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Aproksymacja rozwiązań i oszacowania hiperkontraktywne dla modelu ewolucji dyslokacji

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Approximation of solutions and hypercontractivity estimates for a model of the evolution of dislocations

Doctoral dissertation
written under supervision of
Professor Piotr Biler

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STRESZCZENIE

Dyslokacje, zwane również liniowymi wadami w strukturze krystalicznej, są najważniejszą klasą niedoskonałości występujących we wszystkich ciałach stałych o strukturze krystalicznej takich jak kryształy czy metale. Ich obecność znacząco wpływa na mechaniczne własności ciał stałych, które są wysoce podatne na stopień niedoskonałości. Jedne z najważniejszych to dyfuzja, plastyczność oraz wytrzymałość kryształów. Dyslokacje odgrywają tak ważną rolę w teorii materiałów, ponieważ nawet jeden defekt odpowiedniego typu może powodować istotne jakościowo i nieodwracalne zmiany w strukturze materiałów.

Przedmiotem badań w pracy są dwa modele opisujące ewolucję dyslokacji w kryształach. Pierwszy jednowymiarowy model ciągły składa się z nieliniowego i nielokalnego równania, gdzie nielokalny składnik jest reprezentowany przez ułamkowy laplasjan. Dla tak przedstawionego problemu badamy istnienie słabych rozwiązań oraz dowodzimy pewnych oszacowań hiperkontraktywnych tzn. pokazujących regularność rozwiązań. Ponadto jesteśmy zainteresowani zbadaniem istnienia oraz asymptotyki rozwiązań samopodobnych. Drugi z modeli, model dyskretny, składa się z układu równań różniczkowych zwyczajnych, gdzie każde z równań opisuje ewolucję dokładnie jednej dyslokacji. W tym modelu naszym celem jest zbadanie procesu gromadzenia się dyslokacji i tworzenia tak zwanych ścian dyslokacji.

W Rozdziale 1 przedstawiamy wstęp do teorii dyslokacji, gdzie opisujemy m.in. zjawisko przemieszczania się dyslokacji. Dodatkowo opisujemy szczegółowo wyżej wspomniane modele, którymi będziemy się zajmować. Celem

Rozdziału 2 jest zebranie potrzebnych informacji na temat ułamkowego laplasjanu oraz ułamkowych pochodnych, które są wykorzystywane w pracy. Następnie w Rozdziale 3 korzystając z metody regularyzacji parabolicznej udowadniamy istnienie słabych rozwiązań modelu ciągłego i badamy asymptotykę norm rozwiązań dla dużych czasów. Dalej w Rozdziale 4 zajmujemy się rozwiązaniami samopodobnymi pewnych szczególnych zagadnień, gdzie dowodzimy ich istnienia bądź nieistnienia oraz oraz badamy asymptotykę rozwiązań w jednym szczególnym przypadku. Rozdział 5 jest poświęcony wygasaniu w skończonym czasie rozwiązań nieliniowego modelu, gdzie ułamkowy laplasjan jest zastąpiony przez zwykły laplasjan. Na koniec w Rozdziale 6 badamy drugi z naszych modeli oraz dowodzimy, że dyslokacje kumulują się tworząc tak zwane ściany dyslokacji. Dowód ten poprzedzamy serią eksperymentów numerycznych, które potwierdzają rozważaną hipotezę.

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ABSTRACT

Dislocations, also called linear defects, are the most important class of imperfections occurring in all crystalline solids like crystals and metals. Their presence has a significant influence on mechanical properties of solids which are highly sensitive to solid perfection. The most relevant ones are diffusion phenomena, plasticity and crystal strength. The reason why dislocations are so essential is that even one defect of a proper type may cause irreversible effects.

The aim of the dissertation is to investigate two models describing evolution of dislocations in crystals. The first model involves a non-linear and non-local equation in one space dimension, where the non-local term is represented by the fractional Laplace operator. For such a problem we study existence of weak solutions and we prove hypercontractivity estimates. Besides, we also study existence and non-existence of self-similar solutions. The second model consists of a system of ordinary differential equations, where each equation describes evolution of exactly one dislocation. Our main goal is to study a long time behaviour of the dynamics of dislocations leading to the creation of so-called walls of dislocations.

In Chapter 1 we give an extended introduction to dislocation theory, where we describe a phenomenon of the motion of dislocations. Moreover, we provide a deeper presentation of the models, which are under the investigation in the dissertation. The main purpose of Chapter 2 is to introduce the fractional Laplace operator occurring in the first model and properties of the fractional derivative. Furthermore, in Chapter 3 we prove existence of weak

solutions, by considering approximating solutions of the regularised problem via the vanishing viscosity method, and study the long time behaviour of the L^p-norms of solutions. In Chapter 4 we study existence of self-similar solutions, and we show the intermediate asymptotics in one particular case. Chapter 5 is devoted to the extinction of solutions of the local counterpart of the first model, where the fractional Laplace operator is replaced by the usual one. Finally, in Chapter 6 we investigate the second model, and we prove the accumulation of dislocations.

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Contents

Chapter 1

Introduction

Dislocations – linear defects. Mathematically, dislocations can be explained as topological defects (imperfections) or irregularities occurring in all crystalline solids, *i.e.* crystals, metals and other non-metallic materials whose atoms are arranged in a pattern that repeats periodically creating a so-called *crystal structure*. We can distinguish several possible classes of imperfections occurring in crystals, see [64]. The most important ones are linear defects, which can be described as groups of atoms in irregular positions. Such defects are commonly called *dislocations*.

Dislocations were discovered independently in 1934 by Taylor [72], Orowan [57] and Polyani [61], who attempted to understand what atoms do when the solid deforms and where such deformations come from. They interpreted plastic deformation by what is now called an *edge dislocation*. However, the theory describing the elastic fields of the defects was originally developed by an Italian mathematician Volterra at the end of the 19th century.

In nature the crystal structure in most crystalline materials is not perfect. Usually the regular patterns are interrupted by some crystallographic defects. By a such defect (imperfection) we mean a small region where the regular pattern breaks down and some atoms are not properly surrounded by its neighbours, see Figure 1.1.

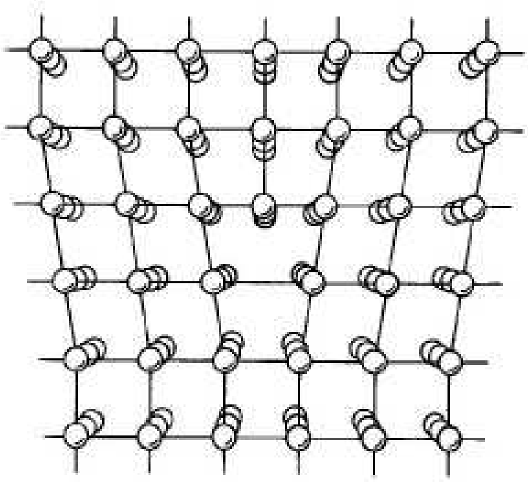


Figure 1.1: An example of an edge dislocation. Notice that a dislocation can also exist when the extra half plane of atoms is inserted from below. These two types are called positive and negative dislocations.

Mechanical properties. The presence of dislocations has a big influence on the mechanical properties of solids which are highly sensitive to solid perfection. The most important *sensitive* properties are: diffusion phenomena, the ionic and electronic conductivity in insulating crystal and semiconductors, plasticity and crystal strength. On the contrary, heat, elasticity and the principal features of optical absorption and dispersion are examples of *insensitive* properties, see [77]. The reason why dislocations form a very important class of imperfections in solids is that even one defect of a proper type may cause irreversible effects, see [37, 39, 64, 77].

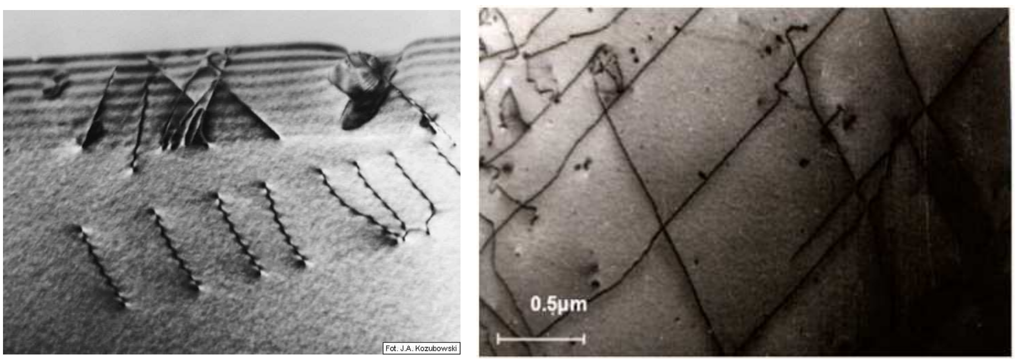


Figure 1.2: Examples of dislocations in real materials. For more examples we refer to the book of Hirth and Lothe [37].

Motion of dislocations. Whenever the material is submitted to shear stress, dislocations lines can move through the lattice structure, see Figure 1.3. This only needs to be a small force since a very small fraction of the bonds is being broken at any given time. As presented, the dislocation moves a small amount to the right. The movement of the dislocation across the plane eventually causes the upper half of the crystal to move with respect to the lower half.

By motion of dislocations one can understand the motion of a large number of atoms. A certain simple form of dislocations can generate a large number of additional dislocations by a process of expansion and subdivision. This mechanism is better known as the Frank-Read mechanism, which is the most plausible explanation for the origin of dislocations that accumulate during deformation.

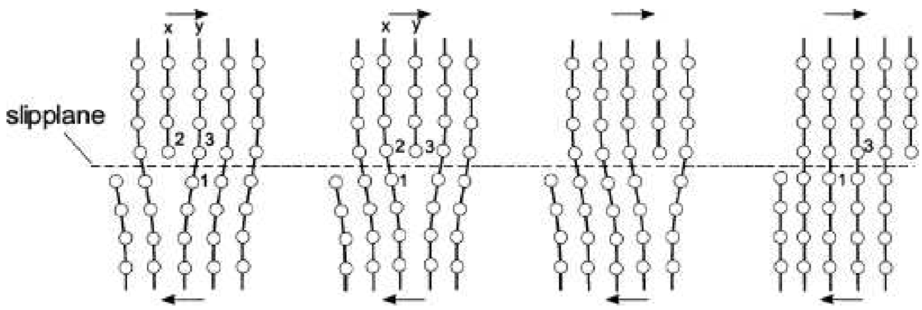


Figure 1.3: An example of motion of a single dislocation under the applied stress.

1.1 One-dimensional model

One possible and simplified model describing the dislocation dynamics (see [78, 79]) is given by an interacting particle system described by

$$x_i' = b_i\sigma - V_0'(x_i) + b_i \sum_{j\neq i} b_j V'(x_i - x_j) \quad \text{for} \quad i = 1, \dots, N, \tag{1.1}$$

where σ is a given constant force, V_0 is a given potential and V is a potential of two-body interactions. One can think of x_i as a position of dislocation line of an infinite length and parallel to the z-axis. In this model dislocations can be either positive or negative, depending on the parameters (the so-called Burgers vector, see [37]) which are denoted by b_i and can be simplified to ± 1. Dislocations of the the same sign repel each other, whereas dislocations with opposite sings attract each other accordingly to the previously defined potential V. Furthermore, if the distance between dislocations with opposite signs drops down to zero, we can observe the annihilation of the dislocations. However, if the external force overcomes the forces tending to keep dislocations together, then they separate.

Related results. For a particular potential $V'(z) = \frac{z}{|z|^2}$ and dislocations being of only one type, self-similar solutions and the asymptotic behaviour of other solutions of (1.1) were already studied by Head [34, 35, 36] sixty years ago.

Motivated by those results obtained by Head, Deslippe *et al.* [26] performed numerical simulations of (1.1) for the same potential $V'(z) = \frac{z}{|z|^2}$ and dislocations of two types. In case of a positive force σ, which is responsible for emitting new pairs of dislocations, they showed that the number of dislocations behaves like $\frac{t}{\ln t}$ for large t.

Recently, Forcadel *et al.* showed in [31] that, under suitable assumptions on V_0, V and with $\sigma = 0$ in (1.1), the rescaled "cumulative distribution function"

$$\rho^\varepsilon(x,t) = \varepsilon\Big(-\frac{1}{2} + \sum_{i=1}^{N} H\Big(x - \varepsilon x_i(\tfrac{t}{\varepsilon})\Big)\Big) \tag{1.2}$$

(where H is the Heaviside function: $H(x) = 1\!\!1_{\{x \geqslant 0\}}$) converges towards the (linear) density of dislocations u which is a unique solution of the corresponding initial value problem for the non-linear diffusion equation

$$u_t + |u_x|\Lambda u = 0 \quad \text{on} \quad \mathbb{R} \times (0,\infty). \tag{1.3}$$

Here for $\alpha \in (0,2)$, $\Lambda^\alpha = \Big(-\frac{\partial^2}{\partial x^2}\Big)^{\frac{\alpha}{2}}$ is the pseudo-differential operator on the real line defined via the Fourier transform

$$\widehat{\Big(\Lambda^\alpha w\Big)}(\xi) = |\xi|^\alpha \widehat{w}(\xi). \tag{1.4}$$

For a more detailed presentation of the operator $\Lambda = \Lambda^1$ see Chapter 2.

Equation (1.3) has been recently studied by Biler, Karch, Monneau with u_x instead of $|u_x|$. This change is justified by the fact of occurrence of dislocations of only one type: either positive or negative. In [9] they proved the existence and uniqueness of viscosity solutions and constructed explicit positive self-similar solutions of (1.3).

Additionally, Caffarelli and Vázquez in [16] considered a multidimensional counterpart of (1.3) written in a slightly more general form

$$\partial_t v = \nabla \cdot (v \nabla p), \tag{1.5}$$

with the non-local pressure law $p = (-\Delta)^{-s} v$, where $s \in (0,1)$. Equation (1.3) can be easily derived from (1.5), with $s = \frac{1}{2}$ and in one-dimension case, by taking as u a one-dimensional primitive of v. For bounded and compactly supported initial conditions, the authors of [16] proved the existence of weak and bounded solutions of (1.5) that propagate with finite speed.

Moreover, most recently, Biler, Imbert and Karch in [10] investigated a generalisation of the porous medium equation (1.5) with non-linear and non-local pressure $p = (-\Delta)^{-s}(|v|^{m-2}v)$, with $s \in (0,1)$ and $m > 1$. They showed the existence of weak solutions to the corresponding Cauchy problem, and compactly supported positive self-similar solutions were constructed explicitly.

Presentation of the problem. Motivated by the physical considerations described above and by the recent results, it is reasonable to study the following initial value problem involving non-linear and non-local equation with $u = u(x,t)$ in one space dimension ($x \in \mathbb{R}$)

$$u_t = |u_x|\Big(-\sigma - \Lambda^\alpha u\Big) \quad \text{on} \quad \mathbb{R} \times (0, +\infty), \tag{1.6a}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{1.6b}$$

The system above can be considered as a continuous counterpart of the interacting particle system (1.1) with the two-body interaction force $V'(z) = \frac{z}{|z|^{\alpha+1}}$ and as a natural extension of (1.3) with exterior forces. The occurrence of the absolute value $|u_x|$ in the equation allows dislocations of the opposite signs to vanish after interaction. Additionally, here σ denotes an external force acting on the system which can either create new pairs of dislocations or annihilate them.

Furthermore, the positive part of u_x, $\max\{u_x, 0\}$, describes the density of positive dislocations; on the other hand, $\max\{-u_x, 0\}$ describes the density of negative dislocations. Due to the fact that positive and negative dislocations are located on the positive and negative half-line respectively, we can assume that the initial condition u_0 is non-positive and satisfies: $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

However, we shall see that for proving existence of solutions it is much more reasonable to consider equation (1.6) in terms of the derivative of u, *i.e.* the density of dislocations. Namely, setting $v = u_x$, (1.6) can be transformed into the following problem:

$$v_t + \frac{\partial}{\partial x}\Big(|v|\big(\sigma - \nabla^{\alpha-1}v\big)\Big) = 0, \tag{1.7a}$$

$$v(x,0) = v_0(x) \quad \text{for} \quad x \in \mathbb{R}, \tag{1.7b}$$

with the initial condition satisfying $\int v_0 \,\mathrm{d}x = 0$ and the fractional gradient defined in dimension one as

$$\nabla^{s-1} f(x) = C_s \int_{\mathbb{R}} \big(f(x) - f(x+z)\big) \frac{z}{|z|^{1+s}} \,\mathrm{d}z, \tag{1.8}$$

with some constant $C_s > 0$. Here, in the transformation we used the identity (2.7) for the fractional gradient, *i.e.* $(-\Delta)^{\frac{\alpha}{2}} f = -\nabla \cdot \nabla^{\alpha-1} f$.

1.2 Two-dimensional model

Let us consider a model describing horizontal motion of dislocation lines parallel to the z-axis. Considering the cross section of these lines we can reduce the problem to its two-dimensional counterpart where each dislocation line is represented by its position $(x_i(t), i) \in \mathbb{R} \times \mathbb{Z}$. Finally, such horizontal evolution can be characterized as follows

$$x_i' = \sum_{j \neq i} f(x_j - x_i, j - i) \quad \text{for} \quad i \in \mathbb{Z}. \tag{1.9}$$

Here $f\colon \mathbb{R}\times\mathbb{Z}\backslash\{0\}\to\mathbb{R}$ is an anisotropic force of two-body interactions. An example of such a force, according to [33], is

$$f(x,y)=\frac{x(y^2-x^2)}{(y^2+x^2)^2}. \tag{1.10}$$

An important aspect of interatomic interactions is that atoms can attract each other at longer distances and repel at short distances aggregating into various bulk forms. Such behaviour, of course, depends on the form of the considered potentials.

One of the forces describing both long-range attraction and short-range repulsion between atoms is the interaction force given by (1.10). In such an example two particles attract each other if the vertical angle between them is less then $\frac{\pi}{4}$ and, on the other hand, repel each other if the angle is greater then $\frac{\pi}{4}$, see Figure 1.4 and Figure 1.5.

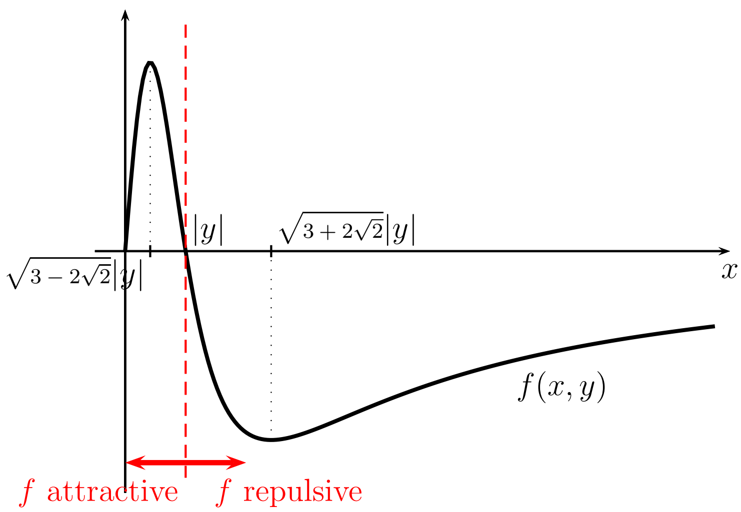


Figure 1.4: Interaction force $f(x,y)$ as a function of the distance between two atoms for some fixed $y\in\mathbb{Z}\backslash\{0\}$ with the property $f(-x,y)=-f(x,y)$. A vertical angle between two particles corresponds to $\arctan(\frac{x}{y})$. Thus $\frac{\pi}{4}$ reads as $x=|y|$.

In the literature, however, there is a convention to express force in terms of energy potentials commonly called *interatomic potentials*. Thus, a general force acting on an atom can be seen as the negative derivative of some potential function with respect to its position: $f(r)=-\phi'(r)$.

Related results. Another possible model, first proposed in 1924 and repeatedly improved in subsequent years, involves the Lennard-Jones potential [49]

$$\phi(r) = 4\varepsilon\left[\left(\frac{r}{\sigma_0}\right)^{-12} - \left(\frac{r}{\sigma_0}\right)^{-6}\right],$$

where r is a distance between two atoms, ε is the depth (minimum) of the energy and σ_0 is the finite distance at which the interparticle potential is zero. Due to its computational simplicity and relatively good approximations, the Lennard-Jones potential is extensively used to describe the properties of gases and in computer simulations [49, 50].

There is no necessity to deal only with two-body potentials. One approach to represent the many-body potentials energy is to consider it as a sum of two-body, three-body, $\ldots$, N-body terms. An example of such constructed energy potential is the Stillinger-Weber potential [71] for semiconductor silicon containing only two- and three-body terms.

More facts about dislocations, examples of potentials used in various models and numerical simulations performed on these models can be found in the book of Bulatov and Cai [15].

A similar model to ours, where a finite number of dislocations of different types occur (for instance positive and negative ones), was considered by El Hajj, Ibrahim and Monneau [33]. The authors studied horizontal motion of dislocation lines and they derived formally a two-dimensional mean field model called Groma-Balogh model. In the same paper, they also investigated a model with additional boundary conditions. They observed that positive dislocations move to the right, whereas the negative ones move to the left. In particular, numerical simulations of deformations of a slab under an external shear stress were performed.

Related models called *individual cell-based* models occur not only in the theory of dislocations but also in the study of, *e.g.* chemotherapy, where x_i denotes a center of a tumour cell [28], chemotaxis [70] and many others. Moreover, particles may also evolve according to stochastic differential equations, see [12] and references therein for numerical simulations.

Presentation of the problem. The system of all particles acting together under the above defined force can be rewritten in the following way

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t) = F(X(t)), & t > 0, \\ X(0) = X^0, \end{cases} \tag{1.11}$$

where $X(t) = (x_i(t))_{i\in\mathbb{Z}}, F(X) = (F_i(X))_{i\in\mathbb{Z}}$ and X^0 is some given initial position of dislocations. Moreover, $F_i(X)$ describes a resultant force acting on the i-th particle, *i.e.* $F_i(X) \stackrel{\text{def}}{=} \sum_{j\neq i} f(x_j - x_i, j - i)$ for each $i \in \mathbb{Z}$.

Since our aim is to study a long time behaviour of the dynamics of particles which creates walls of dislocations, the property of the force f described in (1.10) forces us to consider the problem (1.11) with the initial data in the following set

$$\Omega = \left\{ X : |x_i - x_j| \leqslant \sqrt{3 - 2\sqrt{2}}\, |i - j| \right\}. \tag{1.12}$$

Notice here that $\arctan\left(\sqrt{3 - 2\sqrt{2}}\right) = \frac{\pi}{8}$ guarantees that the force f restricted to Ω is not only attractive but also non-decreasing with respect to the first variable. Therefore, we are able to prove a comparison principle, which helps us to conclude *e.g.* existence of global-in-time solutions.

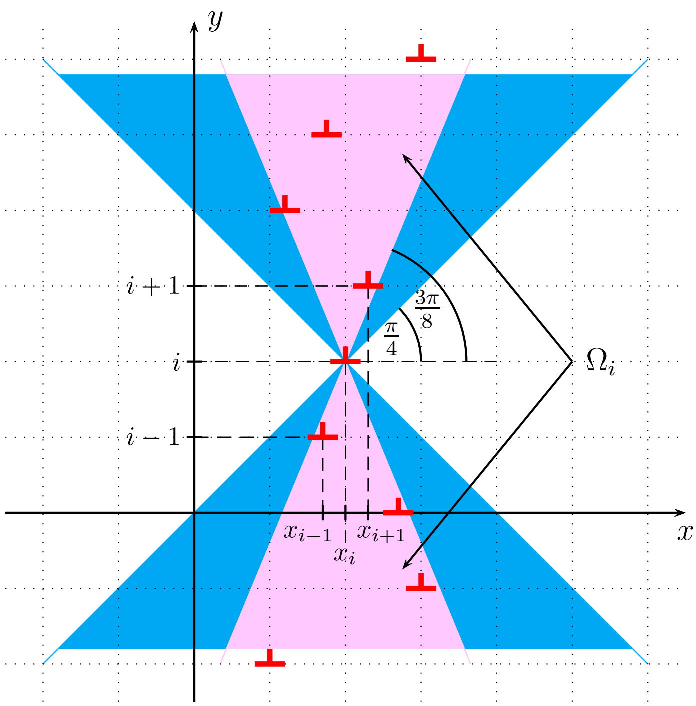


Figure 1.5: A fixed particle x_i attracts all other particles if they are placed in a region marked in blue and pink. However, the force f is nondecreasing only if the particles are located in the region marked in pink. Such a domain we call Ω_i, and we can present Ω, defined in (1.12), as $\Omega = \cap_{i\in\mathbb{Z}}\Omega_i$.

1.3 Notation

In this work the usual norm of the Lebesgue space $L^p(\mathbb{R})$ with respect to the spatial variable is denoted by $\|\cdot\|_p$ for any $p \in [1,\infty]$, and $W^{k,p}(\mathbb{R})$ is the corresponding Sobolev space. Fractional order Sobolev spaces (in the literature also called Aronszajn, Gagliardo or Slobodeckij spaces), $W^{s,p}(\mathbb{R})$ with $s \in (0,1)$ and $p \in [1,+\infty)$, we define as follows

$$W^{s,p}(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R})\colon \frac{|f(x)-f(y)|}{|x-y|^{\frac{1}{p}+s}} \in L^p(\mathbb{R}^2) \right\} \tag{1.13}$$

endowed with the natural norm

$$\|f\|_{W^{s,p}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}}. \tag{1.14}$$

In the case when $s > 1$ and is not an integer, we can write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$. Then the space $W^{s,p}(\mathbb{R})$ consists of all functions $u \in W^{m,p}(\mathbb{R})$ whose distributional derivatives $D^\alpha u \in W^{\sigma,p}(\mathbb{R})$ with $|\alpha| = m$. In this paper, however, we shall need only the fractional Sobolev space with the exponent $s \in (0,1)$. Furthermore, using the definition (1.13)-(1.14), it is possible to define the fractional Sobolev space over any open and bounded domain, which is not the case of the definition via the Fourier transform. However, there remains a problem of boundary conditions.

Let us first mention that the $W^{s,p}$-norm with $s \in (0,1)$, defined in (1.14), is equivalent to the following definition (see [54] for the proof)

$$\|f\|_{W^{s,p}(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})} + \|\nabla^s f\|_{L^p(\mathbb{R})}, \tag{1.15}$$

which in some cases is easier to apply.

Additionally, for $s \in (0,1)$ and $p = 2$ the fractional Sobolev space $W^{s,p}(\mathbb{R})$ coincides with the space $H^{s,p}(\mathbb{R})$, see for instance [27, Proposition 3.4], which is not true in the case $p \neq 2$, see [80]. The space $H^{s,p}(\mathbb{R})$ is defined via the Fourier transform

$$H^{s,p}(\mathbb{R}) = \Big\{ f \in L^p(\mathbb{R})\colon \mathcal{F}^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\Big) \in L^p(\mathbb{R}) \Big\} \tag{1.16}$$

supplemented with the norm

$$\|f\|_{H^{s,p}(\mathbb{R})} = \|\mathcal{F}^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L^p(\mathbb{R})}. \tag{1.17}$$

Moreover, $f * g$ denotes the usual convolution, *i.e.*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y.$$

The letter C corresponds to a generic constant (always independent of x and t) which may vary from line to line. Sometimes we write, e.g. $C = C(\alpha, \beta, \ldots)$ to emphasise the dependence of C on particular parameters $\alpha, \beta, \ldots$.

Chapter 2

The Fractional Laplace operator

Diffusion refers to the process by which matter is transported from one part of a system to another as a result of random molecular motion. An example of a phenomenon dominated by diffusion is the Brownian motion first observed in 1827 by a botanist Robert Brown. Later, due to the work of Einstein and Smoluchowski from the beginning of the 20th century, such phenomena like diffusion and Brownian motion can be explained in terms of the heat equation and, in particular, of the Laplace operator.

Recently, however, researchers have started replacing the standard, well-known Laplace operator by its generalisation, namely by the fractional Laplace operator $(-\Delta)^{\frac{s}{2}}$. The aim of so introduced operator is to take into account the so-called long range interactions. A derivation of that operator from a simple random walks with possibly long jumps can be found in, *e.g.* [75].

The new operator does not act by pointwise differentiation but by a global integration with respect to a very singular kernel. It arises in diverse applications such as fluid mechanics, image processing, machine learning, kinetic equations, phase transitions, non-local heat conduction and the peridynamic model for mechanics, see for instance [25] and references therein.

Usually the fractional Laplace operator is defined via the Fourier variables. Namely, if f is in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and $(-\Delta)^{\frac{s}{2}} f = g$, then

$\hat{g} = |\xi|^s \hat{f}$. However, if $0 < s < 2$, then the fractional Laplace operator can be also defined via an integral representation.

Singular operator. Let f be a regular enough function, *e.g.* $f \in \mathcal{S}(\mathbb{R}^n)$, and let $s \in (0, 2)$, then following for instance [47, p. 43], [69, p. 117], we can write

$$(-\Delta)^{\frac{s}{2}} f(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+s}} \, \mathrm{d}y, \tag{2.1}$$

where $C_{n,s}$ is a normalisation constant and is precisely given by

$$C_{n,s} = \frac{2^{s-1} s \Gamma((n+s)/2)}{\pi^{n/2} \Gamma(1 - s/2)} > 0,$$

see [76]. The Cauchy principal value denoted here by $P.V.$ is used to compute values of certain improper integrals with cancellations, which otherwise would diverge because of the singularity in the integrands and is given by

$$P.V. \int_{\mathbb{R}^n} f(x) \, \mathrm{d}x = \lim_{\varepsilon \to 0} \left(\int_{B_\varepsilon(c)} f(x) \, \mathrm{d}x + \int_{\mathbb{R}^n \setminus B_\varepsilon(c)} f(x) \, \mathrm{d}x \right),$$

where f has a singularity at the point $a < x = c < b$.

Furthermore, one can notice that the integral on the right-hand side of (2.1) with f smooth is not really singular near x, provided $s \in (0, 1)$. Indeed, for any smooth enough function f (*e.g.* bounded and globally Lipschitz) and $s \in (0, 1)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+s}} \, \mathrm{d}y &\leqslant C \int_{B_R(x)} \frac{1}{|x - y|^{n+s-1}} \, \mathrm{d}y \\ + \|f\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_R(x)} \frac{1}{|x - y|^{n+s}} \, \mathrm{d}y &= C \left(\int_0^R \frac{1}{r^s} \, \mathrm{d}r + \int_R^\infty \frac{1}{r^{s+1}} \, \mathrm{d}y \right) < \infty. \end{aligned}$$

Additionally, there is also a possibility to interpret the singular integral form (2.1) as a weighted second order differential quotient. Indeed, by changing of variables twice, $y \mapsto y - x$ and $y \mapsto x - y$, one may show that for $f \in \mathcal{S}(\mathbb{R}^n)$, $s \in (0, 2)$ and each $x \in \mathbb{R}^n$ the fractional Laplace operator assumes the following form

$$(-\Delta)^{\frac{s}{2}} f(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{f(x + y) + f(x - y) - 2f(x)}{|y|^{n+s}} \, \mathrm{d}y. \tag{2.2}$$

Notice that, due to the Taylor expansion, we have

$$|f(x+y)+f(x-y)-2f(x)| \leqslant y^2 \max_{|\alpha|=2} \max_{x\in\mathbb{R}^n} |D^\alpha u(x)|$$

and the term under the integral is integrable near 0, see the argument in [27]; therefore, the integral is convergent and there is no need to take the principal value $P.V.$

Riesz transforms. The Riesz potential defines the inverse of the fractional Laplace operator on Euclidean space. Namely, if $s > 0$, then the Riesz potential $\mathcal{I}_s f$ is defined by the convolution, see [53, p. 492] or [69, p. 117], in the following way

$$\Big(\mathcal{I}_s f\Big)(x) = \bar{C}_{n,s} \int_{\mathbb{R}^n} K_{s,n}(x-y) f(y)\, \mathrm{d}y, \tag{2.3}$$

where $K_{s,n}(x)$ is known as the Riesz kernel

$$K_{s,n}(x) = \begin{cases} |x|^{s-n}, & s-n \neq 0,2,4,6,\ldots, \\ |x|^{s-n} \ln \frac{1}{|x|}, & s-n = 0,2,4,6,\ldots, \end{cases}$$

and $\bar{C}_{n,s}$ is the normalization constant [53, p. 490]. For $k = 0,1,2,3,\ldots$, the constant is given by

$$\bar{C}_{n,s} = \begin{cases} \Gamma((n-s)/2)\Big[\pi^{n/2} 2^s \Gamma(s/2)\Big]^{-1}, & s \neq n+2k, \\ \Big[(-1)^{(n-s)/2} \pi^{n/2} 2^{s-1} (\frac{s-n}{2})! \Gamma(s/2)\Big]^{-1}, & s = n+2k. \end{cases}$$

This singular integral is well-defined if f decays rapidly enough at infinity. In particular, for $0 < s < n$, the condition $f \in L^p(\mathbb{R}^n)$ is sufficient due to the relation between norms of f and $\mathcal{I}_s f$ given by the Hardy-Littlewood-Sobolev theorem, recalled below as Theorem 3.12 in dimension one. On the other hand, for $s \geqslant n$ we expect $f \in \mathcal{S}(\mathbb{R}^n)$.

The Fourier multiplier of $\mathcal{I}_s$ is $|\xi|^{-s}$, [69, Lemma 2, p. 117], [53, Lemma 25.2, p. 490]. Therefore, now for any $f \in \mathcal{S}(\mathbb{R}^n)$ we can consider the operator

$$\mathcal{I}_s f = (-\Delta)^{-\frac{s}{2}} f \quad \text{for } s \in (0,n). \tag{2.4}$$

For this reason the fractional Laplace operator is often called the Riesz fractional derivative. Moreover, the value of the Riesz fractional derivative at certain point x is easily computable numerically for $s \in (0,1) \cup (1,2)$, see for instance [56, Chapter 8.2].

Markov semigroups. Fix $0 < \alpha \leqslant 2$ and $n \in \mathbb{N}$. Let us define a new non-negative function

$$p_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} \,\mathrm{d}\xi, \quad x \in \mathbb{R}^n, t > 0,$$

which has the property $\|p_t\|_{L^1} = 1$. Notice that the assumption on α is essential because p_t is not anymore non-negative for $\alpha > 2$.

For each $t \geqslant 0$ and $x \in \mathbb{R}^n$ we can define the Markov semigroup $(P_t, t \geqslant 0)$ (associated with the Lévy process)

$$P_t f(x) = \int_{R^n} p_t(x-y) f(y) \,\mathrm{d}y$$

satisfying the following properties

$$\begin{aligned} P_t \colon L^1 \to L^1, \qquad & \int_{\mathbb{R}^n} P_t u(x) \,\mathrm{d}x = \int u(x) \,\mathrm{d}x, \\ u \geqslant 0 \Rightarrow P_t u \geqslant 0, \qquad & \|P_t u\|_{L^1(\mathbb{R}^n)} \leqslant \|u\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{2.5}$$

Moreover, exploiting the definition of the Fourier transform together with Fubini's theorem, we are able to rephrase the above expression and for $f \in \mathcal{S}(\mathbb{R}^n)$ write it in a more convenient way

$$P_t f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} \hat{f}(\xi) \,\mathrm{d}\xi. \tag{2.6}$$

Now from (2.6) we can derive its generator. It is known [29, Definition 1.2, p. 49] that the infinitesimal generator, denoted by A, of a C_0-semigroup $(T_t, t \geqslant 0)$ is given by the equation: $Af = \lim_{t \searrow 0} \frac{1}{t}(T_t f - f)$ for f in the domain of the operator A, $f \in \mathcal{D}(A)$. Here, in particular, for each $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we can directly compute

$$\begin{aligned} \lim_{t \searrow 0} \frac{1}{t}(P_t f - f)(x) &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} |\xi|^\alpha \hat{f}(\xi) \,\mathrm{d}\xi \\ &= -\mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F} f)(x) = -(-\Delta)^{\frac{\alpha}{2}} f(x). \end{aligned}$$

Thus, we see that the fractional Laplace operator $-(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of the Markov semigroup $(P_t, t \geqslant 0)$.

Properties of fractional derivatives. Now we shall show some properties that are frequently used in further calculations involving a non-local vector operator denoted by ∇^s whose Fourier multiplier is $i\xi|\xi|^{s-1}$.

Lemma 2.1. *Let us assume that $f \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,2)$. Then the following property*

$$(-\Delta)^{\frac{s}{2}} f \stackrel{\text{def}}{=} \Lambda^s f = -\nabla^{\frac{\alpha}{2}} \cdot \nabla^{\frac{\alpha}{2}} f = -\nabla \cdot \nabla^{s-1} f \tag{2.7}$$

holds true with the fractional gradient ∇^{s-1} written as $\nabla \mathcal{I}_{2-s}$.

Proof. Applying the Fourier transform, we have

$$(-\Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f) = \mathcal{F}^{-1}(-i\xi \cdot i\xi|\xi|^{s-2}\mathcal{F} f) = -\nabla \cdot \nabla^{s-1} f,$$

where in the last equality we use the definition of the fractional gradient in the Fourier variables. ■

Let us first emphasise here that notations for fractional gradients are consistent with the usual gradient: $\nabla^1 = \nabla$. Furthermore, ∇^0 is the Riesz transform and in dimension one is the inverse Hilbert transform. Moreover, due to its definition given in the Fourier variables, we can rewrite it for $s \in (0,2)$ as $\nabla^{s-1} f = \nabla \mathcal{I}_{2-s} f$. Thus, for smooth and bounded functions the fractional gradient can be expressed via the following singular integral formula

$$\nabla^{s-1} f(x) = \tilde{C}_{n,s} \int_{\mathbb{R}^n} \Big(f(x) - f(x+z) \Big) \frac{z}{|z|^{n+s}} \, \mathrm{d}z, \tag{2.8}$$

with some constant $\tilde{C}_{n,s} = \frac{\Gamma((n+s)/2)}{2^{1-s}\pi^{n/2}\Gamma((2-s)/2)} > 0$.

Lemma 2.2. *Let us assume that $f \in \mathcal{S}(\mathbb{R})$. Then the following property*

$$(-\Delta)^{\frac{1}{2}} f = \mathcal{H} f' \tag{2.9}$$

holds true. By $\mathcal{H}$ we denote the Hilbert transform defined as

$$\mathcal{H} f(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y} \, \mathrm{d}y.$$

Chapter 3

Existence of solutions

There are many concepts of defining generalised solutions that occur naturally. Of course, it is preferable to find classical solutions, but it might be difficult to prove their existence. Hence, one may weaken the notion of solutions so that the existence would be easier to prove. However, the question of uniqueness of such solutions is often difficult to answer in a satisfactory way.

One possible way of getting a definition of generalised solutions is to multiply the equation by a suitably smooth test function and integrate by parts some of the terms. This brings us to the question of regularity of solutions that allows the expression to make sense and leads to the definition of a *weak solution*.

Definition 3.1 (Weak solutions)**.** *Let $Q_T = \mathbb{R} \times (0,T)$. Assume that $v \in L^1(Q_T)$, $|v|\nabla^{\alpha-1}v \in L^1_{\text{loc}}(Q_T)$, $v \in L^\infty((0,T), W^{(\alpha-1)_+,p}(\mathbb{R}))$. A function $v : Q_T \to \mathbb{R}$ is called a weak solution of the problem* (1.7) *if v satisfies*

$$\int_0^T \int_{\mathbb{R}} \Big(- v\phi_t + |v|\nabla^{\alpha-1}v\phi_x - \sigma|v|\phi_x \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} v_0\phi(\cdot,0)\,\mathrm{d}x = 0, \quad (3.1)$$

for any test function $\phi \in C^\infty(Q_T) \cap C(\overline{Q_T})$ such that ϕ has compact support in the space variable x and vanishes near $t = T$.

For arbitrary initial data, the results obtained in Theorem 3.2 are similar to the ones in the paper of Biler, Imbert, Karch [10]. However, for the regularised problem (3.3) with zero-integral initial data (i.e. $\int v_0 = 0$) we expect a better decay rate as in the case of the heat equation, where the L^2-norm of solutions decays like $t^{-\frac{3}{4}}$ instead of $t^{-\frac{1}{4}}$. In order to improve the decay of the L^2-norm of approximating solutions we exploited an analogue of the Nash inequality involving the L^2-norm of the first moment of solutions. It appeared that this approach, however, does not lead to a better result. Moreover, such an improvement is not directly possible for the original problem (1.7) since the better decay of the approximate solutions would depend evidently on ε.

Theorem 3.2 (Existence and decay of L^p-norms)**.** *Let $\alpha \in (0, 2)$ and $\sigma \in \mathbb{R}$. Given $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} v_0 \, \mathrm{d}x = 0$ there exists a global-in-time weak solution v of the problem* (1.7)*. This solution v satisfies $\int_{\mathbb{R}} v(x,t) \, \mathrm{d}x = 0$ for all $t > 0$.*

Moreover, for any $1 \leqslant p \leqslant \infty$, the following decay estimate of L^p-norms of each solutions v of (1.7) *holds true*

$$\|v(t)\|_p \leqslant C_\alpha t^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)} \|v_0\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} \tag{3.2}$$

for any $t > 0$ and a constant C_α independent of p.

The proof of Theorem 3.2 is based on [10] and will be split into two parts. In the first part, we consider approximating solutions of the regularised problem, *cf.* (3.3), and we show global-in-time existence and uniqueness of solutions. Afterwards, in Section 3.5, we shall pass to the limit with the parameter $\varepsilon \searrow 0$, and we shall see that the limit satisfies (1.7) in a weak sense according to Definition 3.1.

3.1 Approximate equation and hypercontractivity estimates

A very frequent approach to construct solutions of the Cauchy problem for non-linear parabolic equations is the vanishing viscosity method. It consists of approximation of the solution u of a problem $L(u) = 0$ by solutions u^ε of a more regular problem $L_\varepsilon(u^\varepsilon) = 0$. The key idea of the method is to add a small amount of diffusion, namely $\varepsilon\Delta u$, which makes the regularised equation act somewhat like the heat equation; thus, the solution u^ε is infinitely differentiable despite the non-linearity. Moreover, proving the existence of solutions of the regularised problem is often much easier because the viscous term prevents formation of singularities that would occur in the case $\varepsilon = 0$. We expect that vanishing viscosity technique should allow us to recover the correct solution of the original problem as $\varepsilon \searrow 0$. This idea comes from the analysis of the most classical example of the Burgers equation $u_t = \varepsilon u_{xx} + uu_x$.

In order to prove existence of weak solutions of (1.7), we define first the following regularised problem

$$v_t = \varepsilon v_{xx} - \frac{\partial}{\partial x}\Big(|v|\Big(\sigma - \nabla^{\alpha-1}v\Big)\Big), \tag{3.3a}$$

$$v(x,0) = v_0(x) \quad \text{for} \quad x \in \mathbb{R}, \tag{3.3b}$$

with a fixed $\varepsilon > 0$.

Remark 3.3. *A function $v = v(x,t)$ satisfies the problem* (3.3a)-(3.3b) *in a weak sense if the following relation*

$$\iint \Big(-v\phi_t + \varepsilon v_x\phi_x + |v|\nabla^{\alpha-1}v\phi_x - \sigma|v|\phi_x\Big)\,\mathrm{d}x\,\mathrm{d}t + \int v_0\phi(\cdot,0)\,\mathrm{d}x = 0 \tag{3.4}$$

holds for all test functions $\phi \in C^\infty(Q_T) \cap C(\overline{Q_T})$ such that ϕ has compact support in the space variable x and vanishes near $t = T$.

Since at that moment we do not have any information concerning existence of solutions of such regularised problem, all decay estimates presented below are formal. However, they are needed to prove the existence of global-in-time solutions which will be constructed by considering the so-called mild solutions, see Section 3.3 and Section 3.4.

Theorem 3.4 (Decay of L^p-norms of solutions). *Assume that $v = v(x,t)$ is a solution of the regularised problem* (3.3). *Suppose also that $v_0 \in L^1(\mathbb{R})$.*

Then for every $p \in [1,\infty]$ there exists a constant $C_{p,\alpha}$ such that the following inequality

$$\|v(t)\|_p \leqslant C_{p,\alpha} t^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)} \|v_0\|_1^{\frac{p\alpha+1}{p(\alpha+1)}} \tag{3.5}$$

holds true for all $t > 0$. Moreover, for all $t > 0$ the following relation is satisfied

$$\int_{\mathbb{R}} v(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} v_0(x)\,\mathrm{d}x, \tag{3.6}$$

which means that mass is preserved.

Remark 3.5. *Notice that for $p = 1$ the inequality* (3.5) *reads as $\|v(t)\|_1 \leqslant \|v_0\|_1$. In particular, this means that a solution $v_0 \mapsto v(t)$ defines a non-linear semigroup for which there is a regularisation effect known for the semigroup generated by $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$, see the "hypercontractivity" estimate* (3.5)*.*

Before we go further, we recall some useful functional inequalities which will be used in the proof of Theorem 3.4. They generalise well-known Nash, Gagliardo-Nirenberg etc. inequalities to the case of fractional derivatives considered in this work.

Lemma 3.6 (Nash inequality)**.** *Let $\alpha > 0$. Assume that $w \in L^1(\mathbb{R})$ and $\nabla^{\frac{\alpha}{2}} w \in L^2(\mathbb{R})$, then there exists a constant $C_N > 0$ such that*

$$\|w\|_2^{2(1+\alpha)} \leqslant C_N \|\nabla^{\frac{\alpha}{2}} w\|_2^2 \, \|w\|_1^{2\alpha} \tag{3.7}$$

holds true.

Proof. For every $R > 0$ we can write the L^2-norm of w splitting it in an appropriate manner

$$\begin{aligned}\|w\|_2^2 = \|\widehat{w}\|_2^2 &= \int\limits_{|\xi|\leqslant R} |\widehat{w}(\xi)|^2\,\mathrm{d}\xi + \int\limits_{|\xi|>R} |\widehat{w}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \int\limits_{|\xi|\leqslant R} |\widehat{w}(\xi)|^2\,\mathrm{d}\xi + R^{-\alpha} \int\limits_{|\xi|>R} |\xi|^\alpha |\widehat{w}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant C\|w\|_1^2 R + CR^{-\alpha}\|\nabla^{\frac{\alpha}{2}} w\|_2^2\end{aligned}$$

Now choosing the optimal $R = \left(\frac{\|\nabla^{\frac{\alpha}{2}} w\|_2^2}{\|w\|_1^2}\right)^{\frac{1}{1+\alpha}}$ we arrive at (3.7). ■

The proof of the classical Nash inequality, *i.e.* involving the usual gradient instead of the fractional one, can be found in a book [51] (or originally in [18]), where the authors present a different approach to prove the inequality.

Lemma 3.7 (Gagliardo-Nirenberg type inequality)**.** *Let $p \in (1, \infty)$ and $\alpha > 0$ be fixed. For all $v \in L^1(\mathbb{R})$ such that $\nabla^{\frac{\alpha}{2}} |v|^{\frac{p+1}{2}} \in L^2(\mathbb{R})$, the following inequality is valid*

$$\|v\|_p^a \leqslant C_N \left\| \nabla^{\frac{\alpha}{2}} |v|^{\frac{p+1}{2}} \right\|_2^2 \|v\|_1^b, \tag{3.8}$$

with

$$a = \frac{p(p+\alpha)}{p-1}, \quad b = \frac{p\alpha + 1}{p-1}. \tag{3.9}$$

Proof. Let us first define a new function w as follows: $w = |v|^{p+1}$. Now for such a defined function we apply a Nash-type inequality (3.7) obtaining

$$\|v\|_{p+1}^{(p+1)(1+\alpha)} \leqslant C_N \left\| \nabla^{\frac{\alpha}{2}} |v|^{\frac{p+1}{2}} \right\|_2^2 \|v\|_{\frac{p+1}{2}}^{(p+1)\alpha}. \tag{3.10}$$

Next, using the following interpolation of Hölder inequalities

$$\|v\|_p \leqslant \|v\|_{p+1}^{\gamma} \|v\|_1^{1-\gamma} \quad \text{with } \gamma = \frac{(p-1)(p+1)}{p^2},$$
$$\|v\|_{\frac{p+1}{2}} \leqslant \|v\|_p^{\delta} \|v\|_1^{1-\delta} \quad \text{with } \delta = \frac{p}{p+1}$$

in (3.10) and carefully collecting all the parameters together we arrive at

$$\|v\|_p^a \leqslant C_N \left\| \nabla^{\frac{\alpha}{2}} |v|^{\frac{p+1}{2}} \right\|_2^2 \|v\|_1^b,$$

with coefficients

$$a = \frac{(p+1)(\alpha+1)}{\gamma} - (p+1)\alpha\delta = \frac{p^2}{p-1}(\alpha+1) - \alpha p = \frac{p(p+\alpha)}{p-1}$$

and

$$b = (1-\delta)(p+1)\alpha + \frac{1-\gamma}{\gamma}(p+1)(\alpha+1) = -\alpha p + \frac{(\alpha+1)p^2}{p-1} - (p+1) = \frac{\alpha p + 1}{p-1}$$

which ends the proof. ■

The usual integration by parts, extensively used in the derivation of *a priori* estimates for PDE's, should be replaced by the following inequalities involving fractional powers of the Laplace operator

Lemma 3.8 (Kato inequality). *Let $0 < \alpha < 2$. Then for all $w \in L^1(\mathbb{R})$ such that $\Lambda^\alpha w \in L^1(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} (\Lambda^\alpha w) \operatorname{sgn} w \, \mathrm{d}x \geqslant 0. \tag{3.11}$$

Proof. Since, as mentioned in Chapter 2, the operator $-\Lambda^\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ generates the Markov semigroup, we can write

$$\int_{\mathbb{R}} (\Lambda^\alpha w) \operatorname{sgn} w \, \mathrm{d}x = -\int_{\mathbb{R}} \left(\lim_{t \searrow 0} \frac{P_t w - w}{t} \right) \operatorname{sgn} w \, \mathrm{d}x$$
$$= \lim_{t \searrow 0} \frac{1}{t} \int_{\mathbb{R}} \Big(|w| - P_t w \operatorname{sgn} w \Big) \, \mathrm{d}x \geqslant \lim_{t \searrow 0} \int_{\mathbb{R}} \frac{|w| - |w|}{t} \, \mathrm{d}x = 0.$$

In order to justify the inequality let $w = w_+ - w_-$. Then $P_t w = P_t w_+ - P_t w_-$. Applying the sign preservation property and the contraction property (2.5) we obtain the desired result. An original proof can be found in [52]. ∎

Lemma 3.9 (Stroock-Varopoulos inequality). *Let $0 < \alpha < 2$. For every $q > 1$, we have*

$$\int_{\mathbb{R}} (\Lambda^\alpha w) |w|^{q-2} w \, \mathrm{d}x \geqslant \frac{4(q-1)}{q^2} \int_{\mathbb{R}} \left(\nabla^{\frac{\alpha}{2}} |w|^{\frac{q}{2}} \right)^2 \mathrm{d}x \tag{3.12}$$

for all $w \in L^q(\mathbb{R})$ such that $\Lambda^\alpha w \in L^q(\mathbb{R})$.

In the proof the operator Λ^α is taken as a limit of an appropriate difference quotient for the generator of a Markov semigroup, see for instance [52].

Proof of Theorem 3.4. The proof consists of several steps.

Step 1: L^1-estimates. First we prove the L^1-estimate

$$\|v(t)\|_1 \leqslant \|v_0\|_1 \tag{3.13}$$

(see (3.5) for $p = 1$). To this end, we multiply the equation (3.3a) by $\operatorname{sgn} v$ and integrate over $\mathbb{R}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |v| \,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} v_{xx} \operatorname{sgn} v \,\mathrm{d}x - \sigma \int_{\mathbb{R}} (|v|)_x \operatorname{sgn} v \,\mathrm{d}x + \int_{\mathbb{R}} \Big(|v|\nabla^{\alpha-1}v\Big)_x \operatorname{sgn} v \,\mathrm{d}x. \quad (3.14)$$

The first term on the right-hand side is non-positive due to the Kato inequality (3.11) with $\alpha = 2$. Thus, it can be skipped. Now using a smooth approximation of the sign function by, *e.g.* $\operatorname{sgn}_\mu(v) = v(v^2+\mu)^{-\frac{1}{2}}$, integrating by parts and passing to the limit with $\mu \searrow 0$, one can show that the last two quantities of the above equality are zero. This implies (3.5) for $p = 1$.

Step 2: L^p-estimates. Next, assuming $p > 1$ we multiply (1.7a) by $|v|^{p-2}v$ and integrate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |v|^p \,\mathrm{d}x = -\varepsilon(p-1)\int_{\mathbb{R}} v_x^2 |v|^{p-2} \,\mathrm{d}x - \int_{\mathbb{R}} \Big(\sigma|v|\Big)_x |v|^{p-2}v \,\mathrm{d}x + \int_{\mathbb{R}} \Big(|v|\nabla^{\alpha-1}v\Big)_x |v|^{p-2}v \,\mathrm{d}x. \quad (3.15)$$

One can notice that the first term on the right-hand side is non-positive; hence, we can skip it. Now integrating by parts and after simple algebraic manipulations one can show that the second term on the right-hand side is zero. Indeed,

$$-\int_{\mathbb{R}} \Big(\sigma|v|\Big)_x |v|^{p-2}v \,\mathrm{d}x = \sigma\int_{\mathbb{R}} |v|\Big(|v|^{p-2}v\Big)_x \,\mathrm{d}x = \sigma\frac{p-1}{p}\int_{\mathbb{R}} \Big(|v|^{p-1}v\Big)_x \,\mathrm{d}x = 0.$$

The last quantity on the right-hand side of (3.15) can be transformed, integrating by parts twice and using the identity (2.7), into

$$\int_{\mathbb{R}} \Big(|v|\nabla^{\alpha-1}v\Big)_x |v|^{p-2}v \,\mathrm{d}x = \frac{p-1}{p}\int_{\mathbb{R}} \Big(\nabla^{\alpha-1}v\Big)_x |v|^{p-1}v \,\mathrm{d}x = -\frac{p-1}{p}\int_{\mathbb{R}} \Big(\Lambda^\alpha v\Big)|v|^{p-1}v \,\mathrm{d}x. \quad (3.16)$$

Up to now all the calculations are formal. In order to make them rigorous, one should multiply an equation (1.7a) not by $|v|^{p-2}v$ but rather by its regular counterpart: $\operatorname{sgn} v\Big((v^2+\varepsilon^2)^{\frac{p-1}{2}} - \varepsilon^{p-1}\Big)$ as has been done in [10].

Consequently, by the Stroock-Varopoulos inequality (3.12) with $q = p+1$ applied to (3.16), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_p^p \leqslant -\frac{4p(p-1)}{(p+1)^2}\left\|\nabla^{\frac{\alpha}{2}}|v|^{\frac{p+1}{2}}\right\|_2^2. \tag{3.17}$$

Thus, by Lemma 3.7 combined with the L^1-estimate (3.13), we arrive at the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_p^p \leqslant -\frac{4p(p-1)}{(p+1)^2}C_N^{-1}\|v_0\|_1^{-\frac{p\alpha+1}{p-1}}\|v\|_p^{\frac{p(p+\alpha)}{p-1}}. \tag{3.18}$$

The above differential inequality can be easily solved setting $f(t) = \|v(t)\|_p^p$ and solving the following ordinary differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leqslant -Kf^{\frac{p+\alpha}{p-1}},$$

with a constant $K > 0$. Integrating this over 0 and t, we get

$$-\frac{p-1}{\alpha+1}\int_0^t f(s)^{-\frac{\alpha+1}{p-1}}\,\mathrm{d}s \leqslant -Kt.$$

Hence, any non-negative solution has to satisfy the algebraic decay

$$f(t) \leqslant \left(\frac{K(1+\alpha)}{p-1}\right)^{-\frac{p-1}{\alpha+1}} t^{-\frac{p-1}{\alpha+1}}.$$

This implies (3.5) for each $p \in (1,\infty)$ with the constant

$$C_{p,\alpha} = \left(\frac{4p(\alpha+1)}{(p+1)^2}C_N^{-1}\right)^{-\frac{1}{\alpha+1}\left(1-\frac{1}{p}\right)}. \tag{3.19}$$

Here C_N is a constant appearing in the Gagliardo-Nirenberg inequality, see Lemma 3.7.

Step 3: Improvement of the constants. Let us notice here that we cannot simply pass to the limit directly in (3.5) since the constant $C_{p,\alpha} \to \infty$ as $p \to \infty$ even in the simplest case of the linear heat equation. Therefore, the case $p = \infty$ has to be treated more carefully. We shall use the Moser-Alikakos technique of estimating the L^p-norms with $p = 2^n$ recursively in order to improve the constant and to handle the limit case. We refer the reader to *e.g.* [5, 38, 45], where this technique is extensively used and to [4], where the essential features of the method were originally introduced.

Our starting point is the already obtained estimate (3.5) with $n = 1$ ($p = 2$). By a constant C_1 we denote $C_1 = C_{2,\alpha} = \left(\frac{8}{9}(\alpha+1)C_N^{-1}\right)^{-\frac{1}{2(\alpha+1)}}$.

Suppose now that we have the following inequality for $p = 2^n$

$$\|v(t)\|_{2^n} \leqslant C_n t^{-\frac{1}{\alpha+1}(1-2^{-n})} \|v_0\|_1^{\frac{2^n\alpha+1}{2^n(\alpha+1)}}. \tag{3.20}$$

Let us consider (3.17) with $p = 2^{n+1}$. Then we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{2^{n+1}}^{2^{n+1}} &\leqslant -K\|v(t)\|_{2^{n+1}+1}^{(2^{n+1}+1)(\alpha+1)} \|v(t)\|_{\frac{2^{n+1}+1}{2}}^{-\alpha(2^{n+1}+1)} \\ &\leqslant -K\|v(t)\|_{2^{n+1}}^{2^{n+1}(1+\alpha+2^{-n})} \|v(t)\|_{2^n}^{-2^n(2\alpha+2^{-n})} \\ &\leqslant -\kappa\|v(t)\|_{2^{n+1}}^{2^{n+1}(1+\alpha+2^{-n})} t^{\frac{2^n}{\alpha+1}(1-2^{-n})(2\alpha+2^{-n})}, \end{aligned} \tag{3.21}$$

with the constants

$$\kappa = KC_n^{-2^n(2\alpha+2^{-n})} \|v_0\|_1^{\frac{2^n\alpha+1}{2^n(\alpha+1)}2^n(2\alpha+2^{-n})} \quad \text{and} \quad K = \frac{2^{n+3}(2^{n+1}-1)}{(2^{n+1}+1)^2 C_N}.$$

In order to get the first inequality we use (3.10) with $p = 2^n$. Next, using the following interpolation of Hölder inequalities

$$\|v\|_{2^{n+1}} \leqslant \|v\|_{2^{n+1}+1}^{\gamma} \|v\|_{2^n}^{1-\gamma} \quad \text{with } \gamma = \frac{2^{n+1}+1}{2^{n+1}+2},$$
$$\|v\|_{\frac{2^{n+1}+1}{2}} \leqslant \|v\|_{2^{n+1}}^{\delta} \|v\|_{2^n}^{1-\delta} \quad \text{with } \delta = \frac{2}{2^{n+1}+1}$$

and collecting carefully all the indices together, we obtain the second inequality. Finally, we reach the third inequality in (3.21) using the inductive hypothesis (3.20) to estimate the L^{2^n}-norm of $v(t)$. Now letting $f(t) = \|v(t)\|_{2^{n+1}}^{2^{n+1}}$, we can transform (3.21) into the differential inequality for $f(t)$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t)^{\alpha+2^{-n}} \geqslant \frac{K(\alpha+2^{-n})}{C_n^{2^n(2\alpha+2^{-n})}} \|v_0\|_1^{\frac{2^n\alpha+1}{2^n(\alpha+1)}2^n(2\alpha+2^{-n})} t^{\frac{2^n}{\alpha+1}(1-2^{-n})(2\alpha+2^{-n})},$$

which can be solved integrating it over $[0, t]$. Hence, we obtain

$$\|v(t)\|_{2^{n+1}} \leqslant C_{n+1} t^{\frac{1}{\alpha+1}(1-2^{-n-1})} \|v_0\|_1^{\frac{2^{n+1}\alpha+1}{2^{n+1}(\alpha+1)}}, \tag{3.22}$$

with the constant

$$C_{n+1} = \left(\frac{\frac{2^n}{\alpha+1}(1-2^{-n})(2\alpha+2^{-n})+1}{K(\alpha+2^{-n})} C_n^{2^n(2\alpha+2^{-n})} \right)^{\frac{2^{-n-1}}{\alpha+2^{-n}}}. \tag{3.23}$$

The inequality (3.22) is nothing else but (3.5) for any $p = 2^{n+1}$ and $n \in \mathbb{N}$.

Step 4: Boundedness of the constants. In order to show that the constant C_n is bounded it is enough to prove that $\log C_n$ is bounded. Thus, we write

$$\log C_{n+1} \leqslant A_n + B_n \log C_n,$$

where, for $n \geqslant 1$, the constants are

$$\begin{aligned} A_n &= \frac{1}{\alpha 2^{n+1}+2} \log \left(\frac{\frac{2^n}{\alpha+1}(1-2^{-n})(2\alpha+2^{-n})+1}{K(\alpha+2^{-n})} \right) \leqslant C\frac{n}{2^n}, \\ B_n &= (1 - \frac{1}{\alpha 2^{n+1}+2}) \leqslant 1 - \frac{1}{4\cdot 2^n} < 1. \end{aligned}$$

Using the fact that $B_n \leqslant 1$, we can conclude

$$\lim_{n\to\infty} \log C_n \leqslant \sum_{n\geqslant 1} A_n + \log C_1 \prod_{n\geqslant 1} B_n < \infty.$$

Hence, the constant C_n itself is bounded.

Step 5: Conclusions. The Hölder inequality

$$\|v\|_p \leqslant \|v\|_{2^n}^{\frac{2^{n+1}}{p}-1} \|v\|_{2^{n+1}}^{2-\frac{2^{n+1}}{p}}$$

completes the proof for every $p \in (2^n, 2^{n+1})$. ■

3.2 Mild solutions

The concept of mild solutions comes from the non-linear semigroup theory. When an operator $A\colon \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ generates a strongly continuous C_0-semigroup $T(t)$, one can define another abstract version of generalised solutions constructed via the integral representation. Indeed, notions of mild solutions were introduced by Browder [14] and Kato [46] in the early 60's of the 20th century. The idea is to construct the solution of the corresponding general nonhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \\ u(0) = u_0, \end{cases}$$

as a solution of the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\,\mathrm{d}s$$

whenever the integral does make sense. In other words, to search for the fixed point of the following transform

$$v \mapsto T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\,\mathrm{d}s.$$

This is the so-called Picard iterative method already used by Oseen [58] at the beginning of the 20th century in order to establish the (local) existence of a classical solution to the Navier-Stokes equations for a regular initial value. Later, at the turn of the 20th and 21st century, the framework of mild solutions was widely exploited in the theory of Navier-Stokes equations by, *e.g.* Cannone [17], Meyer [55], Lemarié-Rieusset [48].

In the sequel we are mainly interested in studying *mild solutions* of the regularised problem. To this end, we need to first define what it means to be a mild solution of our problem. Thus, we start with the following definition.

Definition 3.10 (Mild solutions). *A function $u \in C([0,T], X)$ is called a mild solution (or equivalently: solution in the sense of theory of semi-groups) to the initial value problem* (3.3) *if the following integral equation (often called the Duhamel formula)*

$$v(t) = G(\cdot, \varepsilon t) * v_0 + \int_0^t \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) * \Big(|v(s)|\nabla^{\alpha-1} v(s)\Big)\,\mathrm{d}s - \int_0^t \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s))\Big(\sigma|v(s)|\Big)\,\mathrm{d}s. \quad (3.24)$$

is satisfied.

The main problem in such an approach is that we should find an appropriate space X and we do not know a priorily what the relations of mild solutions to weak and strong ones are. In fact, any classical solution is a mild solution; however, the reverse implication is not always true.

In our definition of mild solutions by $G(x,t)$ we denote the heat kernel $G(x,t) = (4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right)$. Therefore, let us recall the well known estimates for the heat kernel, which will be frequently used from now on. The proof of the following Lemma is a direct consequence of the Young inequality for the convolution.

Lemma 3.11. *Let $1 \leqslant q \leqslant p \leqslant +\infty$, and let $f \in L^q(\mathbb{R})$. Then the estimates*

$$\|G(\cdot,\varepsilon t) * f\|_p \leqslant Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_q, \tag{3.25}$$

$$\left\|\frac{\partial}{\partial x}G(\cdot,\varepsilon t) * f\right\|_p \leqslant Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|f\|_q \tag{3.26}$$

hold true for some positive constants $C = C(p,q,\varepsilon)$.

Proof. First, we shall prove the first inequality. From the Young inequality for the convolution, see [51], we get $\|G(\cdot,\varepsilon t) * f\|_p \leqslant \|G(\cdot,\varepsilon t)\|_r\|f\|_q$, where p,q,r are connected with the following relation $\frac{1}{p}+1 = \frac{1}{r}+\frac{1}{q}$. Therefore, it is enough to calculate the L^r-norm of the heat kernel. Namely, for $1 \leqslant r < \infty$ we obtain

$$\begin{aligned}\|G(\cdot,\varepsilon t)\|_r^r &= (4\pi\varepsilon t)^{-\frac{r}{2}} \int_{\mathbb{R}} e^{-\frac{rx^2}{4\varepsilon t}}\,\mathrm{d}x \\ &= (4\pi\varepsilon t)^{-\frac{r}{2}}(4\varepsilon t)^{\frac{1}{2}}r^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-z^2}\,\mathrm{d}z = (4\pi\varepsilon t)^{\frac{1}{2}(1-r)}r^{-\frac{1}{2}}.\end{aligned}$$

The second equality of the above calculations we obtain by a simple change of variables, $z = \frac{\sqrt{r}x}{\sqrt{4\varepsilon t}}$. This implies (3.25) with the constant

$$C = (4\pi\varepsilon)^{-\frac{1}{2}\left(\frac{1}{r}-1\right)}r^{-\frac{1}{2r}}. \tag{3.27}$$

One can notice that $C \geqslant 1$, and the equality is satisfied only for $p = q$. On the other hand, for $r = \infty$ (then $p = \infty$ and $q = 1$) the calculations are even simpler. Indeed,

$$\|G(\cdot,\varepsilon t)\|_\infty = (4\pi\varepsilon t)^{-\frac{1}{2}} \sup_{x\in\mathbb{R}} \exp\left(-\frac{x^2}{4\varepsilon t}\right) \leqslant (4\pi\varepsilon t)^{-\frac{1}{2}}.$$

To prove the second inequality we again use the Young inequality for the convolution. This leads us to calculate the L^r-norm of the derivative of the heat kernel. Namely, let assume for a moment that $1 \leqslant r < \infty$, then we have

$$\begin{aligned}\left\|\tfrac{\partial}{\partial x}G(\cdot,\varepsilon t)\right\|_r^r &= \int_{\mathbb{R}} \left|(4\pi\varepsilon t)^{-\frac{1}{2}} e^{-\frac{x^2}{4\varepsilon t}}\,\tfrac{2x}{4\varepsilon t}\right|^r \,\mathrm{d}x \\ &= (\varepsilon t)^{-r}(\varepsilon t)^{\frac{1}{2}} \int_{\mathbb{R}} \left|(4\pi)^{-\frac{1}{2}} e^{-y^2/4}\, y/2\right|^r \,\mathrm{d}y = (\varepsilon t)^{-r}(\varepsilon t)^{\frac{1}{2}} \left\|\tfrac{\partial}{\partial x}G(\cdot,1)\right\|_r^r \\ &= Ct^{r(-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2})}.\end{aligned}$$

The second equality has been obtained by a simple change of variables, $y = \frac{x}{\sqrt{\varepsilon t}}$. Now let us assume that $p = \infty$ and $q = 1$ since, $r = \infty$ only for such chosen p, q. Meanwhile, changing variables, *i.e.* $z = \frac{x}{\sqrt{4\varepsilon t}}$, we get

$$\begin{aligned}\left\|\tfrac{\partial}{\partial x}G(\cdot,\varepsilon t)\right\|_\infty &= (4\pi\varepsilon t)^{-\frac{1}{2}} \sup_{x\in\mathbb{R}} \left|\exp\left(-\frac{x^2}{4\varepsilon t}\right)\frac{x}{2\varepsilon t}\right| \\ &= (4\pi\varepsilon t)^{-\frac{1}{2}}(\varepsilon t)^{-\frac{1}{2}} \sup_{z\in\mathbb{R}} \left|\exp(-z^2)z\right|.\end{aligned}$$

Fortunately, $z\exp(-z^2)$ is a bounded function in $z \geqslant 0$ which attains its maximum $\frac{1}{\sqrt{2e}}$ at $z = \frac{1}{\sqrt{2}}$. Thus, we obtain (3.26). ■

Let us also recall inequalities for the Riesz potential $\mathcal{I}_\beta = (-\Delta)^{-\frac{\beta}{2}}$, which show a smoothing effect of those singular integrals.

Theorem 3.12 (Hardy-Littlewood-Sobolev inequality)**.** *Let $\beta \in (0,1)$ and let $1 < p < q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \beta$. Then for $f \in L^p(\mathbb{R})$ the following inequality*

$$\|\mathcal{I}_\beta f\|_q \leqslant C\|f\|_p \tag{3.28}$$

is satisfied with a constant $C = C(p, q, \beta)$.

In our problem (1.7) a slightly different operator, namely $\nabla^{\alpha-1} = \nabla\mathcal{I}_{2-\alpha}$, appears. Thus, by Theorem 3.12 we obtain the following inequality

$$\|\nabla^{\alpha-1}f\|_q \leqslant C(p,q,\alpha)\|f\|_p \tag{3.29}$$

for all $\alpha \in (0,1)$ and $1 < p < q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - 1 + \alpha$. One can notice that the inequality (3.29) is valid if, for $\alpha \in (0,1]$, we rewrite the operator $\nabla^{\alpha-1}$ as $\nabla^{\alpha-1} = \nabla^0 \mathcal{I}_{1-\alpha}$, where the components of ∇^0 are the Riesz transforms.

The proof of Theorem 3.12 can be found, *e.g.* in [69, p. 120].

In order to prove Theorems 3.16 and 3.18 below, it is a standard practice to show first the local-in-time existence of mild solutions and then continue them to the global-in-time ones. It is useful to recall the following abstract approach (extending the so-called Kato approach) proposed by Meyer [55] as well as its generalisations.

Lemma 3.13. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a bilinear form satisfying*

$$\|B(x_1, x_2)\|_{\mathcal{X}} \leqslant C\|x_1\|_{\mathcal{X}}\|x_2\|_{\mathcal{X}} \tag{3.30}$$

for some $C > 0$ and all $x_1, x_2 \in \mathcal{X}$.

Then for all $y \in \mathcal{X}$ such that $4C\|y\|_{\mathcal{X}} < 1$ there exists a solution $x \in \mathcal{X}$ of the equation $x = y+B(x, x)$ satisfying the estimate $\|x\|_{\mathcal{X}} \leqslant 2\|y\|_{\mathcal{X}}$. Moreover, the solution is unique in the ball in $\mathcal{X}$ of radius $\frac{2}{\|y\|}$.

The above Lemma gives the existence and the uniqueness of the solution via a contraction mapping argument. Its proof can be found, for instance in [55] or [48, Theorem 13.2]. Such a scheme is as well applicable in different models, for instance chemotaxis [62].

For our problem, however, this Lemma could be used in a direct way only in the case where the value of the parameter σ is 0 and dislocations of only one type are considered (*i.e.* the term $|v|$ is replaced by v in (1.7) and (3.3)). Therefore, it is more appropriate to use a modification of the above lemma, where the form B is not necessary bilinear but is of the special form which still allows us to adopt a contraction mapping argument by using simple absolute value bounds under the integrals defining B.

Lemma 3.14. *Let $\mathcal{X} = \{x : \langle x, \nabla^s x\rangle \in \mathcal{X}_1 \times \mathcal{X}_2\}$ be a Banach space supplemented with the norm $\|x\|_{\mathcal{X}} = \|x\|_{\mathcal{X}_1} + \|\nabla^s x\|_{\mathcal{X}_2}$, with Banach spaces $\mathcal{X}_1, \mathcal{X}_2$*

and a parameter $s \in (0,1)$. Moreover, let $B\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ be an operator of the form $B(x_1,x_2) = \int G_1 * \Big(|x_1| G_2 * x_2\Big)$. Let also $\bar{B}\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}_1$ and $\tilde{B}\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}_2$ be continuous, linear (or at least linear with respect to the first variable) forms such that

$$\begin{aligned}
&\|B(x_1,x_2)\|_{\mathcal{X}} \leqslant \|\bar{B}(|x_1|,x_2)\|_{\mathcal{X}_1} + \|\tilde{B}(|x_1|,x_2)\|_{\mathcal{X}_2} \leqslant C\|x_1\|_{\mathcal{X}}\|x_2\|_{\mathcal{X}},\\
&\|\nabla^s(B(x_1,x_1)-B(x_2,x_2))\|_{\mathcal{X}_2} \leqslant \|\tilde{B}(|x_1|,x_1-x_2)\|_{\mathcal{X}_2} + \|\tilde{B}(|x_1-x_2|,x_2)\|_{\mathcal{X}_2},\\
&\|B(x_1,x_1)-B(x_2,x_2)\|_{\mathcal{X}_1} \leqslant \|\bar{B}(|x_1|,x_1-x_2)\|_{\mathcal{X}_1} + \|\bar{B}(|x_1-x_2|,x_2)\|_{\mathcal{X}_1}
\end{aligned} \tag{3.31}$$

for some $C>0$, all $x_1,x_2\in\mathcal{X}$ and some kernels G_1, G_2.

Let $L\colon \mathcal{X}\to\mathcal{X}$ be an operator of the form $L(x) = \int G_3 * |x|$. Let $\bar{L}\colon \mathcal{X}\to\mathcal{X}_1$ and $\tilde{L}\colon \mathcal{X}\to\mathcal{X}_2$ be two continuous, linear operators such that

$$\begin{aligned}
&\|L(x)\|_{\mathcal{X}} \leqslant \|\bar{L}|x|\|_{\mathcal{X}_1} + \|\tilde{L}|x|\|_{\mathcal{X}_2} \leqslant K\|x\|_{\mathcal{X}},\\
&\|L(x_1)-L(x_2)\|_{\mathcal{X}} \leqslant \|\bar{L}|x_1-x_2|\|_{\mathcal{X}_1},\\
&\|\nabla^s(L(x_1)-L(x_2))\|_{\mathcal{X}} \leqslant \|\tilde{L}|x_1-x_2|\|_{\mathcal{X}_2}
\end{aligned} \tag{3.32}$$

with some $0<K<1$, all $x\in\mathcal{X}$ and some kernel G_3.

Then for each $y\in\mathcal{X}$ such that $\|y\|_{\mathcal{X}} < \frac{(1-K)^2}{4C}$ there exists a solution $x\in\mathcal{X}$ of the equation

$$x = y + B(x,x) + Lx \tag{3.33}$$

satisfying the estimate

$$\|x\|_{\mathcal{X}} \leqslant \frac{1-K-\sqrt{(1-K)^2-4C\|y\|_{\mathcal{X}}}}{2C} < \frac{1-K}{2C}. \tag{3.34}$$

Moreover, the solution is unique in the ball in $\mathcal{X}$ of radius $\frac{2}{(1-K)\|y\|_{\mathcal{X}}}$.

Remark 3.15. *Notice that such a defined space $\mathcal{X}$ has been introduced in order to deal with the fractional gradient in a proper way. However, if there is no need to consider the fractional gradient separately, we may define the space $\mathcal{X}$ as $\mathcal{X}_1$ with the $\mathcal{X}_1$-norm, i.e. $\mathcal{X}_2=\{0\}$. Then we still get the existence result where the operators $\tilde{B},\tilde{L}$ are not anymore required.*

Proof of Lemma 3.14. The proof is a direct consequence of the Banach fixed point theorem. Let $T\colon \mathcal{X} \to \mathcal{X}$ be a map such that $T(x) = y + B(x,x) + L(x)$. Now we shall show that the map T satisfies $T[B_R] \subset B_R$ for some ball $B_R \subset \mathcal{X}$ centred at the origin with a radius R and that T is a contraction over that ball using the assumptions (3.31), (3.32) imposed on the operators B and L.

Step 1: Invariance. For $x \in B_R$, we get

$$\begin{aligned}\|Tx\|_{\mathcal{X}} &\leqslant \|y\|_{\mathcal{X}} + \|B(x,x)\|_{\mathcal{X}} + \|L(x)\|_{\mathcal{X}} \\ &\leqslant \frac{(1-K)^2}{4C} + C\|x\|^2_{\mathcal{X}} + K\|x\|_{\mathcal{X}} < R,\end{aligned}$$

which is satisfied if $R = \frac{1-K}{2C}$.

Step 2: Contraction. Let $x_1, x_2 \in B_R$ with R as already determined. Then

$$\begin{aligned}\|Tx_1 - Tx_2\|_{\mathcal{X}} &\leqslant \|B(x_1,x_1) - B(x_2,x_2)\|_{\mathcal{X}_1} + \|L(x_1) - L(x_2)\|_{\mathcal{X}_1} \\ &\quad + \|\nabla^s(B(x_1,x_1) - B(x_2,x_2))\|_{\mathcal{X}_2} + \|\nabla^s(L(x_1) - L(x_2)\|_{\mathcal{X}_2} \\ &\leqslant \|\bar{B}(|x_1|,|x_1 - x_2|)\|_{\mathcal{X}_1} + \|\bar{B}(|x_1 - x_2|,|x_2|)\|_{\mathcal{X}_1} + \|\bar{L}|x_1 - x_2|\|_{\mathcal{X}_1} \\ &\quad + \|\tilde{B}(|x_1|,|x_1 - x_2|)\|_{\mathcal{X}_2} + \|\tilde{B}(|x_1 - x_2|,|x_2|)\|_{\mathcal{X}_2} + \|\tilde{L}|x_1 - x_2|\|_{\mathcal{X}_1} \\ &\leqslant C\Big(\|x_1\|_{\mathcal{X}} + \|x_2\|_{\mathcal{X}}\Big)\|x_1 - x_2\|_{\mathcal{X}} + K\|x_1 - x_2\|_{\mathcal{X}} < \gamma\|x_1 - x_2\|_{\mathcal{X}},\end{aligned}$$

with $\gamma = K + C\Big(\|x_1\|_{\mathcal{X}} + \|x_2\|_{\mathcal{X}}\Big) < 1$. As the Banach fixed point theorem shows, there exists a unique solution of the equation in the ball $B((1-K)/2C)$. ∎

In the following two sections we shall prove the existence of the solution of the regularised problem (3.3). However, due to the appearance of the fractional gradient in the equation, we are forced to distinguish two cases: $\alpha \in (0,1]$ and $\alpha \in (1,2)$, and to prove the existence result for each case separately. In the first one, as a result of the Hardy-Sobolev-Littlewood inequality, we can apply Lemma 3.14 taking into account Remark 3.15, see Section 3.3. On the other hand, the inequality (3.29) for $\alpha \in (1,2)$ is no longer satisfied. Thus, we need to find a different way to deal with the fractional gradient. Therefore, we look for solutions in the fractional Sobolev space (as defined in (1.13)-(1.14)) and apply Lemma 3.14 directly, see Section 3.4.

3.3 Existence of approximating solutions in the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for arbitrary initial data and $\alpha \in (0,1]$

Theorem 3.16 (Existence of solutions)**.** *Let $\varepsilon > 0$, $\sigma \in \mathbb{R}$ and $\alpha \in (0,1]$. For every $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique global-in-time solution v of the regularised Cauchy problem* (3.3) *in the space*

$$v \in \mathcal{C}([0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})). \tag{3.35}$$

Remark 3.17. *Since the heat semigroup is not strongly continuous in L^∞ (in general $\lim_{t \searrow 0} \|e^{t\Delta} v_0 - v_0\|_\infty \neq 0$), we consider $e^{t\Delta}$ in the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in order to have at least $e^{t\Delta} v_0 \rightharpoonup v_0$ (in the sense of weak convergence of measures) as $t \searrow 0$.*

Proof of Theorem 3.16. As mentioned earlier, we are going to apply Lemma (3.14) with the space $\mathcal{X} = \mathcal{X}_1 = C([0,T], L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, which is a Banach space with the norm $\|v\|_{\mathcal{X}} \equiv \sup_{0 \leqslant t \leqslant T} \|v(t)\|_{L^1} + \sup_{0 \leqslant t \leqslant T} \|v(t)\|_{L^\infty}$, to the equation (3.24) written in the form $v(t) = G(\cdot, \varepsilon t) * v_0 + B(v,v)(t) + L(v)(t)$. Here the form B is defined by

$$B(u,v)(t) = \int_0^t \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) * \Big(|u|\nabla^{\alpha-1} v\Big)(s)\, \mathrm{d}s, \tag{3.36}$$

which is linear only with respect to second variable. Let define also the operator L by

$$L(v)(t) = \int_0^t \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) * \Big(\sigma |v(s)|\Big)\, \mathrm{d}s. \tag{3.37}$$

To overcome difficulties resulting from the fact that neither B is bilinear nor L is linear, we define (a kind of majorant) $\bar{B}$ by setting

$$\bar{B}(u,v)(t) = \int_0^t \left| \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) \right| * \Big(u|\nabla^{\alpha-1} v|\Big)(s)\, \mathrm{d}s. \tag{3.38}$$

We show now that so defined operator satisfies $\bar{B} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. More precisely, we show the continuity with a bound that depends on T. Namely, there exists a constant C_1 such that for all $T > 0$ and all $u, v \in \mathcal{X}$, we have

$$\|\bar{B}(|u|, v)\|_{\mathcal{X}} \leqslant C_1\Big(T^{\frac{1}{2}} + T^{\frac{1}{2}-\frac{1}{2q}}\Big)\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}. \tag{3.39}$$

We observe that indeed all the properties mentioned in (3.31) hold.

In the estimates below we shall use systematically the interpolation of the L^p-norms with $f \in L^1 \cap L^\infty$

$$\|f\|_p \leqslant \|f\|_1^{\frac{1}{p}}\|f\|_\infty^{1-\frac{1}{p}} \leqslant \|f\|_1 + \|f\|_\infty \tag{3.40}$$

which comes from the Young inequality and the general interpolation result for the Lebesgue spaces. Namely, let $1 \leqslant q < p < r \leqslant \infty$ and $\frac{1}{p} = \frac{\lambda}{q} + \frac{1-\lambda}{r}$, we have

$$\|f\|_p \leqslant \|f\|_q^\lambda \, \|f\|_r^{1-\lambda} \tag{3.41}$$

for every $f \in L^q \cap L^r$.

Assume that $u, v \in \mathcal{X}$. By the estimates for the heat kernel (3.26) combined with the Hardy-Sobolev-Littlewood inequality (3.29) and the Hölder inequality, we have

$$\begin{aligned}
\|\bar{B}(|u|, v)(t)\|_{L^1} &\leqslant \int_0^t \left\| \left|\frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s))\right| * \Big(|u||\nabla^{\alpha-1}v|\Big)(s)\right\|_{L^1} \mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{-\frac{1}{2}} \left\|\Big(|u||\nabla^{\alpha-1}v|\Big)(s)\right\|_{L^1} \mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{-\frac{1}{2}} \left\|u(s)\right\|_{L^{q^*}} \left\|\nabla^{\alpha-1}v(s)\right\|_{L^q} \mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^{q^*}} \|v(s)\|_{L^p} \,\mathrm{d}s
\end{aligned}$$

for each $p \in (0, \infty)$ and $\frac{1}{q} + \frac{1}{q^*} = 1$, and $\frac{1}{q} = \frac{1}{p} - (1-\alpha)$. Consequently, by the interpolation of L^p-norms (3.40), we obtain

$$\sup_{0 \leqslant t \leqslant T} \|\bar{B}(|u|, v)\|_{L^1} \leqslant CT^{\frac{1}{2}}\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}. \tag{3.42}$$

To deal with the L^∞-norm of $\bar{B}(|u|,v)$, we proceed similarly: for each $q \in (1,\infty)$ we have

$$\begin{aligned}
\|\bar{B}(|u|,v)\|_{L^\infty} &\leqslant \int_0^t \left\| \left|\frac{\partial}{\partial x} G(\cdot,\varepsilon(t-s))\right| * \Big(|u||\nabla^{\alpha-1}v|\Big)(s)\right\|_{L^\infty} \mathrm{d}s \\
&\leqslant C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} \Big\|\Big(|u||\nabla^{\alpha-1}v|\Big)(s)\Big\|_{L^q} \mathrm{d}s \\
&\leqslant C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} \Big\|u(s)\Big\|_{L^p} \Big\|\nabla^{\alpha-1}v(s)\Big\|_{L^r} \mathrm{d}s \\
&\leqslant C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} \|u(s)\|_{L^p} \|v(s)\|_{L^p} \mathrm{d}s.
\end{aligned}$$

with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$ and $\frac{1}{r} = \frac{1}{p} - (1-\alpha)$. Thus, by the interpolation of L^p-norms (3.40) we have

$$\sup_{0\leqslant t\leqslant T} \|\bar{B}(|u|,v)\|_{L^\infty} \leqslant CT^{\frac{1}{2}-\frac{1}{2q}} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}. \tag{3.43}$$

Estimates (3.42) and (3.43) imply that the form $\bar{B}$ satisfies (3.39) and that the original operator B is in fact continuous: $B\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$.

Similarly, for the sublinear operator L we define a linear operator $\bar{L}$ by letting

$$\bar{L}(v)(t) = \int_0^t \left|\frac{\partial}{\partial x} G(\cdot,\varepsilon(t-s))\right| * \Big(|\sigma| v(s)\Big) \mathrm{d}s. \tag{3.44}$$

We show that such a defined operator $\bar{L}$ is continuous: $\bar{L} : \mathcal{X} \to \mathcal{X}$ and there exists a constant C_2 such that for ech $T > 0$ and all $v \in \mathcal{X}$ we have

$$\|\bar{L}v\|_{\mathcal{X}} \leqslant C_2 |\sigma| T^{\frac{1}{2}} \|v\|_{\mathcal{X}}. \tag{3.45}$$

Again, we observe that all the relations (3.32) are satisfied.

Assume that $v \in \mathcal{X}$, then by the heat kernel estimate (3.26) we get for each $p \in [1,\infty]$

$$\begin{aligned}\|\bar{L}(v)(t)\|_{L^p} &\leqslant \int_0^t \left\| \left| \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) \right| * |\sigma| v(s) \right\|_{L^p} \mathrm{d}s \\ &\leqslant C|\sigma| \int_0^t (t-s)^{-\frac{1}{2}} \|v(s)\|_{L^p} \,\mathrm{d}s.\end{aligned}$$

Hence, by the interpolation of L^p-norms (3.40), and the above inequality with $p=1$ and $p=\infty$, we obtain (3.45). Moreover the original operator L is in fact continuous: $L\colon \mathcal{X} \to \mathcal{X}$.

Now by Lemma 3.14, if we choose $T >$ so small that

$$C_2|\sigma|T^{\frac{1}{2}} < 1 \quad \text{and} \quad 4C_1\Big(T^{\frac{1}{2}} + T^{\frac{1}{2}-\frac{1}{2q}}\Big)\|v_0\|_{\mathcal{X}} < \Big(1 - C_2|\sigma|T^{\frac{1}{2}}\Big)^2, \tag{3.46}$$

then there exists a solution in the space $\mathcal{X}$ with $\|v\|_{\mathcal{X}} < \frac{1-K}{2C}$. Moreover, this solution is unique in the ball in $\mathcal{X}$ of radius $\frac{1-K}{2C}$ where $K = C_2|\sigma|T^{\frac{1}{2}}$ and $C = C_1\Big(T^{\frac{1}{2}} + T^{\frac{1}{2}-\frac{1}{2q}}\Big)$.

Now every local-in-time solution v, constructed by the method described above, can be extended step-by-step to the global-in-time one since each of its L^p-norms decays, see Theorem (3.4), and therefore our solution always remains in the ball of the radius $\frac{1-K}{2C}$ *i.e.* $B\Big(\frac{1-K}{2C}\Big) \subset \mathcal{X}$. ■

3.4 Existence of approximating solutions in the space $L^1(\mathbb{R}) \cap W^{\alpha-1,p}(\mathbb{R})$ for arbitrary initial data and $\alpha \in (1,2)$

Theorem 3.18 (Existence of solutions)**.** *Let $\varepsilon > 0$, $\sigma \in \mathbb{R}$, $\alpha \in (1,2)$ and $p \geqslant 1$ such that $(\alpha-1)p > 1$. For every $v_0 \in L^1(\mathbb{R}) \cap W^{\alpha-1,p}(\mathbb{R})$, there exists a unique global-in-time solution v of the regularised Cauchy problem* (3.3) *satisfying*

$$v \in \mathcal{C}([0,\infty), L^1(\mathbb{R}) \cap W^{\alpha-1,p}(\mathbb{R})). \tag{3.47}$$

Remark 3.19. *Here Remark 3.17 on the weak continuity of the heat semigroup applies mutatis mutandis to the above case.*

Before we go to the proof of the existence of solutions, we first recall a continuous embedding result [27, Theorem 8.2] between $W^{s,p}$ and L^∞, which will be used in the proof.

Proposition 3.20 (Continuous embedding). *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp > 1$. Then the fractional space $W^{s,p}(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, i.e. there exists a constant $C > 0$ depending on s, p such that*

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant C\|f\|_{W^{s,p}(\mathbb{R})}$$

for any function $f \in W^{s,p}(\mathbb{R})$.

Proof of Theorem 3.18. In order to prove the theorem, we proceed similarly as in the previous proof. Let $\mathcal{X} = \{x : \langle x, \nabla^s x\rangle \in \mathcal{X}_1 \times \mathcal{X}_2\}$ be a Banach space with the norm $\|x\|_{\mathcal{X}} = \|x\|_{\mathcal{X}_1} + \|\nabla^s x\|_{\mathcal{X}_2}$, where $\mathcal{X}_1, \mathcal{X}_2$ are Banach spaces defined in the following way

$$\begin{aligned} \mathcal{X}_1 &= C([0,T], L^1(\mathbb{R}) \cap L^p(\mathbb{R})), \quad && \|\cdot\|_{\mathcal{X}_1} = \sup_{0\leqslant t\leqslant T} \Big(\|\cdot\|_{L^1} + \|\cdot\|_{L^p}\Big), \\ \mathcal{X}_2 &= C([0,T], L^p(\mathbb{R})), && \|\cdot\|_{\mathcal{X}_2} = \sup_{0\leqslant t\leqslant T} \Big(\|\cdot\|_{L^p}\Big). \end{aligned}$$

Since the Hardy-Sobolev-Littlewood inequality is no longer valid, we apply Lemma 3.14 with the spaces $\mathcal{X}, \mathcal{X}_1$ and $\mathcal{X}_2$ to the equation (3.24) written in the form $v(t) = G(\cdot, \varepsilon t) * v_0 + B(v,v)(t) + L(v)(t)$. The operators B, L are defined by (3.36), (3.37) respectively. The operators $\bar{B}, \bar{L}$, used as majorants, are defined by (3.38), (3.44) respectively. Additionally, we define another two operators in the following way

$$\tilde{B}(u,v)(t) = \int_0^t \Big|\nabla^\alpha G(\cdot, \varepsilon(t-s))\Big| * \Big(u|\nabla^{\alpha-1}v|\Big)(s)\,\mathrm{d}s, \tag{3.48}$$

and

$$\tilde{L}(v)(t) = \int_0^t \Big|\nabla^\alpha G(\cdot, \varepsilon(t-s))\Big| * \Big(|\sigma| v(s)\Big)\,\mathrm{d}s. \tag{3.49}$$

First, we show that the operators defined by (3.38) and (3.48) satisfy $\bar{B} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}_1$ and $\tilde{B} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}_2$, and that there exists a constant C_1

such that for all $T > 0$ and all $u, v \in \mathcal{X}$ we have

$$\|\bar{B}(|u|,v)\|_{\mathcal{X}_1} + \|\tilde{B}(|u|,v)\|_{\mathcal{X}_2} \leqslant C_1(T^{\frac{1}{2}} + T^{1-\frac{\alpha}{2}})\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}. \tag{3.50}$$

We observe that all the properties (3.31) are satisfied.

Assume that $u, v \in \mathcal{X}$. By inequality (3.26) and Proposition 3.20, we have

$$\begin{aligned}
\|\bar{B}(|u|,v)(t)\|_{L^q} &\leqslant \int_0^t \left\| \left| \frac{\partial}{\partial x} G(\cdot, \varepsilon(t-s)) \right| * \Big(|u||\nabla^{\alpha-1}v|\Big)(s) \right\|_{L^q} \mathrm{d}s \\
&\leqslant \bar{C} \int_0^t (t-s)^{-\frac{1}{2}} \left\| \Big(|u||\nabla^{\alpha-1}v|\Big)(s) \right\|_{L^q} \mathrm{d}s \\
&\leqslant \bar{C} \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^\infty} \|\nabla^{\alpha-1}v(s)\|_{L^q} \mathrm{d}s \\
&\leqslant \bar{C} \int_0^t (t-s)^{-\frac{1}{2}} (\|u(s)\|_{L^q} + \|\nabla^{\alpha-1}u(s)\|_{L^q}) \|v(s)\|_{L^q} \mathrm{d}s,
\end{aligned}$$

where $\bar{C}$ is a positive constant. Thus, combining the above inequalities with $q = 1$ and $q = p$, we obtain the continuity of the form $\bar{B}$, *i.e.*

$$\|\bar{B}(|u|,v)\|_{\mathcal{X}_1} \leqslant \bar{C}T^{\frac{1}{2}}\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}. \tag{3.51}$$

To deal with the L^p-norm of $\tilde{B}(|u|,v)$ we proceed similarly. Indeed

$$\begin{aligned}
\left\|\tilde{B}(|u|,v)(t)\right\|_{L^p} &\leqslant \int_0^t \left\| \left| \nabla^\alpha G(\cdot, \varepsilon(t-s)) \right| * \Big(|u||\nabla^{\alpha-1}v|\Big)(s) \right\|_{L^p} \mathrm{d}s \\
&\leqslant \tilde{C} \int_0^t (t-s)^{-\frac{\alpha}{2}} \left\| \Big(|u||\nabla^{\alpha-1}v|\Big)(s) \right\|_{L^p} \mathrm{d}s \\
&\leqslant \tilde{C} \int_0^t (t-s)^{-\frac{\alpha}{2}} \|u(s)\|_{L^\infty} \|\nabla^{\alpha-1}v(s)\|_{L^p} \mathrm{d}s \\
&\leqslant \tilde{C} \int_0^t (t-s)^{-\frac{\alpha}{2}} (\|u(s)\|_{L^p} + \|\nabla^{\alpha-1}u(s)\|_{L^p}) \|v(s)\|_{L^p} \mathrm{d}s.
\end{aligned}$$

Hence, we have

$$\|\tilde{B}(u,v)\|_{\mathcal{X}_2} \leqslant \tilde{C}T^{1-\frac{\alpha}{2}}\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}} \tag{3.52}$$

for a positive constant $\tilde{C}$.

Estimates (3.51) and (3.52) imply that (3.50) is satisfied; thus, the bilinear form B is continuous, *i.e.* $B\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$.

Next, we show that the linear operators defined by (3.44) and (3.49) satisfy $\bar{L}:\mathcal{X}\to\mathcal{X}_1$ and $\tilde{L}:\mathcal{X}\to\mathcal{X}_2$, and that there exists a constant K_1 such that for all $T>0$ and all $v\in\mathcal{X}$, we have

$$\|\bar{L}|v|\|_{\mathcal{X}_1}+\|\tilde{L}|v|\|_{\mathcal{X}_2}\leqslant K_1|\sigma|(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}})\|v\|_{\mathcal{X}}. \tag{3.53}$$

Notice, however, that $\sup_{0\leqslant t\leqslant T}\|\bar{L}v\|_p$ and $\sup_{0\leqslant t\leqslant T}\|\bar{L}v\|_1$ have been already estimated in the previous proof in Section 3.3, see the estimate (3.45). Therefore, we need to compute the L^p-norm of $\tilde{L}|v|$ only. Assume that $v\in\mathcal{X}$, then by inequality (3.26) we get

$$\begin{aligned}\|\tilde{L}|v|(t)\|_{L^p}&\leqslant\int_0^t\Big\|\big|\nabla^\alpha G(\cdot,\varepsilon(t-s))\big|*\sigma|v(s)|\Big\|_{L^p}\,\mathrm{d}s\\&\leqslant\tilde{K}|\sigma|\int_0^t(t-s)^{-\frac{\alpha}{2}}\|v(s)\|_{L^p}\,\mathrm{d}s.\end{aligned}$$

Hence, for a positive constant $\tilde{K}$, we have

$$\|\tilde{L}|v|\|_{\mathcal{X}_2}\leqslant\tilde{K}|\sigma|T^{1-\frac{\alpha}{2}}\|v\|_{\mathcal{X}}. \tag{3.54}$$

Thus, by the estimate (3.45) from the previous proof and the above inequality, we arrive at (3.53) and the continuity of the operator $L\colon\mathcal{X}\to\mathcal{X}$ is obtained.

Now it follows from Lemma 3.14 that if we choose $T>0$ sufficiently small so that

$$\begin{aligned}&K_1|\sigma|\Big(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}}\Big)<1\quad\text{and}\\&\qquad 4C_1\Big(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}}\Big)\|v_0\|_{\mathcal{X}_T}<\Big(1-K_1|\sigma|\Big(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}}\Big)\Big)^2,\end{aligned} \tag{3.55}$$

then there exists a solution in the space $\mathcal{X}_T$ with $\|v\|_{\mathcal{X}_T}\leqslant\frac{1-K}{2C}$. Moreover, this solution is unique in the ball in $\mathcal{X}_T$ of radius $\frac{1-K}{2C}$ where $K=K_1|\sigma|\Big(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}}\Big)$ and $C=C_1\Big(T^{\frac{1}{2}}+T^{1-\frac{\alpha}{2}}\Big)$.

Now every local-in-time solution v, constructed by the method described above, can be extended step-by-step to the global-in-time one since each of its L^p norms for $p \in [1, +\infty)$ is bounded uniformly for $t > 0$. ∎

3.5 Passage to the limit in the definition of weak solutions

Now we are in a position to pass to the limit with $\varepsilon \searrow 0$ in the weak formulation (3.4) of the regularised problem to obtain a weak solution of (1.6) in the sense of Definition 3.1.

In order to perform the passage to the limit, we need to prove some compactness result. Usually one establishes compactness in the target space imitating the scheme of the Arzelà-Ascoli theorem, where some regularity of the time derivative in the same space is required. If we do not have enough regularity in the target space, there is still a possibility to get compactness exploiting the Aubin-Lions lemma or one of its numerous variations [7, 68], where the time derivative is estimated in some weaker norm. In our case, however, any regularity of time derivatives is hard to prove. Therefore, we shall use a compactness criterion proposed by Rakotoson and Temam see [63] where a simpler condition is sufficient (at least in the framework of Hilbert spaces).

Theorem 3.21 (Rakotoson-Temam)**.** *Let $(V, \|\cdot\|_V)$, $(H, \|\cdot\|_H)$ be two separable Hilbert spaces. Assume that $V \subset H$ with a compact and dense embedding. Consider a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ converging weakly to a function u in $L^2\Big((0,T), V\Big)$, for some $T \in (0, \infty)$. Then u^ε converges strongly to u in $L^2\Big((0,T), H\Big)$ if and only if*

i) $u^\varepsilon(t)$ converges weakly to $u(t)$ in H for a.e. $t \in (0, T)$,

ii) $\lim_{|E|\to 0} \sup_{\varepsilon>0} \int_{E\subset[0,T]} \Big\|u^\varepsilon(t)^2\Big\|_H \,\mathrm{d}t = 0$.

Now we show first estimates of the sequence $\{v^\varepsilon\}_{\varepsilon\in(0,1]}$ of solutions of (3.3) with a fixed $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Later, having these estimates, we prove compactness of the sequence $\{v^\varepsilon\}_{\varepsilon\in(0,1]}$ checking all conditions included in the Rakotoson-Temam lemma. Finally, we shall be able to pass to the limit $\varepsilon \searrow 0$.

Estimates of the sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$. One can notice that by (3.17) we have obtained the inequality $\|v(t)\|_p \leqslant \|v_0\|_p$ for all $t > 0$ and $p \in [1,\infty)$. By taking the limit $p \to \infty$ the bound $\|v(t)\|_\infty \leqslant \|v_0\|_\infty$ is also obtained. Thus, we get

$$\{v^\varepsilon\}_{\varepsilon\in(0,1]} \quad \text{is bounded in} \quad L^q\Big((0,T), L^p(\mathbb{R})\Big) \tag{3.56}$$

for each $T > 0$ and $q, p \in [1,\infty]$. Moreover, integrating the equation (3.15) over $[0,T]$ with $p = 2$ and dropping the last two terms of (3.15) which are negative imply that

$$\int_0^T \left\|\nabla^{\frac{\alpha}{2}} |v(t)|^{\frac{3}{2}}\right\|_2^2 \mathrm{d}t \leqslant \frac{9}{8}\|v_0\|_2^2. \tag{3.57}$$

Hence, we get the following property

$$\{\sqrt{\varepsilon}\, v_x\}_{\varepsilon\in(0,1]} \quad \text{is bounded in} \quad L^2\Big((0,T), L^2(\mathbb{R})\Big). \tag{3.58}$$

Next, from (3.17) for $p = 2$, we have

$$\left\{|v^\varepsilon|^{\frac{3}{2}}\right\}_{\varepsilon\in(0,1]} \quad \text{is bounded in} \quad L^2\Big((0,T), W^{\frac{\alpha}{2},2}(\mathbb{R})\Big). \tag{3.59}$$

Compactness of the sequence $\{v^\varepsilon\}_{\varepsilon\in(0,1]}$. The properties (3.56)-(3.59) permit us to use the Rakotoson-Temam compactness result recalled in Theorem 3.21 above.

Let $\omega^\varepsilon := |v^\varepsilon|^{\frac{3}{2}}$. We apply Theorem 3.21, with $V = W^{\frac{\alpha}{2},2}(\Omega)$, $H = L^2(\Omega)$ and an arbitrary bounded domain $\Omega \subset \mathbb{R}$, to the sequence $\{\omega^\varepsilon\}_{\varepsilon\in(0,1]}$. For such a choice of spaces we have

$$V \subset\subset H \quad \text{and} \quad L^2\Big((0,T), V\Big) \subset L^2\Big((0,T), H\Big) \tag{3.60}$$

with $\subset\subset$ denoting dense compact embedding, see [59, Lemma 10] or [27, Theorem 7.1] in more general case ($1 \leqslant q \leqslant p < \infty$). By (3.59) one can see that $\{\omega^\varepsilon\}_{\varepsilon\in(0,1]}$ converges weakly to a limit function in $L^2\Big((0,T),V\Big)$. Next, due to (3.60), a subsequence of the functions ω^ε converges weakly to some function $\omega(t)$ for almost every $t \in (0,T)$ in $L^2\Big((0,T),H\Big)$. Thus, the first condition of Theorem 3.21 is satisfied. Now we turn our attention to the second condition. By (3.56) we easily check that

$$\lim_{|E|\to 0}\sup_{\varepsilon\in(0,1]}\int_{E\subset[0,T]}\|\omega^\varepsilon(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t = \lim_{|E|\to 0}\sup_{\varepsilon\in(0,1]}\int_{E\subset[0,T]}\Big\|v^\varepsilon(t)\Big\|^3_{L^3(\Omega)} = 0. \tag{3.61}$$

Thus, by Theorem 3.21, we obtain the strong convergence of the sequence $\{\omega^\varepsilon\}_{\varepsilon\in(0,1]}$ in the space $L^2\Big((0,T),H\Big)$. Hence, since $\Omega\subset\mathbb{R}$ was chosen arbitrarily, we have the following results

$$\omega^\varepsilon \to \omega \quad \text{strongly in} \quad L^2_{\text{loc}}\Big((0,T)\times\mathbb{R}\Big), \tag{3.62}$$

$$v^\varepsilon \to v \quad \text{almost everywhere in} \quad (0,T)\times\mathbb{R}. \tag{3.63}$$

Passage to the limit. To find the equation satisfied by the limit function $v(t)$, we recall the weak formulation of the regularised problem (3.3) with the initial data v_0 either in $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ for $\alpha\in(0,1]$ or in the space $L^1(\mathbb{R})\cap W^{\alpha-1,p}(\mathbb{R})$ for $\alpha\in(1,2)$.

$$\begin{aligned}-\iint v^\varepsilon\phi_t\,\mathrm{d}x\,\mathrm{d}t + \int v_0\,\phi(\cdot,0)\,\mathrm{d}x &= -\varepsilon\iint v^\varepsilon_x\,\phi_x\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\iint |v^\varepsilon|\nabla^{\alpha-1}v^\varepsilon\phi_x\,\mathrm{d}x\,\mathrm{d}t + \sigma\iint |v^\varepsilon|\phi_x\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.64}$$

for all test functions $\phi\in C^\infty(Q_T)\cap C(\overline{Q_T})$ such that ϕ has compact support in the space variable x and vanishes near $t=T$.

First, let us notice that it is easy to pass to the limit $\varepsilon\searrow 0$ in the first and the last terms of (3.64), namely by (3.56), we have

$$\iint v^\varepsilon\phi_t\,\mathrm{d}x\,\mathrm{d}t \to \iint v\phi_t\,\mathrm{d}x\,\mathrm{d}t \tag{3.65}$$

$$\sigma\iint |v^\varepsilon|\phi_x\,\mathrm{d}x\,\mathrm{d}t \to \sigma\iint |v|\,\phi_x\,\mathrm{d}x\,\mathrm{d}t. \tag{3.66}$$

Next, by (3.58), the following relation

$$\sqrt{\varepsilon}\iint \sqrt{\varepsilon}\,v^\varepsilon_x\,\phi_x\,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{3.67}$$

holds true as $\varepsilon \searrow 0$.

To pass to the limit in the non-linear term we need to treat this more carefully. At the beginning we use the following decomposition

$$\iint |v^\varepsilon| \nabla^{\alpha-1} v^\varepsilon \phi_x \, \mathrm{d}x \, \mathrm{d}t - \iint |v| \nabla^{\alpha-1} v \phi_x \, \mathrm{d}x \, \mathrm{d}t =$$
$$\iint \Big(|v^\varepsilon| - |v| \Big) \nabla^{\alpha-1} v^\varepsilon \phi_x \, \mathrm{d}x \, \mathrm{d}t + \iint |v| \Big(\nabla^{\alpha-1} v^\varepsilon - \nabla^{\alpha-1} v \Big) \phi_x \, \mathrm{d}x \, \mathrm{d}t$$
$$=: \mathfrak{I}_1 + \mathfrak{I}_2 \quad (3.68)$$

and we show that those quantities converge to zero as $\varepsilon \searrow 0$.

Step 1: Passage to the limit in $\mathfrak{I}_1$ *cf.* (3.68). Since $v^\varepsilon \to v$ almost everywhere in $(0, T) \times \mathbb{R}$, it is sufficient to show that the term $\nabla^{\alpha-1} v^\varepsilon$ is bounded. Then by the Lebesgue dominated convergence theorem the first quantity on the right-hand-side of (3.68) tends to zero.

Let $\alpha \in (0, 1)$, then by (3.29) the operator $\nabla^{\alpha-1}$ is bounded form $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for every $p \in [1, \infty)$ and q such that $\frac{1}{q} = \frac{1}{p} - (1 - \alpha)$. Therefore,

$$\Big\{ \nabla^{\alpha-1} v^\varepsilon \Big\}_{\varepsilon \in (0,1]} \quad \text{is bounded in} \quad L^\infty\Big((0, T), L^q(\mathbb{R}) \Big). \qquad (3.69)$$

On the other hand, for $\alpha \in (1, 2)$, first we need to use the following result

$$\nabla^{\alpha-1} f = \nabla \mathcal{I}_{2-\alpha} f = \mathcal{I}_{1-\alpha} f = \nabla^{\frac{\alpha}{2}} \mathcal{I}_{1-\frac{\alpha}{2}} f = \mathcal{I}_{1-\frac{\alpha}{2}} \nabla^{\frac{\alpha}{2}} f \qquad (3.70)$$

with $1 - \frac{1}{2} \in (0, 1)$. Hence, by (3.28), the Riesz potential $\mathcal{I}_{1-\frac{\alpha}{2}}$ is bounded from $L^2(\mathbb{R})$ to $L^r(\mathbb{R})$, where $r = \frac{2}{\alpha-1}$. Now by (3.63) and the Lebesgue dominated convergence theorem, we get

$$\iint v^\varepsilon \nabla^{\frac{\alpha}{2}} \phi \, \mathrm{d}x \, \mathrm{d}t \to \iint v \nabla^{\frac{\alpha}{2}} \phi \, \mathrm{d}x \, \mathrm{d}t, \quad \text{as } \varepsilon \searrow 0,$$

for each test function ϕ with compact support in the space variable x vanishing near $t = T$. Hence, we obtain that $\Big\{ \nabla^{\frac{\alpha}{2}} v^\varepsilon \Big\}_{\varepsilon \in (0,1]}$ is weakly convergent and thus $\Big\| \nabla^{\frac{\alpha}{2}} v^\varepsilon \Big\|$ is bounded in $L^2\Big((0, T) \times \mathbb{R} \Big)$. Finally, this implies that

$$\Big\{ \nabla^{\alpha-1} v^\varepsilon \Big\}_{\varepsilon \in (0,1]} \quad \text{is bounded in} \quad L^2\Big((0, T), L^{\frac{2}{\alpha-1}}(\mathbb{R}) \Big). \qquad (3.71)$$

Step 2: Passage to the limit in $\mathfrak{I}_2$ *cf.* (3.68). One can notice that by (3.69) and (3.71), $\Big\{ \nabla^{\alpha-1} v^\varepsilon \Big\}_{\varepsilon \in (0,1]}$ converges weakly, along a subsequence, to some function $f \in L^2\big((0,T), L^p(\mathbb{R})\big)$ with $p = 2,\ \alpha \in (0,1]$ or $p = \frac{2}{\alpha-1},\ \alpha \in (1,2)$. To identify f as $\nabla^{\alpha-1}v$ observe that, by (3.63) and by the Lebesgue dominated convergence theorem, we have

$$\iint v^\varepsilon \nabla^{\alpha-1}\phi \,\mathrm{d}x\,\mathrm{d}t \to \iint v\nabla^{\alpha-1}\phi \,\mathrm{d}x\,\mathrm{d}t, \quad \text{as } \varepsilon \searrow 0,$$

for each test function ϕ with compact support in the space variable x vanishing near $t = T$. Accordingly, we get the convergence

$$\nabla^{\alpha-1}v^\varepsilon \to \nabla^{\alpha-1}v \quad \text{weakly in } L^2\big((0,T), L^p(\mathbb{R})\big), \tag{3.72}$$

with $p = 2,\ \alpha \in (0,1]$ or $p = \frac{2}{\alpha-1},\ \alpha \in (1,2)$. Now we see that the second term on the right-hand side of (3.68) tends to zero due to (3.72) and the Lebesgue dominated convergence theorem.

Thus, we have proved that the limit function $v \in L^\infty\big((0,T), L^p(\mathbb{R})\big)$ is a weak solution of (3.3) with $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Chapter 4

Analysis of self-similarity

Scale invariance is a very basic concept originally coming from physics. This is a property of objects that does not change if the variables are multiplied by a suitably related factors. In mathematics it can be viewed as a quite particular case of study of the invariance of objects (e.g. differential equations) under general groups of transformations.

Let us examine the application of scaling transformations to our model of dynamics of dislocations in a more detailed way. Let $u = u(x,t)$ be a solution of

$$\partial_t u = |\partial_x u|\Big(-\sigma - \Lambda^\alpha u\Big). \tag{4.1}$$

We apply the group of dilations in all the variables

$$u' = Ku, \quad x' = Lx, \quad t' = Tt, \tag{4.2}$$

and impose the condition on a new u' defined in terms of the new variables x' and t', i.e.

$$u'(x',t') = Ku\left(\frac{x'}{L}, \frac{t'}{T}\right) \tag{4.3}$$

such that it has to be again a solution of (4.1). Here, however, the symbol $'$ is just an index, and not the symbol of derivation. Then

$$\partial_{t'} u' = \frac{K}{T}\partial_t u\left(\frac{x'}{L}, \frac{t'}{T}\right), \ \partial_{x'} u' = \frac{K}{L}\partial_x u\left(\frac{x'}{L}, \frac{t'}{T}\right), \ \Lambda^\alpha_{x'} u' = \frac{K}{L^\alpha}\Lambda^\alpha_x u\left(\frac{x'}{L}, \frac{t'}{T}\right),$$

imply a relation between K, T, L. Namely, if $\sigma = 0$, then (4.3) is a solution if and only if $KT = L^{\alpha+1}$. On the other hand, if $\sigma \neq 0$, we get $L = T$ and

$K = T^\alpha$. Thus, we have a two-parameter family of functions in the case $\sigma = 0$ and one-parameter family in the case $\sigma \neq 0$. Hence, we may freely choose parameters L and T such that the rescaled function can be written as

$$u'(x', t') = \begin{cases} L^{\alpha+1} T^{-1} u\left(\frac{x'}{L}, \frac{t'}{T}\right) & \text{if } \sigma = 0, \\ L^\alpha u\left(\frac{x'}{L}, \frac{t'}{T}\right) & \text{if } \sigma \neq 0. \end{cases} \tag{4.4}$$

In practice one of the free parameters is often used to warrant that u' preserves some important property, e.g. *conservation of mass, conservation of momentum* or *invariance of the equation*. This allows to classify the family of all scale-invariant solutions according to a new relation.

The basic idea is to introduce another relation between the two independent parameters in order to reduce the transformation to a one-parameter family of rescaled functions. One possible relation is $K = L^{-\gamma}$ for some $\gamma \in \mathbb{R}$ which comes from physical considerations. Then considering the case $\sigma = 0$ we are able to express K and L in terms of T:

$$K = T^{-(1-\beta(\alpha+1))}, \quad L = T^\beta, \tag{4.5}$$

where the parameter β can be represented in terms of γ.

Consequently, substituting $\lambda = 1/T$ the solution has the following form written in the standard notation

$$u_\lambda(x, t) = \lambda^{1-\beta(\alpha+1)} u(\lambda^\beta x, \lambda t) \quad \text{for } \sigma = 0, \tag{4.6}$$

$$u_\lambda(x, t) = \lambda^{-\alpha} u(\lambda x, \lambda t) \quad \text{for } \sigma \neq 0. \tag{4.7}$$

The parameter β is to be determined later. The conclusions of our calculations lead to the following three definitions.

Definition 4.1 (Forward self-similar solutions)**.** *Let $\sigma = 0$ and $\alpha \in (0, 2]$. If $u(x, t)$ is a solution of* (1.6a)*, then $u^\lambda(x, t) = \lambda^{1-(\alpha+1)\beta} u(\lambda^\beta x, \lambda t)$ for each $\lambda > 0$ is so. Thus, the scale invariant solutions are of the form*

$$u(x, t) = \frac{1}{(t+1)^{1-\beta(\alpha+1)}} \Phi(\xi) \quad \textit{with} \quad \xi = \frac{x}{(t+1)^\beta}, \tag{4.8}$$

for some $0 < \beta < \frac{1}{\alpha+1}$ and a function $\Phi : \mathbb{R} \to \mathbb{R}$.

We look for a non-positive, compactly supported function Φ *satisfying* $\xi\Phi' \geqslant 0$ *and the following elliptic-type equation*

$$(\beta(\alpha+1)-1)\Phi - \beta\xi\Phi' = -|\Phi'|\Lambda^\alpha\Phi \quad \textit{for all} \quad \xi \in \mathbb{R}. \tag{4.9}$$

Definition 4.2 (Backward self-similar solutions)**.** *Let* $\sigma < 0$ *and* $\alpha \in (0, 2]$*. If* $u(x,t)$ *is a solution of* (1.6a)*, then* $u^\lambda(x,t) = \lambda^{-\alpha}u(\lambda x, \lambda t)$ *for each* $\lambda > 0$ *is so. Thus, the scale invariant solutions are of the form*

$$u(x,t) = (T-t)^\alpha \Phi(\xi) \quad \textit{with} \quad \xi = \frac{x}{T-t}, \tag{4.10}$$

for some $T > 0$ *and a function* $\Phi : \mathbb{R} \to \mathbb{R}$*.*

We look for a non-positive, compactly supported function Φ *satisfying* $\xi\Phi' \geqslant 0$ *and the following elliptic-type equation*

$$-\alpha\Phi + \xi\Phi' = |\Phi'|\Big(-\sigma - \Lambda^\alpha\Phi\Big) \quad \textit{for all} \quad \xi \in \mathbb{R}. \tag{4.11}$$

Here, however, a backward solution means that the solution either blows up, or extincts at time $t = T$.

Definition 4.3 (Forward self-similar solutions)**.** *Let* $\sigma > 0$ *and* $\alpha \in (0, 2]$*. If* $u(x,t)$ *is a solution of* (1.6a)*, then* $u^\lambda(x,t) = \lambda^{-\alpha}u(\lambda x, \lambda t)$ *for each* $\lambda > 0$ *is so. Thus, the scale invariant solutions are of the form*

$$u(x,t) = (t+1)^\alpha \Phi(\xi) \quad \textit{with} \quad \xi = \frac{x}{t+1}, \tag{4.12}$$

for some function $\Phi : \mathbb{R} \to \mathbb{R}$*.*

We look for a non-positive, compactly supported function Φ *satisfying* $\xi\Phi' \geqslant 0$ *and the following elliptic-type equation*

$$\alpha\Phi - \xi\Phi' = |\Phi'|\Big(-\sigma - \Lambda^\alpha\Phi\Big) \quad \textit{for all} \quad \xi \in \mathbb{R}. \tag{4.13}$$

Proposition 4.4 (Scaling property)**.** *Let* $\sigma \in \mathbb{R}$ *and let* $\Phi(\xi)$ *be a solution of one of the equations* (4.9), (4.11) *or* (4.13)*. Then for each* $\lambda > 0$ *the rescaled function*

$$\Phi^\lambda(\xi) = \frac{1}{\lambda^{\alpha+1}}\Phi(\lambda\xi) \quad \textit{for all} \quad \xi \in \mathbb{R}, \tag{4.14}$$

is a solution of (4.9), (4.11) *or* (4.13) *respectively, and the rescaled constant exterior force is of the form* $\sigma^\lambda = \frac{\sigma}{\lambda}$*. Notice that if* $\sigma = 0$*, then* $\sigma^\lambda = 0$*.*

4.1 Viscosity solutions and the comparison principle

The concept of viscosity solutions was introduced in the early 80's of the 20th century by Michael Crandall and Pierre–Louis Lions [21] in the context of the Hamilton-Jacobi equation

$$H(x, u, Du, D^2u) = 0, \quad \text{in a domain } \Omega \subset \mathbb{R}^n, \tag{4.15}$$

to generalise the notion of classical solutions of partial differential equations. The viscosity solution received its name from the vanishing viscosity method used to prove the existence result in certain cases such as the Burgers equation or inviscid Hamilton-Jacobi equation.

Due to the non-linearity, the Hamilton-Jacobi equations often do not have classical solutions even if the Hamiltonian H is an analytic function. Moreover, since the derivatives occur in a genuinely non-linear way, it would even be difficult to apply the notion of weak solutions. Thus, the concept of viscosity solutions has become increasingly prevalent in the analysis of degenerate elliptic equations where the vanishing viscosity method itself no longer works. However, for elliptic equations (linear in the derivatives) the notions of viscosity solutions and weak solutions are equivalent, see [42].

Nowadays the existence of solutions is often obtained by Perron's method introduced in 1923 by Oskar Perron in order to find solutions of the Laplace equation. It consists in building a solution as the supremum of a suitable family of viscosity subsolutions. The first who used the Perron's method to solve the non-linear first-order equation was H. Ishii in his paper [41].

Following [22], we shall give a definition of the viscosity solutions for the second-order PDE

$$F(x, u, Du, D^2u) = 0, \quad x \in \Omega, \tag{4.16}$$

which will be used in subsequent sections in various contexts. We shall also exploit it to prove the comparison principle (Theorem 4.10) for the local counterpart of the problem (1.6), *i.e.* the problem (1.6) with $\alpha = 2$. Then the local equation reads as

$$u_t = |u_x|(u_{xx} - \sigma) \quad \text{on} \quad \mathbb{R} \times (0, +\infty). \tag{4.17}$$

First, however, let us define sub- and superdifferentials of semicontinuous functions which will be used in a definition of viscosity solutions and in the classical Ishii's Lemma recalled below. In all what follows, we denote by $\mathbb{S}^n = \mathbb{S}^n(\mathbb{R})$ the set of symmetric $n \times n$ matrices.

Definition 4.5 (Elliptic sub- and superdifferential of order two)**.** *Let Ω be a locally compact subset of $\mathbb{R}^n$ and $u \in USC(\Omega)$. Then the supperdifferential $\mathcal{D}^+u$ of order two of the function u is defined by: $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ belongs to $\mathcal{D}^+u(x)$ if $x \in \Omega$ and*

$$u(y) \leqslant u(x) + \langle p, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2)$$

as $\Omega \ni y \to x$. In a similar way, we define the subdifferential of order two by $\mathcal{D}^-u = -\mathcal{D}^+(-u)$. We also define

$$\bar{\mathcal{D}}^+u(x) = \left\{ \begin{array}{l} (p, X) \in \mathbb{R}^n \times \mathbb{S}^n \ \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{S}^n \\ \textit{such that } (p_n, X_n) \in \mathcal{D}^+u(x_n) \\ \textit{and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \end{array} \right\}.$$

The set $\bar{\mathcal{D}}^-u(x)$ is defined in a similar way.

Example. Let $u(x) = |x|$ and $\Omega = \mathbb{R}$. Then the sub- and superdifferentials of the function u are given by

$$\mathcal{D}^+u(x) = \left\{ \begin{array}{ll} \{(1, X) : X \geqslant 0\}, & x > 0, \\ \emptyset, & x = 0, \\ \{(-1, X) : X \geqslant 0, & x < 0, \end{array} \right.$$

and

$$\mathcal{D}^-u(x) = \left\{ \begin{array}{ll} \{(1, X) : X \leqslant 0\}, & x > 0, \\ ((-1, 1) \times \mathbb{R}) \cup (\{-1, 1\} \times [0, \infty)), & x = 0, \\ \{(-1, X) : X \leqslant 0, & x < 0. \end{array} \right.$$

Before we precise a definition of viscosity solutions, let us recall here a definition of upper semi-continuous (USC for short) and lower semi-continuous (LSC for short) functions. A function $f\colon \Omega \to \mathbb{R}$ is said to be upper (resp. lower) semi-continuous if the following inequality

$$f(x_0) \geqslant \limsup_{x\to x_0} f(x) \quad \Big(\text{resp. } f(x_0) \leqslant \limsup_{x\to x_0} f(x)\Big)$$

holds true for every $x_0 \in \Omega$.

Definition 4.6 (Viscosity solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*
i) *We say that $u \in USC(\Omega)$ is a* ***viscosity subsolution*** *of* (4.16) *(equivalently, a viscosity solution of $F \leqslant 0$) if*

$$F(x_0, u(x_0), p, X) \leqslant 0 \quad \textit{for all } x_0 \in \Omega \textit{ and } (p, X) \in \mathcal{D}^+u(x_0).$$

ii) *We say that $u \in LSC(\Omega)$ is a* ***viscosity supersolution*** *of* (4.16) *(equivalently, a viscosity solution of $F \geqslant 0$) if*

$$F(x_0, u(x_0), p, X) \geqslant 0 \quad \textit{for all } x_0 \in \Omega \textit{ and } (p, X) \in \mathcal{D}^-u(x_0).$$

iii) *We say that u is a* ***viscosity solution*** *of* (4.16) *if it is a viscosity sub- and supersolution of* (4.16)*.*

Remark 4.7. *Notice that Definition 4.5 of sub- and superdifferentials is equivalent with: if $\bar{x} \in \Omega$, then for all $\phi \in C^2(\Omega)$*

$$\mathcal{D}^+u(\bar{x}) = \Big\{\Big(D\phi(\bar{x}), D^2\phi(\bar{x})\Big) : u - \phi \textit{ has a local maximum at } \bar{x}\Big\},$$
$$\mathcal{D}^-u(\bar{x}) = \Big\{\Big(D\phi(\bar{x}), D^2\phi(\bar{x})\Big) : u - \phi \textit{ has a local minimum at } \bar{x}\Big\}.$$

Hence, the definition of the viscosity solutions given above is consistent with, e.g. [8, Definition 1.1, p. 25] where the function u is tested form below and from above by smooth test functions.

Now we shall state the classical Ishii's lemma [22, Theorem 3.2] which will help us to conclude the proof of the comparison principle. The proof of the Ishii's lemma can be found in [20].

Theorem 4.8 (Elliptic version of Ishii's lemma). *Let U and V be subsets of $\mathbb{R}$, $u \in USC(U)$ and $v \in LSC(V)$. Let $\phi\colon U \times V \to \mathbb{R}$ be of class C^2. Assume, moreover, that $(x, y) \mapsto u(x) - v(y) - \phi(x, y)$ reaches a local maximum at $(\bar{x}, \bar{y}) \in U \times V$. We denote $p_1 = D_x\phi(\bar{x}, \bar{y}), p_2 = -D_y\phi(\bar{x}, \bar{y})$ and $A = D^2\phi(\bar{x}, \bar{y})$.*

Then for each $\mu > 0$ such that $\mu A < I$ there exist $q_1, q_2 \in \mathbb{R}$ such that $(p_1, q_1) \in \bar{\mathcal{D}}^+u(\bar{x})$, $(p_2, q_2) \in \bar{\mathcal{D}}^-v(\bar{y})$ and

$$-\Big(\frac{1}{\mu} + \|A\|\Big)I \leqslant \begin{pmatrix} q_1 & 0 \\ 0, & -q_2 \end{pmatrix} \leqslant A + \mu A^2, \tag{4.18}$$

where I is the identity matrix in $\mathbb{R}^2$. The norm of the symmetric matrix A used in (4.18) is

$$\|A\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} = \sup\{\langle A\xi, \xi\rangle : |\xi| \leqslant 1\}.$$

Corollary 4.9 (Consequence of Ishii's lemma). *Given $T > 0$. Let U and V be locally compact subset of $\mathbb{R}$, $u \in USC(U \times [0, T))$ and $v \in LSC(V \times [0, T))$. Let $\phi\colon U \times [0, T) \times V \times [0, T) \to \mathbb{R}$ be of class C^2. Moreover, assume that $(x, t, y, s) \mapsto u(x, t) - v(y, s) - \phi(x, t, y, s)$ reaches a local maximum at $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in U \times [0, T) \times V \times [0, T)$. Computing the following quantities at point $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$, we set $\tau_1 = D_t\phi, \tau_2 = -D_s\phi, p_1 = D_x\phi, p_2 = -D_y\phi$ and*

$$\bar{A} = \begin{pmatrix} A & A_1 \\ A_1^T & A_2 \end{pmatrix} \tag{4.19}$$

with

$$A = \begin{pmatrix} D^2_{xx}\phi & D^2_{xy}\phi \\ D^2_{yx}\phi & D^2_{yy}\phi \end{pmatrix}, \; A_1 = \begin{pmatrix} D^2_{xt}\phi & D^2_{xs}\phi \\ D^2_{yt}\phi & D^2_{ys}\phi \end{pmatrix}, \; A_2 = \begin{pmatrix} D^2_{tt}\phi & D^2_{ts}\phi \\ D^2_{st}\phi & D^2_{ss}\phi \end{pmatrix}.$$

Let $\bar{I}$ be the identity matrix in $\mathbb{R}^4$. Then for every $\mu > 0$ such that $\mu\bar{A} \leqslant \bar{I}$ there exist $q_1, q_2 \in \mathbb{R}$ such that (where by $*$ we denote some elements of $\mathbb{R}$ that we do not precise)

$$(\tau_1, p_1, \begin{pmatrix} q_1 & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^+u(\bar{x}, \bar{t}), \quad (\tau_2, p_2, \begin{pmatrix} q_2 & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^-v(\bar{y}, \bar{s})$$

and

$$-\Big(\frac{1}{\mu}+\|\bar{A}\|\Big)I \leqslant \begin{pmatrix} q_1 & 0 \\ 0, & -q_2 \end{pmatrix} \leqslant A + 2\mu\Big(A^2 + A_1\cdot A_1^T\Big),$$

where I is the identity matrix in $\mathbb{R}^2$.

The above corollary is nothing else but Ishii's lemma with new variables $\hat{x}=(x,t)$ and $\hat{y}=(y,s)$. Moreover, we obtain a proper matrix inequality after relabeling the vectors of the basis (going from coordinates (x,t,y,s) to coordinates (x,y,t,s)), see [30, Corollary 9.3].

Theorem 4.10 (Comparison principle)**.** *Let u be a viscosity subsolution and v be a viscosity supersolution of the equation* (4.17) *in the sense of Definition 4.6. Assume, moreover, that the initial conditions satisfy $u_0(x)\leqslant v_0(x)$ for $x\in\mathbb{R}$. Then $u(x,t)\leqslant v(x,t)$ for all $t>0$ and $x\in\mathbb{R}$.*

Proof. Let u,v be a viscosity sub- and supersolution of (4.17) respectively and let $u_0(x)\leqslant v_0(x)$ for all $x\in\mathbb{R}$.

Step 1: Doubling of variables. Suppose by contradiction that

$$M=\sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}}} u(x,t)-v(x,t)>0.$$

Let $\varepsilon,\delta,\gamma,\eta>0$ and define

$$\begin{aligned} M_{\varepsilon,\delta,\gamma,\eta} &\overset{\text{def}}{=} \sup_{\substack{t,s\in[0,T]\\ x,y\in\mathbb{R}}}\left(u(x,t)-v(y,s)-\frac{|x-y|^2}{2\varepsilon}-\frac{|t-s|^2}{2\delta}-\frac{\eta}{T-t}-\gamma|x|^2\right)\\ &= u(\bar{x},\bar{t})-v(\bar{y},\bar{s})-\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon}-\frac{|\bar{t}-\bar{s}|^2}{2\delta}-\frac{\eta}{T-\bar{t}}-\gamma|\bar{x}|^2. \end{aligned} \tag{4.20}$$

for certain $\bar{x},\bar{y}\in\mathbb{R}$ and $\bar{t},\bar{s}\in[0,T]$. Note that the supremum is attained since the mapping

$$(x,y,t,s)\mapsto\psi(x,y,t,s)=u(x,t)-v(y,s)-\frac{|x-y|^2}{2\varepsilon}-\frac{|t-s|^2}{2\delta}-\frac{\eta}{T-t}-\gamma|x|^2$$

is continuous and satisfies $\psi(x,y,t,s)\to-\infty$ as $|x|,|y|\to\infty, t\to T$. Moreover, for all $\kappa>0$, there exist $x_\kappa\in\mathbb{R}$ and $t_\kappa\in[0,T]$ such that

$$M\geqslant u(x_\kappa,t_\kappa)-v(x_\kappa,t_\kappa)\geqslant M-\kappa.$$

Hence, we arrive at

$$\begin{aligned} M_{\varepsilon,\delta,\gamma,\eta} &\geqslant \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} \left(u(x,t) - v(x,t) - \frac{\eta}{T-t} - \gamma|x|^2 \right) \\ &\geqslant u(x_\kappa,t_\kappa) - v(x_\kappa,t_\kappa) - \frac{\eta}{T-t_\kappa} - \gamma|x_\kappa|^2 \\ &\geqslant M - \kappa - \frac{\eta}{T-t_\kappa} - \gamma|x_\kappa|^2 \\ &\geqslant \frac{M}{2} > 0, \end{aligned}$$

for $\kappa = \frac{M}{6}$ and η, γ chosen sufficiently small: $\gamma \leqslant \frac{M}{6|x_\kappa|^2}$ and $\eta \leqslant \frac{M(T-t_\kappa)}{6}$.

Step 2: Avoiding $\bar{t} = 0$ and $\bar{s} = 0$. Assume that either $\bar{t} = 0$, or $\bar{s} = 0$. After computations in Step 1 of the proof we still have two free parameters: δ and ε. Notice that if $\delta \to 0, \varepsilon \to 0$, then by (4.20) we would get $|\bar{t} - \bar{s}| \to 0$ and $|\bar{x} - \bar{y}| \to 0$. We would then deduce

$$0 < M_{\varepsilon,\delta,\gamma,\eta} \to \underbrace{u(\bar{x},0) - v(\bar{x},0)}_{\leqslant 0} - \frac{\eta}{T} - \gamma|\bar{x}|^2 \leqslant -\frac{\eta}{T} < 0,$$

which is a contradiction.

Step 3: Avoiding $\bar{s} = T$ for small δ. By (4.20), we have $\bar{t} < T - c_\eta < T$. Again notice that if $\delta \to 0$, then $|\bar{t} - \bar{s}| \to 0$. Thus, $\bar{s} < T - \tilde{c}_\eta < T$ for sufficiently small δ.

Step 4: Viscosity inequalities. We know form (4.20) that $u(\bar{x},\bar{t}) - v(\bar{y},\bar{s}) - \Phi(\bar{x},\bar{t},\bar{y},\bar{s})$ has a local maximum at $(\bar{x},\bar{t},\bar{y},\bar{s})$, where

$$\Phi(x,t,y,s) = \frac{|x-y|^2}{2\varepsilon} + \frac{|t-s|^2}{2\delta} + \frac{\eta}{T-t} + \gamma|x|^2. \tag{4.21}$$

Then it is natural to apply the classical Ishii's Lemma in the elliptic case with the new coordinates $(\tilde{x} = (x,t), \tilde{y} = (y,s))$. Indeed, we only use a corollary of Ishii's Lemma, namely Corollary 4.9. Applying that result, we get for every $\mu > 0$ satisfying $\mu\bar{A} \leqslant \bar{I}$ (with $\bar{A}$ defined in (4.19) and $\bar{I}$ is the identity matrix in $\mathbb{R}^4$) the existence of $q_1, q_2 \in \mathbb{R}$ such that

$$(\tau_1, p_1, \begin{pmatrix} q_1 & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^+ u(\bar{x},\bar{t}), \quad (\tau_2, p_2, \begin{pmatrix} q_2 & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^- v(\bar{y},\bar{s}) \tag{4.22}$$

and

$$-\Big(\frac{1}{\mu}+\|\bar{A}\|\Big)I \leqslant \begin{pmatrix} q_1 & 0 \\ 0, & -q_2 \end{pmatrix} \leqslant A+2\mu A^2, \tag{4.23}$$

where I is the identity matrix in $\mathbb{R}^2$ and other terms are given by $\tau_1 = \frac{\bar{t}-\bar{s}}{\delta}+\frac{\eta}{(T-\bar{t})^2}$, $p_1=\frac{\bar{x}-\bar{y}}{\varepsilon}+2\gamma\bar{x}$, $\tau_2=\frac{\bar{t}-\bar{s}}{\delta}$, $p_2=\frac{\bar{x}-\bar{y}}{\varepsilon}$ and

$$A=\frac{1}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}+2\gamma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, we can write the viscosity inequalities for the limit sub- superdifferentials $\bar{\mathcal{D}}^+u(\bar{x},\bar{t})$ and $\bar{\mathcal{D}}^-v(\bar{y},\bar{s})$. This gives, according to Definition 4.6, $\tau_1 \leqslant |p_1|(q_1-\sigma)$ and $\tau_2 \geqslant |p_2|(q_2-\sigma)$ since u is a viscosity subsolution of (4.17), and v, supersolution. Precisely, these inequalities reads as

$$\frac{\eta}{(T-\bar{t})^2}+\frac{\bar{t}-\bar{s}}{\delta} \leqslant \left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right|(q_1-\sigma)+2\gamma|\bar{x}|\Big(|q_1|+|\sigma|\Big)$$

and

$$\frac{\bar{t}-\bar{s}}{\delta} \geqslant \left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right|(q_2-\sigma).$$

Subtracting the viscosity inequalities, we arrive at

$$\frac{\eta}{(T-\bar{t})^2} \leqslant (q_1-q_2)\left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right|+2\gamma|\bar{x}|\Big(|q_1|+|\sigma|\Big). \tag{4.24}$$

Step 5: Conclusions. First, notice that due to (4.20), we know that $\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon}$ and $\gamma|\bar{x}|^2$ are bounded. Thus, we conclude $\frac{|\bar{x}-\bar{y}|}{\varepsilon} \leqslant \frac{C}{\sqrt{\varepsilon}}$ and $\gamma|\bar{x}| \leqslant C\sqrt{\gamma}$ for some constant $C>0$. Moreover, taking the matrix inequality (4.23) on vectors $\begin{pmatrix} \xi \\ \xi \end{pmatrix}^T$ and $\begin{pmatrix} \xi \\ \xi \end{pmatrix}$, we obtain

$$(q_1-q_2)\xi^2 \leqslant 2\gamma\xi^2+4\mu\|A^2\|\xi^2, \quad \forall \xi \in \mathbb{R}.$$

Additionally, taking the same matrix inequality but this time applied to vectors $\begin{pmatrix} \xi \\ 0 \end{pmatrix}^T$ and $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$, we get

$$q_1\xi^2 \leqslant \frac{1}{\varepsilon}\xi^2+2\gamma\xi^2+2\mu\|A^2\|\xi^2, \quad \forall \xi \in \mathbb{R}.$$

Thus, from (4.24) with all the estimates calculated above, we obtain

$$\frac{\eta}{(T-\bar{t})^2} \leqslant \frac{2\gamma C}{\sqrt{\varepsilon}} + \frac{4\mu\|A^2\|C}{\sqrt{\varepsilon}} + 2C\sqrt{\gamma}\Big(\frac{1}{\varepsilon} + 2\gamma + 2\mu\|A^2\| + |\sigma|\Big).$$

Now passing to the limit with $\gamma \searrow 0$ and $\mu \searrow 0$ the right-hand side of the above inequality tends to zero, while the quantity on the left-hand side is constant. Thus, we get a contradiction. ■

4.2 Existence versus non-existence of self-similar solutions in the case $\alpha = 2$

A solution $u = u(x,t)$ of a non-linear evolution partial differential equations is called self-similar if the knowledge of u at the instance of time t_0 is sufficient to obtain u for all $t > 0$ by a suitable rescaling. For instance in the case of the porous medium equation

$$u_t = \Delta(|u|^{m-1}u) \tag{4.25}$$

with $m > 1$, it is known that every non-negative solution eventually converges to a Barenblatt-Pattle solution, *i.e.* a solution of (4.25) with the Dirac delta function as the initial data. Such solutions are called fundamental solutions of (4.25). Later on, Kamin and Vázquez [44] extended this result to solutions not necessarily positive but with positive integrals by showing that those solutions become positive in a finite time. Moreover, in the same paper [44, Section 5] they proved convergence of solutions with zero mass to so-called dipole solutions, *i.e.* solutions of (4.25) with $u(x,0) = \delta_0'(x)$.

Additionally, in [40] Hulshof studied similarity solutions of the form

$$u(x,t) = t^{\alpha}U(xt^{-\beta}), \tag{4.26}$$

with the following relation imposed on the parameters: $2\beta = (m-1)\alpha + 1$. He proved the existence of the strictly decreasing sequence α_n such that compactly supported similarity solutions of the type above exist if and only if $\alpha = \alpha_k$ for some integer $k \geqslant 0$. Moreover, the solution changes its sign exactly k times and is symmetric (antisymmetric) if k is even (odd). This result includes the Barenblatt-Pattle ($\alpha_0 = -\frac{1}{m+1}$) and dipole ($\alpha_1 = -\frac{1}{m}$) solutions.

This section is devoted to study of the existence of self-similar solutions of (4.17) and the intermediate asymptotics, *i.e.* the property which can be described in the language of dynamical systems as the global asymptotic stability of self-similar profiles.

Theorem 4.11 (Existence of self-similar solutions). *Let $\sigma = 0$ and $\beta = \frac{1}{4}$. Consider a function $\Phi : \mathbb{R} \to \mathbb{R}$ defined as*

$$\Phi(\xi) = -\frac{1}{18}\Big(1 - |\xi|^{\frac{3}{2}}\Big)_+^2. \tag{4.27}$$

Then the function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ defined by (4.8) *is a unique self-similar solution of* (4.17).

Remark 4.12. *Let $\sigma = 0$. Since mass $M = -\int u(x,t)\ \mathrm{d}x$ is conserved in time for any $M \in [0, \infty)$, we can find, applying Proposition 4.4, i.e. suitably scaling, a self-similar solution u with any $M > 0$*

$$u(x,t) = -\frac{1}{18(t+1)^{\frac{1}{4}}}\left(R^{\frac{3}{2}} - \left|\frac{x}{(t+1)^{\frac{1}{4}}}\right|^{\frac{3}{2}}\right)_+^2, \tag{4.28}$$

where $R = \sqrt[4]{20M}$.

Lemma 4.13 (Uniqueness of β). *Let $\sigma = 0$. Then $\beta = \frac{1}{4}$ is uniquely prescribed value of the parameter β for which there exists a self-similar solution of* (4.17) *defined by* (4.8).

As we have already mentioned, mass of a solution of (4.17) is conserved. It is the natural consequence since Equation (4.17) with $\sigma = 0$, which is the case in our consideration, can be written in a divergent form. Namely, the right-hand side of (4.17) is of the form $\frac{1}{2}\Big((|u_x|u_x)^2\Big)_x$ which allows us to conclude that mass, integral of u, is conserved. Therefore, we choose the value of β such that the mass of the rescaled solution does not change in time. Indeed, by change of variables, we have

$$\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{(t+1)^{1-3\beta}}\Phi\left(\frac{x}{(t+1)^{\beta}}\right)\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{(t+1)^{1-4\beta}}\Phi(y)\,\mathrm{d}y.$$

The above calculations prescribed $\beta = \frac{1}{4}$ uniquely.

Proof of Lemma 4.13. Contrarily, let us assume that there exist $\tilde{\beta} \neq \frac{1}{4}$ and a function $\tilde{\Phi}(\xi) : \mathbb{R} \to \mathbb{R}$ with compact support satisfying (4.9). Then u and $\tilde{u}_{\tilde{\beta}}$ are two different solutions of (4.17) defined, according to Definition 4.1, as

$$u(x,t) = \frac{1}{(t+1)^{\frac{1}{4}}}\Phi\left(\frac{x}{(t+1)^{\frac{1}{4}}}\right), \quad \tilde{u}_{\tilde{\beta}}(x,t)\frac{1}{(t+1)^{1-3\tilde{\beta}}}\tilde{\Phi}\left(\frac{x}{(t+1)^{\tilde{\beta}}}\right).$$

Applying Proposition 4.4 to the function $\tilde{u}_{\tilde{\beta}}$, we can define a new function

$$\tilde{u}^{\lambda}_{\tilde{\beta}}(x,t) = \frac{1}{(t+1)^{1-3\tilde{\beta}}}\frac{1}{\lambda^3}\tilde{\Phi}\left(\frac{\lambda x}{(t+1)^{\tilde{\beta}}}\right), \tag{4.29}$$

which also is a solution of (4.17) for each $\lambda > 0$.

Let us assume for a moment that $\frac{1}{4} < \tilde{\beta}$. Taking functions u and $\tilde{u}^{\lambda}_{\tilde{\beta}}$ at time $t = 0$ and varying the value of λ, we are able to rescale a solution $\tilde{u}^{\lambda}_{\tilde{\beta}}$ such that it will fit between a solution u and the zero function. Thus, we obtain

$$u(x,0) \leqslant \tilde{u}^{\lambda}_{\tilde{\beta}}(x,0) \leqslant 0 \quad \text{for all} \quad x \in \mathbb{R}, \tag{4.30}$$

for some λ sufficiently big. From the comparison principle, Theorem 4.10, we can see that the inequality in (4.30) is satisfied not only for initial time $t = 0$ but for all $t > 0$, namely

$$u(x,t) \leqslant \tilde{u}^{\lambda}_{\tilde{\beta}}(x,t) \leqslant 0 \quad \text{for all} \quad x \in \mathbb{R},\, t > 0. \tag{4.31}$$

On the other hand, the inequality (4.31) implies

$$0 \leqslant \frac{1}{(t+1)^{\frac{1}{4}}} \leqslant \frac{1}{(t+1)^{1-3\tilde{\beta}}} \quad \text{for all} \quad t > 0, \tag{4.32}$$

which is a contradiction. Since $1 - 3\tilde{\beta} < \frac{1}{4}$, the inequality (4.31) cannot be satisfied for $t \gg 1$. In the case $\tilde{\beta} < \frac{1}{4}$ we can repeat our reasoning scaling, according to the Proposition 4.4, a solution u instead of $\tilde{u}_{\tilde{\beta}}$. Again, varying the value of λ we can locate the rescaled u between $\tilde{u}_{\tilde{\beta}}$ and the zero function which leads to a contradiction as before. This ends the proof. ∎

Proof of Theorem 4.11. Let $\sigma = 0$ and $\beta = \frac{1}{4}$. Assume that a solution u of the problem (4.17) is of the form (4.8). Then a function $\phi(\xi)$ satisfies the following equation

$$\phi(\xi) + \xi\phi'(\xi) = -4|\phi'(\xi)|\phi''(\xi), \quad \text{for } \xi \in \mathbb{R}.$$

Since we are looking for an even, non-positive and compactly supported function ϕ satisfying $\xi\phi'(\xi) \geqslant 0$ for each $\xi \in \mathbb{R}$, we may consider the above equation only on the positive half-line. Thus, integrating it from 0 to x, we get

$$\xi\phi(\xi) = -2\Big(\phi'(\xi)\Big)^2, \quad \text{for } \xi \geqslant 0.$$

Due to Proposition 4.4, we may assume that $\operatorname{supp}\phi = [-1, 1]$. Hence, integrating the equation over the interval $[0, x]$ once more we obtain, after simple algebraic computations, that the function ϕ is given by (4.27) which completes the proof. ∎

Theorem 4.14 (Solution defined almost everywhere)**.** *Let $\alpha = 2$ and $\sigma = -1$. Consider a function $\Phi\colon \mathbb{R} \to \mathbb{R}$ defined as*

$$\Phi(\xi) = -\frac{1}{18}(1 - |\xi|)_+^3. \tag{4.33}$$

Then the function $u\colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ defined according to Definition 4.2 is an almost everywhere, self-similar solution of (4.17)*. Indeed, it is a solution except $x = 0$.*

However, the function u is not a viscosity solution in the sense of Definition 4.6.

Proof. Due to Proposition 4.4 we can assume that $\sigma = -1$. Then it is straightforward to check that the function Φ given by (4.33) satisfies the equation

$$-2\Phi(\xi) + \xi\Phi'(\xi) = |\Phi'(\xi)|\Big(\Phi''(\xi) + 1\Big), \quad \text{for } \xi \in \mathbb{R}\backslash\{0\}. \tag{4.34}$$

Hence, the function $u(x,t) = (T-t)^2\Phi(x/(T-t))$ is a solution of (4.17) except one point: $x = 0$.

To prove that u is not a viscosity solution of (4.17), it is enough to prove that Φ is not a viscosity solution of (4.34) over the whole line $\mathbb{R}$. Indeed, according to Definition 4.6, it suffices to construct a smooth test function ψ such that $\psi(0) = \Phi(0)$ and $\Phi - \psi$ has a local minimum at $x = 0$, and the following relation holds:

$$-2\psi(0) < |\psi'(0)|\Big(\psi''(0) + 1\Big). \tag{4.35}$$

Notice that if we construct such a test function, then, according to Definition 4.6, the function Φ (defined by (4.33)) is not a viscosity supersolution at the point $x = 0$. Thus, it is not a viscosity solution either.

Let us define a test function $\psi(\xi) = -\frac{1}{18} + C_1\xi + \frac{C_2}{2}\xi^2$ with $C_1, C_2 > 0$. Now since a test function is supposed to be below the function Φ (otherwise the difference $\Phi - \psi$ does not attain a local minimum at zero), we obtain the estimate $C_1 = \psi'(0) \leqslant |\Phi'(0^{\pm})| \leqslant \frac{1}{6}$. Furthermore, calculating derivatives one can check that indeed the difference $\Phi - \psi$ has a local minimum at zero over the interval $[-\bar{x}, \bar{x}]$, where

$$\bar{x} = \frac{2 + 6C_2 - \sqrt{((2+6C_2)^2 - 4(1-6C_1))}}{2} > 0.$$

Taking $C_2 > \max\{\frac{1}{9C_1} - 1, 0\}$ the estimate (4.35) is valid and we conclude that the function ϕ given by (4.33) is not a viscosity solution of (4.34) over $\mathbb{R}$. ∎

The above proof shows that whenever a viscosity, self-similar solution of the problem (4.17) exists, it must be a $C^1(\mathbb{R})$ function. However, due to the non-linearity, we cannot expect uniqueness of solutions of the problem (4.34). Thus, we cannot guarantee that there is no $C^1(\mathbb{R})$ self-similar solution of (4.17). Nevertheless, we expect such a result.

Now there are only two possibilities. Either the solution has different scaling than we proposed in Definition 4.2, or the solution is not self-similar at all. To exclude the first potential possibility we can assume that the solution scales in the following way

$$u(x,t) = \gamma(t)\Phi\left(\frac{x}{\delta(t)}\right)$$

for some unknown functions $\gamma(t)$ and $\delta(t)$. Then, however, if we put such u in the equation (4.17), we notice that whatever is a function Φ it must satisfy the following equation

$$\gamma'\Phi - \frac{\gamma\delta' y}{\delta}\Phi' = \frac{\gamma}{\delta}|\Phi'|\left(\frac{\gamma}{\delta^2}\Phi'' - \sigma\right),$$

where $\Phi = \Phi(y)$ with a new variable $y = \frac{x}{\delta(t)}$. Nevertheless, the above equation is to be independent of t. Thus, a function $\gamma(t)$ is required to be of the form $\gamma(t) = \delta^2(t)$ and we can write

$$\delta'(t)\Big(2\Phi - y\Phi'\Big) = |\Phi'|(\Phi'' - \sigma). \tag{4.36}$$

Consequently, $\delta'(t) = const$ and hence either $\delta(t) = t$ or $\delta(t) = T - t$ for some positive value of T. Though, in the first case $\delta(t) = t$, the left-hand side of (4.36) is strictly negative in a neighbourhood of zero since we are looking for a negative, compactly supported solution such that $y\Phi'(y) \geqslant 0$, whereas the right-hand side of (4.36) is strictly positive provided σ negative. Therefore, the only reasonable self-similar scaling would be the one proposed at the beginning in Definition 4.2 for which, however, we expect the non-existence of solutions. Nonetheless, we are still able to show the extinction of solutions of (4.17) in a finite time, see below Chapter 5.

Theorem 4.15 (Existence of self-similar solutions)**.** *Let $\sigma > 0$. Consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined as*

$$\Phi_A(\xi) = -\frac{A-\sigma}{2}(A - |\xi|)_+^2, \tag{4.37}$$

with $A \geqslant \sigma$. Then a function $u = (t+1)^2\Phi_A(\xi)$ defined by (4.12) is a unique self-similar solution of the problem

$$u_t = |u_x|(u_{xx} - \sigma) \quad on \quad (\mathbb{R}\backslash\{0\}) \times (0, +\infty), \tag{4.38}$$

$$u(0,t) = t^2\Phi(0). \tag{4.39}$$

The parameter A is uniquely determined by the initial condition (4.39)*.*

Remark 4.16. *Here $-t^2\Phi(0)$ is the total amount of dipoles and the rate of production of dipoles is $-2t\Phi'(0)$.*

Proof of Theorem 4.15. It is straightforward to check that the function Φ_A, defined above, satisfies the equation (4.13) which for $\alpha = 2$ assumes the form

$$|\Phi'|\Phi'' - \sigma|\Phi'| - 2\Phi + \xi\Phi' = 0 \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{4.40}$$

To show the uniqueness let us assume contrarily that there exists another function $\tilde{\Phi}$ satisfying (4.40) and denote $\Phi = \Phi_A$ for simplicity. Then we define

$$M = \sup(\Phi - \tilde{\Phi}) = (\Phi - \tilde{\Phi})(\xi_0) \geqslant (\Phi - \tilde{\Phi})(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{4.41}$$

Hence, we get

$$\begin{aligned} 0 &= |\tilde{\Phi}'(\xi_0)|\tilde{\Phi}''(\xi_0) - \sigma|\tilde{\Phi}'(\xi_0)| - 2\tilde{\Phi}(\xi_0) + \xi_0\tilde{\Phi}'(\xi_0) \\ &\geqslant |\Phi'(\xi_0)|\Phi''(\xi_0) - \sigma|\Phi'(\xi_0)| - 2\Phi(\xi_0) + \xi_0\Phi'(\xi_0) + 2(\Phi - \tilde{\Phi})(\xi_0) \\ &= 2(\Phi - \tilde{\Phi})(\xi_0) = 2M. \end{aligned} \tag{4.42}$$

This implies that $\tilde{\Phi} \geqslant \Phi$ for all $\xi \in \mathbb{R}$. Conversely, defining $M = \sup(\tilde{\Phi} - \Phi)$ the inequality $\Phi \geqslant \tilde{\Phi}$ can be easily shown, which leads to the conclusion that $\Phi = \tilde{\Phi}$ for all $\xi \in \mathbb{R}$. ∎

4.3 The entropy method for proving global asymptotic stability of self-similar solutions — intermediate asymptotics

Entropy is a special kind of the Lyapunov functional that combines well with other *a priori* estimates and is often used in the analysis of the time decay of solutions of the initial value problems towards their thermal equilibrium states. This method has been developed by, for instance Carrillo and Toscani for the classical Fokker-Planck equation [19, 73], and for general Fokker-Planck-type equations by Arnold *et al.* [6].

The entropy method consists essentially in deriving an equation for the evolution of the convex relative entropy and connecting it with with the non-negative entropy dissipation. Usually from this equation one can conclude the decay of the relative entropy but without the rate. Sometimes it is possible to obtain the rate of convergence by studying the time evolution of the entropy dissipation. In the literature such an approach is called the *entropy-entropy dissipation method*. However, in most cases computing the time derivative of the entropy dissipation does not provide the answer what the relation between the relative entropy and the entropy dissipation is. Therefore, to get the exponential decay of the relative entropy one needs to use some Sobolev-type inequality.

Let us consider the following non-linear Fokker-Planck equation

$$v_\tau = \nabla\Big(|\nabla v|^{p-2}\nabla v + yv\Big), \quad y \in \mathbb{R}^n,\ \tau > 0, \tag{4.43a}$$

$$v(y,0) = v_0(y), \tag{4.43b}$$

where we assume $p > 1$ and the initial condition satisfies $v_0 \in L^1(\mathbb{R}^n)$. In 1988 Kamin and Vázquez [43] proved an L^1 and L^∞ convergence of the non-negative solution to the fundamental solution with no rates. Later, Del Pino and Dolbeault [23, 24] obtained the rates of convergence for $\frac{2n+1}{n+1} < p < n$, using the entropy method described above, which were next improved by Agueh [1, 3]. Moreover, Agueh proved his results for $\frac{2n+1}{n+1} < p$.

It is worth emphasizing that after a suitable rescaling the equation (4.9) for the self-similar profile Φ is exactly the one-dimensional Fokker-Planck equation with $p = 3$ and a function v replaced by Φ. Since the results proved by Agueh were extended for a general equation for which the Fokker-Planck equation is a particular example, we shall rewrite its proof for this particular case, *i.e* (4.43) with $p = 3$ and $n = 1$ which reads as

$$v_\tau = \Big(|v_y|v_y + yv\Big)_y, \quad y \in \mathbb{R},\ \tau > 0, \tag{4.44a}$$

$$v(y,0) = v_0(y) \geqslant 0. \tag{4.44b}$$

Definition 4.17 (Relative entropy)**.** *Let $v = v(y, \tau)$ be a solution of* (4.44)*. Then we denote*

$$\Sigma(v) = \int_{\mathbb{R}} \tfrac{4}{3} v^{\frac{3}{2}} + \tfrac{2}{3} v |y|^{\frac{3}{2}} \, \mathrm{d}y \tag{4.45}$$

the free energy of the solution v. Moreover, we denote the relative entropy of v with respect to v_∞ by $\Sigma(v, v_\infty) = \Sigma(v) - \Sigma(v_\infty)$, where v_∞ is the stationary solution of (4.44) *given by*

$$v_\infty(y) = \Big(C - \tfrac{1}{3}|y|^{\frac{3}{2}}\Big)_+^2.$$

Theorem 4.18 (Convergence to the equilibrium)**.** *Let $v = v(y, \tau)$ be a solution of the problem* (4.44)*. Then the following estimate*

$$\|v(\cdot, \tau) - v_\infty\|_{L^1(\mathbb{R})} \leqslant C e^{-\frac{\tau}{2}} \sqrt{(\Sigma(v_0, v_\infty)} \tag{4.46}$$

holds provided $\Sigma(v_0, v_\infty) < \infty$. Moreover, any solution $u = u(x,t)$ of (4.17) *with $\sigma = 0$ defined as*

$$u(x,t) = (1+4t)^{-\frac{1}{4}} v\Big(\sqrt[3]{2} x (1+4t)^{-\frac{1}{4}}, \ln 4t\Big), \tag{4.47}$$

satisfies the convergence rate

$$\|u(\cdot, t) - u_\infty\|_{L^1(\mathbb{R})} \leqslant \frac{C \sqrt{(\Sigma(v_0, v_\infty)}}{(1+4t)^{\frac{1}{8}}}. \tag{4.48}$$

Proof. First, let us calculate the time derivative of the relative entropy. Namely, using the integration by parts and the equation itself, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \Sigma(v(\tau), v_\infty) &= - \int_{\mathbb{R}} \frac{|v_y|^3}{v^{\frac{1}{2}}} + y v_y v^{\frac{1}{2}} + \frac{y}{|y|^{\frac{1}{2}}} |v_y| v_y + |y|^{\frac{3}{2}} v \, \mathrm{d}y \\ &= -(I_1 + I_2 + I_3 + I_4) \stackrel{\text{def}}{=} -I(v(\tau)). \end{aligned}$$

Using the Hölder and Young inequalities, we shall show that $I(v(\tau))$, often called *entropy dissipation* (or the *generalised relative Fisher information of $v(\tau)$ with respect to v_∞*), is non-negative. Indeed, we have

$$\begin{aligned} |I_2| &\leqslant \int_{\mathbb{R}} v^{-\frac{1}{6}} v_y v^{\frac{2}{3}} y \, \mathrm{d}y \leqslant \left(\int_{\mathbb{R}} v^{-\frac{1}{2}} |v_y|^3 \, \mathrm{d}y \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}} v |y|^{\frac{3}{2}} \, \mathrm{d}y \right)^{\frac{2}{3}} \\ &\leqslant \frac{1}{3} \int_{\mathbb{R}} v^{-\frac{1}{2}} |v_y|^3 \, \mathrm{d}y + \frac{2}{3} \int_{\mathbb{R}} v |y|^{\frac{3}{2}} \, \mathrm{d}y = \frac{1}{3} I_1 + \frac{2}{3} I_4. \end{aligned}$$

Thus, we conclude $I_2 \geqslant -\frac{1}{3}I_1 - \frac{2}{3}I_4$. Similarly, we get

$$|I_3| \leqslant \int_{\mathbb{R}} |v_y|^2 v^{-\frac{1}{3}} v^{\frac{1}{3}} |y|^{\frac{1}{2}} \, \mathrm{d}y \leqslant \left(\int_{\mathbb{R}} |v_y|^3 v^{-\frac{1}{2}} \, \mathrm{d}y \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} |y|^{\frac{3}{2}} v \, \mathrm{d}y \right)^{\frac{1}{3}}$$
$$\leqslant \frac{2}{3} \int_{\mathbb{R}} |v_y|^3 v^{-\frac{1}{2}} \, \mathrm{d}y + \frac{1}{3} \int_{\mathbb{R}} |y|^{\frac{3}{2}} v \, \mathrm{d}y = \frac{2}{3} I_1 + \frac{1}{3} I_4$$

and conclude $I_3 \geqslant -\frac{2}{3}I_1 - \frac{1}{3}I_4$. Due to these two estimates, we prove that the dissipation I is non-negative and hence the relative entropy is a nonincreasing function in time.

Using the Sobolev-type inequality (Lemma 4.19 below) and integrating the intermediate inequality, we arrive at

$$\Sigma(v(\tau), v_\infty) \leqslant e^{-\tau} \Sigma(v_0, v_\infty).$$

Hence, we obtain the exponential rate of decay of the relative entropy. Now applying the Csiszár-Kullback-type inequality (Lemma 4.20 below) we conclude the decay rate (4.46) for the function v. To complete the proof we exploit the already proved estimate (4.46) with the definition (4.47) of the function u. ■

Lemma 4.19 (Log-Sobolev-type inequality)**.** *Let* $v = v(y, \tau)$ *be a solution of the problem* (4.44)*. Then we have*

$$\Sigma(v(\tau), v_\infty) \leqslant I(v(\tau)), \quad \textit{for all } \tau > 0,$$

where I *is as in the proof of Theorem 4.18.*

For the proof of the above Lemma see for instance [3, Theorem 2.2]. It consists of the Hölder inequality applied to some energy inequality defined in [2, Proposition 2.7].

Lemma 4.20 (Csiszár-Kullback-type inequality)**.** *Let* $v = v(y, \tau)$ *be a solution of the problem* (4.44)*. Then we have*

$$\|v(\cdot, \tau) - v_\infty\|^2_{L^1(\mathbb{R})} \leqslant C\Sigma(v(\tau), v_\infty),$$

with the constant $C = C(v_\infty) > 0$*.*

For the proof, see [3, Theorem 4.1].

4.4 Non-existence of self-similar solutions in the case $\sigma = 0$ and $\alpha = 1$

Theorem 4.21 (Non-existence of self-similar solutions)**.** *Let* $\sigma = 0, \beta = \frac{1}{2}$. *Then there does not exist a self-similar solution of* (1.6a) *defined according to Definition 4.1.*

Assume $\sigma = 0$, then we notice that as long as the value of the parameter β is chosen appropriately, the equation (4.9) can be rewritten in a more accessible form. Indeed, if we choose $\beta = \frac{1}{\alpha+1}$ and apply Proposition 4.4, we obtain the new Dirichlet problem

$$\begin{cases} \Lambda^\alpha \Phi(\xi) = f(\xi), & \xi \in (-1,1), \\ \Phi(\xi) = 0, & \xi \in \mathbb{R}\backslash(-1,1), \end{cases} \tag{4.49}$$

with $f(\xi) = \frac{1}{\alpha+1}|\xi|$.

There is an extensive literature devoted to the problem (4.49). The existence result in [65, 11] says that for a given function $f \in L^2(-1,1)$ there exists a unique solution $\Phi \in H^{\frac{\alpha}{2}}(\mathbb{R})$. Moreover, the solution is given explicitly by the Green function for the fractional derivative operator, *i.e.*

$$\Phi(\xi) = \int_{-1}^{1} G(\xi,y) f(y)\, \mathrm{d}y, \tag{4.50}$$

where

$$G(x,y) = C_{n,s}|x-y|^{\alpha-1} \int_0^{r_0(x,y)} \frac{r^{\frac{\alpha}{2}-1}}{(r+1)^{\frac{1}{2}}}\, \mathrm{d}r \tag{4.51}$$

with

$$r_0(x,y) = \frac{(1-x^2)(1-y^2)}{(x-y)^2}. \tag{4.52}$$

which also holds in multidimensional case with slight modifications.

In particular for $\Omega = \mathbb{R}_+$ and $f \equiv 0$, the solution of (4.49) is given explicitly by $\Phi(x) = x_+^{\frac{\alpha}{2}}$. We call Φ to be $\frac{\alpha}{2}$-harmonic in $\mathbb{R}_+$. Another example comes from [32], where Getoor showed that the function $\Phi(x) = C_{n,\alpha}\Big(r^2 - |x|^2\Big)^{\frac{\alpha}{2}}$ is the solution of (4.49) with $f \equiv 1$ and $\Omega = B_r \subset \mathbb{R}^n$.

Furthermore, for $\Omega = \mathbb{R}^n$ the author of [67] proved the regularity result: if $f \in C^\beta(\mathbb{R}^n)$, then $\Phi \in C^{\alpha+\beta}(\mathbb{R}^n)$ as long as $\beta + \alpha$ is not an integer. Later in 2012, Ros-Otton and Serra proved in [66] that for f bounded over a $C^{1,1}$ domain Ω, the solution of (4.49) satisfies $\Phi \in C^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and $\Phi(x)\Big(\mathrm{dist}(x, \partial\Omega)\Big)^{-\frac{\alpha}{2}} \in C^\beta(\overline{\Omega})$ for some $\beta \in (0, 1)$.

However, there is still a different way to get the explicit solution of (4.49) in the particular case $\alpha = 1$. Henceforth, assume $\alpha = 1$ then applying Lemma 2.2 and using the fact that the Hilbert transform $\mathcal{H}$ and $\frac{\mathrm{d}}{\mathrm{d}x}$ commute, see for instance [60], we can again rewrite the above equation such that we obtain the following problem

$$\begin{cases} \mathcal{H}\Phi(\xi) = \frac{1}{4}|\xi|\xi, & \forall \xi \in (-1, 1), \\ \Phi(\xi) = 0, & \xi \in \mathbb{R}\backslash(-1, 1), \end{cases} \tag{4.53}$$

where $\mathcal{H}$ is the Hilbert transform. Now taking the inverse operator to $\mathcal{H}$, which is nothing else but $-\mathcal{H}$, we would like to solve the equation. However, we do not know how the right-hand side of (4.53) looks like outside the interval $(-1, 1)$ which is required by the Hilbert transform. Therefore, we cannot use it directly. Instead, we exploit the general result given by Tricomi [74, Chapter 4.3, p. 173] on the so called finite Hilbert transform. Original applications of that singular integral transform included problems in aerodynamics related to wing profiles.

Theorem 4.22. *If the function $f(x)$ belongs to the class $L^p(-1, 1)$ for $p > \frac{4}{3}$, then the so-called airfoil equation*

$$f(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\Phi(y)}{y - x}\,\mathrm{d}y \stackrel{\text{def}}{=} \mathcal{F}_x[\Phi(y)] \tag{4.54}$$

has the solution

$$\Phi(x) = -\frac{1}{\pi}P.V.\int_{-1}^{1} \sqrt{\frac{1 - y^2}{1 - x^2}}\frac{f(y)}{y - x}\,\mathrm{d}y + \frac{C}{\sqrt{1 - x^2}}, \tag{4.55}$$

where C is an arbitrary constant.

Here $\mathcal{F}_x$ is called the finite Hilbert transform. The proof of the theorem as well as the justification why p should be greater than $\frac{4}{3}$ can be found in [74, Chapter 4.3, p. 173].

Remark 4.23. *The first term on the right-hand side belongs to the class* $L^p(-1,1)$ *with* $1 < p < \frac{4}{3}$*. Moreover, The second term belongs to* $L^p(-1,1)$ *with* $1 < p < 2$ *and represents a general solution of the corresponding homogeneous equation in the space* $L^p(-1,1)$ *for* $p > 1$*.*

Due to Proposition 4.4, we can always rescale a function Φ to get its support contained in $[-1,1]$. Therefore, we arrive at the problem (4.54) with a function $f(\xi) = \frac{1}{4}|\xi|\xi$. Since the function f is continuous, it is integrable over a compact interval and satisfies the assumption of Theorem 4.22. Hence, we get the following result

Lemma 4.24. *Let* $f(x) = -\frac{1}{4}|x|x$*. Then the airfoil equation* (4.54) *has explicit solutions*

$$\Phi(x) = \begin{cases} \frac{1-3x^2+6\pi C}{6\pi\sqrt{1-x^2}} + \frac{1}{4\pi}x^2 \ln\left(\frac{1+\sqrt{1-x^2)}}{1-\sqrt{1-x^2)}}\right), & x \in (-1,1)\backslash\{0\} \\ \frac{1}{6\pi} + C, \quad x = 0, \end{cases} \tag{4.56}$$

with an arbitrary constant C*.*

Proof. By Theorem 4.22, we know that solution is given by Equation (4.55). Thus, for $x \in (-1,1)\backslash\{0\}$ fixed, we can write

$$\begin{aligned} P.V. \int_{-1}^{1} \sqrt{\frac{1-y^2}{1-x^2}} \frac{y|y|}{4(y-x)} \, \mathrm{d}y &= \frac{1}{2\sqrt{1-x^2}} \, P.V. \int_0^1 \frac{y^3\sqrt{1-y^2}}{y^2-x^2} \, \mathrm{d}y \\ &= \frac{1}{6\sqrt{1-x^2}} + \frac{x^2}{2\sqrt{1-x^2}} \, P.V. \int_0^1 \frac{r^2}{1-r^2-x^2} \, \mathrm{d}r. \end{aligned} \tag{4.57}$$

The second equality comes simply from the change of variables, $\sqrt{1-y^2} = r$. Next, keeping in mind that x is a fixed number, we calculate the term on the right-hand side of (4.57). Namely, we have

$$\begin{aligned} P.V. \int_0^1 \frac{r^2}{1-r^2-x^2} \, \mathrm{d}r &= -1 + (1-x^2) \, P.V. \int_0^1 \frac{1}{1-r^2-x^2} \, \mathrm{d}r \\ &= -1 + \frac{1}{2}\sqrt{1-x^2}\Big(\ln\left(1+\sqrt{1-x^2}\right) - \ln\left(1-\sqrt{1-x^2}\right)\Big). \end{aligned} \tag{4.58}$$

Now we compute the value of the function Φ at the origin. One can notice that due to the form of the function f, the integral in (4.55) at $x = 0$ is integrable and thus we can simply conclude that

$$\Phi(0) = \frac{1}{4\pi} \int_{-1}^{1} \sqrt{1-y^2}\, |y| \,\mathrm{d}y + C = \frac{1}{6\pi} + C. \tag{4.59}$$

Collecting the results obtained in (4.57), (4.58) and (4.59) with the equation for the general form of solutions, *i.e.* (4.55), we obtain (4.56). ■

Proof of Theorem 4.21. In Lemma 4.24, we have already constructed a family of solutions of the problem (4.53). Since we started our analysis in Lemma 4.56 with considering compact supported functions, we would like to find the one solution such that it is continuous at $x = \pm 1$, *i.e.* to find an appropriate value of the constant C.

First, we notice that the limit when $x \to \pm 1$ of the second term, the one containing a logarithmic function, of (4.56) is zero. Next, we need to choose $C = \frac{1}{3\pi}$ to get

$$\lim_{x \to \pm 1} \frac{1 - 3x^2 + 6\pi C}{6\pi\sqrt{1-x^2}} = \lim_{x \to \pm 1} \frac{1}{2\pi}\sqrt{1-x^2} = 0,$$

otherwise the limit is not finite.

However, such solution does not satisfy the assumption $x\Phi(x) \geqslant 0$; moreover, it is non-negative contrarily to the assumptions imposed on the solution we looked for. ■

Chapter 5

Finite time extinction

As we have already shown there does not exist a self-similar solution defined by Definition 4.2 of the problem

$$u_t = |u_x|(u_{xx} - \sigma) \quad \text{for } x \in \mathbb{R} \tag{5.1a}$$

$$u(x, 0) = u_0(x) \tag{5.1b}$$

with $\sigma < 0$. However, we shall show the finite time extinction of solutions provided u_0 is a compactly supported initial condition. By the extinction we mean that the solution will disappear.

Definition 5.1 (Extinction). *We say that the solution u of* (5.1) *extincts in a finite time if there exists a finite time T such that $\|u(\cdot, t)\|_\infty > 0$ for $t \in [0, T)$ but $\|u(\cdot, T)\|_\infty = 0$, where $\|u(\cdot, t)\|_\infty = \sup_{x\in\mathbb{R}} |u(x, t)|$. The time T is called the extinction time of the solution u.*

Theorem 5.2 (Extinction in finite time). *Let u be a solution of* (5.1)*, then there exists a finite time T, controlled form below and from above by some well defined constants $\underline{T}, \overline{T}$ (which will appear in the proof), such that a solution u of* (5.1) *extincts at time T.*

5.1 Extinction time

In order to prove Theorem 5.2 we first recall the definition of viscosity solutions. Then we construct sub- and supersolutions of the problem (5.1) such that the initial condition will fit exactly between them. Next, we justify that such a relation is satisfied not only at the initial time but for each time concluding the extinction of the solution of (5.1).

Definition 5.3 (Viscosity solutions). *Let v be a locally bounded function. Then the upper semi-continuous (resp. lower semi-continuous) function v is called a viscosity subsolution (resp. supersolution) of*

$$v_t = F(x, v, v_x, v_{xx}) \quad \textit{for } x \in \mathbb{R} \tag{5.2}$$

if for any point x_0 and test function $\psi \in C^2(\mathbb{R})$ such that $v - \psi$ reaches its local maximum (resp. local minimum) at x_0, we have

$$\psi_t(x_0) \leqslant F(x, v(x_0), \psi_x(x_0), \psi_{xx}(x_0))$$
$$(\textit{resp. } \psi_t(x_0) \geqslant F(x, v(x_0), \psi_x(x_0), \psi_{xx}(x_0)).$$

A continuous function v is a viscosity solution of (5.2) if it is a viscosity subsolution and supersolution.

Proposition 5.4 (Subsolution). *Let $\sigma < 0$ be fixed and, according to Definition 4.2, let us consider a function $\underline{u} = (T-t)^2 \underline{\Phi}\Big(\frac{x}{(T-t)}\Big)$ such that $\underline{\Phi}$ is defined in the following way*

$$\underline{\Phi}(\xi) = -\frac{1}{18}\Big(|\sigma| - |\xi|\Big)_+^3, \tag{5.3}$$

then $\underline{u}$ is a viscosity subsolution of (5.1) for all $T > 0$.

Proof. The function $\underline{u}$ is a viscosity subsolution of (5.1) if and only if $\underline{\Phi}$ is a viscosity subsolution of the following problem

$$-2\Phi + \xi\Phi' = |\Phi'|(\Phi'' - \sigma) \quad \text{for } \xi = \frac{x}{T-t} \in \mathbb{R}. \tag{5.4}$$

However, the above defined function $\underline{\Phi}$ is actually a classical solution of (5.4) for $\xi \neq 0$ which is straightforward to check. Finally, by Definition (5.3) there does not exist any test function $\psi \in C^1(\mathbb{R})$ such that $\Phi - \psi$ attains a local maximum at point $x_0 = 0$. Thus, $\underline{\Phi}$ is a viscosity subsolution of (5.4); consequently, $\underline{u}$ is a viscosity subsolution of (5.1). ∎

Proposition 5.5 (Supersolution)**.** *Let $\sigma < 0$ be fixed and, according to Definition 4.2, let us consider a function $\overline{u} = (T-t)^2\overline{\Phi}\Big(\frac{x}{(T-t)}\Big)$ such that $\overline{\Phi}$ is defined in the following way*

$$\overline{\Phi}(\xi) = -\frac{1}{9}\Big(|\sigma|^{\frac{3}{2}} - |\xi|^{\frac{3}{2}}\Big)_+^2, \tag{5.5}$$

then $\overline{u}$ is a viscosity supersolution of (5.1) *for all $T > 0$.*

Proof. As previously, we state that the function $\overline{u}$ is a viscosity supersolution of (5.1) if and only if $\overline{\Phi}$ is a viscosity supersolution of (5.4). Let us denote $\Phi = -v^2$ for some function $v(\xi) = A(1 - |\xi|^{\frac{3}{2}})_+$ and some constant $A > 0$. Then due to the fact that Φ is symmetric, we can consider the problem only on the positive half-line. Thus, let $\xi \in [0, 1)$ (for $\xi \geqslant 1$ the function $\Phi \equiv 0$ which obviously satisfies (5.4) since zero is a trivial solution), then v satisfies the inequality

$$2(v')^3 + v\Big((v')^2\Big)' + \xi v' - v \leqslant |\sigma|v' \quad \text{for } \xi \in [0, 1).$$

Consequently, from the definition of v we can calculate

$$v'(\xi) = -\frac{3A}{2}\xi^{\frac{1}{2}}, \qquad \Big((v')^2\Big)'(\xi) = \frac{9}{4}A^2$$

obtaining a new inequality written in terms of $z = \sqrt{\xi} \in [0, 1)$. Indeed, we have

$$18A^2\Big(\frac{1}{4} - z^3\Big) - 2 - z^3 + 3|\sigma|z \leqslant 0, \tag{5.6}$$

where the left-hand side of the above inequality we denote by $f_A(z)$. First, we get an inequality on A, namely $A \leqslant \frac{3}{2}$, otherwise $f_A(0) > 0$. Next, we calculate the first derivative of $f_A(z)$:

$$f_A'(z) = -54A^2z^2 - 3z^2 + 3|\sigma|$$

in order to check where the function f_A reaches its maximum. One can notice that $\overline{z}_A = \sqrt{\frac{|\sigma|}{1+18A^2}}$ is a solution of $f_A'(z) = 0$. Now there are two cases to consider. The first one: $\overline{z}_A \geqslant 1$. Then for $z \in [0, 1)$

$$f_A(z) \leqslant f_A(1) = 3\Big(|\sigma| - 1 - \frac{18}{4}A^2\Big) \leqslant 0$$

implies $|\sigma| \leqslant 1 + \frac{18}{4}A^2$, which is a contradiction to the assumption $\overline{z}_A \geqslant 1$. On the other hand, in the second case: $\overline{z}_A < 1$, we obtain

$$f_A(z) \leqslant f_A(\overline{z}_A) = 2|\sigma|\sqrt{\frac{|\sigma|}{1+18A^2}} - \left(2 - \frac{9}{2}A^2\right) \leqslant 0,$$

which implies $|\sigma| \leqslant \left(\left(1 - \frac{9}{4}A^2\right)\sqrt{1+18A^2}\right)^{\frac{2}{3}}$.

Hence, so far we have proved that a function $\overline{\Phi} = -A^2(1 - |\xi|^{\frac{3}{2}})_+^2$ is a supersolution of (5.4) as long as the relation

$$|\sigma| \leqslant \left(\left(1 - \tfrac{9}{4}A^2\right)\sqrt{1+18A^2}\right)^{\frac{2}{3}} \tag{5.7}$$

is satisfied.

However, we can still construct a supersolution for any given negative σ. To do so, we first choose an appropriate pair (σ, A) satisfying the relation (5.7), *e.g.* $\sigma = -1$ and $A = \frac{1}{3}$. Indeed, the right-hand side of (5.7) reaches its maximum $\left(\frac{3\sqrt{3}}{4}\right)^{\frac{2}{3}} > 1$ at point $A = \frac{1}{3}$.

Next, applying Proposition 4.4 with $\lambda = \frac{1}{|\sigma|}$, we can rescale $\overline{\Phi}$ such that this is still a supersolution of (5.4) for any negative σ and has the desired form. ∎

Proof of Theorem 5.2. Let as denote by u a solution of (5.1) with a compactly supported initial condition u_0. Let $\mathrm{supp}(u_0) = [-A, A]$ for some $A > 0$. By Propositions 5.4 and 5.5 we constructed explicit sub- and supersolutions of (5.1); however, we have not expressed yet what the value of T is.

One can notice that by the definition of $\overline{u}$ we can always choose T small enough such that u_0 is below $\overline{u}(\cdot, 0)$. Therefore, let us denote

$$\underline{T} = \max\left\{T : T \leqslant \frac{A}{|\sigma|},\ T^2\,\overline{\Phi}\left(\frac{x}{T}\right) \geqslant u_0(x),\ \forall x \in \mathbb{R}\right\}. \tag{5.8}$$

On the other hand, we can always choose T big enough such that u_0 is above $\underline{u}(\cdot, 0)$. Hence, again we set

$$\overline{T} = \min\left\{T : T \geqslant \frac{A}{|\sigma|},\ T^2\,\underline{\Phi}\left(\frac{x}{T}\right) \leqslant u_0(x),\ \forall x \in \mathbb{R}\right\}. \tag{5.9}$$

Now applying the comparison principle, see Theorem 4.10, we obtain the following relation

$$\underline{u}(x,t) \leqslant u(x) \leqslant \overline{u}(x,t) \leqslant 0, \quad \forall x \in \mathbb{R} \text{ and } t > 0.$$

Therefore, the solution u will extinct in finite time but not earlier than $\overline{u}$ and not later than $\underline{u}$. This implies that extinction time T of the solution u satisfies $\overline{T} \geqslant T \geqslant \underline{T}$ which completes the proof. ■

5.2 Numerical results

In this section we shall perform some numerical experiments which confirm the result obtained in Theorem 5.2 and give the approximate time of extinction of solutions of the problem (5.1). In general, we are not able to write a numerical scheme over the whole space. However, notice that since we are searching for non-positive solutions with compact support, it is sufficient to consider the following initial-boundary value problem

$$\begin{cases} u_t = |u_x|(u_{xx} - \sigma), & (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, & x \in \partial\Omega, \ t \in \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.10}$$

with $\Omega = [-A, A]$ a subset of $\mathbb{R}$ "large" enough and such that it contains the $\operatorname{supp}(u_0)$, where A will be specified later.

To discretise the domain $\Omega \times \mathbb{R}_+$, we introduce a space step $\Delta x = \frac{A}{N} > 0$ (with N a positive integer) and a family of time steps $\Delta t^n > 0$, and we define the nodes of the regular mesh by

$$(x_i, t_n) = (i\Delta x, \Delta t^n + t_{n-1}), \quad \text{for } n \geqslant 1, \ i \in \{-N, \dots, N\}.$$

By U_i^n we denote the value of a discrete approximate solution at the point (x_i, t_n) and $u(x,t)$, the exact solution of (5.10). The initial data is discretised in the following way

$$U_i^0 = u_0(x_i), \quad \text{for } i \in \{-N, \dots, N\}$$

and the Dirichlet boundary conditions imply

$$U_{-N}^n = U_N^n = 0, \quad \text{for all } n > 0.$$

Consequently, at each time step we have to calculate the values of the approximate solution $U^n = (U_{-N}^n, \ldots, U_N^n)^T$. Moreover, since our problem is non-linear, we use the simplest numerical integrator which is the Euler forward method

$$U_i^{n+1} = U_i^n + \Delta t^n B_i^n \cdot G_i^n, \tag{5.11}$$

for $i = -N+1, \ldots, N-1$ and $n \in \mathbb{N}$. Unfortunately, we can encounter difficulties with the Euler forward method. Most notably it is unclear how large the time step Δt^n can be. A too small time step is obviously inefficient but a too large time step can make the numerical integration inaccurate and even unstable. To avoid such problems and other numerical errors, we define B_i^n, G_i^n in such a way that the proposed scheme (5.11) is monotone. Namely, for $i = -N+1, \ldots, N-1$ and $n \in \mathbb{N}$ we set

$$B_i^n = \left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{(\Delta x)^2} + 1 \right),$$

$$G_i^n = \begin{pmatrix} \max\left\{0, \frac{U_{i+1}^n - U_i^n}{\Delta x}, \frac{U_{i-1}^n - U_i^n}{\Delta x}\right\}, & \text{if } B_i \geqslant 0 \\ \max\left\{0, \frac{U_i^n - U_{i+1}^n}{\Delta x}, \frac{U_i^n - U_{i-1}^n}{\Delta x}\right\}, & \text{if } B_i < 0 \end{pmatrix}.$$

Lemma 5.6 (Monotonicity of the scheme)**.** *Let* $\mathcal{N} = \{-N+1, \ldots, N-1\}$. *The scheme derived in* (5.11) *is monotone if and only if the condition*

$$\Delta t^n \leqslant \frac{\Delta x}{\frac{2}{\Delta x} \max\limits_{i \in \mathcal{N}} G_i^n + \max\limits_{i \in \mathcal{N}} |B_i^n|}$$

is satisfied.

Proof. Fix $i \in \mathcal{N}$. Let us denote $F_i(U^n) = U_i^n + \Delta t^n B_i^n \cdot G_i^n$. We say that the scheme is monotone if the partial derivative of F_i in a U_k^n direction for $k \in \mathcal{N}$ is non-negative. Thus, we see that $\frac{\partial F_i(U^n)}{\partial U_k^n} = 0$ for all k such that $|k - i| > 1$. Now choose $k = i \pm 1$. Then we have

$$\frac{\partial F_i(U^n)}{\partial U_k^n} = \frac{\Delta t^n G_i^n}{(\Delta x)^2} + \frac{\Delta t^n}{\Delta x} |B_i^n| \geqslant 0.$$

Hence, only one derivative to calculate is left. Indeed, for $k = i$ we obtain

$$\begin{aligned}\frac{\partial F_i(U^n)}{\partial U_i^n} &= 1 - \Delta t^n \left(\frac{2G_i^n}{(\Delta x)^2} + \frac{|B_i^n|}{\Delta x} \right) \\ &\geqslant 1 - \Delta t^n \left(\frac{2}{(\Delta x)^2} \max_{i \in \mathcal{N}} G_i^n + \frac{1}{\Delta x} \max_{i \in \mathcal{N}} |B_i^n| \right) \geqslant 0\end{aligned}$$

by the assumption imposed on Δt^n. ■

One can notice that the monotonicity of the scheme implies the following property: if $U^1 \geqslant U^0$, then $U^{n+1} \geqslant U^n$ for all $n > 0$. Moreover, for all initial data, in Proposition 5.4 we have already constructed a subsolution of (5.1). Let us denote by $\underline{U}$ the approximation of that subsolution. Then it satisfies $\underline{U} \leqslant F(\underline{U})$. Furthermore, it immediately yields $U^n \geqslant \underline{U}$ for all $n > 0$. Thus, it suffices to take any $A \geqslant \overline{T}$, where $\overline{T}$ is defined by (5.9). We can as well conclude $U^n \leqslant 0$ for all $n > 0$ since zero is a trivial solution of (5.11) and also a supersolution.

Let us consider the following initial data $u_0(x) = -\frac{1}{10}(1 - x^2)_+^2$. Due to (5.8) and (5.9) we can approximate the lower and the upper bound of the extinction time T. Namely, we arrive at $\overline{\mathbf{T}} = \mathbf{1.6980353}$ and $\underline{\mathbf{T}} = \mathbf{0.9486833}$. Applying the numerical scheme (5.11) we conclude that the extinction time is $\mathbf{T} \approx \mathbf{1,446}$.

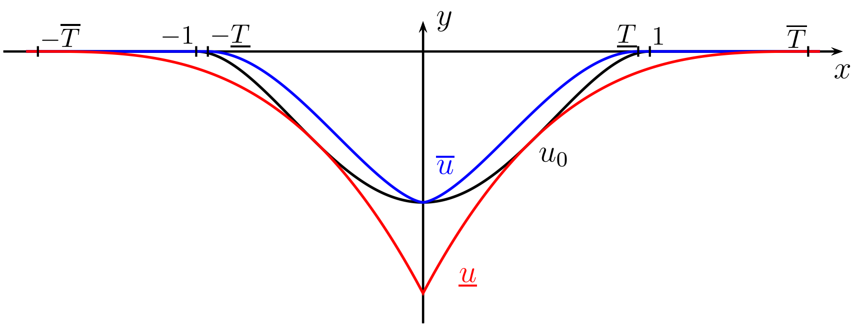


Figure 5.1: A graph of the initial data u_0 together with a suitable supersolution $\overline{u}(\cdot, 0)$ (defined by Proposition 5.5 with $T = \underline{T}$) and a subsolution $\underline{u}(\cdot, 0)$ (defined by Proposition 5.4 with $T = \overline{T}$).

		Stop condition		
		10^{-10}	10^{-12}	10^{-14}
Space-step Δx	10^{-1}	2.56085155210465	2.66085155188506	2.76085155187872
	10^{-2}	1.69279370766363	1.75279367369390	1.80279367344999
	10^{-3}	1.47417932486352	1.48517897415510	1.49517897108321
	10^{-4}	1.44318934839620	1.44468553097471	1.44608550064828

Table 5.1: Extinction times for different space-steps Δx and different stop conditions. A better approximation can be obtained for a smaller Δx. However, since the problem is non-linear, the computations are expensive.

Chapter 6

Creation of dislocation walls in the discrete model

Let us first recall the problem which will be considered in this section and which describes the horizontal motion of dislocations $(x_i(t), i) \in \mathbb{R} \times \mathbb{Z}$. By $x_i(t)$ we mean a position of i-th dislocation living on i-th level, see Figure 1.5, with the assumption that there is exactly one dislocation on each level. Then the problem can be described by the system of ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t) = F(X(t)), \quad t > 0, \\ X(0) = X^0, \end{cases} \tag{6.1}$$

where $X(t) = (x_i(t))_{i \in \mathbb{Z}}, F(X) = (F_i(X))_{i \in \mathbb{Z}}$ and X^0 is some given initial position of dislocations. Moreover, $F_i(X)$ stands for a resulting force acting on the i-th particle, *i.e.* $F_i(X) \stackrel{\text{def}}{=} \sum_{j \neq i} f(x_j - x_i, j - i)$ for each $i \in \mathbb{Z}$ with an interparticle interaction force $f \colon \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$. Examples of such forces have been already stated in Section 1.2. In the sequel, however, we are mainly interested in a one particular force and a special condition for initial data. Namely, we assume

$$f(x, y) = \frac{x(y^2 - x^2)}{(y^2 + x^2)^2}, \tag{6.2a}$$

$$X^0 \in \Omega \cap \ell^\infty, \tag{6.2b}$$

where

$$\Omega = \left\{ X : |x_i - x_j| \leqslant \sqrt{3 - 2\sqrt{2}}\, |i - j| \right\} \tag{6.3}$$

and $\ell^\infty = \ell^\infty(\mathbb{R})$ is the Banach space of all bounded sequences over $\mathbb{R}$ supplemented with the norm $\|\cdot\|_\infty = \sup_{n\in\mathbb{Z}} |x_n|$.

Our first result deals with the existence of solutions to the considered problem. More precisely, it reads as follows

Theorem 6.1 (Existence of a unique solution)**.** *Let* (6.2) *hold. Then there exists a unique solution* $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$ *of the Cauchy problem* (6.1)*. Moreover, if the initial data* X^0 *is* N*-periodic (i.e.* $x_i^0 = x_{i+N}^0$*, for every* $i \in \mathbb{Z}$*), then the solution remains* N*-periodic for every time* $t > 0$*.*

The proof of the theorem consists of the application of the classical Cauchy-Lipschitz theorem and the comparison principle result. Notice that in general the locally Lipschitz condition with respect to the first variable of the function f is sufficient to obtain a unique local-in-time solution. In order to extend it to the global-in-time one we need to provide an apriori estimate, *e.g.* the comparison principle that ensures us that ℓ^∞-norm of the solution does not blow up.

However, if the function f satisfies the Lipschitz condition globally, which happens when f is defined by (6.2a), we immediately obtain a unique global-in-time solution by extending it with the universal step $T > 0$, see [13, Thm 7.3, p. 184]. Thus, in that case the comparison principle is needed only to ensure that the solution belongs to Ω for all times $t > 0$.

To prove the comparison principle for the problem (6.1), the monotonicity of a function f is a necessary assumption. Hence, the reason why we consider the initial condition in the special domain Ω is that the function f defined by (6.2a) is indeed monotone over that set.

Our second result is the long time behaviour of the dynamics of particles where we prove that dislocations accumulate creating so-called walls of dislocations. This result can be stated in the following way

Theorem 6.2 (Convergence to flat walls)**.** *Let* $X(t)$ *be the* N*-periodic solution of the problem* (6.1)-(6.2)*. Then it converges to a constant stationary* *solution of the problem* (6.1)-(6.2)*, i.e. for every* $i \in \mathbb{Z}$*, we have* $\lim_{t\to\infty} x_i(t) = c$*, where* $c = \frac{1}{N}\sum_{i=1}^{N} x_i^0$ *is the barycenter of the initial data.*

For the proof of the above theorem we refer to Section 6.3, and to Section 6.4 for numerical experiments which show among others the convergence result for periodic initial data.

6.1 Comparison principle

In this section we are going to prove the comparison principle for the problem (6.1) with a continuous, globally nondecreasing, with respect to first variable, function $f\colon \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ and any initial data $X^0 \in \ell^\infty$.

Lemma 6.3 (Comparison principle)**.** *Let* $f = f(x, y)$ *be a function as described above. Let* $T > 0$ *and* X, Y *be two solutions of* (6.1) *such that* $X^0 \leqslant Y^0$*. Then* $X(t) \leqslant Y(t)$ *for any* $t \in [0, T]$*.*

Proof. Notice first that the assumption $X^0 = X(0) \leqslant Y(0) = Y^0$ reads as $x_n(0) \leqslant y_n(0)$ for every $n \in \mathbb{Z}$. We shall prove that $x_n(t) \leqslant y_n(t)$ for every $n \in \mathbb{Z}$ and $t \in [0, T]$.

Let us first define new functions $Z(t) = (z_n(t))_{n\in\mathbb{Z}}$ and $M(t)$ in the following way

$$z_n(t) = x_n(t) - y_n(t), \quad M(t) = \sup_{n\in\mathbb{Z}} z_n(t). \tag{6.4}$$

The second equality of (6.4) implies

$$\forall_{t^*\in[0,T]}\, \exists_{n^*(t^*)}\ M(t^*) = z_{n^*(t^*)}(t^*),$$

where $n^*(t^*)$ may not be necessarily finite. Our goal is to show that $M(t) \leqslant 0$ for all times $t \in [0, T]$.

In view of Definition 4.6 of viscosity solutions and Remark 4.7, let ϕ be an arbitrary, sufficiently smooth test function such that it satisfies the following properties

$$\begin{cases} M(t) \leqslant \phi(t), \\ M(t^*) = \phi(t^*). \end{cases}$$

We say that M satisfies $M'(t) \leqslant 0$ in the viscosity sense if and only if $\phi'(t^*) \leqslant 0$.

Moreover, due to the following relations

$$z_{n^*(t^*)}(t) \leqslant M(t) \leqslant \phi(t), \quad z_{n^*(t^*)}(t^*) = M(t^*) = \phi(t^*), \tag{6.5}$$

which hold by the definitions of M and ϕ, we can compute $\phi'(t) = \frac{\mathrm{d}}{\mathrm{d}t} z_{n^*(t^*)}(t)$ at $t = t^*$, since $z_{n^*(t^*)}, \phi$ are sufficiently smooth.

Next, by the Gronwall inequality (which in the viscosity solutions framework is nothing else but the comparison principle) we deduce

$$M(t) \leqslant M(0) \leqslant 0.$$

Therefore, our aim now is to show that indeed $\frac{\mathrm{d}}{\mathrm{d}t} z_{n^*(t)}(t) \leqslant 0$ at $t = t^*$.

Step 1: $n^*(t^*) < \infty$. Let $n^* = n^*(t^*)$. Using the Taylor expansion of the function F from (6.1), we have

$$\begin{aligned} \frac{\mathrm{d}z_{n^*}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}x_{n^*}(t)}{\mathrm{d}t} - \frac{\mathrm{d}y_{n^*}(t)}{\mathrm{d}t} = F_{n^*}(X(t)) - F_{n^*}(Y(t)) \\ &= \sum_{m\in\mathbb{Z}} \partial_m F_{n^*}(\Theta(t))(x_m(t) - y_m(t)), \end{aligned}$$

where $\Theta(t) = \alpha X(t) + (1-\alpha)Y(t)$ for some $\alpha \in (0,1)$. Here $\partial_m F_n(X)$ is to be understood as

$$\partial_m F_n(X) := \frac{\mathrm{d}F_n(X)}{\mathrm{d}x_m}.$$

In particular for $t = t^*$ we obtain

$$\begin{aligned} \frac{\mathrm{d}z_{n^*}(t^*)}{\mathrm{d}t} &= \partial_{n^*} F_{n^*}(\Theta(t^*)) z_{n^*}(t^*) + \sum_{\substack{m\in\mathbb{Z} \\ m\neq n^*}} \partial_m F_{n^*}(\Theta(t^*))(x_m(t^*) - y_m(t^*)) \\ &\leqslant \partial_{n^*} F_{n^*}(\Theta(t^*)) z_{n^*}(t^*) + z_{n^*}(t^*) \sum_{\substack{m\in\mathbb{Z} \\ m\neq n^*}} \partial_m F_{n^*}(\Theta(t^*)) \\ &= z_{n^*}(t^*) \sum_{m\in\mathbb{Z}} \partial_m F_{n^*}(\Theta(t^*)) = 0. \end{aligned} \tag{6.6}$$

The inequality in the middle line and the last equality in the above computations can be justified as follow.

First, by the assumption on monotonicity of the function f, we notice that for every $m \neq n^*$ we have

$$\partial_m F_{n^*}(\Theta(t^*)) = f_x(\Theta_m(t) - \Theta_{n^*}(t), m - n^*)\alpha \geqslant 0.$$

Here f_x denotes the partial derivative of $f = f(x, y)$ with respect to the first variable x.

Second, we can calculate explicitly $\partial_{n^*} F_{n^*}(\Theta(t^*))$. Namely, by the structure of the function F, we get

$$\partial_{n^*} F_{n^*}(\Theta(t^*)) = -\sum_{m \neq n^*} f_x(\Theta_m(t) - \Theta_{n^*}(t), m - n^*)\alpha.$$

Summing up all the derivatives of F, we arrive at the last equality of (6.6).

Step 2: $n^*(t^*) = \infty$. Then there exists a subsequence n_k such that

$$z := \sup_{\substack{n \in \mathbb{Z} \\ t \in [0,T]}} z_n(t) = \lim_{k \to \infty} z_{n_k}(t^*). \tag{6.7}$$

Let us define new functions by shifting indices

$$\begin{cases} x_n^k(t) = x_{n+n_k}(t), \\ y_n^k(t) = y_{n+n_k}(t), \\ z_n^k(t) = z_{n+n_k}(t). \end{cases}$$

Then the functions $x_n^k(t)$ and $y_n^k(t)$ are bounded, since the functions x_m and y_m are bounded for every $m \in \mathbb{Z}$. We get, up to a subsequence of k,

$$\lim_{k \to \infty} z_n^k(t) = z_n^\infty(t) = x_n^\infty(t) - y_n^\infty(t).$$

Furthermore, due to the definition of $z_n^k(t)$ and (6.7), we arrive at

$$z_0^k(t^*) = z_{n_k}(t^*)$$
$$\lim_{k \to \infty} z_0^k(t^*) = z = z_0^\infty(t^*) = x_0^\infty(t^*) - y_0^\infty(t^*).$$

Now we have

$$z = \sup_{\substack{n \in \mathbb{Z} \\ t \in [0,T]}} z_n(t) \geqslant \sup_{n \in \mathbb{Z}} z_n(t^*) \geqslant z_{n+n_k}(t^*) = z_n^k(t^*).$$

Passing to the limit with $k \to \infty$, we arrive at $z \geqslant z_n^\infty$ for all $n \in \mathbb{Z}$.

On the other hand, by (6.7) we know that

$$z = z_0^\infty(t^*) \leqslant \sup_{n\in\mathbb{Z}} z_n^\infty(t^*) \leqslant z. \tag{6.8}$$

Hence, we obtain

$$z = z_0^\infty(t^*) = \sup_{n\in\mathbb{Z}} z_n^\infty(t^*). \tag{6.9}$$

Additionally, by the assumption on initial data, we check that

$$z_0^\infty(0) = \lim_{k\to\infty} z_0^k(0) = \lim_{k\to\infty} x_{n_k}(0) - y_{n_k}(0) \leqslant 0. \tag{6.10}$$

Applying the result of Step 1 for $z_n^k(t)$ with $n^* = 0$ we prove the desired result. ■

6.2 Existence and uniqueness of solutions

In this section we give the proof of Theorem (6.1) which combines the classical Cauchy-Lipschitz theorem [13, Thm 7.3, p. 184] and the comparison principle, Lemma 6.3.

Proof of Theorem 6.1. In the proof we argue in several steps.

Step 1: Properties of the function f. Let the function f be defined by (6.2a). Clearly, since $f(\cdot, y) \in C^\infty(\mathbb{R})$ for every $y \in \mathbb{Z}\backslash\{0\}$ fixed, there exists $x_y = \sqrt{3-\sqrt{2}}|y|$ such that

$$f(x_y, y) = \max_{x\in\mathbb{R}} f(x, y)$$

and $f(\cdot, y)$ is nondecreasing over $[-x_y, x_y]$, see for instance Figure 1.4.

Moreover, we see that for fixed $y \in \mathbb{Z}\backslash\{0\}$

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} f(x, y)\right| \leqslant \frac{\mathrm{d}}{\mathrm{d}x} f(0, y) = \frac{1}{y^2}. \tag{6.11}$$

Hence, f is globally Lipschitz continuous over $\mathbb{R}$ with $\frac{1}{y^2}$ Lipschitz constant depending on fixed y.

Step 2: Existence of a unique global solution of (6.1) *in* $C([0,\infty),\ell^\infty)$. Let $X=(x_i)_{i\in\mathbb{Z}},\ Y=(y_i)_{i\in\mathbb{Z}}\in\ell^\infty$. Exploiting (6.11), we have

$$\begin{aligned}\|F(X)-F(Y)\|_{\ell^\infty} &= \max_{i\in\mathbb{Z}}|F_i(X)-F_i(Y)|\\ &= \max_{i\in\mathbb{Z}}\Big|\sum_{j\neq i} f(x_j-x_i,j-i)-f(y_j-y_i,j-i)\Big|\\ &\leqslant \max_{i\in\mathbb{Z}}\sum_{j\neq i}|f(x_j-x_i,j-i)-f(y_j-y_i,j-i)|\\ &\leqslant \max_{i\in\mathbb{Z}}\sum_{j\neq i}\frac{1}{(j-i)^2}\Big(|x_j-y_j|+|x_i-y_i|\Big)\\ &\leqslant 4\|X-Y\|_{\ell^\infty}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{2}{3}\pi^2\|X-Y\|_{\ell^\infty}.\end{aligned}$$

Therefore, using the classical Cauchy-Lipschitz theorem [13, Thm 7.3, p. 184], there exists a unique solution $X\in C^1([0,\infty),\ell^\infty)$ of (6.1).

Step 3: Extended problem. For $y\in\mathbb{Z}\backslash\{0\}$ fixed, let us define a new function $\tilde f=\tilde f(x,y)$ in the following way

$$\begin{cases}\tilde f(x,y)=f(x,y) & \text{for } x\in[-x_y,x_y],\\ \tilde f(x,y)=f(x_y,y) & \text{for all } x\geqslant x_y,\\ \tilde f(x,y)=f(-x_y,y) & \text{for all } x\leqslant -x_y.\end{cases}\tag{6.12}$$

Note that this function is also Lipschitz with respect to the first variable over the whole space and let us consider the following extended system

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\tilde X(t)=\tilde F(\tilde X(t)) \quad t\geqslant 0,\\ \tilde X(0)=X(0)\in\Omega\cap\ell^\infty,\end{cases}\tag{6.13}$$

where $\tilde X(t)=(\tilde x_i(t))_{i\in\mathbb{Z}}$ and $\tilde F(\tilde X)=(\tilde F_i(\tilde X))_{i\in\mathbb{Z}}$, with

$$\tilde F_i(\tilde X(t)):=\sum_{j\neq i}\tilde f(\tilde x_j-\tilde x_i,j-i).\tag{6.14}$$

Similarly to Step 1, we can prove that for every $\tilde X=(\tilde x_i)_{i\in\mathbb{Z}},\ \tilde Y=(\tilde y_i)_{i\in\mathbb{Z}}$, we have

$$\|\tilde F(\tilde X)-\tilde F(\tilde Y)\|_{\ell^\infty}\leqslant\frac{2}{3}\pi^2\|\tilde X-\tilde Y\|_{\ell^\infty}.$$

Therefore, again by the classical Cauchy-Lipschitz theorem [13, Thm 7.3, p. 184], there exists a unique solution $\tilde{X} \in C^1([0,+\infty), \ell^\infty)$ of the extended problem (6.13).

Step 3: Invariance. In this step we are going to show that if the initial data belongs to Ω, then the solution $\tilde{X}(t)$ is also in Ω. The condition $\tilde{X}^0 \in \Omega$ reads as follows

$$-\sqrt{3-2\sqrt{2}}|i-j| \leqslant \tilde{x}_i^0 - \tilde{x}_j^0 \leqslant \sqrt{3-2\sqrt{2}}|i-j|, \quad \forall i,j \in \mathbb{Z}.$$

Setting $m = i - j$, we obtain

$$\underline{x}_i^m(0) := \tilde{x}_{i-m}^0 - \sqrt{3-2\sqrt{2}}|m| \leqslant \tilde{x}_i^0 \leqslant \tilde{x}_{i-m}^0 + \sqrt{3-2\sqrt{2}}|m| =: \overline{x}_i^m(0),$$

for every $i, m \in \mathbb{Z}$.

Moreover, from the definition of the function $\tilde{F}$, see (6.14), it is clear that the problem (6.13) is invariant by translations. Hence, by the previous step, $\underline{X}^m = (\underline{x}_i^m)_{i\in\mathbb{Z}}$ and $\overline{X}^m = (\overline{x}_i^m)_{i\in\mathbb{Z}}$ are solutions of (6.13) for each $m \in \mathbb{Z}$. Now since $\tilde{f}$ is nondecreasing, we can apply the comparison principle, Lemma 6.3, and deduce that for every $i, m \in \mathbb{Z}$

$$\underline{x}_i^m(t) \leqslant \tilde{x}_i^m(t) \leqslant \overline{x}_i^m(t) \quad \text{for } t > 0.$$

Hence $\tilde{X}(t) \in \Omega$ for all $t > 0$.

Step 4: Conclusions. Since the solution of the extended problem, constructed in Step 2, satisfies $\tilde{X}(t) \in \Omega$ for all $t > 0$, then by the definition we have $\tilde{f}(\tilde{x}_i - \tilde{x}_j, i-j) = f(\tilde{x}_i - \tilde{x}_j, i-j)$. As a consequence

$$\tilde{F}(\tilde{X}(t)) = F(\tilde{X}(t)) \quad \text{over} \quad t > 0.$$

Therefore, $\tilde{X}$ solves the original problem (6.1). Moreover, by the uniqueness of the solution of the problem (6.1), see Step 1, we obtain

$$X(t) = \tilde{X}(t) \in \Omega.$$

Thereupon, we have proved that $X(t) \in C([0,\infty), \Omega \cap \ell^\infty)$ is the unique global solution of the problem (6.1)-(6.2).

Step 5: Periodicity. Assume that the initial data $X(0) = X^0$ is N-periodic ($i.e.$ $x_i^0 = x_{i+N}^0$) and define $Y^0 = (x_{i+N}^0)_{i\in\mathbb{Z}}$. Then we conclude that $X(t)$ and $Y(t)$ are two global-in-time solutions of (6.1) and belong to Ω. However, since $Y^0 = X^0$, by the uniqueness we have $X(t) = Y(t)$ for each time $t > 0$. Thus, $X(t)$ is N-periodic for $t > 0$. ∎

6.3 Convergence to flat walls

The aim of this section is to prove that under the periodicity assumption imposed on the initial data, the solution constructed in Theorem 6.1 converges to a special stationary solution of the problem (6.1).

Proof of Theorem 6.2. Let $X = (x_i)_{i\in\mathbb{Z}} \in C([0,\infty), \Omega \cap \ell^\infty)$ be a N-periodic ($i.e.$ $x_{i+N} = x_i$) solution of (6.1). Without loss of generality, we may transform (1.9) into the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} g(x_j - x_i, j-i), \quad i = 1, \ldots, N, \tag{6.15}$$

with the function $g(x,y) = \sum\limits_{k\in\mathbb{Z}} f(x, y+kN)$ uniformly bounded in x. In order to prove the convergence of the solution we argue by steps.

Step 1. Let us define a new function

$$M(t) = \frac{1}{2}\sum_{i=1}^{N} x_i^2(t). \tag{6.16}$$

We shall show that $M(t)$ decreases in time. Indeed, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} x_i^2(t) &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} x_i(t)\, g(x_j(t) - x_i(t), j-i) \\
&= \sum_{i=1}^{N}\sum_{j=i+1}^{N} x_i(t)\, g(x_j(t) - x_i(t), j-i) + \sum_{i=1}^{N}\sum_{j=1}^{i-1} x_i(t)\, g(x_j(t) - x_i(t), j-i) \\
&= \sum_{i=1}^{N}\sum_{k=1}^{N-i} x_i(t)\, g(x_{i+k}(t) - x_i(t), k) - \sum_{j=1}^{N}\sum_{i=j+1}^{N} x_i(t)\, g(x_i(t) - x_j(t), i-j)
\end{aligned}$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{N-i} x_i(t)\, g(x_{i+k}(t) - x_i(t), k) - \sum_{j=1}^{N}\sum_{k=1}^{N-j} x_{j+k}(t)\, g(x_{j+k}(t) - x_j(t), k)$$
$$= \sum_{i=1}^{N}\sum_{k=1}^{N-i} (x_i(t) - x_{i+k}(t))\, g(x_{i+k}(t) - x_i(t), k) \leqslant 0.$$

First, let us mention that due to the fact that the function $f = f(x,y)$, defined by (6.2a) is symmetric in y and antisymmetric in x, the function $g = g(x,y)$ possesses such property as well. Moreover, as a result of the boundedness of the function $g(\cdot, y)$ (which comes form the Lipschitz condition of $f(\cdot, y)$) and the fact that only finite sums are considered, we are allowed to use Fubini's theorem and change the order of summation. These facts justify the third equality.

The inequality is obtained by the fact that each single expression under the sums is non-positive due to the definitions of the functions g, f and the fact that $X(t) \in \Omega \cap \ell^\infty$ for $t \geqslant 0$.

Finally, we conclude that $M(t) \to M_0$ as $t \to \infty$ since $M(t)$ is non-negative and non-increasing.

Step 2. Let us define $X^n(t) := X(t+n)$. Then X^n is a solution of (6.1), and hence $\frac{\mathrm{d}}{\mathrm{d}t}X^n(t) \in \ell^\infty$. By the Arzelà-Ascoli theorem, up to some subsequence, $X^n(t) \to X^\infty(t)$ as $n \to \infty$ for every $t > 0$. Thus, we can write

$$M_0 = \lim_{n\to\infty} M(t+n) = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{N} \Big(x_i^n(t)\Big)^2 = \frac{1}{2}\sum_{i=1}^{N} \Big(x_i^\infty(t)\Big)^2. \tag{6.17}$$

Since X^n is a solution of (6.15), then the limit X^∞ is a classical solution and $X^\infty(t) \in \Omega \cap \ell^\infty$. Therefore, repeating all the computations performed in Step 1 for X^∞, we arrive at

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} M_0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\sum_{i=1}^{N}\Big(x_i^\infty(t)\Big)^2$$
$$= \sum_{i=1}^{N}\sum_{k=1}^{N-i} (x_i^\infty(t) - x_{i+k}^\infty(t))\, g(x_{i+k}^\infty(t) - x_i^\infty(t), k).$$

Since the solution lives in Ω, every term under the sums is non-positive. Thus, either $x_i^\infty(t) = x_{i+k}^\infty(t) = x_j^\infty(t)$ for all $i, j = 1, \ldots, N$, or $g = 0$. However, the condition $g = 0$ immediately implies the first case.

Next, we plug X^∞ into the equation (6.15) to see that indeed $\frac{\mathrm{d}}{\mathrm{d}t}x_i^\infty(t) = 0$ and thus $x_i^\infty(t) = x_i^{0,\infty} = c$, for all $i = 1, \dots, N$ and some $c \in \mathbb{R}$. Moreover, we can write the explicit value of M_0, i.e. $M_0 = \frac{1}{2}Nc^2$.

Step 3. Take another convergent subsequence, $X^{n_k}(t)$, of $X^n(t)$ such that $X^{n_k}(t) \to \overline{X}^\infty(t)$ as $k \to \infty$. Repeating all the calculations performed in Step 2, we may show that $\overline{x}_i^\infty(t) = b$ for all $i = 1, \dots, N$, and $t \geqslant 0$ and some $b \in \mathbb{R}$. As before we conclude that $M_0 = \frac{1}{2}Nb^2$. Thus $b = c$ because we may assume, without loss of generality, that $b, c \geqslant 0$ since the problem (6.15) is invariant by translations, and we can shift initial data to be positive.

This implies that the accumulation point of X is unique. Hence, $x_i(t) \to c$ as $t \to \infty$ for all $i = 1, \dots, N$.

Step 4. In this step first we prove that the barycenter is preserved in time, i.e. $\sum\limits_{i=1}^N x_i(t) = \sum\limits_{i=1}^N x_i(0)$ for all $t > 0$, which allows us to determine the value of the constant c.

Since g is a bounded function, changing the order of summation, we may write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N x_i(t) &= \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N g(x_j - x_i, j - i) = \sum_{j=1}^N \sum_{\substack{i=1\\ i\neq j}}^N g(x_j - x_i, j - i) \\ &= -\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N g(x_j - x_i, j - i) = 0. \end{aligned}$$

This implies

$$\frac{1}{N}\sum_{i=1}^N x_i(0) = \frac{1}{N}\sum_{i=1}^N x_i(t) = \frac{1}{N}\sum_{i=1}^N c = c. \tag{6.18}$$

Thus, we have proved the desired result. ∎.

6.4 Numerical experiments

Here we present results of some numerical experiments to confirm the results obtained in Theorem 6.2. We construct an adaptive scheme as follows. Let $N > 0$ denote the total number of the interacting particles, Δt, denote a time-step and let us define an approximate solution of (6.1) by a solution $X^n = (X_1^n, \dots, X_N^n)$ of the following forward Euler scheme

$$X^{n+1} = X^n + \Delta t F(X^n) \stackrel{\text{def}}{=} S(X^n). \tag{6.19}$$

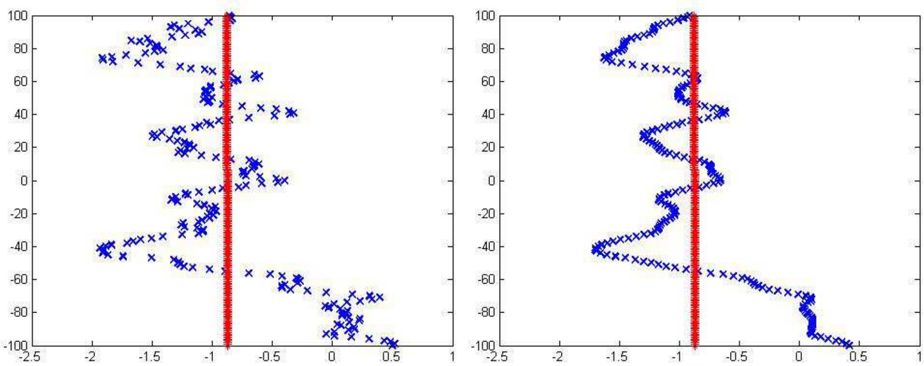
Lemma 6.4 (Monotonicity of the scheme)**.** *The scheme derived in* (6.19) *is monotone if and only if the time-step satisfies* $dt \leqslant \frac{3}{\pi^2}$ *and the initial data* $X^0 \in \Omega$ *defined in* (6.3)*.*

Proof. To prove the monotonicity it is enough to show that $\partial_j S_i(X^n) \geqslant 0$ for all $i, j = 1, \dots, N$. First, we notice that due to Lemma 6.3 we get $X^n \in \Omega$ for all $n \in \mathbb{N}$. Next, since considered function f is nondecreasing with respect to the first variable, it is easy to see that $\partial_j F_i(X^n) = \Delta t\, f_x(X_j^n - X_i^n, j-i) \geqslant 0$, for $j \neq i$. Now consider the case $j = i$. Due to the bound on the derivative of f, see (6.11), we obtain

$$\partial_i F_i(X^n) = 1 - \Delta t \sum_{\substack{j=1 \\ j \neq i}}^{N} f_x(X_j^n - X_i^n, j-i) \geqslant 1 - \Delta t \frac{\pi^2}{3} \geqslant 0.$$

This completes the proof. ∎

On the left-upper plot in Figure 6.1, we denote by ✘ the initial data $X^0 \in \Omega \cap \ell^\infty$ taken in numerical experiments. Furthermore, by ✶ we emphasise what the limit solution (by Theorem 6.2 the limit solution is at the barycenter of initial data) is. In Figure 6.1 we observe the evolution of dislocations which eventually converge.



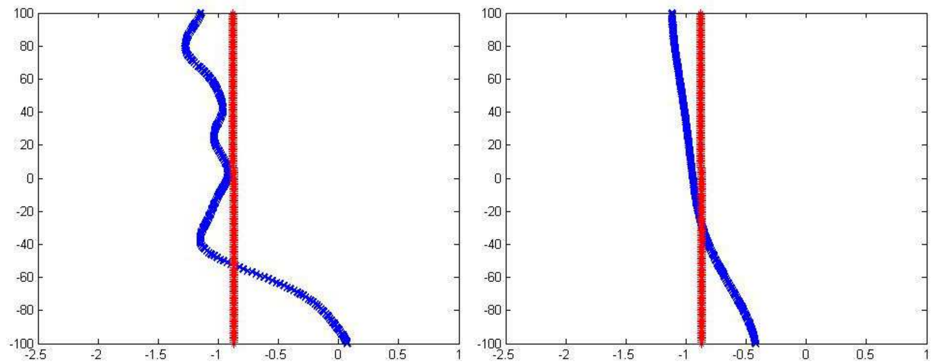


Figure 6.1: Evolution of dislocations of (1.11) with initial data $X^0 \in \Omega$.

However, we may also consider the initial data $X^0 \in \overline{\Omega}$, where

$$\overline{\Omega} = \left\{ X : \sqrt{3 - 2\sqrt{2}}\, |i - j| < |x_i - x_j| < |i - j| \right\}, \qquad (6.20)$$

see the blue region in Figure 1.5. It is worth noticing that for such initial data the force acting on dislocations is still attractive; however, we do not have a comparison principle, and we cannot guarantee that the solution stays in $\overline{\Omega}$. Still numerical experiments can be performed to see what happen with dislocations.

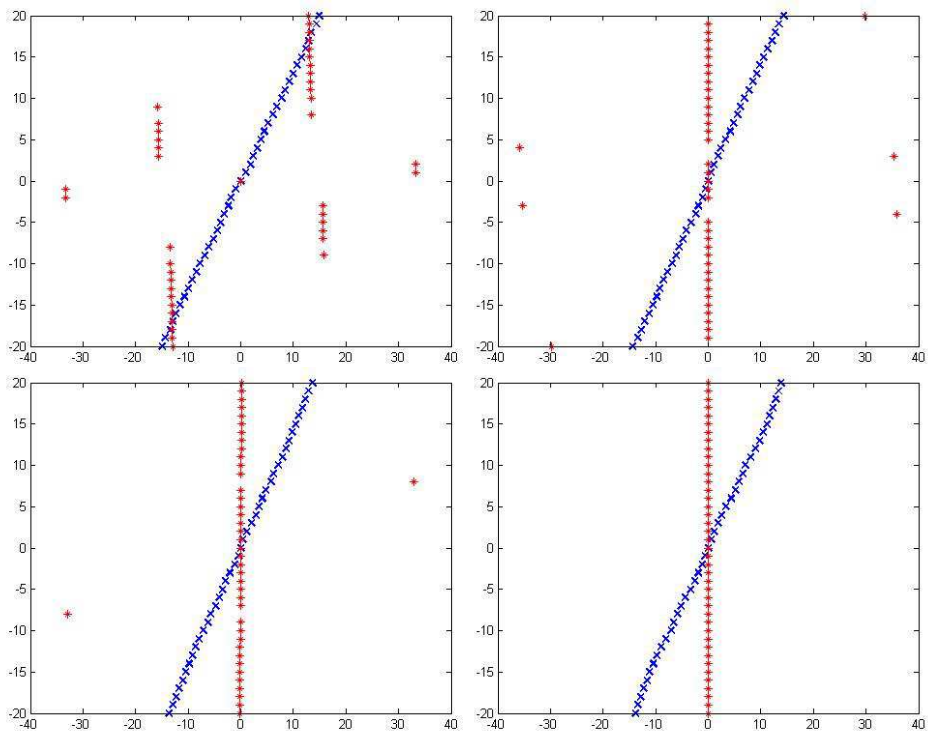


Figure 6.2: Evolution of dislocations of (6.1) with initial data $X^0 \in \overline{\Omega}$. Each simulation starts with different initial data.

In the above pictures we can see that even small perturbation of initial data produces completely different solutions. The only one (right-lower plot in Figure 6.2) converged to a flat wall, while the remainder does not.

Summary of the results

The main goal of the dissertation was to investigate models describing evolution of dislocations in crystals. First we focused on the one-dimensional, continuous model involving a non-local and non-linear equation, where the non-linear term was represented by the fractional Laplace operator. This model was widely presented in the introduction, see Chapter 1, where we mentioned its physical interpretations and the related results.

Hence we began our studies in Chapter 3 by proving existence of weak solutions of the Cauchy problem. Taking the advantage of the vanishing viscosity method, we have considered a regularised problem, for which we have proved the hypercontractivity estimates. We are convinced that for the zero-integral initial data there is a possibility to improve the decay rate as for the heat equation. Furthermore, having used the estimates, we have proved existence of global-in-time mild solutions via the Banach fixed point theorem. Then we have shown that the limit of the family of mild solutions exists and this limit function is indeed a weak solution of the original problem.

The next problem we dealt with were self-similar solutions, where we tried to answer the question concerning existence of such scale invariant solutions. Since the problem seemed to be rather difficult, we first started from a simpler one, namely we considered an equation where the fractional Laplace operator was replaced by the usual one. For such a problem, in Chapter 4 we have proved existence of self-similar solutions in the case of non-negative external force. In a particular case, when the force is negative, we have shown that a constructed explicit, almost everywhere solution is actually not a viscosity solution. Thus, we suppose that in this case there is no self-similar solution at all. Next, we considered the problem with the half Laplace operator, where we have proved non-existence of self-similar solutions in a very particular case.

In Chapter 5 we studied finite-time extinction of solutions of the local problem, *i.e.* the problem with the usual Laplace operator, with the negative external force. First, we have constructed the viscosity sub- and supersolutions and due to the comparison principle have shown that a solution extincts in a finite time. Moreover, numerical experiments have been performed in order to determine an approximate time of extinction.

Finally, in Chapter 6 we focused on the second model which consists of a system of ordinary differential equations, where each equation describes the motion of exactly one dislocation. We have proved existence of the classical solution. Moreover, under the periodicity assumption imposed on the initial data, we have shown that the solution converges to a special stationary solution determined by the barycenter of the initial data. Such behaviour has also been noticed in the performed numerical simulations.

Since dislocations play a significant role in material science, models describing these phenomena deserve a much more extended and diversified presentation. Even for these two models briefly presented in the dissertation there is room for additional analysis and improvements. Firstly, we are convinced that under an additional zero-integral assumption we may expect better convergence rate as in the case of the heat equation. Despite the fact that such an improvement is expected only for the regularised problem, it still would be a profitable and interesting task to study. Furthermore, from the analysis point of view the the question concerning existence of self-similar solutions for the dislocation case (*i.e.* with the fractional Laplace operator) would be highly interesting and important. However, its analysis seems rather difficult since the major issue is to find an exact scaling-invariant family. Additionally, the analysis of the discrete model towards an accumulation of dislocations done by now raises the question of what assumptions different than periodicity need to be imposed on initial data in order to expect similar behaviour like a creation of dislocations walls.

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