nabla \pi(\varrho, \varrho_A) \,\mathrm{d}x + \delta \int_\Omega \varrho \nabla \Delta^{2s+1} \varrho \cdot \nabla \phi(\varrho) \,\mathrm{d}x \\ &\qquad - \int_\Omega \nabla \phi(\varrho) \cdot \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \,\mathrm{d}x - \eta \int_\Omega \Delta^2 \mathbf{u} \cdot \nabla \phi(\varrho) \,\mathrm{d}x - \varepsilon \int_\Omega (\nabla \varrho \cdot \nabla) \mathbf{u} \cdot \nabla \phi(\varrho) \,\mathrm{d}x. \end{aligned} \tag{3.90}$$

Recalling the form of $\phi(\varrho)$ it can be deduced that the combination of (3.85) with (3.88-3.90) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\varrho \mathbf{u} \cdot \nabla \phi(\varrho) + \frac{1}{2} \varrho |\nabla \phi(\varrho)|^2 \right) \,\mathrm{d}x + \int_\Omega \nabla \pi(\varrho, \varrho_A) \cdot \nabla \phi(\varrho) \,\mathrm{d}x + 2\delta \int_\Omega |\Delta^{s+1} \varrho|^2 \,\mathrm{d}x \\ &\quad = -\int_\Omega \nabla \phi(\varrho) \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \,\mathrm{d}x + \int_\Omega (\operatorname{div}(\varrho \mathbf{u}))^2 \phi'(\varrho) \,\mathrm{d}x - 2\lambda \int_\Omega \Delta^s \nabla(\varrho \mathbf{u}) : \Delta^s \nabla^2 \varrho \,\mathrm{d}x \\ &\qquad - \varepsilon \int_\Omega \operatorname{div}(\varrho \mathbf{u}) \phi'(\varrho) \Delta \varrho \,\mathrm{d}x - \eta \int_\Omega \Delta \mathbf{u} \cdot \nabla \Delta \phi(\varrho) \,\mathrm{d}x + \varepsilon \int_\Omega \frac{|\nabla \phi(\varrho)|^2}{2} \Delta \varrho \,\mathrm{d}x \\ &\qquad - \varepsilon \int_\Omega (\nabla \varrho \cdot \nabla) \mathbf{u} \cdot \nabla \phi(\varrho) \,\mathrm{d}x + \varepsilon \int_\Omega \varrho \nabla \phi(\varrho) \cdot \nabla \left(\phi'(\varrho) \Delta \varrho \right) \,\mathrm{d}x. \end{aligned} \tag{3.91}$$

It is then easy to check that the first two terms from the r.h.s of (3.91) can be transformed into

$$\int_\Omega \left[(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho) - \nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\right]\ \mathrm{d}x = \int_\Omega \left[2\varrho|\mathbf{D}(\mathbf{u})|^2 - \frac{1}{2}\varrho|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2\right]\ \mathrm{d}x,$$

and thus, the assertion of Lemma 3.17 follows by adding (3.59) to (3.91). □

Our next aim is to derive uniform estimates from inequality (3.83). To this purpose we will integrate it with respect to time. For any $\psi_m \in \mathcal{D}(0,T)$, the first term from the l.h.s. of (3.83) equals

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho e_c(\varrho)\right)\ \mathrm{d}x\ \psi_m\ \mathrm{d}t$$
$$= -\int_0^T\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho e_c(\varrho)\right)\psi_m'\ \mathrm{d}x\ \mathrm{d}t. \quad (3.92)$$

Now, choosing a sequence of $\psi_m \in \mathcal{D}(0,\tau)$ such that $\psi_m \to 1$ pointwisely in $(0,\tau)$, $\psi_m \to 0$ pointwisely in $[\tau, T)$, $0 < \tau < T$ we see that ψ_m' approximates the inner normal vector to the boundary of the time interval $[0,\tau]$. In other words, it generates two Dirac distributions at the ends of $[0,\tau]$. Thus, using the fact that

$$t \mapsto \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho e_c(\varrho)\right)\ \mathrm{d}x \in C([0,\tau]),$$

we let $m \to \infty$ in (3.92) and from (3.83) we get

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho e_c(\varrho)\right)(\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \nabla\phi(\varrho)\cdot\nabla\pi(\varrho,\varrho_A)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad + \int_0^\tau\int_\Omega \left(\frac{1}{2}\varrho|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2 + 2\delta|\Delta^{s+1}\varrho|^2 + \delta\varepsilon|\Delta^{s+1}\varrho|^2 + \eta|\Delta\mathbf{u}|^2\right)\ \mathrm{d}x\ \mathrm{d}t\\
&\le \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla^{2s+1}\varrho|^2 + \varrho\pi(\varrho)\right)(0)\ \mathrm{d}x + \varepsilon\int_0^\tau\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t \quad (3.93)\\
&\quad + \varepsilon\int_0^\tau\int_\Omega \Delta\varrho\frac{|\nabla\phi|^2}{2}\ \mathrm{d}x\ \mathrm{d}t + \varepsilon\int_0^\tau\int_\Omega \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \varepsilon\int_0^\tau\int_\Omega \operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\ \mathrm{d}x\ \mathrm{d}t - \eta\int_0^\tau\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

The only nonpositive contribution to the l.h.s. of (3.93) is contained in the second integral, as we can not determine the sign of the part corresponding to molecular pressure. However, we have

$$\int_\Omega \nabla\phi\cdot\nabla\pi_m(\varrho,\varrho_A)\ \mathrm{d}x = \int_\Omega \left(\frac{2|\nabla\varrho|^2}{\varrho m_B} + \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\frac{2\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\right)\ \mathrm{d}x$$

moreover,

$$\left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \frac{\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\ \mathrm{d}x = \left(\frac{1}{m_A} - \frac{1}{m_B}\right)\int_\Omega \left(\frac{|\nabla\varrho|^2}{\varrho} + \nabla\varrho\cdot\nabla Y_A\right)\ \mathrm{d}x \quad (3.94)$$

and

$$\left|\int_\Omega \nabla\varrho\cdot\nabla Y_A\ \mathrm{d}x\right| \le c(\varepsilon)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x + \varepsilon\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x. \quad (3.95)$$

To control the second term we proceed by the same lines as in the proof of Lemma 3.7. Mimicking the steps leading to (3.27), we use $Y_A = \frac{\varrho_A}{\varrho} \in L^2(0,T;W^{1,2}(\Omega))$ as a test function in (3.70), we obtain

$$\int_\Omega \frac{1}{2}\varrho Y_A^2(T)\ \mathrm{d}x + \left(\varepsilon + \frac{1}{\max\{m_A, m_B\}}\right)\int_0^T\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho Y_A^2(0)\ \mathrm{d}x + \int_0^T\int_\Omega \varrho|\omega(Y)|Y_A\ \mathrm{d}x\ \mathrm{d}t + c\int_0^T\int_\Omega |\nabla\varrho\cdot\nabla Y_A|\ \mathrm{d}x\ \mathrm{d}t. \quad (3.96)$$

Hence, by the Cauchy inequality, we can justify that the $L^1(\Omega)$ norm of $\varrho|\nabla Y_A|^2$ is controlled by the $L^1(\Omega)$ norm of $\frac{|\nabla\varrho|^2}{\varrho}$ independently of the approximation parameters, so we end up with

$$\int_\Omega |\nabla\phi\cdot\nabla\pi_m(\varrho,\varrho_A)|\ \mathrm{d}x \leq c(m_A,m_B)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x.$$

Finally, the Gronwall-type argument can be applied to absorb this term by the l.h.s. of (3.83). Concerning terms from the r.h.s of (3.93), the first of them can be estimated as follows

$$\left|\varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\right| \leq 2\varepsilon\|\nabla\mathbf{u}\|_{L^6(\Omega)}\|\varrho^{-1}\|_\infty\|\varrho\|^2_{W^{1,6/5}(\Omega)}.$$

The Sobolev imbedding implies that for $c(s)\varepsilon < \eta$ and s sufficiently large we have

$$\left|\varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\right| \leq \frac{\eta}{3}\|\Delta\mathbf{u}\|^2_{L^2(\Omega)} + c(\varepsilon)\|\varrho^{-1}\|^2_{L^\infty(\Omega)}\|\varrho\|^4_{H^{2s+1}(\Omega)}$$

and the last term is bounded uniformly in time due to (3.80) provided $\varepsilon = \varepsilon(\delta)$. For the second term we may write

$$\left|\varepsilon\int_\Omega \Delta\varrho\frac{|\nabla\phi|^2}{2}\ \mathrm{d}x\right| \leq 4\varepsilon\|\varrho\|_{H^2(\Omega)}\|\varrho^{-1}\|^2_{L^\infty(\Omega)}\|\varrho\|^2_{H^1(\Omega)} \leq c(\varepsilon)\|\varrho\|^3_{H^{2s+1}(\Omega)}\|\varrho^{-1}\|^2_{L^\infty(\Omega)}$$

and the same argument leads to boundedness uniformly in time provided ε is sufficiently small with respect to δ.
The third term is even easier since

$$\left|\varepsilon\int_\Omega \varrho\nabla\phi\nabla(\phi'\Delta\varrho)\ \mathrm{d}x\right| = 4\varepsilon\int_\Omega \frac{(\Delta\varrho)^2}{\varrho}\ \mathrm{d}x \leq c(\varepsilon)\|\varrho\|^2_{H^{2s+1}(\Omega)}\|\varrho^{-1}\|_{L^\infty(\Omega)}.$$

By the definition of ϕ the fourth term equals

$$-\varepsilon\int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\ \mathrm{d}x = -\varepsilon\int_\Omega (2\,\mathrm{div}\,\mathbf{u}\Delta\varrho + \mathbf{u}\cdot\nabla\phi\Delta\varrho)\ \mathrm{d}x,$$

hence we have

$$\left|-\varepsilon\int_\Omega \mathrm{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\ \mathrm{d}x\right| \leq c\varepsilon\left(\|\mathbf{u}\|_{W^{1,6}(\Omega)}\|\varrho\|_{H^2(\Omega)} + \|\mathbf{u}\|_\infty\|\varrho^{-1}\|_\infty\|\varrho\|_{H^1}\|\varrho\|_{H^2}\right)$$

and for ε sufficiently small with respect to η the r.h.s. is bounded by

$$\frac{\eta}{3}\|\Delta\mathbf{u}\|_2^2 + c\varepsilon\left(\|\varrho\|^2_{H^{2s+1}(\Omega)} + \|\varrho^{-1}\|^2_{L^\infty(\Omega)}\|\varrho\|^4_{H^{2s+1}(\Omega)}\right).$$

Finally, we estimate the last term in (3.93)

$$\left|\eta\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\ \mathrm{d}x\right| \le \sqrt{\eta}\|\Delta\mathbf{u}\|_{L^2(\Omega)}\sqrt{\eta}\|\nabla\Delta\phi(\varrho)\|_{L^2(\Omega)},$$

where

$$\nabla\Delta\phi(\varrho) = \frac{2\nabla\Delta\varrho}{\varrho} - \frac{2(\nabla\varrho\cdot\nabla)\nabla\varrho}{\varrho^2} - \frac{2(\nabla\varrho\cdot\nabla)\nabla\varrho}{\varrho^2} - \frac{2\Delta\varrho\nabla\varrho}{\varrho^2} + \frac{4|\nabla\varrho|^2\nabla\varrho}{\varrho^3}.$$

For s sufficiently large we may show that

$$\|\nabla\Delta\phi(\varrho)\|_{L^2(\Omega)} \le (1+\|\varrho\|_{H^{2s+1}(\Omega)})^3(1+\|\varrho^{-1}\|_{L^\infty(\Omega)})^6$$

and on account of (3.80) , (3.77) both terms from the r.h.s. are bounded for all time.

Summarizing, from the Bresch-Desjardins relation we can additionally deduce that

$$\nabla\sqrt{\varrho}\in L^\infty(0,T;L^2(\Omega)) \quad \sqrt{\delta}\Delta^{s+1}\varrho\in L^2(0,T;L^2(\Omega))$$

uniformly withe respect to ε,η,δ. Moreover, in view of (3.43) we can write

$$\begin{aligned}\int_\Omega \nabla\phi(\varrho)\cdot\nabla\pi_c(\varrho)\ \mathrm{d}x &= 2\int_\Omega \pi_c'(\varrho)\frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x\\ &= 8k\int_{\{x\in\Omega:\varrho\le 1\}}\varrho^{-4k-2}|\nabla\varrho|^2\ \mathrm{d}x + 2\gamma\int_{\{x\in\Omega:\varrho>1\}}\varrho^{\gamma-2}|\nabla\varrho|^2\ \mathrm{d}x\\ &\ge \int_{\{x\in\Omega:\varrho\le 1\}}|\nabla\xi(\varrho)^{-2k}|^2\ \mathrm{d}x + \int_{\{x\in\Omega:\varrho>1\}}|\nabla\varrho^{\gamma/2}|\ \mathrm{d}x,\end{aligned}$$

where ξ is smooth and such that $\xi(y)=y$ for $y\le 1/2$ and $\xi(y)=0$ for $y>1$. So, by the entropy equality (3.93) we obtain additionally that

$$\nabla\xi(\varrho)^{-2k}\in L^2((0,T)\times\Omega), \quad \nabla\varrho^{\gamma/2}\in L^2((0,T)\times\Omega_2),$$

where $\Omega_2=\{x\in\Omega:\varrho>1\}$. Moreover, via the Sobolev imbedding theorem we show that

$$\frac{1}{c^2}\|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^6(\Omega_2))} \le \|\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;H^1(\Omega_2))} \le \|\nabla\varrho^{\frac{\gamma}{2}}\|^2_{L^2(0,T;L^2(\Omega_2))} + \|\varrho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega_2))}$$

where c is the constant from the Sobolev inequality.
Furthermore, by a simple interpolation one gets

$$\|\varrho^\gamma_{\varepsilon\eta}\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \le \|\varrho^\gamma_{\varepsilon\eta}\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))}\|\varrho^\gamma_{\varepsilon\eta}\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))} \le c.$$

3.5.2 Passage to the limit $\varepsilon,\eta\to 0$

It turns out that the limit passages with ε and η can be done in one step. Indeed, by the previous estimates we can extract subsequences, such that

$$\eta\Delta\mathbf{u}_{\varepsilon\eta},\ \varepsilon\nabla\varrho_{\varepsilon\eta}\to 0 \text{ strongly in } L^2((0,T)\times\Omega),$$

and assuming suitable relation between ε and η also

$$\varepsilon\nabla\varrho_{\varepsilon\eta}\nabla\mathbf{u}_{\varepsilon\eta}\to 0 \text{ strongly in } L^1((0,T)\times\Omega).$$

After remarks from the previous section, the only questionable limit passage at this level is due to the convective term of momentum equation, since we need to justify the strong convergence of the velocity. The argument for this is that the lower bound on the density depends only on δ and is uniform with respect to ε, η. Therefore we have boundedness of $\nabla \mathbf{u}_{\varepsilon\eta}$ in $L^2((0,T)\times\Omega)$. To improve the time regularity observe that from the approximate continuity equation we can bound the norm of $\partial_t(\varrho_{\varepsilon\eta}\mathbf{u}_{\varepsilon\eta})$ in $L^p(0,T;H^{-k}(\Omega))$ for some $k=k(s)>0$ and $p>1$. Then, using the Aubin-Lions lemma we get that $\varrho_{\varepsilon\eta}\mathbf{u}_{\varepsilon\eta}\to m_\delta$ when $\varepsilon,\eta\to 0$, strongly in $L^2((0,T)\times\Omega)$, which, due to the convergence of $\varrho_{\varepsilon\eta}$ to ϱ and of $\varrho_{ep\eta}^{-1}$ to ϱ^{-1} almost everywhere, implies the strong convergence of $\mathbf{u}_{\varepsilon\eta}$ to $\mathbf{u}$ in $L^2(0,T;L^q(\Omega))$, $q<6$.

3.5.3 Passage to the limit $\delta \to 0$

Here we lose the uniform bound from below for the density, so the strong convergence of velocity can not be deduced by the procedure described above. Nevertheless, we can still use the Hölder inequality to verify

$$\|\nabla\mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \le c(\Omega)\left(1+\|\nabla\xi(\varrho)^{-2k}\|_{L^2((0,T)\times\Omega)}\right)\|\sqrt{\varrho}\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega)},$$

where $\frac{1}{p}=\frac{1}{2}+\frac{1}{2\cdot 2k\cdot 2}$, $\frac{1}{q}=\frac{1}{2}+\frac{1}{6\cdot 2k\cdot 2}$. After applying the Sobolev imbedding we thus obtain

$$\mathbf{u}\in L^p(0,T;L^{q^*}(\Omega)),\quad p=\frac{8k}{4k+1},\quad q^*=\frac{24k}{4k+1},\quad k>1. \tag{3.97}$$

This in turn implies that for $0\le\epsilon\le 1/2$ we have the following estimate

$$\|\sqrt{\varrho}\mathbf{u}\|_{L^{p'}(0,T;L^{q'}(\Omega))} \le \|\varrho\|^{1/2-\epsilon}_{L^\infty(0,T;L^\gamma(\Omega))}\|\sqrt{\varrho}\mathbf{u}\|^{2\epsilon}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}\|^{1-2\epsilon}_{L^p(0,T;L^{q^*}(\Omega))},$$

where p', q' are given by $\frac{1}{p'}=\frac{1-2\epsilon}{p}$, $\frac{1}{q'}=\frac{1/2-\epsilon}{\gamma}+\frac{2\epsilon}{2}+\frac{1-2\epsilon}{q^*}$. Taking $\epsilon>1/10$ we have $p',q'>2$ and the argument for strong convergence of $\sqrt{\varrho_\delta}\mathbf{u}_\delta$ from previous section applies verbatim.

Remark 3.18. *The final information about the velocity obtained from this procedure is* (3.97). *Note that it could be improved by assuming faster growth of the barotropic pressure in the areas of small densities than* $-\varrho^{-4k}$. *However, this still would not be sufficient to repeat the logharithmic estimate performed in the section dedicated to sequential stability of weak solutions.*

The result achieved in this section can be stated as follows.

Theorem 3.19. *Let* Ω *be a periodic box* $\mathbb{T}^3$. *Let us assume that the viscosity coefficients* $\mu(\varrho)=\varrho$, $\nu(\varrho)=0$ *and the structural properties (3.3-3.9),* (3.43) *be satisfied. The initial data* $\varrho^0,\mathbf{u}^0,\varrho^0_A$ *satisfy* (3.13) *together with the following bounds*

$$\int_\Omega\left(\frac{1}{2}\frac{\left|(\varrho\mathbf{u})^0\right|^2}{\varrho^0}+\varrho^0 e_c\left(\varrho^0\right)\right)\,\mathrm{d}x<\infty,\qquad \int_\Omega\frac{\left|\nabla\varrho^0\right|^2}{\varrho^0}\,\mathrm{d}x<\infty.$$

Then there exists a global in time weak solution to (3.2) *in the sense of Definition 3.2. Moreover, the following regularity properties hold:*

$$\varrho\in L^\infty(0,T;L^1\cap L^\gamma(\Omega)),\quad \sqrt{\varrho}\in L^\infty(0,T;H^1(\Omega)),\quad \varrho>0 \text{ a.e. on } (0,T)\times\Omega,$$
$$\sqrt{\varrho}\mathbf{u}\in L^\infty(0,T;L^2(\Omega)),\quad \sqrt{\varrho}\nabla\mathbf{u}\in L^2(0,T;L^2(\Omega)),\quad \mathbf{u}\in L^p(0,T;L^{q^*}(\Omega)),$$
$$\sqrt{\varrho}\nabla Y_A\in L^2(0,T;L^2(\Omega)),\quad 0\le Y_A\le 1 \text{ a.e. on } (0,T)\times\Omega,$$

where p,q^* *are given by* (3.97).

Chapter 4

Reaction-diffusion equations for n species

In the following chapter we investigate only the reaction-diffusion equations of species

$$\partial_t \varrho_k + \operatorname{div}(\varrho_k \mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho \omega_k, \quad k = 1, ...n, \tag{4.1}$$

keeping the thermodynamical framework originating from the general theory of mixtures. Our main motivation is to show a possible extension of the procedure from Section 3.3 to treat more complex system describing the motion of n-component chemically reacting mixture, where all the reactions might be reversible.

4.1 Introduction

The basic property of system (4.1) is that the sum of equations gives the time evolution of the total mass of the mixture, the so called continuity equation

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \tag{4.2}$$

which is hyperbolic. So, the mathematical description of the flow of n-component mixture leads to a degenerate parabolic equations with hyperbolic deviation.

The main obstacle to apply the classical approach [2,23,57] for systems of parabolic equations, is the structure of diffusion fluxes. They form an elliptic operator, however not diagonal, even not symmetric. Thus any direct technique of renormalization of the system is not admissible as it is the case for the scalar system [88]. The only possibility is to employ the information concerning the entropy production (1.14); then we deal with the symmetric matrix D_{kl} (1.21) which is positive definite over a subspace of co-dimension 1. It follows that the whole mathematical analysis should be done in terms of $\log p_k$ instead of ϱ_k. This approach is effective, since it guarantees immediately that the densities of gas components will be nonnegative. Using a suitable approximation we are allowed to obtain existence through the Galerkin approximation, and then passing to subsequent limits we find the solution of the original problem. The compactness of the approximate sequence is guaranteed due to uniform $L \log L$ bounds and an extra information about the whole density ϱ. The last fact allows to control the regularity of space derivatives of solutions.

We discuss our model in terms of velocity, which in our case is given and is relatively smooth, particularly we are guaranteed that the continuity equation admits unique solutions. The last feature requires that $\operatorname{div} \mathbf{u} \in L^2(0, T; L^\infty(\Omega))$. On the other hand, the density is assumed to be regular; e.g. the final information derived from Section 3.4 would be enough.

Recall that for the Navier-Stokes-type of compressible fluids models, the thermodynamical concept of entropy is of great importance [36, 64, 72, 84] as it provides majority of all available estimates. Following this path, one may expect the same for the multicomponent flows, still subject to a similar type of conservation laws [5,38]. Regrettably it turns out that the approximation of the diffusion flux by the Fick law and the presence of the species concentrations in the state equation leads to difficulties in determining the sign of production of entropy associated with the diffusive process. Roughly speaking, one should be able to deduce directly from the form of $\mathbf{F}_k$ that the part of entropy production (1.14) associated with species diffusion is nonnegative. In particular, the following condition must be fulfilled

$$-\int_\Omega \sum_{k=1}^n \frac{\mathbf{F}_k}{m_k} \cdot \nabla \log p_k \; \mathrm{d}x \geq 0. \tag{4.3}$$

In consequence, to be physically consistent, one has to deal with a more complicated form of diffusion (1.18) leading to degeneration in system (4.1).

The systems of parabolic PDEs with strong cross-diffusion are also present in the population or the chemotaxis models [19, 43], for which the existence of certain Lyapunov functional often allows to introduce the entropy variables. Rewriting the system in terms of these variables usually leads to a symmetric and positive diffusion matrix, which may also help in proving nonnegativity or even L^∞ bounds. An overview of these methods can be found in [16].

To finish the introductory part, let us mention three possible interpretation of our result.

- If system (4.1) is a part of the large model, where ϱ, ϱ_k, $\mathbf{u}$ are determined each other like in the previous chapter, then our result can be viewed as an auxiliary tool giving hints how to proceed with the full system.

- For the simplest case $\mathbf{u} \equiv \mathbf{0}$ the density ϱ is a given fixed function and the model takes into account just diffusion, neglecting the effects of transport. A relevant local-in-time existence result for such model has been obtained by Bothe [9].

- If the velocity field is given, the chemical reactions have no influence on the speed of particles, they do not produce any internal force (pressure like force). Such model is admissible for "cold" reactions, where we do not observe any rapid changes of energy.

4.2 Notation and Main Result.

We assume that Ω is a periodic box in $\mathbb{R}^3$, $\Omega = \mathbb{T}^3$ and we supplement system

$$\partial_t \varrho_k + \operatorname{div}(\varrho_k \mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho\omega_k, \quad k = 1, ...n, \tag{4.4}$$

with the initial conditions

$$\varrho_k(0,\cdot) = \varrho_k^0(\cdot), \quad \varrho_k^0 \geq 0, \quad \sum_{k=1}^n \int_\Omega \varrho_k^0 \; \mathrm{d}x = \int_\Omega \varrho^0 \; \mathrm{d}x = M_0. \tag{4.5}$$

Remark 4.1. *From the point of view of present chapter, there are no obstacles to assume the Neumann boundary conditions $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$ together with the impermeability of boundary $\partial\Omega$, meaning $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$. However, the higher regularity of the density that we assume here, has been so far proven only for the periodic domains [113], so we stick to this restriction.*

We now detail our assumptions on the diffusion flux $\mathbf{F}_k$ and the species production terms ω_k, $k = 1, \dots, n$.

General conditions on $\mathbf{F}_k$
In case of isothermal chemical reactions and lack of external forces, the diffusion fluxes can be written in the following form

$$\mathbf{F}_k = -C_0 \sum_{l=1}^{n} C_{kl} \mathbf{d}_l, \quad k = 1, ...n, \tag{4.6}$$

where $C_0,\ C_{kl}$ are multicomponent flux diffusion coefficients satisfying (1.20) and $\mathbf{d}_k = (d_k^1, d_k^2, d_k^3)$ is the species k diffusion force

$$d_k^i = \nabla_{x_i} \left(\frac{p_k}{\pi} \right) + \left(\frac{p_k}{\pi} - \frac{\varrho_k}{\varrho} \right) \nabla_{x_i} \log \pi. \tag{4.7}$$

In the above formula $\pi = \pi_m$ denotes the internal pressure which now consist only of the molecular part

$$\pi(\varrho_1, \dots, \varrho_n) = \sum_{k=1}^{n} p_k(\varrho_k) = \sum_{k=1}^{n} \frac{\varrho_k}{m_k}.$$

Since the chemical reactions considered in this chapter may be completely reversible, the symmetric role is given to all of the species. Therefore, to fix the idea, we shall concentrate on the following explicit form of C

$$C = \begin{pmatrix} Z_1 & -Y_1 & \dots & -Y_1 \\ -Y_2 & Z_2 & \dots & -Y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -Y_n & -Y_n & \dots & Z_n \end{pmatrix}, \tag{4.8}$$

where $Z_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} Y_i$. For the sake of simplicity, we assume that $C_0 = \pi$.

Remark 4.2. *Note that the matrix C is singular since $CY = 0$ and is not symmetric in general.*

Remark 4.3. *It is easy to check that by the expressions for the diffusion forces* (4.7) *and the properties of C one can rewrite* (4.6) *into the following form*

$$\mathbf{F}_k = -\left(\nabla p_k - Y_k \nabla \pi \right) = -\sum_{l=1}^{n} C_{kl} \nabla p_l. \tag{4.9}$$

Species production rates
For the isothermal reactions, the species production rates are functions of the species mass fractions only. We will additionally assume that they are Lipschitz continuous with respect to $\varrho_1, \dots, \varrho_n$ and that there exist positive constants $\underline{\omega}$ and $\overline{\omega}$ such that

$$-\underline{\omega} \leq \omega_k(\varrho_1, \dots, \varrho_n) \leq \overline{\omega}, \qquad \text{for all} \quad 0 \leq Y_k \leq 1, \quad k = 1, \dots, n; \tag{4.10}$$

moreover, we suppose that

$$\omega_k(Y_1, \dots, Y_n) \geq 0 \quad \text{whenever} \quad Y_k = 0. \tag{4.11}$$

We also anticipate the mass constraint between the chemical source terms

$$\sum_{k=1}^{n} \omega_k = 0. \tag{4.12}$$

Another restriction that we postulate for chemical sources is dictated by the second law of thermodynamics, which asserts that the entropy production associated with any adimissible chemical reaction is nonnegative. In particular, ω_k must enjoy condition (1.24). For fixed positive ϑ and equal constant-pressure specific heats for all the species, this condition may be translated into the following one

$$\int_\Omega \sum_{k=1}^{n} \frac{\log p_k \omega_k \varrho}{m_k} \, \mathrm{d}x \le c, \tag{4.13}$$

which allows us to control the source term in the main estimate (4.26). Here and subsequently c denotes a constant that may differ throughout the paper and, if it is not marked otherwise, depends only on the data.

Remark 4.4. *From* (4.6) *and* (1.20) *it follows that* $\mathbf{F} = (\mathbf{F}_1, \dots, \mathbf{F}_n)^T \in R(C) = U^\perp$*, therefore, taking the scalar product between* $\mathbf{F}$ *and* U *it can be deduced that*

$$\sum_{k=1}^{n} \mathbf{F}_k = 0, \tag{4.14}$$

which together with (4.12) *leads to the continuity equation for* ϱ

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad \varrho^0 = \sum_{k=1}^{n} \varrho_k^0. \tag{4.15}$$

In particular, the total mass of the mixture is conserved

$$\int_\Omega \varrho(t) \, \mathrm{d}x = M_0, \quad \text{for a.a. } t \in (0,T).$$

Main result

The main result of this paper is the following.

Theorem 4.5. *Let* ϱ *be a sufficiently smooth solution of* (4.15) *such that* ϱ *is bounded in* $L^2(0,T;W^{1,2}(\Omega))$*. Moreover, let* $0 < \inf_\Omega \varrho^0 \le \sup_\Omega \varrho^0 < \infty$*. Let* $\mathbf{u} \in L^\infty((0,T)\times\Omega)$ *be fixed such that* $\operatorname{div}\mathbf{u} \in L^2((0,T);L^\infty(\Omega))$*. Assuming (4.10–4.8) problem (4.4-4.5) admits a global in time weak solution, such that*

$$\varrho_k \ge 0 \quad \text{a.e. in } (0,T)\times\Omega, \qquad \sum_{k=1}^{n} \int_\Omega \varrho_k(t) \, \mathrm{d}x = M_0.$$

Furthermore, the following regularity properties hold:

$$\varrho_k \in C([0,T]; L\log L_{\text{weak}^*}(\Omega)) \quad \text{and} \quad \nabla\sqrt{\varrho_k} \in L^2((0,T)\times\Omega), \quad k = 1,\dots,n.$$

Remark 4.6. *To maintain consistency with the existence results from the previous chapter one should rather work with* $\sqrt{\varrho}$ *instead of* ϱ*, since it is only possible to show integrability of* $\nabla\sqrt{\varrho}$*. Note that we actually need that* $\sqrt{\varrho} \in L^2(0,T;W^{1,2}(\Omega))$ *but due to assumptions on* $\boldsymbol{u}$ *and* ϱ^0 *it is equivalent with the assumption in the theorem.*

Remark 4.7. *Note that assuming* $\mathbf{u}$ *much more regular, we would be able to recover the regularity of the density* ϱ *required in Theorem 4.5. Note, however, that in practical applications, except for very special situations, such a regularity is not possible to obtain.*

The rest of this chapter is devoted to the proof of this theorem. The strategy is to add the standard regularization terms and then to employ the Galerkin method for the system rewritten in terms of so-called entropy variables.

4.3 Existence of solutions

4.3.1 Galerkin approximation

Our aim is to construct the weak solution to the semi-linear parabolic problem

$$\begin{gathered}(\delta+\mathrm{e}^{r_k})\partial_t r_k+\operatorname{div}(\mathrm{e}^{r_k}\mathbf{u})-\operatorname{div}\left((\delta+\varepsilon\mathrm{e}^{r_k})\nabla r_k\right)+\frac{\operatorname{div}\mathbf{F}_k}{m_k}=\frac{\varrho\omega_k}{m_k},\\ r_k(0,x)=r_k^0,\end{gathered}\tag{4.16}$$

for every $k=1,\dots,n$, $(t,x)\in[0,T]\times\Omega$, and for any given, smooth (as in Theorem 4.5) vector $\mathbf{u}$. To this purpose we will employ the Galerkin technique. We denote by N the (finite) dimension of the approximation. The aim (achieved in the next section) will be to pass with $N\to\infty$. More precisely, we assume that $r_{k,N}$ has the following structure

$$r_{k,N}=\sum_{i=1}^{N}a_{k,N}^i(t)h_i(x),\tag{4.17}$$

where the functions $\{h_i\}_{i\in\mathbb{N}}$ form an orthogonal basis of the Hilbert space $W^{1,2}(\Omega)$, they are smooth and orthonormal with respect to the scalar product $(\cdot,\cdot)$ in $L^2(\Omega)$. We look for the coefficients $a_{k,N}^i(t)$, $t\in[0,T]$, $k=1,\dots,n$, $i=1,\dots,N$ such that

$$a_{k,N}^i(0)=(r_k^0,h_i)\tag{4.18}$$

and the following equality is satisfied

$$\begin{aligned}&\int_\Omega\partial_t(\delta r_{k,N}+\mathrm{e}^{r_{k,N}})h_l\ \mathrm{d}x\\ &\quad=-\int_\Omega\operatorname{div}\left(\mathrm{e}^{r_{k,N}}\mathbf{u}-(\delta+\varepsilon\mathrm{e}^{r_{k,N}})\nabla r_{k,N}+\frac{\mathbf{F}_{k,N}}{m_k}\right)h_l\ \mathrm{d}x+\int_\Omega\frac{\varrho_N\omega_k}{m_k}h_l\ \mathrm{d}x,\end{aligned}\tag{4.19}$$

for any $l=1,\dots,N$. Here,

$$\varrho_N=\sum_{k=1}^{N}m_k\mathrm{e}^{r_{k,N}}\tag{4.20}$$

and $\mathbf{F}_{k,N}$ is given in (4.23) below. We have

Theorem 4.8. *For any* $N\in\mathbb{N}$ *there exist uniquely determined functions* $r_{1,N},\dots,r_{n,N}$ *of the form* (4.17) *satisfying* (4.18) *and* (4.19). *Moreover, there exists a constant* c *depending only on* T *and the initial data, such that*

$$\begin{aligned}&\sqrt{\delta}\|r_{k,N}\|_{L^\infty(0,T;L^2(\Omega))}+\|e^{r_{k,N}}r_{k,N}\|_{L^\infty(0,T;L^1(\Omega))}+\sqrt{\delta}\|\nabla r_{k,N}\|_{L^2((0,T)\times\Omega)}\\ &\qquad+\sqrt{\varepsilon}\|\nabla\sqrt{e^{r_{k,N}}}\|_{L^2((0,T)\times\Omega)}+\sum_{k=1}^{n}\left\|\frac{\mathbf{F}_{k,N}}{\sqrt{m_k e^{r_{k,N}}}}\right\|_{L^2((0,T)\times\Omega)}\le c.\end{aligned}\tag{4.21}$$

Proof. This task is equivalent with solving the set of N ODEs corresponding to each of n equations of the system (4.19)

$$\delta \dot{a}^l_{k,N}(t) + \sum_{i=1}^{N} \dot{a}^i_{k,N}(\mathrm{e}^{r_{k,N}} h_i, h_l) = -\mathbb{X}_{k,N}(r_{k,N}, h_l) + \left(\frac{\varrho_N \omega_k}{m_k}, h_l \right), \tag{4.22}$$

with the initial conditions given by (4.18). In the above formula ϱ_N is given by (4.20), $\dot{a}(t)$ is time derivative of $a(t)$, and

$$\mathbb{X}_{k,N}(r_{k,N}, h_l) = \int_\Omega \mathrm{div} \left(\mathrm{e}^{r_{k,N}} \mathbf{u} - (\delta + \varepsilon \mathrm{e}^{r_{k,N}}) \nabla r_{k,N} + \frac{\mathbf{F}_{k,N}}{m_k} \right) h_l \; \mathrm{d}x,$$

$$\mathbf{F}_{k,N} = -(\nabla \mathrm{e}^{r_{k,N}} - Y_{k,N} \sum_{j=1}^{n} \mathrm{e}^{r_{j,N}} \nabla r_{j,N}) = -\sum_{j=1}^{n} C_{kj,N} \mathrm{e}^{r_{j,N}} \nabla r_{j,N}. \tag{4.23}$$

The matrix C_N is given by

$$C_N = \begin{pmatrix} Z_{1,N} & -Y_{1,N} & \dots & -Y_{1,N} \\ -Y_{2,N} & Z_{2,N} & \dots & -Y_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ -Y_{n,N} & -Y_{n,N} & \dots & Z_{n,N} \end{pmatrix},$$

where $Y_{j,N} = \frac{m_j \mathrm{e}^{r_{j,N}}}{\varrho_N}$, and $Z_{k,N} = \sum_{\substack{i=1 \\ i \neq k}}^{n} Y_{i,N}$.

Observe that since the matrix $\mathbb{X}_{k,N}$ involves all n functions $r_{k,N}$, we should solve the system of $N \cdot n$ ODEs simultaneously. To this purpose, we rewrite system (4.22) using the vector $A_{k,N}(t) = (a^1_{k,N}(t), \dots, a^N_{k,N}(t))^T$ into the following form

$$(\delta \mathbb{I} + \mathbb{B}_{k,N}(t)) \, \dot{A}_{k,N}(t) = -\mathbb{X}_{k,N}(r_{k,N}, h^N) + \left(\frac{\varrho_N \omega_k}{m_k}, h^N \right), \quad k = 1, \dots, n, \tag{4.24}$$

where $\mathbb{I}$ is the identity matrix, $(\mathbb{B}_{k,N})_{ij} = \int_\Omega \mathrm{e}^{r_{k,N}} h_i h_j \; \mathrm{d}x$,

$$\mathbb{X}_{k,N}(r_{k,N}, h^N) = (\mathbb{X}_{k,N}(r_{k,N}, h_1), \dots, \mathbb{X}_{k,N}(r_{k,N}, h_N))^T$$

and $(\frac{\varrho_N \omega_k}{m_k}, h^N) = ((\frac{\varrho_N \omega_k}{m_k}, h_1), \dots, (\frac{\varrho_N \omega_k}{m_k}, h_N))^T$.

It is easy to see that the matrix $\delta \mathbb{I} + \mathbb{B}_{k,N}(t)$ is invertible for any $\delta > 0$; indeed $\mathrm{e}^{r_{k,N}}$ is a nonnegative function, thus the time-dependent bilinear form

$$\mathbb{B}_{k,N}[h_i, h_j; t] = (\mathbb{B}_{k,N})_{ij} = \int_\Omega \mathrm{e}^{r_{k,N}} h_i h_j \; \mathrm{d}x$$

is symmetric and positive-semidefinite. Next, using the following property of the block diagonal matrixes

$$\begin{pmatrix} \mathbb{A}_{1,N} & 0 & \dots & 0 \\ 0 & \mathbb{A}_{2,N} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{A}_{n,N} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{A}^{-1}_{1,N} & 0 & \dots & 0 \\ 0 & \mathbb{A}^{-1}_{2,N} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{A}^{-1}_{n,N} \end{pmatrix},$$

we invert the $(n \cdot N) \times (n \cdot N)$ matrix that stands in front of the time derivative of system (4.24) and therefore, problem (4.22) can be replaced by the following one

$$\dot{A}_{k,N}(t) = -\left(\delta \mathbb{I} + \mathbb{B}_{k,N}(t)\right)^{-1} \mathbb{X}_{k,N}(r_{k,N}, h^N) + (\delta \mathbb{I} + \mathbb{B}(t))^{-1} \left(\frac{\varrho_N \omega_k}{m_k}, h^N \right). \tag{4.25}$$

Before we apply the classical result on solvability of the ODE system we check whether the right hand side of (4.25) is Lipschitz with respect to $a_{k,l}(t)$ for all $k = 1, \dots, n$, $l = 1, \dots, N$. This is true on account of the fact that a finite composition of the polynomials and the exponent functions is a Lipschitz function, at least locally with respect to time. Therefore, for sufficiently short time interval $[0, \tau)$ there exists uniformly continuous (with respect to time) solution to problem (4.22).

In order to obtain the global in time solution any a priori estimate on $r_{k,N}$ is needed, because on a finite dimensional space all norms are equivalent. To this end we use in (4.19) the test function $r_{k,N}$ (we multiply each of the equations by $a^l_{k,N}(t)$ first, and then we sum them with respect to $l = 1, \dots, N$). Integrating by parts we obtain the following equality

$$\sum_{k=1}^n \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\delta \frac{r_{k,N}^2}{2} + \mathrm{e}^{r_{k,N}} r_{k,N} - \mathrm{e}^{r_{k,N}} \right) \mathrm{d}x + \sum_{k=1}^n \int_\Omega \left((\delta + \varepsilon \mathrm{e}^{r_{k,N}}) |\nabla r_{k,N}|^2 - \frac{\mathbf{F}_{k,N}}{m_k} \nabla r_{k,N} \right) \mathrm{d}x$$
$$= - \sum_{k=1}^n \int_\Omega \mathrm{e}^{r_{k,N}} \operatorname{div} \mathbf{u} \ \mathrm{d}x + \sum_{k=1}^n \int_\Omega \frac{\varrho_N \omega_k}{m_k} r_{k,N} \ \mathrm{d}x. \quad (4.26)$$

The only problematic term on the left hand side is the last one. However, since $\sum_{k=1}^n \mathbf{F}_{k,N} = 0$ – see (4.14) – we write

$$\sum_{k=1}^n \left(\frac{\mathbf{F}_{k,N}}{m_k \mathrm{e}^{r_{k,N}}} Y_{k,N} \sum_{j=1}^n \mathrm{e}^{r_{j,N}} \nabla r_{j,N} \right) = 0$$

and therefore the last term on the left hand side of (4.26) may be written as follows

$$\begin{aligned} - \sum_{k=1}^n \frac{\mathbf{F}_{k,N}}{m_k} \nabla r_{k,N} &= - \sum_{k=1}^n \frac{\mathbf{F}_{k,N}}{m_k \mathrm{e}^{r_{k,N}}} \nabla \mathrm{e}^{r_{k,N}} \\ &= - \sum_{k=1}^n \frac{\mathbf{F}_{k,N}}{m_k \mathrm{e}^{r_{k,N}}} \left(\nabla \mathrm{e}^{r_{k,N}} - Y_{k,N} \sum_{j=1}^n \mathrm{e}^{r_{j,N}} \nabla r_{j,N} \right) \\ &= \sum_{k=1}^n \frac{\mathbf{F}_{k,N}^2}{m_k \mathrm{e}^{r_{k,N}}} \geq 0. \end{aligned} \quad (4.27)$$

Thus, to get the estimates one only needs to control the right hand side of (4.26). Substituting in (4.13) $p_k = \mathrm{e}^{r_{k,N}}$ and $\varrho = \varrho_N$ we deduce that the last term on the right hand side of (4.26) is bounded

$$\sum_{k=1}^n \int_\Omega \frac{\varrho_N \omega_k}{m_k} r_{k,N} \ \mathrm{d}x \leq c.$$

For the remaining one we have

$$\sum_{k=1}^n \int_\Omega |\mathrm{e}^{r_{k,N}} \operatorname{div} \mathbf{u}| \ \mathrm{d}x \leq \| \operatorname{div} \mathbf{u} \|_\infty \sum_{k=1}^n \int_\Omega \mathrm{e}^{r_{k,N}} \ \mathrm{d}x.$$

We are now at the position to deduce that (4.26) implies

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\delta r_{k,N}^2 + \mathrm{e}^{r_{k,N}} r_{k,N} - \mathrm{e}^{r_{k,N}}\|_{L^1(\Omega)} \right) \\
&\quad + \sum_{k=1}^{n} \left(\delta \|\nabla r_{k,N}\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \sqrt{\mathrm{e}^{r_{k,N}}}\|_{L^2(\Omega)}^2 + \left\| \frac{\mathbf{F}_{k,N}}{\sqrt{m_k \mathrm{e}^{r_{k,N}}}} \right\|_{L^2(\Omega)}^2 \right) \\
&\quad \leq c \sum_{k=1}^{n} (1 + \|\operatorname{div} \mathbf{u}\|_{L^\infty(\Omega)} \|\mathrm{e}^{r_{k,N}}\|_{L^1(\Omega)}).
\end{aligned} \tag{4.28}$$

Because the term $r_{k,N}^2$ is nonnegative and $\mathrm{e}^{r_{k,N}} r_{k,N}$ is bounded from below, $\operatorname{div} \mathbf{u} \in L^2(0,T;L^\infty(\Omega))$, we get using the Gronwall argument estimate (4.21). As was already announced, this estimate allows us to repeat the procedure described before in order to extend the solution to the whole time interval $[0,T]$. □

Although the above construction corresponds only to particular projection of the original problem it is clear that the final estimate is completely independent of N. This is the key argument in the limit passage; derivation of the other uniform estimates is a purpose of the next subsection.

4.3.2 Passage to the limit $N \to \infty$.

Our next goal is to derive bounds uniform with respect to N for fixed $\delta, \varepsilon > 0$ and $\mathbf{u}$ as in Theorem 4.5. We have already mentioned that estimate (4.21) obtained in the previous subsection does not depend on the dimension of Galerkin approximations. In particular, we have that

$$|\nabla \mathrm{e}^{r_{k,N}}| \leq 2|\nabla \sqrt{\mathrm{e}^{r_{k,N}}}| \sqrt{\mathrm{e}^{r_{k,N}}} \tag{4.29}$$

is bounded in $L^2(0,T;L^1(\Omega))$, thus, by the Sobolev imbedding, $\mathrm{e}^{r_{k,N}}$ is bounded in $L^2(0,T;L^{\frac{3}{2}}(\Omega))$. Returning to (4.29) we get

$$\|\nabla \mathrm{e}^{r_{k,N}}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega))} \leq c; \tag{4.30}$$

using once more the Sobolev imbedding theorem and the bound in $L^\infty(0,T;L^1(\Omega))$ we end up with

$$\|\mathrm{e}^{r_{k,N}}\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq c(\varepsilon). \tag{4.31}$$

Having this, we can return to (4.29) to deduce

$$\|\nabla \mathrm{e}^{r_{k,N}}\|_{L^{\frac{5}{4}}((0,T)\times\Omega)} \leq c(\varepsilon). \tag{4.32}$$

Apart from that, the limit passage requires also some further estimates providing compactness with respect to time.

Lemma 4.9. *There exists a constant c depending on the initial data, T, and the parameter ε such that*

$$\delta \|\partial_t r_{k,N}\|_{L^{\frac{5}{4}}(0,T;W^{-1,\frac{5}{4}}(\Omega))} \leq c. \tag{4.33}$$

Proof. We take any $\phi \in W^{1,5}(\Omega) \subset W^{1,2}(\Omega)$ such that $\|\phi\|_{W^{1,5}(\Omega)} \leq 1$ and decompose it into $\phi = \phi_1 + \phi_2$, where ϕ_1 is an orthogonal projection of ϕ (with respect to the scalar product induced by the norm of the space $L^2(\Omega)$) onto the subspace spanned by the vectors $\{h_1, \dots, h_N\}$. Using ϕ_1 as a test function in (4.19) we show that

$$\begin{aligned}
&\int_\Omega \partial_t(\delta r_{k,N} + \mathrm{e}^{r_{k,N}})\phi_1 \,\mathrm{d}x \\
&\quad = \int_\Omega \left(\mathrm{e}^{r_{k,N}}\mathbf{u} - (\delta + \varepsilon \mathrm{e}^{r_{k,N}})\nabla r_{k,N} + \frac{\mathbf{F}_{k,N}}{m_k}\right)\cdot\nabla\phi_1 \,\mathrm{d}x + \int_\Omega \frac{\varrho_N\omega_k}{m_k}\phi_1 \,\mathrm{d}x \\
&\quad \le \sum_{k=1}^n \left(\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathrm{e}^{r_{k,N}}\|_{L^{\frac{5}{4}}(\Omega)} + \delta\|\nabla r_{k,N}\|_{L^{\frac{5}{4}}(\Omega)} + \varepsilon\|\nabla \mathrm{e}^{r_{k,N}}\|_{L^{\frac{5}{4}}(\Omega)}\right)\|\phi_1\|_{W^{1,5}(\Omega)} \\
&\qquad + c\sum_{k=1}^n \left(\left\|\frac{\mathbf{F}_{k,N}}{\sqrt{m_k \mathrm{e}^{r_{k,N}}}}\right\|_{L^2(\Omega)}\|\sqrt{\mathrm{e}^{r_{k,N}}}\|_{L^{\frac{10}{3}}(\Omega)} + \|\mathrm{e}^{r_{k,N}}\|_{L^{\frac{5}{4}}(\Omega)}\right)\|\phi_1\|_{W^{1,5}(\Omega)}.
\end{aligned} \tag{4.34}$$

Then we have

$$\begin{aligned}
\|\partial_t r_{k,N}(t,\cdot)\|_{W^{-1,\frac{5}{4}}(\Omega)} &= \sup_{\phi\in W^{1,5}(\Omega);\|\phi\|\le 1}\left|\int_\Omega \partial_t r_{k,N}(t,\cdot)\phi \,\mathrm{d}x\right| \\
&= \sup_{\phi\in W^{1,5}(\Omega);\|\phi\|\le 1}\left|\int_\Omega \partial_t r_{k,N}(t,\cdot)\phi_1 \mathrm{d}x\right| = \int_\Omega \left|\partial_t r_{k,N}(t,\cdot)\varphi_1\right|\mathrm{d}x
\end{aligned}$$

for some $\varphi_1 \in W_0^{1,5}(\Omega)\cap Lin\{h_1,\dots,h_N\}$. Hence

$$\|\partial_t r_{k,N}(t,\cdot)\|_{W^{-1,\frac{5}{4}}(\Omega)} \le \sup_{\phi\in W^{1,5}(\Omega)\cap Lin\{h_1\dots h_N\};\|\phi\|\le 1}\frac{1}{\delta}\left|\int_\Omega (\delta + \mathrm{e}^{r_{k,N}(t,\cdot)})\partial_t r_{k,N}(t,\cdot)\phi \,\mathrm{d}x\right| \tag{4.35}$$

and due to estimate (4.34) we end up with $\|\partial_t r_{k,N}\|_{L^{\frac{5}{4}}(0,T;W^{-1,\frac{5}{4}}(\Omega))} \le \frac{c(\varepsilon)}{\delta}$. □

Our goal in the remaining part of this subsection is to examine the limit for $N\to\infty$. The above lemma allows us to apply the Aubin-Lions lemma in order to extract the subsequences which satisfy (4.16) in the limit. Indeed, for the sequence $r_{k,N}$ we deduce from (4.21) that it is possible to extract a subsequence such that

$$\begin{aligned}
r_{k,N} &\to r_k && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \\
\nabla r_{k,N} &\to \nabla r_k && \text{weakly in } L^2((0,T)\times\Omega), \\
\partial_t r_{k,N} &\to \partial_t r_k && \text{weakly in } L^{\frac{5}{4}}(0,T;W^{-1,\frac{5}{4}}(\Omega)), \\
r_{k,N} &\to r_k && \text{strongly in } L^2(0,T;L^p(\Omega)),\ p<6;
\end{aligned} \tag{4.36}$$

in particular, there exists a subsequence $r_{k,N}$ which converges to r_k a.e. on $(0,T)\times\Omega$. This, together with (4.33) and the boundedness of space gradient of $\mathrm{e}^{r_{k,N}}$ implies

$$\begin{aligned}
\nabla \mathrm{e}^{r_{k,N}} &\to \nabla \mathrm{e}^{r_k} && \text{weakly in } L^2(0,T;L^1(\Omega))\cap L^{\frac{5}{4}}((0,T)\times\Omega), \\
\mathrm{e}^{r_{k,N}} &\to \mathrm{e}^{r_k} && \text{strongly in } L^q((0,T)\times\Omega),\ q<\tfrac{5}{3},
\end{aligned} \tag{4.37}$$

at least for a chosen subsequence.
Next we recall basics facts from the theory of Orlicz spaces. For more details as well as proofs of results below see e.g. [55], Chapter 3 or [1], Chapter 8.
For the following pair of complementary Young functions

$$\Phi(t) = \mathrm{e}^t - 1; \qquad \Psi = \begin{cases} 0 & \text{for } 0\le t<1, \\ t(\log t - 1) + 1 & \text{for } t\ge 1, \end{cases}$$

we consider the Orlicz spaces $L_\Phi(\Omega)$, $L_\Psi(\Omega) := L\log L(\Omega)$ and the space $E_\Phi(\Omega)$, which is the closure of bounded measurable functions on Ω in $L_\Phi(\Omega)$. In particular, $E_\Phi(\Omega)$ is separable and $(E_\Phi(\Omega))^* = L_\Psi(\Omega)$. Therefore, we can extract a subsequence such that

$$\mathrm{e}^{r_{k,N}} \to \mathrm{e}^{r_k} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L\log L(\Omega)),$$

where by weak* convergence we mean that $\langle \mathrm{e}^{r_{k,N}}, \phi\rangle \to \langle \mathrm{e}^{r_k}, \phi\rangle$ for each $\phi \in L^1(0,T;E_\Phi(\Omega))$. Moreover, by (4.21) we deduce that for all $\eta \in \mathcal{D}(\Omega)$ the functions $t \to \int_\Omega r_{k,N}(t)\eta \,\mathrm{d}x$ form a bounded equicontinuous sequence in $C[0,T]$. Hence, the weak* convergence of $r_{k,N}$, $\mathrm{e}^{r_{k,N}}$ may be improved, using the Arzelá-Ascoli theorem and the density of $\mathcal{D}(\Omega)$ in L^p for $p \in [1,\infty)$ and in $E_\Phi(\Omega)$, to the following

$$\begin{aligned} r_{k,N} \to r_k \quad &\text{in } C([0,T];L^2_{\text{weak}}(\Omega)),\\ \mathrm{e}^{r_{k,N}} \to \mathrm{e}^{r_k} \quad &\text{in } C([0,T];L\log L_{\text{weak}^*}(\Omega)), \end{aligned} \tag{4.38}$$

which gives sense to the initial conditions for r_k and e^{r_k}.
Finally, as $\mathrm{e}^{r_{k,N}} > 0$ on $[0,T]\times\Omega$, $\frac{m_j \mathrm{e}^{r_{j,N}}}{\varrho_N}$ is bounded in $L^\infty((0,T)\times\Omega)$ and so, the pointwise convergence of $\mathrm{e}^{r_{k,N}}$ for any $k = 1,\ldots,n$ implies that

$$\frac{m_j \mathrm{e}^{r_{j,N}}}{\varrho_N} \to \frac{m_j \mathrm{e}^{r_j}}{\sum_{k=1}^n m_k \mathrm{e}^{r_k}} \quad \text{strongly in } L^p((0,T)\times\Omega),\ p<\infty.$$

Therefore we get also the convergence of $\mathbf{F}_{k,N}$. Summarizing, the result achieved in this section can be stated as follows.

Lemma 4.10. *Let the assumptions of Theorem 4.5 be fulfilled and let $\varrho_{k,N}$, $k = 1,\ldots,n$ be the unique solution to the approximate problem* (4.19) *constructed in Theorem 4.8. Then there exists a subsequence $N_l \to +\infty$ such that the limit functions $r_k = \lim_{N_l\to\infty} r_{k,N_l}$ satisfy system* (4.16) *in the sense of distributions.*

4.3.3 Estimates independent of δ, passage to the limit $\delta \to 0$.

In what follows, we will denote by $r_{k,\delta}$, $k = 1,\ldots,n$, the solution to the approximate problem (4.16), constructed in the previous subsection. The next step of the proof is to let $\delta \to 0^+$ in order to eliminate the artificial time derivative as well as the δ-dependent parabolic regularization in (4.16). To this aim, we first need to derive some uniform bounds sufficient to deduce compactness of the nonlinear terms, which is the subject of the present subsection. We start by proving the energy inequality.

Lemma 4.11. *Let $\delta, \epsilon > 0$, then the solution to* (4.16) *enjoys the following estimate*

$$\begin{aligned} &\sup_{t\in(0,T)} \sum_{k=1}^n \| \left(\delta r_{k,\delta}^2 + e^{r_{k,\delta}} r_{k,\delta}\right)(t)\|_{L^1(\Omega)} \\ &\quad + \sum_{k=1}^n \left\{ \int_0^T \delta \|\nabla r_{k,\delta}\|_{L^2(\Omega)} + \varepsilon \|\nabla\sqrt{e^{r_{k,\delta}}}\|^2_{L^2(\Omega)} + \left\| \frac{\mathbf{F}_{k,\delta}}{\sqrt{m_k e^{r_{k,\delta}}}} \right\|^2_{L^2(\Omega)} \mathrm{d}t \right\} \le c, \end{aligned} \tag{4.39}$$

for a constant c that depends only on the initial data and T.

Proof. Due to the pointwise convergence of $r_{k,N}$ and $\mathrm{e}^{r_{k,N}}$, see (4.36),(4.37), we have

$$\int_\Omega \left(\delta\frac{r_{k,N}^2}{2} + \mathrm{e}^{r_{k,N}} r_{k,N} - \mathrm{e}^{r_{k,N}}\right) \mathrm{d}x \to \int_\Omega \left(\delta\frac{r_{k,\delta}^2}{2} + \mathrm{e}^{r_{k,\delta}} r_{k,\delta} - \mathrm{e}^{r_{k,\delta}}\right) \mathrm{d}x$$

in the sense of distributions on $(0,T)$. Moreover, we know that $\nabla r_{k,N}$ converges to $\nabla r_{k,\delta}$ weakly in $L^2((0,T)\times\Omega)$, thus, due to lower semicontinuity of convex functions we have

$$\int_0^T\int_\Omega |\nabla r_k|^2 \ \mathrm{d}x \ \mathrm{d}t \le \liminf_{N\to\infty}\int_0^T\int_\Omega |\nabla r_{k,N}|^2 \ \mathrm{d}x \ \mathrm{d}t.$$

The same argument can be also applied for the nonlinear terms $\nabla\sqrt{\mathrm{e}^{r_{k,N}}}$ and $\frac{\mathbf{F}_{k,N}}{\sqrt{m_k\mathrm{e}^{r_{k,N}}}}$. □

Having obtained the uniform estimates, we can return to our original problem. We define the solution to (4.4) in the following way

$$\varrho_{k,\delta} := m_k\mathrm{e}^{r_{k,\delta}}, \quad k=1,\dots,n.$$

Hence, (4.39) gives rise to the following estimate

$$\operatorname*{ess\ sup}_{t\in(0,T)}\int_\Omega (\log p_{k,\delta}(t))^2 \ \mathrm{d}x + \int_0^T\int_\Omega |\nabla \log p_{k,\delta}|^2 \ \mathrm{d}x \ \mathrm{d}t \le c(\delta),$$

which is equivalent to positivity of partial densities

$$\varrho_{k,\delta} > 0 \quad \text{a.e. in } (0,T)\times\Omega,\ k=1,\dots,n.$$

We may now repeat arguments leading to (4.33). More precisely, as we can now test by any function from the space $L^5(0,T;W^{1,5}(\Omega))$, due to a similar argument as in (4.34) and (4.35) we control $\partial_t r_{k,\delta}$ and $\partial_t(\delta r_{k_\delta}+\mathrm{e}^{r_{k,\delta}})$; thus we also control its difference. Hence, uniformly with respect to δ we have

$$\delta\|\partial_t \log p_{k,\delta}\|_{L^{\frac54}(0,T;W^{-1,\frac54}(\Omega))} + \|\partial_t p_{k,\delta}\|_{L^{\frac54}(0,T;W^{-1,\frac54}(\Omega))} \le c. \tag{4.40}$$

From (4.39) it follows also that the sequence $\sqrt{p_{k,\delta}}$ is uniformly bounded in $L^2(0,T;W^{1,2}(\Omega))$. Therefore, using again the Aubin-Lions lemma, we show that $\varrho_{k,\delta}=p_{k,\delta}m_k$ converges to ϱ_k for $\delta\to0$ pointwisely on $(0,T)\times\Omega$. This together with the uniform estimates from (4.39) allows us to deduce the following convergences when $\delta\to0$:

$$\begin{array}{rll} \delta\log\frac{\varrho_{k,\delta}}{m_k} \to 0 & \text{strongly} & \text{in } L^\infty(0,T;L^2(\Omega)),\\ \delta\nabla\log\frac{\varrho_{k,\delta}}{m_k} \to 0 & \text{strongly} & \text{in } L^2((0,T)\times\Omega),\\ \varrho_{k,\delta}\to\varrho_k & \text{strongly in } L^q((0,T)\times\Omega), q<\frac53, & \\ \nabla\varrho_{k,\delta}\to\nabla\varrho_k & \text{weakly in } L^2(0,T;L\log L_{\text{weak}^*}(\Omega))\cap L^{\frac54}((0,T)\times\Omega), & \\ \varrho_{k,\delta}\to\varrho_k & \text{in } C([0,T];L\log L_{\text{weak}^*}(\Omega)). & \end{array} \tag{4.41}$$

Moreover

$$\begin{array}{c} \frac{\varrho_{j,\delta}}{\sum_{k=1}^n\varrho_{k,\delta}} \to \frac{\varrho_j}{\sum_{k=1}^n\varrho_k} \quad \text{strongly in } L^p((0,T)\times\Omega),\ p<\infty,\\ \varrho_k\ge0 \quad \text{a.e. in } (0,T)\times\Omega,\ k=1,\dots,n, \end{array} \tag{4.42}$$

and due to a similar argument as in the previous section we obtain that

$$\mathbf{F}_{k,\delta}\to\mathbf{F}_k \quad \text{weakly in } L^p((0,T)\times(\Omega)),$$

for some $p>1$, where $\mathbf{F}_k$ depends on the limit functions $\varrho_1,\dots,\varrho_n$ as specified in (4.6).

This is the last argument in favor to let $\delta\to0$ in the approximate system (4.16), we have thus proved the following result.

Lemma 4.12. *The limit quantities* ϱ_k, $k = 1, \dots, n$ *satisfy*

$$\partial_t \varrho_k + \operatorname{div}(\varrho_k \mathbf{u}) - \varepsilon \Delta \varrho_k + \operatorname{div}(\mathbf{F}_k) = \varrho \omega_k, \quad k = 1, ...n, \tag{4.43}$$

in the sense of distributions on $(0, T) \times \Omega$.

In addition, denoting $\varrho = \sum_{k=1}^n \varrho_k$, and summing (4.43) with respect to $k = 1, \dots, n$, the properties (4.12), (4.14) lead to the following equation

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0. \tag{4.44}$$

This equation is, due to the previous lemma, satisfied in the same sense as system (4.43), together with the initial condition $\varrho(0, x) = \varrho^0 = \sum_{k=1}^n \varrho_k^0(x)$ for a.a. $x \in \Omega$.
On the other hand, provided $\mathbf{u}$ and the initial data are sufficiently smooth (as in Theorem 4.5 is enough), we can identify ϱ with ϱ_ε – the unique classical and positive solution to the initial-value problem

$$\partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{u}) - \varepsilon \Delta \varrho_\varepsilon = 0, \quad \varrho_\varepsilon(0, x) = \varrho^0,$$

constructed by means of the usual Galerkin approach within the standard L^2 theory, the bootstrap argument and the maximal $L^p - L^q$ regularity applied to the problem (cf. [85], Sections 7.6.3-7.6.7)

$$\partial_t \varrho_\varepsilon - \varepsilon \Delta \varrho_\varepsilon = f := -\operatorname{div}(\varrho_\varepsilon \mathbf{u}), \quad \varrho_\varepsilon(0, x) = \varrho^0. \tag{4.45}$$

In particular, we know that any solution of (4.43) satisfies

$$\sum_{k=1}^n \varrho_k = \varrho_\varepsilon \quad \text{a.e. in } (0, T) \times \Omega.$$

4.3.4 Estimates independent of ε

This part of the proof is dedicated to derivation of estimates independent of ε and to the last limit passage $\varepsilon \to 0$. The departure point is an analogue of (4.39)

$$\sup_{t \in (0,T)} \sum_{k=1}^n \|p_{k,\varepsilon} \log p_{k,\varepsilon}(t)\|_{L^1(\Omega)} + \sum_{k=1}^n \left\{ \varepsilon \int_\Omega \|\nabla \sqrt{p_{k,\varepsilon}}\|^2_{L^2(\Omega)} \, \mathrm{d}x + \int_0^T \left\| \frac{\mathbf{F}_{k,\varepsilon}}{\sqrt{\varrho_{k,\varepsilon}}} \right\|^2_{L^2(\Omega)} \mathrm{d}t \right\} \leq c, \tag{4.46}$$

where the constant c does not depend on ε, and the fact that $Y_{k,\varepsilon} = \frac{\varrho_{k,\varepsilon}}{\varrho_\varepsilon}$, $k = 1, \dots, n$ satisfy

$$0 \leq Y_{k,\varepsilon} \leq 1, \quad \sum_{k=1}^n Y_{k,\varepsilon} = 1.$$

As we see, it is not so clear whether we have any additional space regularity of solutions. Indeed, to repeat the arguments from the previous limit passage, one needs to show that the quantities $\nabla \sqrt{\varrho_{k,\varepsilon}}$, $k = 1, \dots, n$ are controllable independently of ε. To this end, we investigate more carefully the last term in (4.46). We have the following result.

Lemma 4.13. *Let assumptions of Theorem 4.5 be fulfilled and let estimate* (4.46) *be valid. Then, for any* $k = 1, \dots, n$ *the solution to the approximate problem* (4.43) *satisfies*

$$\|\nabla \sqrt{\varrho_{k,\varepsilon}}\|_{L^2((0,T) \times \Omega)} \leq c, \tag{4.47}$$

where the constant c *does not depend on* ε.

Proof. Using (4.9) we deduce from (4.46) that

$$\sum_{k=1}^{n}\int_0^T\int_\Omega \frac{\mathbf{F}_{k,\varepsilon}^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\ \mathrm{d}t = \sum_{k=1}^{n}\int_0^T\int_\Omega \frac{(\pi_\varepsilon \mathbf{d}_{k,\varepsilon})^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\ \mathrm{d}t \le c. \tag{4.48}$$

To exploit this estimate we first recall the following property of the species diffusion forces

$$\pi d_\varepsilon^i = \pi\begin{pmatrix} d_{1,\varepsilon}^i\\ d_{2,\varepsilon}^i\\ \vdots\\ d_{n,\varepsilon}^i\end{pmatrix} = \begin{pmatrix} Z_{1,\varepsilon} & -Y_{1,\varepsilon} & \dots & -Y_{1,\varepsilon}\\ -Y_{2,\varepsilon} & Z_{2,\varepsilon} & \dots & -Y_{2,\varepsilon}\\ \vdots & \vdots & \ddots & \vdots\\ -Y_{n,\varepsilon} & -Y_{n,\varepsilon} & \dots & Z_{n,\varepsilon}\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x_i}p_{1,\varepsilon}\\ \nabla_{x_i}p_{2,\varepsilon}\\ \vdots\\ \nabla_{x_i}p_{n,\varepsilon}\end{pmatrix} = C_\varepsilon \nabla_{x_i} p_\varepsilon, \tag{4.49}$$

where p_ε denotes the n dimensional vector $(p_{1,\varepsilon},\dots,p_{n,\varepsilon})^T$ and $\nabla p_\varepsilon = (\nabla p_{1,\varepsilon},\dots,\nabla p_{n,\varepsilon})^T$. So, inserting it to (4.48) we obtain

$$\int_0^T\int_\Omega\sum_{k=1}^{n}\frac{(C_\varepsilon\nabla p_\varepsilon)_k^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\ \mathrm{d}t \le c. \tag{4.50}$$

The matrix C_ε is degenerated as the vector $Y_\varepsilon = (Y_{1,\varepsilon},\dots,Y_{n,\varepsilon})^T$ belongs to its kernel, so estimate (4.50) does not imply integrability of $\nabla p_{k,\varepsilon}$ for all $k = 1,\dots,n$, at the same time. However, as we know that $\sum_{k=1}^n \varrho_{k,\varepsilon}$ coincides with the classical unique solution to (4.44) ϱ_ε, we can use the assumption on higher regularity of ϱ_ε to control the full gradient of p_ε. Indeed, note that the matrix C_ε possesses $n-1$ eigenvectors $v^m = (v_1^m,\dots,v_n^m)^t$, $m\in\{1,\dots,n-1\}$

$$v_l^m = \begin{cases} -1 \text{ for } l = m,\\ \ \ 1 \text{ for } l = n,\\ \ \ 0 \text{ for } l \neq m,n,\end{cases}$$

corresponding to the eigenvalue 1. Therefore, denoting

$$C_\varepsilon(t,x)\nabla_{x_i}p_\varepsilon(t,x) := (\nabla_{x_i}p_\varepsilon(t,x))^I, \quad \text{for } (t,x)\in[0,T)\times\Omega,$$

we have, for every (k,i)-th coordinate of $\nabla p_\varepsilon(t,x)$, $(k,i)\in\{1,\dots,n\}\times\{1,2,3\}$, the following decomposition

$$(\nabla_{x_i}p_\varepsilon(t,x))_k = (\nabla_{x_i}p_\varepsilon(t,x))_k^I + \alpha_i(t,x)Y_{k,\varepsilon}(t,x), \tag{4.51}$$

where $\alpha_i(t,x)Y_{k,\varepsilon}(t,x)$ is the k-th coordinate of the projection of vector $\nabla_{x_i}p_\varepsilon(t,x)\in\mathbb{R}^n$ on the nullspace of matrix $C_\varepsilon(t,x)$, which is spanned by the vector $Y_\varepsilon(t,x)$. Next, multiplying (4.51) by m_k and summing over $k\in\{1,\dots,n\}$ one gets

$$\alpha_i = \frac{\nabla_{x_i}\varrho_\varepsilon}{\sum_{k=1}^n m_k Y_{k,\varepsilon}} - \frac{\sum_{k=1}^n m_k(\nabla_{x_i}p_\varepsilon)_k^I}{\sum_{k=1}^n m_k Y_{k,\varepsilon}}.$$

Combining this with (4.51) we can express each of the gradients of partial pressures in terms of known quantities

$$\nabla_{x_i}p_\varepsilon = (\nabla_{x_i}p_\varepsilon)^I + \left(\frac{\nabla_{x_i}\varrho_\varepsilon}{\sum_{k=1}^n m_k Y_{k,\varepsilon}} - \frac{\sum_{k=1}^n m_k(\nabla_{x_i}p_\varepsilon)_k^I}{\sum_{k=1}^n m_k Y_{k,\varepsilon}}\right)Y_\varepsilon. \tag{4.52}$$

Next, due to the first equality in (4.9) we can write

$$\frac{\mathbf{F}_{k,\varepsilon}^2}{\varrho_{k,\varepsilon}} = \frac{|\nabla p_{k,\varepsilon}|^2}{\varrho_{k,\varepsilon}} - 2\frac{Y_{k,\varepsilon}\nabla p_{k,\varepsilon}\cdot\nabla\pi_\varepsilon}{\varrho_{k,\varepsilon}} + \frac{Y_{k,\varepsilon}^2|\nabla\pi_\varepsilon|^2}{\varrho_{k,\varepsilon}},$$

which is bounded $L^1((0,T)\times\Omega)$ for every $k=1,\dots,n$, on account of (4.48). Therefore, by the Cauchy inequality,

$$\int_0^T\int_\Omega \frac{|\nabla p_{k,\varepsilon}|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t \le c\left(1+\int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2|\nabla\pi_\varepsilon|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{4.53}$$

Since $\nabla\pi_\varepsilon = \sum_{k=1}^n(\nabla p_\varepsilon)_k$ and $\sum_{k=1}^n(Y_\varepsilon)_k = 1$, we can use (4.52) and (4.50) to estimate the right hand side of the above inequality. We have

$$\begin{aligned}\int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2|\nabla\pi_\varepsilon|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2|\sum_{k=1}^n(\nabla p_\varepsilon)_k|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2}{\varrho_{k,\varepsilon}}\left(|\sum_{k=1}^n(\nabla p_\varepsilon)_k^I|^2 + \frac{|\nabla\varrho_\varepsilon|^2}{(\sum_{k=1}^n m_kY_{k,\varepsilon})^2} + \frac{|\sum_{k=1}^n m_k(\nabla p_\varepsilon)_k^I|^2}{(\sum_{k=1}^n m_kY_{k,\varepsilon})^2}\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.54}$$

Next, denoting $m_{max} = \max\{m_1,\dots,m_n\}$ and $m_{min} = \min\{m_1,\dots,m_n\}$, the first term from the right hand side of (4.54) can be estimated as follows

$$\begin{aligned}&\int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2}{\varrho_{k,\varepsilon}}\left(\left|\sum_{k=1}^n(\nabla p_\varepsilon)_k^I\right|^2 + \frac{|\sum_{k=1}^n m_k(\nabla p_\varepsilon)_k^I|^2}{(\sum_{k=1}^n m_kY_{k,\varepsilon})^2}\right)\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^T\int_\Omega \frac{Y_{k,\varepsilon}}{\varrho_\varepsilon}\left(\sum_{k=1}^n\left|(\nabla p_\varepsilon)_k^I\right|^2 + \frac{m_{max}^2\sum_{k=1}^n\left|(\nabla p_\varepsilon)_k^I\right|^2}{m_{min}^2}\right)\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^T\int_\Omega \frac{Y_{k,\varepsilon}}{\varrho_\varepsilon}\left(1+\frac{m_{max}^2}{m_{min}^2}\right)\sum_{k=1}^n\frac{\varrho_{k,\varepsilon}\left|(\nabla p_\varepsilon)_k^I\right|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t \le c\int_0^T\int_\Omega\sum_{k=1}^n\frac{\left|(\nabla p_\varepsilon)_k^I\right|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.55}$$

which is bounded due to (4.50).
The second integral on the right hand side of (4.54) can be bounded since $Y_{k,\varepsilon}\le 1$ for any $k=1,\dots,n$

$$\int_0^T\int_\Omega \frac{Y_{k,\varepsilon}^2}{\varrho_{k,\varepsilon}}\frac{|\nabla\varrho_\varepsilon|^2}{(\sum_{k=1}^n m_kY_{k,\varepsilon})^2}\,\mathrm{d}x\,\mathrm{d}t \le c\int_0^T\int_\Omega\frac{|\nabla\varrho_\varepsilon|^2}{\varrho_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t.$$

Returning to (4.53), we have thus shown that

$$\int_0^T\int_\Omega \frac{|\nabla p_{k,\varepsilon}|^2}{\varrho_{k,\varepsilon}}\,\mathrm{d}x\,\mathrm{d}t \le c\left(1+\int_0^T\int_\Omega\frac{|\nabla\varrho_\varepsilon|^2}{\varrho_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right).$$

Using Lemma 4.14 below we control $\nabla\varrho_\varepsilon$ in $L^2((0,T)\times\Omega)$ independently of ε; due to the properties of the initial value (strict positivity) we therefore also control $\nabla\sqrt{\varrho_\varepsilon}$ in the same space. The proof of the lemma is finished. □

Lemma 4.14. *Under the assumptions of Theorem 4.5, there exists c independent of ε such that for any $\varepsilon\in(0,1]$*

$$\|\varrho-\varrho_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))}\le c,$$

where ϱ is the unique solution to (4.2) and ϱ_ε is the unique solution to (4.44).

Proof. We have

$$\partial_t(\varrho_\varepsilon-\varrho)+\operatorname{div}(\boldsymbol{u}(\varrho_\varepsilon-\varrho))-\varepsilon\Delta(\varrho_\varepsilon-\varrho)=\varepsilon\Delta\varrho,$$

with $(\varrho_\varepsilon - \varrho)(0,x) = 0$. Testing equation above by $\varrho_\varepsilon - \varrho$ and recalling that $\|\varrho\|_{L^2(0,T;W^{1,2}(\Omega))} < \infty$, we get the result. $\Box$

This is the final argument that allows us to repeat the procedure described for the limit passage $\delta \to 0$ in order to eliminate the last regularizing term from (4.43). The proof of Theorem 4.5 is now complete. $\Box$

Chapter 5

The heat-conducting mixture of n species

The aim of this chapter is to prove sequential stability of the *weak variational entropy solutions* to the general system described in the introduction to this thesis. The task is hence to generalize the result from Chapter 3, Section 3.2 to the case of heat-conducting mixture of n gaseous species. As we have seen, the proof of sequential stability of weak solutions is a last step in the complete proof of existence of solutions and provides the first argument in favor of construction of an approximate scheme. The key mathematical tool here is the entropy dissipation method combined with another energy-inequality, found for the Navier-Stokes system with density-dependent viscosity coefficients by Bresch and Desjardins [12].

5.1 Introduction

Recall that, to describe the flow of n-component chemically reacting compressible mixture, when no external force is present, we use the following system of equations

$$\left.\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) &= 0 \\ \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla\pi &= \mathbf{0} \\ \partial_t(\varrho E) + \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(\pi\mathbf{u}) + \operatorname{div}\mathbf{Q} - \operatorname{div}(\mathbf{S}\mathbf{u}) &= 0 \\ \partial_t\varrho_k + \operatorname{div}(\varrho_k\mathbf{u}) + \operatorname{div}(\mathbf{F}_k) = \varrho\vartheta\omega_k, \quad k &\in \{1,...,n\} \end{aligned}\right\} \quad \text{in } (0,T)\times\Omega. \tag{5.1}$$

Here, $\Omega = \mathbb{T}^3$ and we supplement the above system by the constitutive relations (1.5-1.19) and (1.23-1.24) taken over from Section 1.3.3. The only difference is the "cold" component of the pressure π_c. Similarly to Section 3.3 we introduce particular modification in the neighborhood of small densities. Namely, we assume that π_c is a continuous function satisfying the following growth conditions

$$\pi_c'(\varrho) = \begin{cases} c_1\varrho^{-\gamma^- -1} & \text{for } \varrho \le 1, \\ c_2\varrho^{\gamma^+ -1} & \text{for } \varrho > 1, \end{cases} \tag{5.2}$$

for positive constants c_1, c_2 and $\gamma^-, \gamma^+ > 1$. It was proposed in [13] to encompass plasticity and elasticity effects of solid materials, for which low densities may lead to negative pressures. By this modification the compactness of velocity at the last level of approximation can be obtained without requiring more a priori regularity than expected from the usual energy approach [69].

The global-in-time existence and asymptotic stability of solutions to system (5.1), supplemented with physically relevant constitutive relations, was established by Giovangigli [45], Chapter 9, Theorem 9.4.1. He considered, for instance, generic matrix C_{kl} relating the diffusion deriving forces $\mathbf{d}_k$ to the species diffusion fluxes

$$\mathbf{F}_k = -\sum_{l=1}^{n} C_{kl}\mathbf{d}_l \ + \ \text{Soret effect}, \quad k = 1, ...n, \tag{5.3}$$

which is singular $CY = 0$, $Y = (Y_1, \ldots, Y_n)^T$, $Y_k = \frac{\varrho_k}{\varrho}$, $k = 1, \ldots, n$, and is not symmetric in general. His result holds provided *the initial data are sufficiently close to an equilibrium state*. Our main motivation is to focus on the case of arbitrary large data. It should be however emphasized that many simplifications are assumed in the present model, in comparison to [45]. In particular, we will concentrate only on the diffusion effects due to the mole fractions and pressure gradients and restrict to particular form of matrix C satisfying properties (1.20). Following [110] we also suppose that the matrix $D_{kl} = \frac{C_{kl}}{\varrho Y_k}$, $k, l = 1, \ldots, n$ is symmetric and coercive on the hyperplanes which do not contain the vector Y. This assumption is consistent with the second law of thermodynamics which postulates non-negativity of the entropy production rate associated with diffusive process.

Due to our knowledge, so far there is no result concerning global-in-time existence of solutions for systems with general diffusion (5.3). Here we present the first step in this direction – the proof that, provided the sequence of sufficiently smooth approximate solutions has been constructed, we are able to extract a subsequence, whose limit satisfies a suitable weak formulation of (5.1). Let us emphasize that we use the framework of weak variational solutions, that is to say the total energy balance is replaced by the entropy inequality and the global total energy balance. These solutions were introduced by Feireisl and Novotný to study solvability and various asymptotic limits of the Navier-Stokes-Fourier system [36]. Within this definition weak solutions may dissipate more kinetic energy than, if there are any, the classical ones. However, it can be verified that this "missing energy" is equal to 0 provided the weak solution is regular enough and satisfies the total global energy balance. Moreover, in [37] the authors proved the *weak-strong uniqueness* of such solution, meaning that it coincides with the strong solution, emanating from the same initial data, as long as the latter exists.

The issue of existence of global weak solutions to the Navier-Stokes equations for heat conducting compressible fluids was also addressed in [13]. Under additional assumption on the "cold" component of the pressure, similar to (5.2), the authors proved sequential stability of weak solutions. Unfortunately, obtainment of analogous result for the full system (5.1) seems to be much more involved, mainly due to the strong coupling between the internal energy equation and the degenerated equations for species. This is the main obstacle in preserving sufficient regularity of the temperature required to perform the limit passage in the weak formulation of energy balance.

The outline of this chapter is the following. In the next section we specify the structural properties for the transport coefficients and postulate several simplifications. In Section 5.3, we define the notion of weak variational solutions and state the main result of the paper. The key a priori estimates are derived in Section 5.4 together with some further estimates and positivity of the absolute temperature. Finally, the last step of the proof of Theorem 5.5 – the limit passage – is performed in Section 5.5.

5.2 Main hypotheses

In what follows we give a list of hypotheses and assumptions on the initial conditions and the form of various coefficients appearing in the fluxes of system (5.1).

Initial data

The choice of quantities describing the initial state of system (5.1) is dictated by the weak formulation of the problem, which is specified in Definition 5.4 below. We take

$$\begin{gathered}\varrho(0,\cdot)=\varrho^0,\ \varrho\mathbf{u}(0,\cdot)=(\varrho\mathbf{u})^0,\ \varrho s(0,\cdot)=(\varrho s)^0,\ \textstyle\int_\Omega \varrho E(0,\cdot)\ \mathrm{d}x=\int_\Omega(\varrho E)^0\ \mathrm{d}x,\\ \varrho_k(0,\cdot)=\varrho_k^0,\quad \text{for } k=1,\ldots,n,\ \text{ in } \Omega.\end{gathered}\tag{5.4}$$

In addition, we assume that ϱ^0 is a nonnegative measurable function such that

$$\int_\Omega \varrho^0\ \mathrm{d}x=M^0,\qquad \int_\Omega \frac{1}{\varrho^0}\left|\nabla\mu\left(\varrho^0\right)\right|^2\ \mathrm{d}x<\infty,\tag{5.5}$$

and the initial densities of species satisfy

$$0\le\varrho_k^0(x),\ k=1,\ldots,n,\quad \sum_{k=1}^n\varrho_k^0(x)=\varrho^0(x),\ \text{ a.e. in } \Omega.\tag{5.6}$$

Further, the initial temperature ϑ^0 is a measurable function such that

$$\vartheta^0(x)>0 \text{ a.e. in } \Omega,\quad \vartheta^0\in W^{1,\infty}(\Omega)$$

and the following compatibility condition is satisfied

$$(\varrho s)^0=\varrho^0 s(\vartheta^0,\varrho_1^0,\ldots,\varrho_n^0),\quad (\varrho s)^0\in L^1(\Omega).\tag{5.7}$$

Finally, we require that the initial distribution of the momentum is such that

$$(\varrho\mathbf{u})^0=0 \text{ a.e. on } \{x\in\Omega:\varrho^0(x)=0\}\quad\text{and}\quad \int_\Omega\frac{\left|(\varrho\mathbf{u})^0\right|^2}{\varrho^0}\ \mathrm{d}x<\infty$$

and the global total energy at the initial time is bounded

$$\int_\Omega(\varrho E)^0\ \mathrm{d}x=\int_\Omega\left(\frac{\left|(\varrho\mathbf{u})^0\right|^2}{2\varrho^0}+\varrho^0 e(\varrho^0,\vartheta^0,\varrho_1^0,\ldots,\varrho_n^0)\right)\ \mathrm{d}x<\infty.\tag{5.8}$$

Transport coefficients

(i) The viscosity coefficients μ and ν in the stress tensor $\mathbf{S}$ (1.15) are functions of the density only. The density dependent viscosity appears when one derives the compressible Navier-Stokes equations using the Chapman-Enskog expansion [4]. Starting from the classical Boltzmann equation one obtains an expression for μ which is proportional only to the square root of the absolute temperature. If the flow is isentorpoic, this dependence may be translated into the dependence on the density $\mu(\varrho)=\varrho^{(\gamma-1)/2}$, see [51], [29].

Here, as in Chapter 3, we assume that the viscosity coefficients $\mu=\mu(\varrho),\ \nu=\nu(\varrho)$ are $C^1(0,\infty)$ functions related by

$$\nu(\varrho)=2\varrho\mu'(\varrho)-2\mu(\varrho),\tag{5.9}$$

which is strictly a mathematical constraint allowing to obtain better regularity of ϱ, see [10]. In addition, they enjoy the following bounds

$$\begin{gathered}\underline{\mu}' \leq \mu'(\varrho) \leq \frac{1}{\underline{\mu}'}, \quad \mu(0) \geq 0, \\ |\nu'(\varrho)| \leq \frac{1}{\underline{\mu}'}\mu'(\varrho), \\ \underline{\mu}'\mu(\varrho) \leq 2\mu(\varrho) + 3\nu(\varrho) \leq \frac{1}{\underline{\mu}'}\mu(\varrho),\end{gathered} \tag{5.10}$$

for some positive constant $\underline{\mu}'$.

Remark 5.1. *The assumption $\underline{\mu}' \leq \mu'(\varrho)$ is not optimal, but it makes the proof much easier, however the bound from above is essential in order to get integrability of several important quantities. For further discussion on this topic we refer to [69].*

(ii) The heat conductivity coefficient $\kappa = \kappa(\varrho, \vartheta)$ from definition of the heat flux $\mathbf{Q}$ (1.23) is a $C^1([0,\infty) \times [0,\infty))$ function which satisfies

$$\underline{\kappa_0}(1+\varrho)(1+\vartheta^\alpha) \leq \kappa(\varrho,\vartheta) \leq \overline{\kappa_0}(1+\varrho)(1+\vartheta^\alpha). \tag{5.11}$$

In the above formulas $\underline{\kappa_0}, \overline{\kappa_0}, \alpha$ are positive constants and $\alpha \geq 2$.

(iii) As was announced in the introduction, in definition of the species diffusion fluxes $\mathbf{F}_k$ (1.18), we restrict to an exact form of the flux diffusion matrix C compatible with the set of mathematical assumptions postulated in [45]. The prototype example is the same as studied in the previous chapter, namely

$$C = \begin{pmatrix} Z_1 & -Y_1 & \dots & -Y_1 \\ -Y_2 & Z_2 & \dots & -Y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -Y_n & -Y_n & \dots & Z_n \end{pmatrix}, \tag{5.12}$$

where $Z_k = \sum_{\substack{i=1 \\ i\neq k}}^n Y_i$. Concerning the diffusion coefficient C_0 from (1.18) we assume that it is a continuously differentiable function of ϑ and ϱ and that there exist positive constants $\underline{C_0}$, $\overline{C_0}$ such that

$$\underline{C_0}\varrho(1+\vartheta) \leq C_0 \leq \overline{C_0}\varrho(1+\vartheta). \tag{5.13}$$

Remark 5.2. *One of the main consequences of* (5.12) *is that*

$$\sum_{k=1}^n \mathbf{F}_k = 0. \tag{5.14}$$

Remark 5.3. *Note that* (5.12) *also implies that the vector of species diffusion forces $\mathbf{d} = (\mathbf{d}_1, \dots, \mathbf{d}_n)^T$ is an eigenvector of the matrix C corresponding to the eigenvalue* 1 *and we recast that*

$$\mathbf{F}_k = -C_0 \sum_{l=1}^n C_{kl}\mathbf{d}_l = -C_0\mathbf{d}_k = -\frac{C_0}{\pi_m}(\nabla p_k - Y_k\nabla\pi_m) = -\frac{C_0}{\pi_m}\sum_{l=1}^n C_{kl}\nabla p_l. \tag{5.15}$$

The species production rates

We assume that ω_k are continuous functions of Y bounded from below and above by the positive constants $\underline{\omega}$ and $\overline{\omega}$

$$-\underline{\omega} \leq \omega_k(Y) \leq \overline{\omega}, \qquad \text{for all } k = 1, \ldots, n; \tag{5.16}$$

we also suppose that

$$\sum_{k=1}^{n} \omega_k = 0, \quad \text{and} \quad \omega_k(Y) \geq 0 \quad \text{whenever} \quad Y_k = 0. \tag{5.17}$$

From the second law of thermodynamics, the process is admissible only if the entropy production rate (1.14) is nonnegative, thus necessarily one has

$$\sum_{k=1}^{n} g_k \varrho \omega_k \leq 0. \tag{5.18}$$

Without loss of generality, we may also assume that

- The formation energies and entropies are constant $e_k^{st}, s_k^{st} = const.$
- The perfect gas constant $R = 1$, the constant-volume specific heats are constant equal for all species
$$c_{vk} = c_v \quad \text{for all } k = 1, \ldots, n. \tag{5.19}$$
- The standard quantities: the temperature, pressure and concentration are rescaled and equal one
$$\vartheta^{st} = p^{st} = \Gamma^{st} = 1.$$

5.3 Weak formulation and main result

In this subsection we define a notion of weak variational entropy solutions to system (5.1) and then we formulate our main result.

Definition 5.4. *We will say $\{\varrho, \mathbf{u}, \vartheta, \varrho_1, \ldots, \varrho_n\}$ is a weak variational entropy solution provided the following integral identities hold.*
1. The continuity equation

$$\int_\Omega \varrho_0 \phi(0,x)\ \mathrm{d}x + \int_0^T \int_\Omega \varrho \partial_t \phi\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega \varrho \mathbf{u} \cdot \nabla \phi\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{5.20}$$

is satisfied for any smooth function $\phi(t,x)$ such that $\phi(T,\cdot) = 0$.
2. The balance of momentum

$$\int_\Omega (\varrho \mathbf{u})^0 \cdot \phi(0,x)\ \mathrm{d}x + \int_0^T \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \phi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \phi\right)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \pi(\varrho, \vartheta, Y) \operatorname{div} \phi\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \mathbf{S} : \nabla \phi\ \mathrm{d}t = 0 \tag{5.21}$$

holds for any smooth test function $\phi(t,x)$ *such that* $\phi(T,\cdot)=0$.
3. The entropy equation

$$\int_\Omega \varrho^0 s(\vartheta^0,\varrho_1^0,\ldots,\varrho_n^0)\phi(0,x)\ \mathrm{d}x+\int_0^T\int_\Omega \varrho s\partial_t\phi\ \mathrm{d}x\ \mathrm{d}t+\int_0^T\int_\Omega \varrho s\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t$$
$$+\int_0^T\int_\Omega\left(\frac{\mathbf{Q}}{\vartheta}-\sum_{k=1}^n\frac{g_k}{\vartheta}\mathbf{F}_k\right)\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t+\langle\sigma,\phi\rangle=0 \quad (5.22)$$

is satisfied for any smooth function $\phi(t,x)$*, such that* $\phi\geq 0$ *and* $\phi(T,\cdot)=0$*, where* $\sigma\in\mathcal{M}^+([0,T]\times\Omega)$ *is a nonnegative measure such that*

$$\sigma\geq\frac{\mathbf{S}:\nabla\mathbf{u}}{\vartheta}-\frac{\mathbf{Q}\cdot\nabla\vartheta}{\vartheta^2}-\sum_{k=1}^n\frac{\mathbf{F}_k}{m_k}\cdot\nabla\left(\frac{g_k}{\vartheta}\right)-\sum_{k=1}^n g_k\varrho\omega_k.$$

4. The global balance of total energy

$$\int_\Omega(\varrho E)^0\ \mathrm{d}x\ \phi(0)+\int_0^T\int_\Omega\varrho E\ \mathrm{d}x\ \partial_t\phi(t)\ \mathrm{d}t=0 \quad (5.23)$$

holds for any smooth function $\phi(t)$*, such that* $\phi(T)=0$.
5. The weak formulation of the mass balance equation for the k-th species

$$\int_\Omega\varrho_k^0\cdot\phi(0,x)\ \mathrm{d}x+\int_0^T\int_\Omega(\varrho_k\partial_t\phi+\varrho_k\mathbf{u}\cdot\nabla\phi)\ \mathrm{d}x\ \mathrm{d}t-\int_0^T\int_\Omega\mathbf{F}_k\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t$$
$$=\int_0^T\int_\Omega\varrho\vartheta\omega_k\phi\ \mathrm{d}x\ \mathrm{d}t, \quad (5.24)$$

$k=1,\ldots,n$*, is satisfied for any smooth test function* $\phi(t,x)$ *such that* $\phi(T,\cdot)=0$.
In addition we require that

$$\varrho,\varrho_k\geq 0,\ k=1,\ldots,n,\quad \sum_{k=1}^n\varrho_k=\varrho,\quad \textit{and}\quad \vartheta>0,\quad \textit{a.e. on } (0,T)\times\Omega. \quad (5.25)$$

In this definition the usual weak formulation of the energy equation (1.4) is replaced by the weak formulation of the entropy inequality and the global total energy balance (5.22)+(5.23). Note however that the entropy production rate has now only a meaning of non-negative measure which is bounded from below by the classical value of σ. Nevertheless, a simple calculation employing the Gibbs formula (1.8) shows that whenever the solution specified above is sufficiently regular, both formulations are equivalent. In particular the entropy inequality (5.22) changes into equality, see e.g. Section 2.5 in [34].

We are now in a position to formulate the main result of this chapter.

Theorem 5.5. *Assume that the structural hypotheses (1.5-1.19), (1.23-1.24) and (5.9-5.19) are satisfied. Suppose that* $\{\varrho_N,\mathbf{u}_N,\vartheta_N,\varrho_{k,N}\}_{N=1}^\infty$*,* $k=1,\ldots,n$ *is a sequence of smooth solutions to (5.1) satisfying the weak formulation (5.20-5.25) with the initial data*

$$\varrho_N(0,\cdot)=\varrho_N^0,\ (\varrho\mathbf{u})_N(0,\cdot)=(\varrho\mathbf{u})_N^0,\ (\varrho s)_N(0,\cdot)=(\varrho s)_N^0,\ \textstyle\int_\Omega(\varrho E)_N(0,\cdot)\ \mathrm{d}x=\int_\Omega(\varrho E)_N^0\ \mathrm{d}x,$$
$$\varrho_{k,N}(0,\cdot)=\varrho_{k,N}^0,\quad \textit{for } k=1,\ldots,n,\ \textit{in } \Omega$$

satisfying (5.5-5.8), moreover

$$\begin{gathered}\inf_{x\in\Omega}\varrho_N^0(x)>0,\quad \inf_{x\in\Omega}\vartheta_N^0(x)>0,\\ \varrho_N^0\to\varrho^0,\ (\varrho\mathbf{u})_N^0\to(\varrho\mathbf{u})^0,\ (\varrho s)_N^0\to(\varrho s)^0,\ (\varrho E)_N^0\to(\varrho E)^0,\ \varrho_{k,N}^0\to\varrho_k^0\quad in\ L^1(\Omega).\end{gathered} \tag{5.26}$$

Then, up to a subsequence, $\{\varrho_N,\mathbf{u}_N,\vartheta_N,\varrho_{k,N}\}$ *converges to the weak solution of problem (5.1) in the sense of Definition 5.4.*

The proof of this theorem can be divided into two main steps. The first one is dedicated to derivation of the energy-entropy estimates which are obtained under assumption that all the quantities are sufficiently smooth. The next step is the limit passage, it consists of various integrability Lemmas combined with condensed compactness arguments.

5.4 A priori estimates

In this section we present the a priori estimates for $\{\varrho_N,\mathbf{u}_N,\vartheta_N,\varrho_{k,N}\}_{N=1}^\infty$ which is a sequence of smooth functions solving (5.1). As mentioned above, assuming smoothness of solutions, we expect that all the natural features of the system can be recovered. The following estimates are valid for each $N=1,2,\ldots$ but we skip the subindex when it does not lead to any confusion.

5.4.1 Estimates based on the maximum principle

To begin, observe that the total mass of the fluid is a constant of motion, meaning

$$\int_\Omega \varrho(t,x)\ \mathrm{d}x=\int_\Omega \varrho^0\ \mathrm{d}x=M_0\quad \text{for } t\in[0,T]. \tag{5.27}$$

Moreover if solution is sufficiently smooth, a classical maximum principle can be applied to the continuity equation in order to show that $\varrho_N(t,x)\geq c(N)>0$, exactly as in (3.15).

Next, by a very similar reasoning we can prove non-negativity of ϑ on $[0,T]\times\Omega$.

Lemma 5.6. *Assume that* $\vartheta=\vartheta_N$ *is a smooth solution of* (5.1)*, then*

$$\vartheta(t,x)\geq c(N)>0\quad for\ (t,x)\in[0,T]\times\Omega. \tag{5.28}$$

Proof. Any solution to (5.1) which is sufficiently smooth is automatically a classical solution of the system

$$\begin{cases}\partial_t\varrho+\operatorname{div}(\varrho\mathbf{u})=0,\\ \partial_t(\varrho\mathbf{u})+\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}\mathbf{S}+\nabla\pi=\mathbf{0},\\ \partial_t(\varrho e)+\operatorname{div}(\varrho e\mathbf{u})+\operatorname{div}\mathbf{Q}=-\pi\operatorname{div}\mathbf{u}+\mathbf{S}:\nabla\mathbf{u},\\ \partial_t\varrho_k+\operatorname{div}(\varrho_k\mathbf{u})+\operatorname{div}(\mathbf{F}_k)=\varrho\vartheta\omega_k,\quad k\in\{1,...,n\},\end{cases} \tag{5.29}$$

where we replaced the total energy balance by the internal energy balance, see (1.4). Let us hence reformulate it in order to obtain the equation describing the temperature. Subtracting from the third equation of (5.29) the component corresponding to the formation energy we obtain

$$\begin{aligned}\partial_t(\varrho(e_c+e_m))+\operatorname{div}(\varrho(e_c+e_m)\mathbf{u})+\sum_{k=1}^n\operatorname{div}(c_{pk}\vartheta\mathbf{F}_k)-\operatorname{div}(\kappa(\varrho,\vartheta)\nabla\vartheta)\\ =-\pi\operatorname{div}\mathbf{u}+\mathbf{S}:\nabla\mathbf{u}-\varrho\vartheta\sum_{k=1}^n e_k^{st}\omega_k,\end{aligned} \tag{5.30}$$

where we used the species mass balance equations. Next renormalizing the continuity equation and employing the relation between e_c and π_c (1.7) we get the temperature equation

$$\partial_t(\varrho e_m) + \operatorname{div}(\varrho e_m \mathbf{u}) + \sum_{k=1}^{n} \operatorname{div}(c_{pk}\vartheta \mathbf{F}_k) - \operatorname{div}(\kappa(\varrho,\vartheta)\nabla\vartheta)$$
$$= -\pi_m \operatorname{div}\mathbf{u} + \mathbf{S} : \nabla\mathbf{u} - \varrho\vartheta \sum_{k=1}^{n} e_k^{st}\omega_k, \quad (5.31)$$

where, in accordance with hypotheses (5.10), the second term on the r.h.s. is nonnegative. Consequently, (5.28) is obtained by application of the maximum principle to the above equation, recalling that $\inf_{x\in\Omega}\vartheta_N^0(x) > 0$. □

An analogous result for partial masses is stated in the following lemma.

Lemma 5.7. *For any smooth solution of (5.1) we have*

$$\varrho_k(t,x) \geq 0 \quad \text{for } (t,x) \in [0,T]\times\Omega,\ k \in \{1,\dots,n\}. \quad (5.32)$$

Moreover

$$\sum_{k=1}^{n} \varrho_k(t,x) = \varrho(t,x) \quad \textit{for } (t,x) \in [0,T]\times\Omega. \quad (5.33)$$

Proof. We integrate each of equations of system (5.1) over the set $\{\varrho_k < 0\}$. Assuming that the boundary i.e. $\{\varrho_k = 0\}$ is a regular submanifold we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{\varrho_k<0\}} \varrho_k \,\mathrm{d}x - \int_{\{\varrho_k=0\}} \frac{\partial p_k}{\partial n}\,\mathrm{d}S_x + \int_{\{\varrho_k=0\}} \frac{\varrho_k}{\varrho}\frac{\partial \pi_m}{\partial n}\,\mathrm{d}S_x = \int_{\{\varrho_k<0\}} \varrho\vartheta\omega_k \,\mathrm{d}x.$$

Since $\frac{\partial p_k}{\partial n}\big|_{\{\varrho_k=0\}} \geq 0$ and $\omega_k\big|_{\{\varrho_k<0\}} \geq 0$ we find

$$\int_{\{\varrho_k<0\}} \varrho_k(T)\,\mathrm{d}x \geq \int_{\{\varrho_k<0\}} \varrho_k^0 \,\mathrm{d}x = 0,$$

thus $|\{\varrho_k < 0\}| = 0$, for every $k = 1,\dots,n$. When $\{\varrho_k = 0\}$ is not a regular submanifold we construct a sequence $\{\varepsilon_l\}_{l=1}^{\infty}$ such that $\varepsilon_l \to 0^+$ and $\{\varrho_k = \varepsilon_l\}$ is a regular submanifold and pass with ε_l to zero.
The proof of (5.33) follows by subtracting the sum of species mass balances equations from the continuity equation. The smooth solution of the resulting system must be, due to the initial conditions (5.6), equal to 0 on $[0,T]\times\Omega$. □

As a corollary from this Lemma we recover relation (1.3), moreover we have the following estimate

$$\|Y_k\|_{L^\infty((0,T)\times\Omega)} \leq 1, \quad k = 1\dots,n. \quad (5.34)$$

5.4.2 The energy-entropy estimates

The purpose of this subsection is to derive a priori estimates resulting from the energy and entropy balance equations. The difference comparing to estimates obtained in the previous subsection is that now we look for bounds which are uniform with respect to N. We start with the following Lemma.

Lemma 5.8. *Every smooth solution of* (5.1) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho\left(\frac{1}{2}|\mathbf{u}|^2+e\right)\,\mathrm{d}x=0. \tag{5.35}$$

Proof. Integrate the third equation of (5.1) with respect to the space variable and employ the periodic boundary conditions. □

Assuming integrability of the initial conditions (5.8) the assertion of the above lemma entails several a priori estimates:

$$\begin{gathered}\|\sqrt{\varrho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\le c,\\ \|\varrho e_c(\varrho)\|_{L^\infty(0,T;L^1(\Omega))}+\|\varrho\vartheta\|_{L^\infty(0,T;L^1(\Omega))}+\|\varrho\|_{L^\infty(0,T;L^1(\Omega))}\le c.\end{gathered} \tag{5.36}$$

It is well known that these natural bounds are not sufficient to prove the weak sequential stability of solutions, not even for the barotropic flow. However, taking into account the form of viscosity coefficients (5.9), (5.10), further estimates can be delivered.

Lemma 5.9. *For any smooth solution of* (5.1) *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\int_\Omega \mathbf{S}:\nabla\mathbf{u}\,\mathrm{d}x=\int_\Omega \pi(\varrho,\vartheta,Y)\operatorname{div}\mathbf{u}\,\mathrm{d}x. \tag{5.37}$$

Proof. Multiply the momentum equation by $\mathbf{u}$ and integrate over Ω. □

The above lemma can not be used to deduce the uniform bounds for the symmetric part of the gradient of $\mathbf{u}$ immediately as it was done in Section 3.2.2. The reason for that is lack of sufficient information for ϑ, so far we only know (5.36). However, it is still possible to derive the following analogue of (3.20).

Lemma 5.10. *Any smooth solution of* (5.1) *satisfies the following identity*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega \mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,\mathrm{d}x=&\\ -\int_\Omega \nabla\phi(\varrho)\cdot\nabla\pi(\vartheta,\varrho,Y)\,\mathrm{d}x+\int_\Omega \pi(\varrho,\vartheta,Y)\operatorname{div}\mathbf{u}\,\mathrm{d}x,&\end{aligned} \tag{5.38}$$

for ϕ such that $\nabla\phi(\varrho)=2\frac{\mu'(\varrho)\nabla\varrho}{\varrho}$.

Proof. The rough idea of the proof is the following. The terms from the l.h.s. of this equality can be evaluated by multiplication of the momentum equation by $\nabla\phi(\varrho)$ and the continuity equation by $|\nabla\phi(\varrho)|^2$. Then one has to combine these equivalences with the balance of kinetic energy (5.37) and include (5.9) to see that some unpleasant terms cancel. For more details we refer to the proof of Lemma 3.17 in Chapter 3 or to the original work of Bresch and Desjardins [10] . □

To control the r.h.s. of (5.37) and (5.38) one needs i.a. to estimate the gradient of ϑ. To this purpose we take advantage of the entropy balance (1.13), we have the following inequality

Lemma 5.11. *For any smooth solution of* (5.1) *we have*

$$\begin{aligned}\int_0^T\int_\Omega \frac{\mathbf{S}:\nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \frac{\kappa|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\sum_{k=1}^n\frac{\pi_m\mathbf{F}_k^2}{C_0\vartheta\varrho_k}\,\mathrm{d}x\,\mathrm{d}t&\\ -\int_0^T\int_\Omega\sum_{k=1}^n g_k\varrho\omega_k\,\mathrm{d}x\,\mathrm{d}t\le c.&\end{aligned} \tag{5.39}$$

Proof. Combining the third equation of (5.1) with the Gibbs relation (1.8) we derive the entropy equation

$$\partial_t(\varrho s) + \mathrm{div}(\varrho s \mathbf{u}) + \mathrm{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta}\mathbf{F}_k\right) = \frac{\mathbf{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \sum_{k=1}^{n} g_k \varrho \omega_k. \quad (5.40)$$

Integrating it over space and time we obtain

$$\int_0^T \int_\Omega \left(\frac{\mathbf{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \sum_{k=1}^{n} g_k \varrho \omega_k\right) \mathrm{d}x \ \mathrm{d}t = \int_\Omega \varrho s(T) \ \mathrm{d}x - \int_\Omega (\varrho s)^0 \ \mathrm{d}x,$$

where the l.h.s. can be transformed using (1.9) and (1.23) into

$$\int_0^T \int_\Omega \frac{\mathbf{S} : \nabla \mathbf{u}}{\vartheta} \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2} \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla \log p_k \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \sum_{k=1}^{n} g_k \varrho \omega_k \ \mathrm{d}x \ \mathrm{d}t = \int_\Omega \varrho s(T) \ \mathrm{d}x - \int_\Omega (\varrho s)^0 \ \mathrm{d}x. \quad (5.41)$$

The first two terms on the l.h.s. of (5.41) have a good sign, the same holds for the last one due to (5.18). Non-negativity of the third one follows from (5.14) and (5.15)

$$-\sum_{k=1}^{n} \frac{\mathbf{F}_k}{m_k} \cdot \nabla(\log p_k) = -\sum_{k=1}^{n} \frac{\mathbf{F}_k}{\vartheta \varrho Y_k} \nabla p_k = -\sum_{k=1}^{n} \frac{\mathbf{F}_k}{\vartheta \varrho Y_k} (\nabla p_k - Y_k \nabla \pi_m) = \sum_{k=1}^{n} \frac{\pi_m \mathbf{F}_k^2}{C_0 \vartheta \varrho_k} \geq 0.$$

Thus, it remains to control the positive part of $\varrho s(T)$ and the negative part of $(\varrho s)^0$. From definition of the entropy (1.10) and assumption (5.19) we get

$$\varrho s = \sum_{k=1}^{n} \varrho Y_k s_k^{st} + \sum_{k=1}^{n} c_v \varrho_k \log \vartheta - \sum_{k=1}^{n} \frac{\varrho_k}{m_k} \log \frac{\varrho_k}{m_k}, \quad (5.42)$$

therefore

$$\int_\Omega [\varrho s(T)]_+ \ \mathrm{d}x \leq c \int_\Omega \varrho(T) \ \mathrm{d}x + c \int_\Omega \varrho \vartheta(T) \ \mathrm{d}x - \sum_{k=1}^{n} \int_\Omega \frac{\varrho_k}{m_k} \log \frac{\varrho_k}{m_k}(T) \ \mathrm{d}x. \quad (5.43)$$

The two first terms from the r.h.s. are bounded due to (5.36), whereas to estimate the positive part of the last one we essentially use the assumption that Ω is a bounded domain. Thus, the positive part of $-x \log x$ is bounded by a constant, and thus integrable over Ω. □

In the rest of Section 5.4 we show how to use Lemmas 5.10 and 5.11 in order to derive uniform estimates for the sequence of smooth solutions $\{\varrho_N, \mathbf{u}_N, \vartheta_N, \varrho_{k,N}\}_{N=1}^{\infty}$ to system (5.1).

5.4.3 Estimates of the temperature.

One of the main consequences of (5.39) is that for $\kappa(\varrho, \vartheta)$ satisfying (5.11) we have the following a priori estimates for the temperature

$$(1+\sqrt{\varrho})\nabla \log \vartheta,\ (1+\sqrt{\varrho})\nabla \vartheta^s \in L^2((0,T)\times\Omega), \tag{5.44}$$

where $s \in [0, \frac{\alpha}{2}]$ and $\alpha \geq 2$. To control the full norm of ϑ^s in $L^2(0,T; W^{1,2}(\Omega))$ we will apply the following version on the Korn-Poincaré inequality (see e.g. Theorem 10.17 in [85]):

Theorem 5.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that r is a non-negative function such that*

$$0 < M_0 \leq \int_\Omega r \ \mathrm{d}x, \ \int_\Omega r^\gamma \ \mathrm{d}x \leq K, \quad \textit{for a certain } \gamma > 1.$$

Then

$$\|\xi\|_{W^{1,p}(\Omega)} \leq C(p, M_0, K)\|\nabla\xi\|_{L^p(\Omega)} + \int_\Omega r|\xi| \ \mathrm{d}x,$$

for any $\xi \in W^{1,p}(\Omega)$.

Recalling (1.7), (5.2), (5.36) and (5.44) one can check that the assumptions of the above theorem are satisfied for $\xi = \vartheta$, $r = \varrho$ and $p = 2$. Therefore, the Sobolev imbedding gives the estimate of the norm of ϑ in $L^2(0,T; L^6(\Omega))$, and so, due to the boundedness of $\nabla\vartheta^{\frac{\alpha}{2}}$ in $L^2((0,T)\times\Omega)$, one gets

$$\vartheta^{\frac{\alpha}{2}} \in L^2(0,T; W^{1,2}(\Omega)). \tag{5.45}$$

5.4.4 Estimates following from the Bresch-Desjardin equality

The aim of this subsection is to derive estimates following from (5.37) and (5.38). Summing these two expressions we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 \ \mathrm{d}x + \int_\Omega \mathbf{S} : \nabla\mathbf{u} \ \mathrm{d}x + \frac{1}{2}\int_\Omega \mu(\varrho)|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2 \ \mathrm{d}x \\ = -\int_\Omega \nabla\phi(\varrho)\cdot\nabla\pi(\vartheta,\varrho,Y) \ \mathrm{d}x + 2\int_\Omega \pi(\varrho,\vartheta,Y)\operatorname{div}\mathbf{u} \ \mathrm{d}x.\end{aligned} \tag{5.46}$$

We first need to justify that the terms from the r.h.s. are bounded or have a negative sign so that they can be moved to the l.h.s. The main problem is to control the contribution from the molecular pressure. It will require to couple the entropy estimate (5.39) with the analogue of comparison principle whose mechanism of action was illustrated in Lemma 3.7 for the simplest two-component mixture and then generalized to the case of n isothermally reacting species in Chapter 4. We start with the proof of analogue of (4.52) from the previous chapter.

Denoting

$$C\nabla_{x_i}p = (\nabla_{x_i}p)^I, \tag{5.47}$$

where

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad \nabla p = \begin{pmatrix} \nabla p_1 \\ \vdots \\ \nabla p_n \end{pmatrix}, \tag{5.48}$$

we obtain, for every k-th coordinate $k \in \{1,\dots,n\}$ and every i-th space coordinate $i \in \{1,2,3\}$, the following decomposition

$$(\nabla_{x_i} p)_k = (\nabla_{x_i} p)^I_k + \alpha_i Y_k. \tag{5.49}$$

Next, multiplying the above expression by m_k and summing over $k \in \{1,\dots,n\}$ one gets

$$\alpha_i = \frac{\nabla_{x_i}(\varrho\vartheta)}{\sum_{k=1}^n m_k Y_k} - \frac{\sum_{k=1}^n m_k (\nabla_{x_i} p)^I_k}{\sum_{k=1}^n m_k Y_k}.$$

Returning (5.49) we can express the full gradients of partial pressures in terms of gradients of temperature, density and the gradient of "known" part of the pressure

$$\nabla p = (\nabla p)^I + \left(\frac{\nabla(\varrho\vartheta)}{\sum_{k=1}^n m_k Y_k} - \frac{\sum_{k=1}^n m_k (\nabla p)^I_k}{\sum_{k=1}^n m_k Y_k} \right) Y. \tag{5.50}$$

As was announced, we will use the above expression in order to control the molecular part of the pressure from the r.h.s. of (5.46).

Estimate of $\nabla\pi(\varrho,\vartheta,Y)\cdot\nabla\phi$. Using definition of ϕ (see Lemma 5.10) and (1.5) we obtain

$$\nabla\phi(\varrho)\cdot\nabla\pi(\varrho,\vartheta,Y) = \mu'(\varrho)\pi_c'(\varrho)\frac{|\nabla\varrho|^2}{\varrho} + \frac{\nabla\mu(\varrho)\cdot\nabla\pi_m}{\varrho}. \tag{5.51}$$

The first term is non-negative due to (5.2), so it can be considered on the l.h.s. of (5.46) and we only need to estimate the second one. Since $\nabla\pi_m = \sum_{k=1}^n (\nabla p)_k$ and $\sum_{k=1}^n (Y)_k = 1$, we may use (5.50) to write

$$\begin{aligned}\int_\Omega \frac{\nabla\mu(\varrho)\cdot\nabla\pi_m}{\varrho}\,\mathrm{d}x &= \int_\Omega \frac{\nabla\mu(\varrho)\cdot\sum_{k=1}^n(\nabla p)^I_k}{\varrho}\,\mathrm{d}x + \int_\Omega \frac{\nabla\mu(\varrho)\cdot\nabla\varrho\vartheta}{\sum_{k=1}^n \varrho_k m_k}\,\mathrm{d}x \\ &+ \int_\Omega \frac{\nabla\mu(\varrho)\cdot\nabla\vartheta\varrho}{\sum_{k=1}^n \varrho_k m_k}\,\mathrm{d}x - \int_\Omega \frac{\nabla\mu(\varrho)\cdot\sum_{k=1}^n m_k(\nabla p)^I_k}{\sum_{k=1}^n \varrho_k m_k}\,\mathrm{d}x = \sum_{i=1}^4 I_i.\end{aligned} \tag{5.52}$$

Note that I_2 is non-negative, so we can put it to the l.h.s. of (5.46).
Next, I_1 and I_4 can be estimated in a similar way, we have

$$\int_\Omega \frac{|\nabla\mu(\varrho)||\sum_{k=1}^n(\nabla p)^I_k|}{\varrho}\,\mathrm{d}x \le \varepsilon\int_\Omega \frac{|\nabla\mu(\varrho)|^2\vartheta}{\varrho}\,\mathrm{d}x + c(\varepsilon)\int_\Omega \frac{|\sum_{k=1}^n(\nabla p)^I_k|^2}{\vartheta\varrho}\,\mathrm{d}x, \tag{5.53}$$

so for ε sufficiently small, the first term can be controlled by I_2 thanks to (5.10). Concerning the second integral, from (5.39) we have

$$\int_0^T\int_\Omega \sum_{k=1}^n \frac{\pi_m \mathbf{F}_k^2}{C_0\vartheta\varrho_k}\,\mathrm{d}x\,\mathrm{d}t \le c. \tag{5.54}$$

Using (5.15), the integral may be transformed as follows

$$\int_0^T\int_\Omega \sum_{k=1}^n \frac{C_0(C\nabla p)_k^2}{\pi_m\vartheta\varrho_k}\,\mathrm{d}x\,\mathrm{d}t \le c, \tag{5.55}$$

thus, due to (5.47) and (5.13), the integral over time of the r.h.s. of (5.53) is bounded.

For I_3 we verify that

$$\left|\nabla\mu(\varrho)\cdot\nabla\vartheta\frac{\varrho}{\sum_{k=1}^n\varrho_k m_k}\right|\le c(\varepsilon)\kappa(\varrho,\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}+\varepsilon\frac{\varrho\vartheta^2}{\kappa(\varrho,\vartheta)}\frac{|\nabla\mu(\varrho)|^2}{\varrho},$$

and the first term is bounded in view of (5.44) whereas boundedness of the second one follows from the Gronwall inequality applied to (5.46). Indeed, note that, due to (5.11), $\frac{\varrho\vartheta^2}{\kappa(\varrho,\vartheta)}$ is bounded by some positive constant.

Estimate of $\pi(\varrho,\vartheta,Y)\operatorname{div}\mathbf{u}$**.** By virtue of (1.5) and (1.7) and the continuity equation

$$\int_\Omega\pi(\varrho,\vartheta,Y)\operatorname{div}\mathbf{u}\,\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\varrho e_c(\varrho)\,\mathrm{d}x+\int_\Omega\varrho\vartheta\left(\sum_{k=1}^n\frac{Y_k}{m_k}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x.$$

Furthermore, by the Cauchy inequality

$$\left|\int_\Omega\varrho\vartheta\left(\sum_{k=1}^n\frac{Y_k}{m_k}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x\right|\;\le\;c\|Y_k\|_{L^\infty(\Omega)}\left(\varepsilon\|\sqrt{\mu(\varrho)}\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega)}+c(\varepsilon)\left\|\frac{\varrho\vartheta}{\sqrt{\mu(\varrho)}}\right\|^2_{L^2(\Omega)}\right).$$

Since $\mu(\varrho)\ge\underline{\mu'}\varrho$, we may write

$$\left\|\frac{\varrho\vartheta}{\sqrt{\mu(\varrho)}}\right\|_{L^2(\Omega)}\le c\|\varrho\vartheta^2\|^{\frac12}_{L^1(\Omega)}\le c\|\varrho\|^{\frac12}_{L^{\frac32}(\Omega)}\|\vartheta\|_{L^6(\Omega)}. \tag{5.56}$$

On account of (5.45), $\vartheta\in L^2(0,T;L^6(\Omega))$. Moreover, the Sobolev imbedding theorem implies that $\|\varrho\|_{L^{\frac{p}{2}}(\Omega)}\le c\left\|\frac{\nabla\mu(\varrho)}{\sqrt{\varrho}}\right\|_{L^2(\Omega)}$ for $1\le p\le 6$, hence the Gronwall inequality applied to (5.46) implies boundedness of (5.56), whence the term $\varepsilon\|\sqrt{\mu(\varrho)}\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega)}$ is then absorbed by $\int_\Omega\mathbf{S}:\nabla\mathbf{u}\,\mathrm{d}x$ from the l.h.s. of (5.46).
Resuming, we have proven the following inequality:

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega\left(\frac12\varrho|\mathbf{u}|^2+\varrho e_c(\varrho)+\frac12\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2\right)(t)\,\mathrm{d}x+\int_0^T\int_\Omega\mu'(\varrho)\pi_c'(\varrho)\frac{|\nabla\varrho|^2}{\varrho}\,\mathrm{d}x\,\mathrm{d}t\\&+(1-\varepsilon)\int_0^T\int_\Omega\frac{\vartheta\nabla\mu(\varrho)\cdot\nabla\varrho}{\sum_{k=1}^n\varrho_k m_k}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\left(\mathbf{S}:\nabla\mathbf{u}+\frac12\mu(\varrho)|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\right)\,\mathrm{d}x\,\mathrm{d}t\le c.\end{aligned} \tag{5.57}$$

Uniform estimates. Taking into account all the above considerations, we can complement the so-far obtained estimates as follows

$$\left\|\sqrt{\vartheta\varrho^{-1}}\nabla\varrho\right\|_{L^2((0,T)\times\Omega)}+\left\|\sqrt{\pi_c'(\varrho)\varrho^{-1}}\nabla\varrho\right\|_{L^2((0,T)\times\Omega)}\le c, \tag{5.58}$$

moreover

$$\left\|\frac{\nabla\mu(\varrho)}{\sqrt{\varrho}}\right\|_{L^\infty(0,T;L^2(\Omega))}\le c. \tag{5.59}$$

Concerning the velocity vector field, in addition to (5.36) we have

$$\|\sqrt{\mu(\varrho)}\nabla\mathbf{u}\|_{L^2((0,T)\times\Omega)}+\left\|\sqrt{\mu(\varrho)\vartheta^{-1}}\nabla\mathbf{u}\right\|_{L^2((0,T)\times\Omega)}\le c. \tag{5.60}$$

5.4.5 Estimates of species densities

Finally, we can take advantage of the entropy estimate (5.39) which together with (5.59) may be used to deduce boudedness of gradients of all species densities.

Lemma 5.13. *For any smooth solution of* (5.1) *we have*

$$\left\|\sqrt{1+\vartheta}\nabla\sqrt{\varrho_k}\right\|_{L^2((0,T)\times\Omega)} \leq c. \tag{5.61}$$

Proof. First, using Remark 5.3 we may write

$$\frac{\pi_m \mathbf{F}_k^2}{C_0\vartheta\varrho_k} = \frac{C_0|\nabla p_k|^2}{\pi_m\varrho_k\vartheta} - 2\frac{Y_k C_0\nabla p_k\cdot\nabla\pi_m}{\pi_m\varrho_k\vartheta} + \frac{Y_k^2 C_0|\nabla\pi_m|^2}{\pi_m\varrho_k\vartheta},$$

which is bounded in $L^1((0,T)\times\Omega)$ on account of (5.54). Clearly,

$$\int_0^T\int_\Omega \frac{C_0|\nabla p_k|^2}{\pi_m\varrho_k\vartheta}\,\mathrm{d}x\,\mathrm{d}t \leq c\left(1+\int_0^T\int_\Omega \frac{Y_k^2C_0|\nabla\pi_m|^2}{\pi_m\varrho_k\vartheta}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{5.62}$$

The r.h.s. of above can be, due to (5.50), estimated as follows

$$\begin{aligned}\int_0^T\int_\Omega \frac{Y_k^2C_0|\nabla\pi_m|^2}{\pi_m\varrho_k\vartheta}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \frac{Y_kC_0|\sum_{k=1}^n(\nabla p)_k|^2}{\pi_m\varrho\vartheta}\,\mathrm{d}x\,\mathrm{d}t\\ \leq c\int_0^T\int_\Omega\frac{C_0}{\pi_m\varrho\vartheta}&\left(\left|\sum_{k=1}^n(C\nabla p)_k\right|^2+\frac{|\nabla(\varrho\vartheta)|^2}{\left(\sum_{k=1}^n m_kY_k\right)^2}+\frac{|\sum_{k=1}^n m_k(C\nabla p)_k|^2}{\left(\sum_{k=1}^n m_kY_k\right)^2}\right)\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{5.63}$$

which is bounded thanks to (5.44), (5.55) and (5.58). In consequence, (5.62) is bounded. Recalling assumptions imposed on C_0 (5.13) and the form of molecular pressure π_m, we deduce that

$$\int_0^T\int_\Omega \frac{\underline{\underline{C_0}}(1+\vartheta)|\nabla\varrho_k|^2}{\varrho_k}\,\mathrm{d}x\,\mathrm{d}t \leq c\left(1+\int_0^T\int_\Omega\frac{(1+\vartheta)\varrho_k|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t\right)$$

and the r.h.s. is bounded, again by (5.34) and (5.44). □

5.4.6 Additional estimates.

In this subsection we present several additional estimates based on imbeddings of Sobolev spaces and the simple interpolation inequalities.

Further estimates of ϱ. From (5.2) and (5.58) we deduce that there exist functions $\xi_1(\varrho)=\varrho$ for $\varrho<(1-\delta)$, $\xi_1(\varrho)=0$ for $\varrho>1$ and $\xi_2(\varrho)=0$ for $\varrho<1$, $\xi_2(\varrho)=\varrho$ for $\varrho>(1+\delta)$, $\delta>0$, such that

$$\|\nabla\xi_1^{-\frac{\gamma^-}{2}}\|_{L^2((0,T)\times\Omega)},\ \|\nabla\xi_2^{\frac{\gamma^+}{2}}\|_{L^2((0,T)\times\Omega)} \leq c,$$

additionally in accordance to (5.36) we are allowed to use the Sobolev imbeddings, thus

$$\|\xi_1^{-\frac{\gamma^-}{2}}\|_{L^2(0,T;L^6(\Omega))},\ \|\xi_2^{\frac{\gamma^+}{2}}\|_{L^2(0,T;L^6(\Omega))} \leq c. \tag{5.64}$$

Remark 5.14. *Note in particular that the first of these estimate implies that*

$$\varrho(t,x)>0 \quad a.e.\ on\ (0,T)\times\Omega. \tag{5.65}$$

Similarly, combination of (5.59) with (5.36) leads to

$$\|\varrho^{\frac{1}{2}}\|_{L^6(\Omega)} \leq c \left\| \frac{\nabla \mu(\varrho)}{\sqrt{\varrho}} \right\|_{L^2(\Omega)},$$

and therefore

$$\varrho \in L^\infty(0,T;L^3(\Omega)). \tag{5.66}$$

Estimate of the velocity vector field. We use the Hölder inequality to write

$$\|\nabla \mathbf{u}\|_{L^p(0,T;L^q(\Omega))} \leq c \left(1 + \|\xi_1(\varrho)^{-1/2}\|_{L^{2\gamma^-}(0,T;L^{6\gamma^-}(\Omega))}\right) \|\sqrt{\varrho}\nabla \mathbf{u}\|_{L^2((0,T)\times\Omega)}, \tag{5.67}$$

where $p = \frac{2\gamma^-}{\gamma^-+1}$, $q = \frac{6\gamma^-}{3\gamma^-+1}$. Therefore, Theorem 5.12 together with the Sobolev imbedding imply

$$\mathbf{u} \in L^{\frac{2\gamma^-}{\gamma^-+1}}(0,T;L^{\frac{6\gamma^-}{\gamma^-+1}}(\Omega)). \tag{5.68}$$

Next, by a similar argument

$$\|\mathbf{u}\|_{L^{p'}(0,T;L^{q'}(\Omega))} \leq c \left(1 + \|\xi_1(\varrho)^{-1/2}\|_{L^{2\gamma^-}(0,T;L^{6\gamma^-}(\Omega))}\right) \|\sqrt{\varrho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}, \tag{5.69}$$

with $p' = 2\gamma^-$, $q' = \frac{6\gamma^-}{3\gamma^-+1}$. By a simple interpolation between (5.68) and (5.69), we obtain

$$\mathbf{u} \in L^{\frac{10\gamma^-}{3\gamma^-+3}}(0,T;L^{\frac{10\gamma^-}{3\gamma^-+3}}(\Omega)), \tag{5.70}$$

and since $\gamma^- > 1$, we see in particular that $\mathbf{u} \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega))$.

Strict positivity of the absolute temperature. We now give the proof of uniform with respect to N positivity of ϑ_N.

Lemma 5.15. *Let $\{\vartheta_N\}_{N=1}^\infty$ be the sequence of smooth functions satisfying estimates* (5.36) *and* (5.44)*, then*

$$\vartheta_N(t,x) > 0 \quad \text{a.e. on } (0,T)\times\Omega. \tag{5.71}$$

Proof. The above statement is a consequence of the following estimate

$$\int_0^T \int_\Omega \left(|\log \vartheta_N|^2 + |\nabla \log \vartheta_N|^2\right) \, \mathrm{d}x \, \mathrm{d}t \leq c, \tag{5.72}$$

which can be obtained, again by application of Theorem 5.12 with $\xi = \log \vartheta_N$ and $r = \varrho_N$. It remains to check that we control the $L^1(\Omega)$ norm of $\varrho|\log\vartheta|$. By (5.22) we have

$$\int_\Omega (\varrho_N s_N)^0 \, \mathrm{d}x \leq \int_\Omega \varrho_N s_N(T) \, \mathrm{d}x,$$

thus substituting the form of ϱs from (5.42) we obtain

$$-c_v \int_\Omega \varrho_N \log \vartheta_N(T) \, \mathrm{d}x \leq \sum_{k=1}^n \int_\Omega \varrho_{k,N} s_k^{st}(T) \, \mathrm{d}x - \sum_{k=1}^n \int_\Omega \frac{\varrho_{k,N}}{m_k} \log \frac{\varrho_{k,N}}{m_k}(T) \, \mathrm{d}x - \int_\Omega (\varrho_N s_N)^0 \, \mathrm{d}x$$

and the r.h.s. is bounded on account of (5.34), (5.66) and the initial condition. On the other hand, the positive part of the integrant $\varrho_N \log \vartheta_N$ is bounded from above by $\varrho_N \vartheta_N$ which belongs to $L^\infty(0,T;L^1(\Omega))$ due to (5.36), so we end up with

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |\varrho_N \log \vartheta_N(t)| \, \mathrm{d}x \leq c, \tag{5.73}$$

which was the missing information in order to apply Theorem 5.12. This completes the proof of (5.72). □

5.5 Passage to the limit

In this section we justify that it is possible to perform the limit passage in the weak formulation of system (5.1). We remark that we focus only on the new features of the system, i.e. the molecular pressure and multicomponent diffusion, leaving the rest of limit passages to be performed analogously as in Chapter 3.

5.5.1 Strong convergence of the density and passage to the limit in the continuity equation.

Exactly as in Section 3.2.3 Lemma 3.9 we have

Lemma 5.16. *If $\mu(\varrho)$ satisfies (5.10), then for a subsequence we have*

$$\sqrt{\varrho_N} \to \sqrt{\varrho} \quad \text{a.e. and strongly in } L^2((0,T)\times\Omega). \tag{5.74}$$

Moreover $\varrho_N \to \varrho$ strongly in $C([0,T];L^p(\Omega))$, $p<3$.

In addition, due to (5.70), one can extract a subsequence such that $\mathbf{u}_N \to \mathbf{u}$ weakly in $L^{\frac{5}{3}}((0,T)\times\Omega)$. Thanks to this we can let $N\to\infty$ in the continuity equation to obtain (5.20).

5.5.2 Strong convergence of the species densities.

Analogously we show the strong convergence of species densities. We have

Lemma 5.17. *Up to a subsequence the partial densities $\varrho_{k,N}$, $k=1,\dots,n$ converge strongly in $L^p(0,T;L^q(\Omega))$, $1\le p<\infty, 1\le q<3$ to ϱ_k. In particular*

$$\varrho_{k,N} \to \varrho_k \quad \text{a.e. in } (0,T)\times\Omega. \tag{5.75}$$

Moreover $\varrho_{k,N}\to\varrho_k$ in $C([0,T];L^3_{\text{weak}}(\Omega))$.

Proof. The estimate (5.61) together with (5.34) and (5.66) give the bound for the space gradients of $\varrho_{k,N}$, $k=1,\dots,n$

$$\nabla\varrho_{k,N} = 2\nabla\sqrt{\varrho_{k,N}}\sqrt{\varrho_{k,N}} \quad \text{is bounded in } L^2(0,T;L^{\frac{3}{2}}(\Omega)). \tag{5.76}$$

Moreover, directly from the equation of species mass conservation we obtain

$$\partial_t(\varrho_{k,N}) := -\operatorname{div}(\varrho_{k,N}\mathbf{u}_N) - \operatorname{div}(\mathbf{F}_{k,N}) + \varrho_N\omega_{k,N} \in L^{\frac{2\alpha}{\alpha+1}}(0,T;W^{-1,\frac{6\alpha}{4\alpha+1}}(\Omega)). \tag{5.77}$$

Indeed, the most restrictive term is the diffusion flux, which can be rewritten as

$$\mathbf{F}_{i,N} \;=\; -\frac{C_0}{\pi_{m,N}}\left(\nabla\vartheta_N\frac{\varrho_{i,N}}{m_i} + \nabla\varrho_{i,N}\frac{\vartheta_N}{m_i} - Y_{i,N}\sum_{k=1}^n \nabla\vartheta_N\frac{\varrho_{k,N}}{m_k} - Y_{i,N}\sum_{k=1}^n\nabla\varrho_{k,N}\frac{\vartheta_N}{m_k}\right). \tag{5.78}$$

Due to (5.34) we have that

$$\frac{C_0}{\pi_{m,N}}\left|\nabla\vartheta_N\frac{\varrho_{i,N}}{m_i} - Y_{i,N}\sum_{k=1}^n\nabla\vartheta_N\frac{\varrho_{k,N}}{m_k}\right| \le c\left(|\nabla\log\vartheta_N| + |\nabla\vartheta_N|\right)\varrho_N,$$

which is bounded in $L^2(0,T;L^{\frac{3}{2}})$ on account of(5.44) and (5.66). Similarly,

$$\frac{C_0}{\pi_{m,N}}\left|\nabla\varrho_{i,N}\frac{\vartheta_N}{m_i}-Y_{i,N}\sum_{k=1}^n\nabla\varrho_{k,N}\frac{\vartheta_N}{m_k}\right|\leq c\sqrt{(\vartheta_N+1)\varrho_N}\sum_{k=1}^n\left|\sqrt{\vartheta_N+1}\nabla\sqrt{\varrho_{k,N}}\right|,$$

thus, according to (5.61) it remains to control the norm of $\sqrt{(\vartheta_N+1)\varrho_N}$ in $L^p((0,T)\times\Omega)$ for some $p>2$. By (5.45) and (5.66) we deduce that $\varrho_N\vartheta_N\in L^\alpha(0,T;L^{\frac{3\alpha}{\alpha+1}}(\Omega))$, so (5.77) is verified. In this manner we actually proved that the sequence of functions

$$\{t\to\int_\Omega\varrho_{k,N}\phi\ \mathrm{d}x\}_{N=1}^\infty,\quad \phi\in C_c^\infty(\Omega)$$

is uniformly bounded and equicontinuous in $C([0,T])$, hence, the Arzelá-Ascoli theorem yields

$$\int_\Omega\varrho_{k,N}\phi\ \mathrm{d}x\to\int_\Omega\varrho_k\phi\ \mathrm{d}x\quad\text{in } C([0,T]).$$

Since $\varrho_{k,N}$ is bounded in $L^\infty(0,T;L^3(\Omega))$ and due to density argument, this convergence extends to each $\phi\in L^{\frac{3}{2}}(\Omega)$.
Finally, the Aubin-Lions argument implies the strong convergence of the sequence $\varrho_{k,N}$ to ϱ_k in $L^p(0,T;L^q(\Omega))$ for $p=2, q<3$, but due to (5.34) and (5.66) it can be extended to the case $p<\infty$. □

5.5.3 Strong convergence of the temperature.

From estimate (5.45) we deduce existence of a subsequence such that

$$\vartheta_N\to\vartheta\quad\text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\tag{5.79}$$

however, time-compactness cannot be proved directly from the internal energy equation (1.4). The reason for this is lack of control over a part of the heat flux proportional to $\varrho\vartheta^\alpha\nabla\vartheta$. This obstacle can be overcome by deducing analogous information from the entropy equation (5.40).

We will first show that all of the terms appearing in the entropy balance (5.40) are nonnegative or belong to $W^{-1,p}((0,T)\times\Omega)$, for some $p>1$.
Indeed, first recall that due to (5.42)

$$|\varrho_N s_N|\leq c\left(\varrho_N+\varrho_N|\log\vartheta_N|+\sum_{k=1}^n\varrho_{k,N}|\log\varrho_{k,N}|\right)$$

and

$$|\varrho_N s_N\mathbf{u}_N|\leq c\left(|\varrho_N\mathbf{u}_N|+|\varrho_N\log\vartheta_N\mathbf{u}_N|+\sum_{k=1}^n|\varrho_{k,N}\log\varrho_{k,N}\mathbf{u}_N|\right),$$

whence due to (5.36), (5.66) and (5.72) we deduce that

$$\{\varrho_N s_N\}_{N=1}^\infty\quad\text{is bounded in } L^2((0,T)\times\Omega),\tag{5.80}$$

moreover

$$\{\varrho_N s_N\mathbf{u}_N\}_{N=1}^\infty\quad\text{is bounded in } L^2(0,T;L^{\frac{6}{5}}(\Omega)).\tag{5.81}$$

The entropy flux is due to (1.9) and (1.23) equal to

$$\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta}\mathbf{F}_k = \frac{\kappa(\varrho,\vartheta)\nabla\vartheta}{\vartheta} + \sum_{k=1}^{n} s_k \mathbf{F}_k.$$

The first part can be estimated as follows

$$\left|\frac{\kappa(\varrho_N,\vartheta_N)\nabla\vartheta_N}{\vartheta_N}\right| \leq |\nabla\log\vartheta_N| + |\varrho_N\nabla\log\vartheta_N| + |\vartheta_N^{\alpha-1}\nabla\vartheta_N| + |\varrho_N\vartheta_N^{\alpha-1}\nabla\vartheta_N|,$$

where the most restrictive term can be controlled as follows $|\varrho_N\vartheta_N^{\alpha-1}\nabla\vartheta_N| \leq |\sqrt{\varrho_N}\vartheta_N^{\frac{\alpha}{2}}||\sqrt{\varrho_N}\nabla\vartheta_N^{\frac{\alpha}{2}}|$, which is bounded on account of (5.44) provided $\varrho_N\vartheta_N^{\alpha}$ is bounded in $L^p((0,T)\times\Omega)$ for $p>1$, uniformly with respect to N. Note that for $0 \leq \beta \leq 1$ we have $\varrho_N\vartheta_N^{\alpha} = (\varrho_N\vartheta_N)^{\beta}\varrho_N^{1-\beta}\vartheta_N^{\alpha-\beta}$, where $(\varrho_N\vartheta_N)^{\beta}$, $\varrho_N^{1-\beta}$, $\vartheta_N^{\alpha-\beta}$ are uniformly bounded in $L^{\infty}(0,T;L^{\frac{1}{\beta}}(\Omega))$, $L^{\infty}(0,T;L^{\frac{3}{1-\beta}}(\Omega))$ and $L^{\frac{\alpha}{\alpha-\beta}}(0,T;L^{\frac{3\alpha}{\alpha-\beta}}(\Omega))$, respectively. Therefore

$$\left\{\frac{\kappa(\varrho_N,\vartheta_N)\nabla\vartheta_N}{\vartheta_N}\right\}_{N=1}^{\infty} \quad \text{is bounded in } L^p(0,T;L^q(\Omega)), \tag{5.82}$$

for p and q satisfying $\frac{1}{p} = \frac{\alpha-\beta}{\alpha}$, $\frac{1}{q} = \beta + \frac{1-\beta}{3} + \frac{\alpha-\beta}{3\alpha}$. In particular $p,q>1$ provided $0<\beta<\frac{\alpha}{2\alpha-1}$.

The remaining part of the entropy flux is equal to

$$\sum_{k=1}^{n} s_{k,N}\mathbf{F}_{k,N} = \sum_{k=1}^{n}\frac{\mathbf{F}_{k,N}}{m_k} + c_v\sum_{k=1}^{n}\log\vartheta_N\mathbf{F}_{k,N} - \sum_{k=1}^{n}\frac{\mathbf{F}_{k,N}}{m_k}\log\frac{\varrho_{k,N}}{m_k},$$

where the middle term vanishes due to (5.14). The worst term to estimate is thus the last one, we rewrite it using (5.15) in the following way

$$-\frac{\mathbf{F}_{i,N}}{m_i}\log\frac{\varrho_{i,N}}{m_i} = \frac{C_0\nabla p_{i,N}}{\pi_{m,N}m_i}\log\frac{\varrho_{i,N}}{m_i} - \frac{\varrho_{i,N}}{\varrho_N}\frac{C_0\nabla\pi_{m,N}}{\pi_{m,N}m_i}\log\frac{\varrho_{i,N}}{m_i}$$

for $i=1,\dots,n$. Both parts have the same structure, so we focus only on the first one, we have

$$\begin{aligned}\frac{C_0}{\pi_{m,N}}\left|\frac{\nabla p_{i,N}}{m_i}\log\frac{\varrho_{i,N}}{m_i}\right| &\leq c|\sqrt{(\vartheta_N+1)\varrho_{i,N}}\log\varrho_{i,N}||\sqrt{\vartheta_N+1}\nabla\sqrt{\varrho_{i,N}}| \\ &\quad + c\sqrt{\varrho_N}\left(|\nabla\log\vartheta_N| + |\nabla\vartheta_N|\right)|\sqrt{\varrho_{i,N}}\log\varrho_{i,N}|.\end{aligned}$$

Using (5.44), (5.45), (5.61) and (5.66) we finally arrive at

$$\left\{\sum_{k=1}^{n} s_{k,N}\mathbf{F}_{k,N}\right\}_{N=1}^{\infty} \quad \text{is bounded in } L^p((0,T)\times(\Omega)), \text{ for } 1<p<\frac{4}{3}. \tag{5.83}$$

We are now ready to proceed with the proof of strong convergence of the temperature. To this end we will need the following variant of the Aubin-Lions Lemma.

Lemma 5.18. *Let g^N converges weakly to g in $L^{p_1}(0,T;L^{p_2}(\Omega))$ and let h^N converges weakly to h in $L^{q_1}(0,T;L^{q_2}(\Omega))$, where $1 \leq p_1,\ p_2 \leq \infty$ and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1. \tag{5.84}$$

Let us assume in addition that

$$\frac{\partial g^N}{\partial t} \text{ is bounded in } L^1(0,T;W^{-m,1}(\Omega)) \text{ for some } m \geq 0 \text{ independent of } N \tag{5.85}$$

$$\|h^N - h^N(\cdot + \xi, t)\|_{L_{q_1}(L_{q_2})} \to 0 \text{ as } |\xi| \to 0, \text{ uniformly in } N. \tag{5.86}$$

Then $g^N h^N$ converges to gh in the sense of distributions on $\Omega \times (0,T)$.

For the proof see [62], Lemma 5.1.

Taking $g^N = \varrho_N s_N$ and $h^N = \vartheta_N$ we verify, due to (5.80), (5.45) and (5.79) , that conditions (5.84), (5.86) are satisfied with $p_1, p_2, q_1, q_2 = 2$. Moreover, for m sufficiently large $L^1(\Omega)$ is imbedded into $W^{-m,1}(\Omega)$, thus by the previous considerations, condition (5.85) is also fulfilled. Therefore, passing to the subsequences we may deduce that

$$\lim_{N\to\infty} \varrho_N s(\varrho_N, \vartheta_N, Y_N)\vartheta_N = \overline{\varrho s(\varrho,\vartheta,Y)\vartheta}.$$

On the other hand, ϱ_N converges to ϱ a.e. on $(0,T)\times\Omega$, hence $\overline{\varrho s(\varrho,\vartheta,Y)\vartheta} = \varrho\overline{s(\varrho,\vartheta,Y)\vartheta}$, in particular, we have that

$$\sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\vartheta} + c_v\overline{\varrho\log\vartheta\ \vartheta} - \sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\log\frac{\varrho_k}{m_k}\vartheta} = \sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\vartheta} + c_v\varrho\overline{\log\vartheta\ \vartheta} - \sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\log\frac{\varrho_k}{m_k}\vartheta}. \tag{5.87}$$

Combining Lemma 5.17 with (5.79) we identify

$$\sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\vartheta} - \sum_{k=1}^n \overline{\frac{\varrho_k}{m_k}\log\frac{\varrho_k}{m_k}\vartheta} = \sum_{k=1}^n \frac{\varrho_k}{m_k}\vartheta - \sum_{k=1}^n \frac{\varrho_k}{m_k}\log\frac{\varrho_k}{m_k}\vartheta,$$

so (5.87) implies that $\varrho\overline{\log\vartheta\vartheta} = \varrho\overline{\log\vartheta}\vartheta$. This in turn yields that $\overline{\log\vartheta\vartheta} = \log\vartheta\vartheta$ a.e. on $(0,T)\times\Omega$, since $\varrho > 0$ a.e. on $(0,T)\times\Omega$, which, due to convexity of function $x\log x$, gives rise to

$$\vartheta_N \to \vartheta \quad \text{a.e. on } (0,T)\times\Omega. \tag{5.88}$$

5.5.4 Limit in the momentum equation, the species mass balance equations and the global total energy balance

Having proven pointwise convergence of sequences $\{\varrho_N\}_{N=1}^\infty$, $\{\varrho_{k,N}\}_{N=1}^\infty$ and $\{\vartheta_N\}_{N=1}^\infty$ we are ready to perform the limit passage in all the nonlinear terms appearing in the momentum equation, the species mass balance equations and the total global energy balance.

(i) Limit in the convective term. Estimate (5.68) implies that for $0 \leq \epsilon \leq 1/2$ we have

$$\|\sqrt{\varrho}\mathbf{u}\|_{L^{p'}(0,T;L^{q'}(\Omega))} \leq \|\varrho\|^{1/2-\epsilon}_{L^\infty(0,T;L^3(\Omega))}\|\sqrt{\varrho}\mathbf{u}\|^{2\epsilon}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}\|^{1-2\epsilon}_{L^{\frac{2\gamma^-}{\gamma^-+1}}(0,T;L^{\frac{6\gamma^-}{\gamma^-+1}}(\Omega))}, \tag{5.89}$$

where p', q' are given by $\frac{1}{p'} = \frac{1-2\epsilon}{\frac{2\gamma^-}{\gamma^-+1}}$, $\frac{1}{q'} = \frac{1/2-\epsilon}{3} + \frac{2\epsilon}{2} + \frac{1-2\epsilon}{\frac{6\gamma^-}{\gamma^-+1}}$. Taking $\epsilon > \frac{1}{2(\gamma^-+1)}$ we have $p', q' > 2$, provided $\gamma^- > 1$, so the convective term converges weakly to $\overline{\varrho\mathbf{u}\otimes\mathbf{u}}$ in $L^p((0,T)\times\Omega)$ for some $p > 1$. To identify the limit, we prove the following lemma.

Lemma 5.19. *Let $p > 1$, then up to a subsequence we have*

$$\varrho_N \mathbf{u}_N \to \varrho \mathbf{u} \quad \text{in } C([0,T]; L^{\frac{3}{2}}_{\text{weak}}(\Omega)),$$
$$\varrho_N \mathbf{u}_N \otimes \mathbf{u}_N \to \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^p((0,T)\times\Omega).$$

Proof. We already know that ϱ_N converges to ϱ a.e. on $(0,T)\times\Omega$. Moreover, due to (5.68), up to extracting a subsequence, $\mathbf{u}_N$ converges weakly to $\mathbf{u}$ in $L^p(0,T;L^q(\Omega))$ for $p>1,\ q>3$. Therefore, the uniform boundedness of the sequence $\varrho_N\mathbf{u}_N$ in $L^\infty(0,T;L^{\frac{3}{2}}(\Omega))$ implies that

$$\varrho_N \mathbf{u}_N \to \varrho \mathbf{u} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^{\frac{3}{2}}(\Omega)).$$

Now, we aim at improving the time compactness of this sequence. Using the momentum equation, we show that the sequence of functions

$$\{t \to \int_\Omega \varrho_N \mathbf{u}_N \phi \,\mathrm{d}x\}_{N=1}^\infty$$

is uniformly bounded and equicontinuous in $C([0,T])$, where $\phi \in C_c^\infty(\Omega)$. But since the smooth functions are dense in $L^3(\Omega)$, applying the Arzelà-Ascoli theorem, we show (5.90).

On the other hand, $\mathbf{u}_N$ is uniformly bounded in $L^p(0,T;W^{1,q}(\Omega))$ for $p>1,\ q>\frac{3}{2}$, so it converges to $\mathbf{u}$ weakly in this space. Since $W^{1,q}(\Omega)$, $q>\frac{3}{2}$ is compactly embedded into $L^3(\Omega)$, by (5.90), we obtain (5.90). □

(ii) Limit in the stress tensor.

Lemma 5.20. *If $\mu(\varrho)$, $\nu(\varrho)$ satisfy (5.10), then for a subsequence we have*

$$\begin{array}{ll} \mu(\varrho_N)\mathbf{D}(\mathbf{u}_N) \to \mu(\varrho)\mathbf{D}(\mathbf{u}) & \text{weakly in } L^p((0,T)\times\Omega) \\ \nu(\varrho_N)\operatorname{div}\mathbf{u}_N \to \nu(\varrho)\operatorname{div}\mathbf{u} & \text{weakly in } L^p((0,T)\times\Omega) \end{array} \quad \text{for } p>1. \tag{5.90}$$

Proof. Due to (5.67), there exists a subsequence such that

$$\nabla\mathbf{u}_N \to \nabla\mathbf{u} \quad \text{weakly in } L^p(0,T;L^q(\Omega)) \text{ for } p>1, q>\frac{3}{2}.$$

Moreover $\mu(\varrho_N), \nu(\varrho_N)$ are bounded in $L^\infty(0,T;L^3(\Omega))$, on account of (5.10). Thus, (5.90) follows by application of Lemma 5.16. □

(iii) Strong convergence of the cold pressure. It follows from estimates (5.36) combined with (5.64) and the Sobolev imbedding theorem that

$$\|\pi_c(\varrho_N)\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \le \|\pi_c(\varrho_N)\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))} \|\pi_c(\varrho_N)\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))} \le c. \tag{5.91}$$

Having this, strong convergence of ϱ_N implies convergence of $\pi_c(\varrho_n)$ to $\pi_c(\varrho)$ strongly in $L^p((0,T)\times\Omega)$ for $1\le p<\frac{5}{3}$.

(iv) Convergence of the diffusion terms. In the proof of Lemma (5.17) it was shown in particular that

$$\{\mathbf{F}_{k,N}\}_{N=1}^\infty \quad \text{is bounded in } L^{\frac{4}{3}}((0,T)\times(\Omega)).$$

By the weak convergence of $\nabla\varrho_{k,N}$, $\nabla\vartheta_N$ to $\nabla\varrho_k$, $\nabla\vartheta$, respectively, deduced from (5.76) and (5.79) together with (5.75), (5.88) and (5.65) we check that it is possible to let $N \to \infty$ in all terms of (5.78). In other words, we have

$$\mathbf{F}_k(\varrho_N, \vartheta_N, \varrho_{k,N}) \to \mathbf{F}_k(\varrho, \vartheta, \varrho_k) \quad \text{weakly in } L^{\frac{4}{3}}((0,T)\times\Omega),\ k \in \{1,\dots,n\}.$$

The convergence results established above are sufficient to perform the limit passage in the momentum, the total global energy balance and the species mass balance equations and to validate, that the limit quantities satisfy the weak formulation (5.21),(5.23) and (5.24).

5.5.5 Limit in the entropy inequality

In view of (5.80-5.83) and the remarks from the previous subsection, it is easy to pass to the limit $N \to \infty$ in all terms appearing in (5.22), except the entropy production rate σ.
However, in accordance with (5.39) we still have that

$$\left\{ \sqrt{\frac{\mu(\varrho_N)}{\vartheta_N}} \left(\nabla \mathbf{u}_N + (\nabla \mathbf{u}_N)^T - \frac{2}{3} \operatorname{div} \mathbf{u}_N \right) \right\}_{N=1}^{\infty} \quad \text{is bounded in } L^2((0,T)\times\Omega).$$

Moreover, by virtue of (5.67), (5.74) and (5.88) we deduce

$$\sqrt{\frac{\mu(\varrho_N)}{\vartheta_N}} \left(\nabla \mathbf{u}_N + (\nabla \mathbf{u}_N)^T - \frac{2}{3} \operatorname{div} \mathbf{u}_N \right) \to \sqrt{\frac{\mu(\varrho)}{\vartheta}} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} \operatorname{div} \mathbf{u} \right)$$

weakly in $L^2((0,T)\times\Omega)$. Evidently, we may treat all the remaining terms

$$\left\{ \sqrt{\frac{\frac{2}{3}\mu(\varrho_N) + \nu(\varrho_N)}{\vartheta_N}} \operatorname{div} \mathbf{u}_N \right\}_{N=1}^{\infty}, \quad \left\{ \frac{\sqrt{\kappa(\varrho_N, \vartheta_N)}}{\vartheta_N} \nabla\vartheta_N \right\}_{N=1}^{\infty}, \quad \left\{ \frac{\sqrt{\pi_m(\vartheta_N, Y_N)}}{\sqrt{C_0 \varrho_{k,N} \vartheta_N}} \mathbf{F}_{k,N} \right\}_{N=1}^{\infty}$$

in the similar way using the fact that they are linear with respect to the weakly convergent sequences of gradients of $\mathbf{u}_N$, ϑ_N and $\varrho_{k,N}$. Thus, preserving the sign of the entropy inequality (5.22) in the limit $N \to \infty$ follows by the lower semicontinuity of convex superposition of operators.

Our ultimate goal is to show that the limit entropy ϱs attains its initial value at least in the weak sense. We have the following result

Lemma 5.21. *Let $\varrho, \mathbf{u}, \vartheta, \varrho_1, \dots, \varrho_n$ be a weak variational entropy solution to* (5.1) *in the sense of Definition 5.4. Then the entropy ϱs satisfies*

$$\operatorname*{ess\,lim}_{\tau\to 0^+} \int_\Omega (\varrho \mathrm{s})(\tau)\phi \,\mathrm{dx} \to \int_\Omega (\varrho \mathrm{s})^0 \phi \,\mathrm{dx}, \quad \forall \phi \in \mathrm{C}^\infty(\Omega). \tag{5.92}$$

Proof. As a consequence of (5.22) we know that

$$\int_\Omega (\varrho s(\vartheta, \varrho_k))(\tau^+)\phi \,\mathrm{d}x \ge \int_\Omega (\varrho s(\vartheta, \varrho_k))(\tau^-)\phi \,\mathrm{d}x,$$

where $\phi \in C^\infty(\Omega)$, $\phi \ge 0$ and $(\varrho s(\vartheta, \varrho_k))(\tau^+) \in \mathcal{M}^+(\Omega)$, $\tau \in [0,T)$, $(\varrho s(\vartheta, \varrho_k))(\tau^-) \in \mathcal{M}^+(\Omega)$, $\tau \in (0,T]$ are the one sided limits of $\varrho s(\tau)$. Note that due to (5.73) $\varrho s \in L^\infty(0,T; L^1(\Omega))$, thus for any Lebesgue point of $\tau \mapsto \varrho s(\tau,\cdot)$ these signed measures coincide with a function $\varrho s(\tau,\cdot) \in L^1(\Omega)$ which satisfies (after some manipulations, cf. [85], Theorem 3.2)

$$\int_\Omega \varrho s(\tau)\phi \, \mathrm{d}x - \langle \sigma, \phi \rangle = \int_\Omega (\varrho s)^0 \phi \, \mathrm{d}x - \int_0^\tau \int_\Omega \varrho s \mathbf{u} \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^n \frac{g_k}{\vartheta} \mathbf{F}_k \right) \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t, \quad (5.93)$$

for any test function $\phi \in C^\infty(\Omega)$, $\phi \geq 0$ where $\sigma \in \mathcal{M}^+([0,T]\times\Omega)$ is a nonnegative measure such that

$$\sigma \geq \frac{\mathbf{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{Q}\cdot\nabla\vartheta}{\vartheta^2} - \sum_{k=1}^n \frac{\mathbf{F}_k}{m_k} \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \sum_{k=1}^n g_k \varrho \omega_k.$$

In order to show (5.92) we thus need to justify that σ is absolutely continuous with respect to Lebesgue measure on $[0,\tau]\times\Omega$. To this end we use in (5.93) a test function $\phi = \vartheta^0$, we get

$$\int_\Omega \varrho s(\tau)\vartheta^0 \, \mathrm{d}x - \left\langle \sigma, \vartheta^0 \right\rangle = \int_\Omega (\varrho s)^0 \vartheta^0 \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{H}\cdot\nabla\vartheta^0 \, \mathrm{d}x \, \mathrm{d}t, \quad (5.94)$$

where, on account of (5.81-5.83)

$$\mathbf{H} = \varrho s \mathbf{u} + \frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^n \frac{g_k}{\vartheta}\mathbf{F}_k \in L^p((0,T)\times\Omega), \text{ for some } p>1. \quad (5.95)$$

Now, testing (5.23) with $\phi_m \in C^\infty[0,T)$ such that $\phi_m \to 1$ pointwisely in $[0,\tau)$, $\phi_m \to 0$ pointwisely in $[\tau,T)$, $0<\tau<T$ and passing to the limit with m, we obtain

$$\int_\Omega \left(\frac{\left|(\varrho\mathbf{u})^0\right|^2}{2\varrho^0} + \varrho^0 e(\varrho^0,\vartheta^0,\varrho_1^0,\dots,\varrho_n^0) \right) \mathrm{d}x = \int_\Omega \left(\frac{|\varrho\mathbf{u}|^2}{2\varrho} + \varrho e(\varrho,\vartheta,\varrho_1,\dots,\varrho_n) \right)(\tau) \, \mathrm{d}x. \quad (5.96)$$

Combining (5.94) with (5.96), we thus get

$$\int_\Omega \left(\frac{|\varrho\mathbf{u}|^2}{2\varrho}(\tau) - \frac{\left|(\varrho\mathbf{u})^0\right|^2}{2\varrho^0} \right) \mathrm{d}x + \int_\Omega \underbrace{\left[\varrho e(\tau) - \vartheta^0 \varrho s(\tau)\right] - \left[\varrho^0 e^0 - \vartheta^0 (\varrho s)^0\right]}_{I^*} \mathrm{d}x + \left\langle \sigma, \vartheta^0 \right\rangle = -\int_0^\tau \int_\Omega \mathbf{H}\cdot\nabla\vartheta^0 \, \mathrm{d}x \, \mathrm{d}t. \quad (5.97)$$

By the Fatou lemma

$$\liminf_{\tau\to 0^+} \int_\Omega \left(\frac{|\varrho\mathbf{u}|^2}{2\varrho}(\tau) - \frac{\left|(\varrho\mathbf{u})^0\right|^2}{2\varrho^0} \right) \mathrm{d}x \geq 0,$$

in addition

$$\lim_{\tau\to 0^+} \int_0^\tau \int_\Omega \mathbf{H}\cdot\nabla\vartheta^0 \, \mathrm{d}x \, \mathrm{d}t = 0$$

on account of (5.95). Moreover, recalling (1.6), (1.10) and (5.19), we recast I^* as follows

$$I^* = \sum_{k=1}^n (e_k^{st} - \vartheta^0 s_k^{st})(\varrho_k(\tau) - \varrho_k^0) + \vartheta^0 \sum_{k=1}^n \left[\frac{\varrho_k}{m_k} \log\frac{\varrho_k}{m_k}(\tau) - \frac{\varrho_k^0}{m_k}\log\frac{\varrho_k^0}{m_k} \right] + \left[\varrho e_c(\varrho)(\tau) - \varrho^0 e_c(\varrho^0)\right] + c_v \left[\varrho\vartheta(\tau) - \varrho^0\vartheta^0 - \varrho\log\vartheta(\tau) + \varrho^0 \log\vartheta^0\right] = \sum_{i=1}^4 I_i. \quad (5.98)$$

Evidently $\int_\Omega (I_1 + I_2 + I_3)\ \mathrm{d}x \to 0$ in view of weak continuity of ϱ and ϱ_k, $k = 1, \dots, n$. Concerning the last term, we have

$$I_4 = c_v \underbrace{\varrho(\tau) \left[\vartheta(\tau) - \log \vartheta(\tau) - \vartheta^0 + \log \vartheta^0\right]}_{\geq 0} + c_v (\varrho(\tau) - \varrho^0)(\vartheta^0 - \log \vartheta^0),$$

therefore

$$\lim_{\tau \to 0^+} \int_\Omega I_4\ \mathrm{d}x \geq 0.$$

Since the entropy production rate is always nonnegative, (5.97) together with above remarks yields $\operatorname{ess\,lim}_{\tau \to 0^+} \left\langle \sigma, \vartheta^0 \right\rangle = 0$, whence

$$\operatorname{ess} \lim_{\tau \to 0^+} \sigma[[0, \tau] \times \Omega] = 0. \ \Box$$

Chapter 6

Appendix

In this chapter we give statements of nowadays classical lemmas and theorems, which were used in proofs of the previous results.

The following two theorems are extensively used in whole the thesis:

Theorem 6.1 (Arzelà-Ascoli)**.** *Let $\Omega \subset \mathbb{R}^N$ be compact and X a compact topological metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(\Omega; X)$ which is equi-continuous, that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d_X[v_n(y), v_n(z)] \leq \varepsilon \text{ provided } |y - z| < \delta \text{ independently of } n = 1, 2, \ldots.$$

Then $\{v_n\}_{n=1}^{\infty}$ is precompact in $C(\Omega; X)$, that is, there exists a subsequence (not relabeled) and a function $v \in C(\Omega; X)$ such that

$$\sup_{y \in \Omega} d_X[v_n(y), v(y)] \to 0 \text{ as } n \to 0.$$

For the proof see [53], Chapter 7, Theorem 17.

W now present a version of the celebrated Aubin-Lions lemma [27, 59]. For more criteria of relative compactness of sequence of functions in $L^p(0, T; B)$, where B is a Banach space we refer to [97].

Theorem 6.2 (Aubin-Lions)**.** *Let X, B, Y be Banach spaces and $X \subset B \subset Y$ with compact imbedding $X \to B$. Suppose also that X and Y are reflexive spaces. For $1 < p, q < \infty$ let*

$$W = \{v \in L^p(0, T; X); \partial_t v \in L^q(0, T; Y)\},$$

where the time derivative is defined in the sense of distributions on $(0, T)$.

Then the imbedding $W \subset L^p(0, T; B)$ is compact.

The Bogovskii operator. We first recall definition of spaces introduced by Temam in [102].

Definition 6.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $1 < p, q < \infty$. We set*

$$E^{q,p}(\Omega) = \left\{ \mathbf{g} \in (L^q(\Omega))^N : \ \operatorname{div} \mathbf{g} \in L^p(\Omega) \right\},$$

$$\|\mathbf{g}\|_{E^{q,p}(\Omega)} = \|\mathbf{g}\|_{L^q(\Omega)} + \|\operatorname{div} \mathbf{g}\|_{L^p(\Omega)},$$

and

$$E_0^{q,p}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{E^{q,p}(\Omega)}.$$

Next, let us consider a problem

$$\begin{aligned} \operatorname{div} \mathbf{u} &= f \ \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} & \quad\ \text{ at } \partial\Omega, \end{aligned} \tag{6.1}$$

for a given function f, such that $\int_\Omega f \ \mathrm{d}x = 0$.
Solution to problem (6.1) in a bounded, Lipschitz domain is given by so called Bogovskii operator introduced in [8]. In the following lemma we recall its main properties.

Lemma 6.4. *Let $\Omega \in \mathbb{R}^N$ be a bounded Lipschitz domain.*
Then there exists a linear operator $\mathcal{B}_\Omega = (\mathcal{B}_\Omega^1, \dots, \mathcal{B}_\Omega^N)$ such that:
(i)

$$\mathcal{B}_\Omega : \overline{L^p(\Omega)} \to \left(W_0^{1,p}(\Omega)\right)^N, \quad 1 < p < \infty,$$

where

$$\overline{L^p(\Omega)} = \left\{ f \in L^p(\Omega) : \int_\Omega f \ \mathrm{d}x = 0 \right\};$$

(ii) for $f \in \overline{L^p(\Omega)}$

$$\operatorname{div} \mathcal{B}_\Omega(f) = f \quad a.e. \ in \ \Omega;$$

(iii) for $f \in \overline{L^p(\Omega)}$

$$\|\nabla \mathcal{B}_\Omega(f)\|_{L^p(\Omega)} \le c(p,\Omega)\|f\|_{L^p(\Omega)}, \quad 1 < p < \infty;$$

(iv) if $f = \operatorname{div} \mathbf{g}$, where $\mathbf{g} \in E_0^{q,p}(\Omega)$ with some $1 < q < \infty$, then

$$\|\mathcal{B}_\Omega(f)\|_{L^q(\Omega)} \le c(q,\Omega)\|\mathbf{g}\|_{L^q(\Omega)}, \quad 1 < p < \infty;$$

(v) if $f \in C_0^\infty(\Omega)$ and $\int_\Omega f \ \mathrm{d}x = 0$, then $\mathcal{B}_\Omega(f) \in (C_0^\infty(\Omega))^N$.

For the proof of this lemma, we refer the reader to [85], Lemma 3.17. An extension of the existence theory for (6.1) to the class of solutions which need not have a trace at the boundary can be found in [21].

The double Riesz transform. In what follows we recall some of basic properties of the double Riesz transform $\mathcal{R} = \nabla \otimes \nabla \Delta^{-1}$ and the inverse divergence operator $\mathcal{A} = \nabla \Delta^{-1}$ defined as follows

$$\mathcal{A}_j[v] = \left(\nabla \Delta^{-1}\right)_j v = -\mathcal{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathcal{F}(v)\right), \tag{6.2}$$

$$\mathcal{R}_{i,j}[v] = \partial_i \mathcal{A}_j[v] = \left(\nabla \otimes \nabla \Delta^{-1}\right)_{i,j} v = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)\right). \tag{6.3}$$

Here, the inverse Laplacian is identified through the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ as

$$(-\Delta)^{-1}(v) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\mathcal{F}(v)\right).$$

Lemma 6.5. *The operator $\mathcal{R}$ is a continuous linear operator from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$. In particular, the following estimate holds true:*

$$\|\mathcal{R}[v]\|_{L^p(\mathbb{R}^3)} \le c(p)\|v\|_{L^p(\mathbb{R}^3)} \quad for \ all \ v \in L^p(\mathbb{R}^3).$$

The operator $\mathcal{A}$ is a continuous linear operator from $L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and from $L^p(\mathbb{R}^3)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ for any $1 < p < 3$. Moreover,

$$\|\nabla \mathcal{A}[v]\|_p \le C(p)\|v\|_p, \quad 1 < p < \infty.$$

The proof of this lemma can be found e.g. in [36], Section 10.16. For more information about the operators defined by means of Fourier multiplier we refer the reader to [99], Chapters III and IV.

In what follows we present two important properties of commutators involving Riesz operator. The first result is a straightforward consequence of the *Div-Curl* lemma (see [101]), its proof can be found in [32], Lemma 5.1.

Lemma 6.6. *Let*

$$\mathbf{V}_\varepsilon \rightharpoonup \mathbf{V} \quad \text{weakly in } L^p(\mathbb{R}^3), \quad r_\varepsilon \rightharpoonup r \quad \text{weakly in } L^q(\mathbb{R}^3),$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$\mathbf{V}_\varepsilon \mathcal{R}(r_\varepsilon) - r_\varepsilon \mathcal{R}(\mathbf{V}_\varepsilon) \rightharpoonup \mathbf{V}\mathcal{R}(r) - r\mathcal{R}(\mathbf{V}) \quad \textit{weakly in } L^s(\mathbb{R}^3).$$

The next lemma can be deduced from the general results of B.Bajšanski and R.Coifman [3], and R.Coifman and Y.Meyer [20].

Lemma 6.7. *Let $w \in W^{1,r}(\mathbb{R}^3)$ and $\mathbf{V} \in L^p(\mathbb{R}^3)$ be given, where $1 < r < 3$, $1 < p < \infty$, $\frac{1}{r} + \frac{1}{p} - \frac{1}{3} < \frac{1}{s} < 1$. Then for all such s we have*

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{\alpha,s}(\mathbb{R}^3)} \leq c(s,p,r)\|w\|_{W^{1,r}(\mathbb{R}^3)}\|\mathbf{V}\|_{L^p(\mathbb{R}^3)},$$

where α is given by $\frac{\alpha}{3} = \frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{r}$.

Here, $W^{\alpha,s}(\mathbb{R}^3)$ for $\alpha \in (0,\infty) \setminus \mathbb{N}$ denotes the Sobolev-Slobodeckii space (see e.g. [103]).
For the proof of this fact see [36], Section 10.17 and the references therein.

Renormalized continuity equation. The following result is a consequence of technique introduced and developed by DiPerna and Lions [24]. Applying it to the continuity equation (extended by 0 outside Ω) we obtain the following result

Lemma 6.8. *Let $\varrho \in L^p(\mathbb{R}^3)$, $p \geq 2$, $\varrho \geq 0$, a. e. in Ω, $\mathbf{u} \in W_0^{1,2}(\mathbb{R}^3)$ satisfy the continuity equation*

$$\operatorname{div}(\varrho\mathbf{u}) = 0$$

in the sense of distributions on $\mathbb{R}^3$, then the pair $(\varrho, \mathbf{u})$ solves the renormalized continuity equation (2.14) in the sense of distributions on $\mathbb{R}^3$ where $b(\cdot)$ is specified as follows:

$$\begin{gathered} b \in C([0,\infty) \cap C^1((0,\infty)), \\ \lim_{s\to 0^+}(sb'(s) - b(s)) \in \mathbb{R}, \\ |b'(s)| \leq Cs^\lambda, \quad s \in (1,\infty), \quad \lambda \leq \tfrac{p}{2} - 1. \end{gathered}$$

The best general reference here is [36], Section 10.18, see also [85].

Maximal $L^p - L^q$ regularity of parabolic equations. Below we recall the well known result about the Maximal Sobolev Regularity of parabolic problem in the whole space

$$\begin{cases} \partial_t u - \Delta \mathbf{u} = f & \text{in } (0,T) \times \mathbb{R}^N, \\ u(0,x) = u^0(x) & \text{in } (0,T) \times \mathbb{R}^N. \end{cases} \tag{6.4}$$

The relevant results for systems with general boundary conditions can be found in the book of Amann [2].
In the following, we put

$$E_0 = L^q\left(\mathbb{R}^N\right), \quad E_1 = W^{2,q}\left(\mathbb{R}^N\right) \quad \text{and} \quad E_{1-1/p} = (E_0, E_1)_{1-1/p,p},$$

where the symbol $(\cdot,\cdot)_{1-1/p,p}$ denotes the corresponding $(1-1/p,p)$-interpolation space ($= B^q_{2-2/p,p}\left(\mathbb{R}^N\right)$, cf. [7], Corollary 4.13).

Theorem 6.9. *Let $1 < p, q < \infty$. Given any $u^0 \in E_{1-1/p}$, $f \in L^p(0,T;E_0)$, the Cauchy problem (6.4) has a unique solution $u \in L^p(0,T;E_1) \cap W^{1,p}(0,T;E_0)$, and*

$$\sup_{t\in(0,T)} \|u(t)\|_{E_{1-1/p}} + \|\partial_t u\|_{L^p(0,T;E_0)} + \|\Delta u\|_{L^p(0,T;E_0)} \leq c\left(\|f\|_{L^p(0,T;E_0)} + \|u^0\|_{E_{1-1/p}}\right)$$

for some positive constant $c = c(p,q)$.

This theorem follows from Theorem 4.10.2, Theorem 4.10.7 and Remark 4.10.9 in Amann [2].

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