Mathematical analysis of thermo-visco-elastic models

Filip Z. Klawe

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prof. dr hab. Piotr Gwiazda and dr hab. Agnieszka Świerczewska-Gwiazda

A need to describe processes occurring in various materials in a better way and to build more detailed models of their behaviors under the action of external forces was the main motivation for studying equations which describe thermo-visco-elastic deformation. With the constant development of technology we still gain deeper understanding of various materials, which allows us to create models describing their behaviors under various external conditions in a more precise way. This information can be crucial, when it comes to proper selection of adequate applications. On the one hand, such research are conducted in laboratories where scientists study the properties of materials during experiments. On the other hand, modern technology and mathematical knowledge can significantly influence and improve modeling of such phenomena.

The objective of this dissertation is to show the existence of solution to special class of models which describe deformation of solid material. Solid mechanics is a very complicated problem, hence we focus on macroscopic effects only. We consider material body which occupies the domain Ω and is treated by external forces and heat flux through the boundary. Our goal is to study visco-elastic type of deformation.

Different properties of deformations result in their different naming convention. If the deformation is reversible and the mechanical energy is not dissipate, i.e. after termination of external forces action the body returns to its initial state, we say that this is an elastic deformation. Moreover, if the strain is proportional to stress we say that this is a linear elasticity. To classify the inelastic deformation, we use a book of Duvaut and Lions, see [18].

It is obvious that different properties of materials depend on external factors, such as, e.g. temperature. The same material placed in the different temperature may differ in terms of its properties. The best example here is a rubber pipe. It is elastic in the room temperature, whereas in the temperature of liquid nitrogen ($-195°C$) it turns brittle and every action of external forces ends by rubber pipe's cracking.

Moreover, every elastic material has got a specified threshold after which it loses its elastic properties. Then, we may observe cracking (when material is brittle) or increase of deformation without increasing of stress. Such deformations are irreversible. Elastic deformation, however, stores the mechanical energy of deformation and after vanishing of external forces this energy is returned. In the case of inelastic deformation, some part of energy is dissipated.

Unfortunately, phenomena that happen in the real world do not fully reflect mathematical and physical definitions. Hence, there appear a necessity to consider inelastic (plastic) deformations. By plastic deformations we understand every deformation which is not an elastic one. Let us present classification of plastic deformations, see [18], using their properties

a) visco-elastic deformation is a type of deformation where reversible and irreversible deformations appear at the beginning of external forces actions.

b) visco-plastic deformation is a type of deformation where irreversible deformations appear after some threshold (on Cauchy stress tensor). When external forces start acting on the material, firstly the elastic deformation appears but only till specified threshold, after which it ceased to be elastic.

c) perfectly-plastic deformation its a type of deformation for which all mechanical energy of external forces is dissipated and after its termination the body does not change its shape and does not come back to any previous state. This deformation appears together with elastic deformation, called elasto-perfectly-plastic deformation. It means that before the threshold this deformation is elastic and after threshold it start to be perfectly-plastic.

Our interest lies in the area of visco-elastic deformations. Moreover, occurence of the inelastic deformations in the model implies that mechanical energy is dissipated into thermal energy, hence we also consider the changes of materials temperature. Therefore, considered models are called thermo-visco-elastic.

For visco-elastic materials we may observe many different phenomena, e.g. creeping, stress relaxation or phase shift in stress response if sinusoidal load is applied. There are many visco-elastic materials, e.g. synthetic polymers, wood, human tissue (ligaments, tendons or disc in human spine) or some metals in specific temperature. Many materials which are elastic in the room temperature are visco-elastic in high temperature. On the other hand, since many metals creep in high temperature (greater than $3000°C$), the filaments in bulbs are made from tungsten which is not visco-elastic in this temperature.

Visco-elastic materials have many real-life applications, e.g.

a) as energy absorbers, since they damp the vibrations;

b) as noise reducer (e.g. in HH-53C rescue helicopter produced by Sikorsky);

c) as car bumpers or in computer devices to protect them from mechanical shock;

More examples of visco-elastic materials' applications may be found in [27].

Since the problem of inelastic deformation is a very complex one, we make a list of simplifications. Therefore in this paper we consider:

- quasi-static case. It means that the evolution is slow and we neglect the acceleration term in the equation for balance of momentum. Similar approach was used also in [4, 11, 14, 18, 25, 26, 32, 37–39, 41, 42];

- infinitesimal displacement. The dependence between the Cauchy stress tensor and the elastic part of deformation is linear (generalized Hooke's law, for more details see [32, 33]);

- zero thermal expansion of material (denoted by ZTE). It is expected that with increasing temperature the volume of material also increases. ZTE materials behave differently than it is expected, i.e. their volume is independent on temperature. Zero thermal expansion materials have many technical applications, cf. [15,29,34,34–36,43], for example, in systems that are subject to thermal shock, in functional materials (thermomechanical actuators and space applications), in precision engineered parts and microdevices.

These simplifications are not only mathematical facilitations but they may be justified from physical point of view. In considered case the physical properties of the model are not lost, which means that

a) the energy of the system is conserved. In the quasi-static case, density of energy consists of thermal and potential energy. Neglection of acceleration term in momentum equation implies also that kinetic energy of the system is neglected.

b) there exists a state function - entropy which has a positive rate of production. Function $s(\theta) = \ln\theta$ is an example of the admissible entropy for this system.

c) temperature is positive.

Thus, the system is thermodynamically complete.

Thermo-visco-elastic system of equations, as a consequence of balance of momentum and balance of energy, cf. [19, 21, 28], captures displacement, temperature and visco-elastic strain. Since these two principles do not take into account the material properties of considered body, we may complement it by adding constitutive relations. A standard technique in case of visco-elastic deformation is to work with two constitutive relations. First one describes the dependency between stress and strains, i.e. it is an equation for the Cauchy stress tensor, see [23, 27]. The second one is a constitutive equation which is characterized by evolution of visco-elastic strain tensor, named also the *flow rule*. Such model (with simplifications discussed previously)

$$\left\{\begin{array}{rcll} -\mathrm{div}\,\boldsymbol{T} &=& \boldsymbol{f} & \text{in } \Omega\times(0,T),\\ \boldsymbol{T} &=& \boldsymbol{D}(\varepsilon(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}) & \text{in } \Omega\times(0,T),\\ \boldsymbol{\varepsilon}^{\mathbf{p}}_t &=& \boldsymbol{G}(\theta,\boldsymbol{T}^d) & \text{in } \Omega\times(0,T),\\ \theta_t-\Delta\theta &=& \boldsymbol{T}^d:\boldsymbol{G}(\theta,\boldsymbol{T}^d) & \text{in } \Omega\times(0,T), \end{array}\right. \tag{1}$$

describes quasi-static evolution of the material's displacement $\boldsymbol{u}:\Omega\times\mathbb{R}_+\to\mathbb{R}^3$, the temperature of the material $\theta:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ and the visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}}:\Omega\times\mathbb{R}_+\to\mathcal{S}^3_d$. We denote by $\mathcal{S}^3$ the set of symmetric 3×3-matrices with real entries and by $\mathcal{S}^3_d$ a subset of $\mathcal{S}^3$ which contains traceless matrices. By $\boldsymbol{T}^d$ we mean deviatoric part (traceless) of the tensor $\boldsymbol{T}$, i.e. $\boldsymbol{T}^d = \boldsymbol{T} - \frac{1}{3}tr(\boldsymbol{T})\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix from $\mathcal{S}^3$. Additionally, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the symmetric part of the displacement's gradient $\boldsymbol{u}$, i.e. $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u})$. The volume force is denoted by $\boldsymbol{f}:\Omega\times\mathbb{R}_+\to\mathbb{R}^3$.

We supplement the system of equations (1) by adding the boundary and initial conditions. We assume that we control the shape of the boundary (Dirichlet boundary conditions on displacement) and heat fluxes through the boundary (Neumann boundary conditions on the temperature). It is obvious that the choice of boundary conditions may be more physical but our goal was to focus on the different issue. We consider models with different constitutive functions describing evolution of visco-elastic strain tensor.

We focus on three different models describing thermo-visco-elastic deformations: Mróz model, Norton-Hoff-type model and model with growth conditions in Orlicz spaces. The difference between these models lies in assumptions made on constitutive functions $\boldsymbol{G}(\theta,\boldsymbol{T}^d)$, which in turn leads to different models, e.g. Bodner-Partom [4, 11, 12], Prandtl-Reuss [14] or others. More information about basic models describing the evolution of nonelastic materials may be found in [1, Chapter 2.2].

Papers of Alber & Chełmiński [2] and Hömberg [24] were the motivation for considerations presented in this PhD dissertation. In [2] the authors considered quasi-static visco-elasticity models with Norton-Hoff constitutive function

$$\boldsymbol{G}(\boldsymbol{T}) = c|\boldsymbol{T}|^{p-2}\boldsymbol{T}, \tag{2}$$

where $c>0$ and $p>2$. The idea of proof was to formulate the problem in a way that it fits to the abstract theory of maximal monotone operators, see [3]. In our case that approach cannot be used because of temperature's dependency on function $\boldsymbol{G}(\cdot,\cdot)$. Moreover, in our case function $\boldsymbol{G}(\cdot,\cdot)$ depends only on deviatoric part of Cauchy stress tensor which has technical consequences. Contrary to the proof of Alber and Chełmiński, where they showed that $\boldsymbol{T}$ belongs to $L^p(0,T,L^p(\Omega,\mathcal{S}^3))$ for $p>2$, the estimates conducted in the present situation provide us only with the fact that $\boldsymbol{T}$ belongs to $L^2(0,T,L^2(\Omega,\mathcal{S}^3))$. The volume change is associated only with the elastic response of the material, hence the plastic response is essentially incompressible, cf. [20]. Furthermore, temperature dependence of $\boldsymbol{G}(\cdot,\cdot)$ destroys the monotone character of the model and it requires us to use different approach.

In [24] author considered more general physical phenomena. Besides displacement and temperature Hömberg was interested in electro-magnetic effects and concentrations of different phases of material. Function $\boldsymbol{G}(\cdot,\cdot)$ is defined in the following way

$$\boldsymbol{G}(\boldsymbol{z},\boldsymbol{T}^d) = \Lambda(\boldsymbol{z})\boldsymbol{T}^d, \tag{3}$$

where $\boldsymbol{z}$ is a vector for concentrations of different phases of material and $\Lambda(\cdot)$ is a *good* function. Furthermore, $\boldsymbol{z} = \mathcal{P}[\theta]$ and $\mathcal{P}[\cdot]$ is an operator which has got *good* properties. In the proof of solution existence author used fixed point theorems.

Results obtained in mentioned papers have been generalized to a wider class of models which depend on the temperature. Moreover, mathematical methods used in proofs regarding existence of thermo-visco-elastic models are strictly different then the ones used in proofs presented in [2, 24].

The main problems which appear in the consideration of (1) are

1) solutions for displacement and temperature are coupled. We cannot split the system (1) into two systems which may be considered independently. This coupling is caused by function $\boldsymbol{G}(\cdot,\cdot)$ which depends on temperature and deviatoric part of Cauchy stress tensor;

2) for all models considered the right-hand side of heat equation is only integrable function. Additionally, the initial condition for temperature is also the integrable function. This implies that the consideration of heat equation requires us to use of nontrivial mathematical techniques.

3) in general constitutive function $\boldsymbol{G}(\cdot,\cdot)$ is nonlinear. Hence, all limit passages of approximate solutions of the system (1) are not straightforward.

4) construction of approximate solutions.

We construct the approximate using two-level Galerkin approximations. It means that the approximate parameters for displacement and temperature are independent; similar technique was used in [8–10].

Construction of approximate solutions is presented in Chapter 2. The basis for displacement contains eigenvalue of elastostatic operator $-\mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\cdot)$ with the domain $W^{1,2}_0(\Omega,\mathbb{R}^3)$, i.e. the functions $\boldsymbol{w}_i$ which are the solutions of

$$\int_\Omega (\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_i) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) - \lambda_i \boldsymbol{w}_i \boldsymbol{\psi})\,\mathrm{d}x = 0, \tag{4}$$

for every $\boldsymbol{\phi} \in C^\infty(\Omega,\mathbb{R}^3)$. However, the basis for temperature are eigenvalue functions of Laplace operator, i.e. functions v_i which are the solutions of

$$\int_\Omega (\nabla v_i \cdot \nabla \phi - \mu_i v_i \phi)\,\mathrm{d}x = 0, \tag{5}$$

for every $\phi \in C^\infty(\overline{\Omega})$. The main issue of this chapter is to construct basis for visco-elastic strain. The natural choice is to take as this basis the set $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_i)\}$, i.e. the symmetric gradients of eigenvalue of elastostatic operator. Unfortunately, such choice is not sufficient and it should be supplied by additional functions. Let us take the solutions $\boldsymbol{\zeta}_i$ of the following equation

$$((\boldsymbol{\zeta}_i,\boldsymbol{\Phi}))_s = \lambda_i(\boldsymbol{\zeta}_i,\boldsymbol{\Phi})_{\boldsymbol{D}} \qquad \forall\ \boldsymbol{\Phi} \in V^s_k, \tag{6}$$

where $(\!(\cdot,\cdot)\!)_s$ is a scalar product of $H^s(\Omega,\mathcal{S}^3)$ ($\frac{3}{2} < s \le 2$), V_k^s is a certain subspace $H^s(\Omega,\mathcal{S}^3)$ and

$$(\boldsymbol{\xi},\boldsymbol{\eta})_{\boldsymbol{D}} := \int_\Omega \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\xi} : \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\eta}\,\mathrm{d}x \qquad \text{for } \boldsymbol{\xi},\boldsymbol{\eta} \in L^2(\Omega,\mathcal{S}^3), \tag{7}$$

is a scalar product of $L^2(\Omega,\mathcal{S}^3)$. Finally, we orthogonalise this basis in the proper space.

For each approximation step, visco-elastic strain basis contains symmetric gradients of first k basis function for displacement, i.e. $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_i)\}_{i=1}^k$, and first l functions which are solutions to (6), i.e. $\{\boldsymbol{\zeta}_i\}_{i=1}^l$. Proof of the approximate solutions existence is a consequence of Carathéodory Theorem, see [30, Theorem 3.4].

The subject of Chapter 3 is to consider the heat equation from thermo-visco-elastic model. We deal with low regularity of data in two different ways. The first approach is based on the paper of Boccardo & Gallouët [7] and the second one on papers of Blanchard and Blanchard & Murat [5,6]. The approach of Boccardo & Gallouët was the first solution regarding parabolic equation with low regular right-hand side. Additionally, renormalised solutions give more information than Boccardo & Gallouët ones. Renormalization methods were used firstly to prove the existence of solution to Boltzmann equation, see [17].

There are two main differences between results presented in [5–7] and our work. The first one lies in the use of boundary conditions. In [5–7] authors consider the problems with Dirichlet boundary condition in the contrast to our case, where we use Neumann boundary condition. Two-dimensional case of Boccardo & Gallouët approach with Neumann boundary conditions was considered in [16]. The second difference is that the heat equation is a part of system of equations. Thus, we do not have full information about right-hand side sequence. In [5–7] authors considered only one equation and they did not have the problem with coupling of equations.

We present the proofs regarding existence of solution to considered models in Chapter 4 (Mróz model), in Chapter 5 (Norton-Hoff-type model) and in Chapter 6 (model with growth conditions in Orlicz spaces). The assumptions on constitutive function $\boldsymbol{G}(\cdot,\cdot)$ for each of these models will appear later, in Assumption 1–3.

Moreover, these chapters are completed with necessary mathematical introduction. The existence proofs requires us to use of advanced mathematical tools. At the beginning of Chapter 4 we make a short introduction to Young measures tools. And at the beginning of Chapter 6 we introduce the generalised Orlicz space. In both cases, we present general information and we prove lemmas, which are used later.

For all models considered, it is important to prove the following inequality

$$\limsup_{l\to\infty} \int_0^t \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^t \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k\,\mathrm{d}x\,\mathrm{d}t, \tag{8}$$

where $\boldsymbol{\chi}_k$ is a weak limit of $\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})$ in a proper space. The additional issue is to identify $\boldsymbol{\chi}_k$.

Assumption 1 (Mróz model)**.** *For the function $\boldsymbol{G}(\cdot,\cdot)$ the following conditions hold*

a) $\boldsymbol{G}(\theta,\boldsymbol{T}^d)$ *is continuous with respect to* θ *and* $\boldsymbol{T}^d$*;*

b) $(\boldsymbol{G}(\theta,\boldsymbol{T}^d_1) - \boldsymbol{G}(\theta,\boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) > 0$*, where* $\boldsymbol{T}^d_1 \neq \boldsymbol{T}^d_2$*,* $\boldsymbol{T}_1,\boldsymbol{T}_2$ *belong to* $\mathcal{S}^3$*;*

c) $|\boldsymbol{G}(\theta,\boldsymbol{T}^d)| \le C|\boldsymbol{T}^d|$*, where* $\boldsymbol{T}$ *belongs to* $\mathcal{S}^3$ *and* C *is a positive constant;*

d) $\boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d \ge \beta|\boldsymbol{T}^d|^2$*, where* $\boldsymbol{T}$ *belongs to* $\mathcal{S}^3$ *and* β *is a positive constant;*

Constants C *and* β *are independent of temperature* θ*.*

In the proof regarding existence of solution to Mróz model we use the theory of Young measures tools, see [31]. Thanks to the assumption that function $\boldsymbol{G}(\cdot,\cdot)$ is a strictly monotone with respect to second variable, we may use these tools. Moreover, in the existence proof we use the methods developed in [13, 22, 40] arising from the Young measures tools.

Assumption 2 (Norton-Hoff-type model)**.** *The function $\boldsymbol{G}(\theta,\boldsymbol{T}^d)$ is continuous with respect to θ and $\boldsymbol{T}^d$ and for $p \geq 2$ it satisfies the following conditions:*

a) $(\boldsymbol{G}(\theta,\boldsymbol{T}^d_1) - \boldsymbol{G}(\theta,\boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) \geq 0$*, for all* $\boldsymbol{T}^d_1, \boldsymbol{T}^d_2 \in \mathcal{S}^3_d$ *and* $\theta \in \mathbb{R}_+$*;*

b) $|\boldsymbol{G}(\theta,\boldsymbol{T}^d)| \leq C(1 + |\boldsymbol{T}^d|)^{p-1}$*, where* $\boldsymbol{T}^d \in \mathcal{S}^3_d$*,* $\theta \in \mathbb{R}_+$*;*

c) $\boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d \geq \beta|\boldsymbol{T}^d|^p$*, where* $\boldsymbol{T}^d \in \mathcal{S}^3_d$*,* $\theta \in \mathbb{R}_+$*,*

where C and β are positive constants, independent of the temperature θ.

Non-strictly monotone conditions on function $\boldsymbol{G}(\cdot,\cdot)$ cause that tools used for Mróz model do not work. We obtain the identification of weak limit $\boldsymbol{\chi}_k$ by use of Minty-Browder trick. By using Young measures tools in the proof of Mróz model we obtain the strong convergence of approximate sequence of solutions, which cannot be done for Norton-Hoff-type model. Moreover, the dependency of function $\boldsymbol{G}(\cdot,\cdot)$ on deviatoric part of Cauchy stress tensor causes that the functions which should be used as a test function for approximate equations are not regular enough with respect to time. Hence, to deal with it we use convolutions with time-dependent smooth functions. If function $\boldsymbol{G}(\cdot,\cdot)$ had depended on full Cauchy stress tensor than such problems would not appear.

Assumption 3 (Model with growth conditions in generalised Orlicz spaces)**.** *Function $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d)$ is a Carethéodory function, i.e. is measurable with respect to x and continuous with respect to θ and $\boldsymbol{T}^d$, and what is more, it satisfies the following conditions:*

a) $(\boldsymbol{G}(x,\theta,\boldsymbol{T}^d_1) - \boldsymbol{G}(x,\theta,\boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) \geq 0$*, for all* $\boldsymbol{T}^d_1, \boldsymbol{T}^d_2 \in \mathcal{S}^3_d$ *and* $\theta \in \mathbb{R}_+$*;*

b) $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d \geq c\left(M(x,\boldsymbol{T}^d) + M^*(x,\boldsymbol{G}(x,\theta,\boldsymbol{T}^d))\right)$ *for a.a.* $x \in \Omega$*, where* $\boldsymbol{T}^d \in \mathcal{S}^3_d$*,* $\theta \in \mathbb{R}_+$ *and c is a positive constant independent of temperature θ;*

c) $\boldsymbol{G}(x,\theta,\mathbf{0}) = \mathbf{0}$ *for a.a.* $x \in \Omega$*.*

Moreover, M is an N-function and M^* is an N-function complementary to M. The class of N-functions is restricted as follows:

1) the inequality holds

$$\int_Q M^*(x,\boldsymbol{A}(x,t))\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q |\boldsymbol{A}|^2\,\mathrm{d}x\,\mathrm{d}t; \tag{9}$$

2) M^* satisfies the Δ_2-condition.

The schemes regarding proofs of existence for Norton-Hoff-type model and for model with growth conditions in Orlicz spaces are similar. Unfortunately, mathematical methods used there are strictly different. The main reason lies in the fact that we do not assume that M satisfies Δ_2-condition. It requires us to consider the problem in non-reflexive space. Hence, in the proof regarding existence of solution to model with growth conditions in Orlicz spaces we use e.g. Minty-Browder trick for nonreflexive spaces, biting limit and Young measures tools.

Additionally, making the limit passage in approximate sequences of solutions we should remember about results presented in Chapter 3. It is not trivial reasoning for the renormalised solution. First of all, the uniform boundedness of right hand side implies only that there exist a temperature, but we cannot prove its properties. We do not know if it is a renormalised temperature. Then, having this temperature, we obtain that the sequence of right hand sides of heat equations converge weakly in $L^1(0,T,L^1(\Omega))$. And now, we may prove that this temperature is a renormalised temperature.

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