Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

Rafał Celiński

Własności rozwiązań układów oddziałujących cząstek opisujących agregację

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prof. dr hab. Grzegorz Karch

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Rafał Celiński

Mathematical studies of interacting particle systems modelling aggregation

PhD dissertation

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Professor Grzegorz Karch

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Streszczenie

Celem rozprawy jest matematyczny opis własności kilku modeli pochodzących z biologii matematycznej. Badane zagadnienia dotyczą układów agregujących cząstek, których ruch opisany jest za pomocą pewnego nielokalnego operatora. Modele te, są też w pewnym sensie uogólnieniem układu równań Keller-Segela opisującego proces chemotaksji. Wyniki dotyczą isnienia rozwiązań badanych zagadnień, ich asymptotyki dla dużych czasów oraz stabilności jednorodnych stanów stacjonarnych. Ponadto badamy warunki, dla których rozwiązania, albo istnieją globalnie, albo wybuchają w skończonym czasie.

Abstract

The goal of this doctoral dissertation is to study several models coming from mathematical biology. We deal with systems of aggregating particles, which move due to certain nonlocal interactions. Such models are, in some sense, generalizations of the famous Keller-Segel equations describing chemotaxis. Our results concern the existence of solutions, their asymptotic behaviour for large time, and the stability of homogeneous steady states. Moreover, we investigate conditions for which solutions either exists globally in time or blow up in finite time.

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Podziękowania

Chciałbym wyrazić swoją wdzięczność osobom bez których ta rozprawa nie mogłaby powstać. Przede wszystkim dziękuję mojej wspaniałej żonie Tamarze oraz synowi Pawłowi za niewyczerpane pokłady wiary w moje możliwości, dopingowanie mnie oraz znoszenie moich częstych wyjazdów. Dziękuję również rodzicom za zapewnienie mi bardzo dobrych warunków do studiowania.

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Contents

Chapter 1

Introduction

1.1 Chemotaxis process

Mathematical analysis plays more and more important role in understanding many complex processes in biology. A great example is a phenomenon called *chemotaxis* which describes the change of motion of a population density or of single particles according to chemical signals in their environment. Cells move toward a higher concentration of the chemical stimulus (*positive chemotaxis*) or away from it (*negative chemotaxis*). In the first case, the chemical signal is called *chemoattractant*, which is usually produced by species itself, whereas it is called *chemorepellent* in the second situation. As a consequence, organisms, such as bacteria, can direct their movement either to find food or to avoid poison. Moreover, in the case of the positive chemotaxis, an interesting feature is that particles can organize themselves spatially (sometimes forming patterns) or form a more complex organisms.

It is known, that chemotaxis play an essential role in many biological processes. In the immunology and inflammatory context it is used to describe the movement of neutrophils (a type of white blood cell used by immune system) toward inflammation from tissue injury as a result of chemicals released by the tissue. Moreover, the chemotaxis type movement is observed in many important cell-communication processes, such as: organization of cell positioning during embryonic development, tumor cell migration or bacterial growth colony and formation of aggregates in populations of *e.g. Escherichia coli.*

One can look at the modelling chemotaxis phenomena from two different perspectives: the full population (macroscopic level) or an individual particle (microscopic level). There is a large number of papers considering the microscopic approach as well as the derivation of the macroscopic ones from them. We refer the reader to [15, 43, 47, 48] and references therein for precise statement and mathematical results.

In this dissertation we focus on macroscopic level. The mathematical study on this approach began in 1971. Evelyn F. Keller and Lee A. Segel [28] presented a system of two strongly coupled parabolic partial differential equations, which describes the aggregation of cells. Let $u(t,x)$ denote the density of the cells and $c(t,x)$ denote a chemoattractant concentration at time t in point x. The simplified Keller-Segel model look as follows

$$u_t = \nabla(\nabla u - \chi u \nabla c) \quad \text{for } x \in \Omega,\ t > 0, \tag{1.1}$$

$$\varepsilon c_t = \Delta c + u - c \qquad \text{for } x \in \Omega,\ t > 0, \tag{1.2}$$

where Ω can be either the subset of $\mathbb{R}^d$ or the whole space and $\varepsilon \geq 0$. The main issue which (still) attracts mathematicians attention is the competition between diffusion and aggregation. Roughly speaking, if diffusion wins then we have global-in-time solutions to (1.1)–(1.2) whereas solutions blow up in finite time in the second case. However, there are still lots of open questions concerning this system. Here, we refer the reader to the review works of Horstmann [21], Hillen and Painter [20] and reference therein for more details about chemotaxis model and some part of mathematical results. Moreover, for later purpose, we highlight the simplified, parabolic-elliptic version of (1.1)–(1.2), which was intensively studied at the first stage of mathematical analysis of chemotaxis, namely

$$u_t = \nabla \cdot (\nabla u - u \nabla c), \quad -\Delta c + c = u, \qquad x \in \Omega,\ t > 0. \tag{1.3}$$

1.2 Structure of the dissertation

This dissertation consist of the following three papers:

- R. Celiński, *Asymptotic behaviour in a one dimensional model of interacting particles*, Nonlinear Analysis **75** (2012), 1972–1979.

- R. Celiński, *Stability of solutions to aggregation equation in bounded domains*, submitted, (arXiv:1204.5293).

- R. Celiński, D. Hilhorst, G. Karch, M. Mimura, *Mathematical properties of solutions to the model of formation chemotactic E. coli colonies*, in preparation.

The next three chapters collect all results from those papers and they can be considered separately.

1.3 Asymptotic behaviour in a one dimensional model of interacting particles

In Chapter 2, we study the asymptotic behaviour of solutions to the one dimensional initial value problem

$$u_t = \varepsilon u_{xx} + \left(u\ K' * u\right)_x \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{1.5}$$

where the *interaction* kernel K' is a given function, the initial datum $u_0 \in L^1(\mathbb{R})$ is nonnegative and $\varepsilon \geq 0$.

Equation (1.4) arises in the study of an animal aggregation as well as in some problems in mechanics of continuous media. The unknown function $u = u(x,t)$ represents either the population density of a species or, in the case of material applications, the particle density. The kernel K' in (1.4) can be understood as the derivative of a certain function K, that is, K' stands for $\mathrm{d}K/\,\mathrm{d}x$. We use this notation to emphasise that the cell interaction described by equation (1.4) takes place by means of a potential K. Moreover, our assumptions on the interaction kernel K' imply that equation (1.4) describes particles interacting according to a repulsive force (this will be clarified bellow).

Let us first notice that the one dimensional parabolic-elliptic system of chemotaxis (1.3) can be written as equation (1.4). Indeed, if we put $K(x) = -\frac{1}{2}e^{-|x|}$ into (1.4), which is the fundamental solution of the operator $\partial_x^2 - \mathrm{Id}$ on $\mathbb{R}$, one can rewrite the second equation of (1.3) as $v = -K * u$. Here, however, we should emphasise that below we consider repulsive phenomena where the interaction kernel has the opposite sign, see Remark 2.3 for more details.

This work is motivated by the recent publication [25] by Karch and Suzuki where the authors study the large time asymptotics of solutions to (1.4)-(1.5) under the assumption $K' \in L^1(\mathbb{R})$. They showed that either the fundamental solution of the heat equation or a nonlinear diffusion wave appear in the asymptotic expansion of solutions as $t \to \infty$. Analogous results on the solutions of the one dimensional chemotaxis model (1.3) can be found in [40, 41]. Here, we would like to point out that, in all those results, a diffusion phenomena play a pivotal role in the large time behaviour of solutions to problem (1.4)-(1.5).

The main goal of this work is to show that for a large class of interaction kernels $K' \in L^\infty(\mathbb{R}) \backslash L^1(\mathbb{R})$, the diffusion is completely negligible in the study of the large time asymptotics of solutions. Let us discuss it more precisely. Our assumption on the interaction kernel implies that $K'(x)$ is a sufficiently small perturbation of the function $-\frac{A}{2}H(x)$, where, $A \in (0, \infty)$ is a constant and H is the sign function given by the formula: $H(x) = -1$ for $x < 0$ and $H(x) = 1$ for $x > 0$ (*cf.* Remark 2.2). Under these assumptions, we show that for large values of time, a solution of problem (1.4)-(1.5) looks as a compactly supported self-similar profile, defined as the space derivative of a rarefaction wave, *i.e.* the solution of the Riemann problem for the inviscid Burgers equation $u_t + Auu_x = 0$ (see Corollary 2.6 for the precise statement).

In our reasoning, first, we consider $\varepsilon > 0$, and our result on the large time behaviour are, in some sense, independent of ε. Next, we pass to the limit as $\varepsilon \to 0$ to obtain an analogous result for the inviscid aggregation equation $u_t - (u\, K' * u)_x = 0$. In particular, our assumptions imply that weak nonnegative solutions to the initial value problem for this inviscid equation exist for all $t > 0$.

1.4 Stability of solutions to aggregation equation in bounded domains

In Chapter 3 we consider the initial value problem for the following non-local transport equation

$$u_t = \nabla \cdot (\nabla u - u \nabla \mathcal{K}(u)) \quad \text{for } x \in \Omega \subset \mathbb{R}^d,\ t > 0, \tag{1.6}$$

supplemented with the Neumann boundary conditions, *i.e*

$$\frac{\partial u}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{1.7}$$

and an initial datum

$$u(x,0) = u_0(x) \geq 0. \tag{1.8}$$

Here, $\mathcal{K}(u) = \mathcal{K}(u)(x,t)$ is a linear operator defined via the following integral formula

$$\mathcal{K}(u)(x,t) = \int_\Omega K(x,y)u(y,t)\,\mathrm{d}y$$

for a certain function $K = K(x,y)$ which we call as an *aggregation kernel.*

There is a large number of works considering the *inviscid* aggregation equation

$$u_t + \nabla \cdot (u(\nabla K * u)) = 0 \tag{1.9}$$

in the whole space $\mathbb{R}^d$ which has been used to describe aggregation phenomena in the modelling of animal collective behaviour as well as in some problems in mechanics of continuous media, see for instance, [12, 32, 35]. The unknown function $u = u(x,t) \geq 0$ represents either the population density of a species or, in the case of materials applications, the particle density. Equation (1.9) was derived from the system of ODE called "individual cell-based model" [8, 47] representing behaviour of a collection of self-interacting particles via pairwise potential which is described by the aggregation kernel K. More precisely, equation (1.9) is a continuum limit for a system of particles $X_k(t)$ placed at the point k at time t, and evolving by the system of differential equations:

$$\frac{dX_k(t)}{dt} = -\sum_{i\in\mathbb{Z}\setminus\{k\}} \nabla K(X_k(t) - X_i(t)), \quad k \in \mathbb{Z},$$

where K is the potential.

Questions on the global-in-time well-posedness, finite and infinite time blowups, asymptotic behaviour of solutions to equation (1.9), as well as to the equation with an additional diffusion term, have been extensively studied by a number of authors; see *e.g.* [2, 4, 5, 11, 26, 30, 31] and reference therein.

One introduces the diffusion term in (1.9) to make the model more realistic and to describe the interesting biological (and mathematical, as well) phenomenon: competition between aggregation and diffusion, see *e.g.* [6, 14, 25, 38].

In this work, however, our main motivation to study such models is that, in a particular case, equation (1.6) corresponds to the parabolic-elliptic chemotaxis model (1.3). Here, the function $K(x,y)$ is the Green function of the operator $-\partial_x^2 + I$ on Ω with the Neumann boundary conditions. This is called the Bessel potential and it is singular at the origin if $\Omega = \mathbb{R}^d$, $d \geq 2$. On the other hand, in the one dimensional case, when $\Omega = [0,1]$ this fundamental solution is given by the explicit formula *i.e.*

$$K(x,y) = \frac{1}{2}e^{-|x-y|} + \frac{e^{x+y} + e^{2-x-y} + e^{x-y} + e^{y-x}}{2(e^2-1)}. \tag{1.10}$$

In this work we derive some properties of solutions of aggregation equation in a bounded domain under no-flux boundary condition (1.7). The main goal, is to study stability of homogeneous solutions. In particular, we derive conditions under which homogeneous solutions to problem (1.6)–(1.8) are either stable or unstable. Here, let us point out that instability result does not depend on dimension of the domain, and cover the case when the aggregation kernel comes from the chemotaxis model (1.3). Hence, even though solutions are global-in-time and bounded, a homogeneous steady state can be unstable. This means that even in the one dimensional chemotaxis (where all solutions are global-in-time) we can observe the competition between aggregation and diffusion mentioned above.

For the completeness of exposition we also discuss the existence of solutions to (1.6)–(1.8). In order to do that, we use rather standard and well known techniques. In particular, we show that under some general conditions on aggregation kernel we can always construct local-in-time solution to (1.6)–(1.8). However, an additional regularity assumption on the initial datum have to be imposed if $\nabla_x K$ is, in some sense, too singular. Moreover, for mildly singular kernels (see Definition 3.9 for a precise statement), problem (1.6)–(1.8) has a global-in-time solution for any nonnegative, integrable initial condition.

1.5 Mathematical properties of solutions to the model of formation chemotactic *E. coli* colonies

Experiments performed by Budrene and Berg [9, 10] have shown that chemotactic strains of bacterias *E. coli* form stable and remarkably complex spatial patterns such as swarm rings, radial spots, and interdigitated arrays of spots, when inoculated in semi-solid agar. They suggested that the complexity of patterns depend strongly on the initial concentration of substrate, which determines how long a multicellular aggregate structures remain active. Budrene and Berg expect that a substrate consumption, a cell proliferation, an excretion of attractant, and especially a chemotactic motility, when they are combined in a certain way, can generate complex spatial structures. In particular, a specialized, and more complex morphogenetic program is not required.

In order to understand theoretically such a chemotactic pattern formation Mimura and Tsujikawa [34] first presented the following mesoscopic model based on the chemotaxis and growth of bacteria,

$$
\begin{aligned}
u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(c)) + f(u) \quad \text{for} \quad x \in \Omega \times (0, \infty), \\
c_t &= d_c \Delta c + \alpha u - \beta c \quad \text{for} \quad x \in \Omega \times (0, \infty),
\end{aligned}
$$

in a bounded domain and supplemented with the homogeneous Neumann boundary conditions and nonnegative initial data. Here, $u = u(x,t)$ denotes the density of cells and $c = c(x,t)$ is a concentration of chemoattractant. Note that, in the absence of the function $f(u)$, this system reduces to the famous Keller-Segel equations (1.1)–(1.2). Thereafter, such a simple description appeared to be insufficient, hence, authors from [1, 22, 23] proposed a new system of four equations, where two states of bacteria have been distinguished: active and inactive ones. Moreover, an additional equation for a nutrient (substrate) evolution was added. This model was, in fact, a result of coupling the Keller-Segel equations with a reaction-diffusion system proposed by Mimura *et al.* in [33] to describe the morphological diversity of colony patterns in bacteria *Bacillus subtilis*. Denoting the density of active bacteria by $u(x,t)$, the density of inactive bacteria by $w(x,t)$, the density of nutrient by $n(x,t)$, and the concentration of chemoattractant by $c(x,t)$, all these functions at position $x \in \Omega$ and time $t \in [0, \infty)$, the diffusion-chemotaxis-growth system which we will study in Chapter 4, has the form

$$u_t = \Delta u - \nabla \cdot (u \nabla \chi(c)) + g(u) n u - b(n) u \tag{1.11}$$

$$c_t = d_c \Delta c + \alpha u - \beta c \tag{1.12}$$

$$n_t = d_n \Delta n - \gamma g(u) n u \tag{1.13}$$

$$w_t = b(n) u. \tag{1.14}$$

System (1.11)–(1.14) is considered in a bounded domain $\Omega \subset \mathbb{R}^d$ with a sufficiently smooth boundary $\partial \Omega$ and is supplemented with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \qquad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0, \tag{1.15}$$

as well as with nonnegative initial data

$$\begin{array}{ll} u(x,0) = u_0(x), & c(x,0) = c_0(x), \\ n(x,0) = n_0(x), & w(x,0) = w_0(x) \end{array} \qquad \text{for} \quad x \in \Omega. \tag{1.16}$$

The authors of [1] studied problem (1.11)–(1.16) numerically and for some specific functions g, b and χ, they obtained results which were closely related to biological experiments. It is remarkable especially, that they generated geometrically different patterns depending only on the initial nutrient concentration, as it was done in real-life experiments performed by Budrene and Berg. Even in the one dimensional case, they observed that, in some range of n_0, bacteria colonies exhibit so-called oscillatory propagating pulse behaviour.

However, there is still quite little done in the rigorous analysis on problem (1.11)–(1.16), so far. In the works [22] and [23], the authors proved that in one dimensional case there exist solutions, which are global-in-time and bounded uniformly in $t > 0$. Moreover, for some specific choice of functions g, b, and χ, they found an asymptotic profile of such solutions when $t \to \infty$.

In this work, we generalise one dimensional results from [22, 23], see Theorem 4.2. Next, we prove the global-in-time existence of solutions of this model in dimension two and three, under suitable smallness assumptions on initial conditions, *cf.* Theorem 4.3. We also show that large solutions of problem (1.11)–(1.16) may blow up in finite time. Here, we apply an approach, which is well-known in the case of the Keller-Segel system modelling chemotaxis and we prove that solutions to (1.11)–(1.16), where the equation (1.12) is replaced by its elliptic counterpart, cannot be global-in-time if the initial mass *i.e.* $\int_\Omega u_0 \, dx$ is sufficiently large, see Theorem 4.5 below for more details.

1.6 Notation and preliminary definitions

The goal of this section is to collect some notations systematically used in this dissertation.

- $C_c^\infty(\Omega)$ is the set of all smooth and compactly supported functions in Ω.

- $L^p(\Omega)$ is the Lebesgue space of all measurable functions $u = u(x)$ such that the norm $\|u\|_p = \left(\int_\Omega |u(x)|^p \, \mathrm{d}x\right)^{1/p}$ is finite. Moreover $W^{k,p}(\Omega)$ is the Sobolev space *i.e.*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall |\alpha| \leq k\}.$$

- We used to say that a function, say $f = f(t)$, decays exponentially, if there exist constants $\mu > 0$ and $C > 0$ such that $|f(t)| \leq Ce^{-\mu t}$ for all $t > 0$.

- The letter C corresponds to a generic constant (always independent of x and t) which may vary from line to line. Sometimes, we write, *e.g.* $C = C(\alpha, \beta, \gamma, ...)$ when we want to emphasise the dependence of C on parameters $\alpha, \beta, \gamma, ...$ Moreover, for simplicity, from time to time we avoid to show the dependence of solutions either on x or on t.

Chapter 2

Asymptotic behaviour in a one dimensional model of interacting particles

2.1 Main results and comments

In this Chapter we focus on the following one dimensional initial value problem

$$u_t = \varepsilon u_{xx} + \left(u\ K' * u\right)_x \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{2.2}$$

where the *interaction* kernel K' is a given function, the initial datum $u_0 \in L^1(\mathbb{R})$ is nonnegative and $\varepsilon \geq 0$.

We begin our study of large time behaviour of solution by recalling that, for $\varepsilon > 0$, the initial value problem (2.1)–(2.2) is known to have a unique and global-in-time solution for a large class of initial conditions u_0 and interaction kernels K'. Such results are more-or-less standard and the detailed reasoning can be found in [26]. In particular, our assumptions (see Theorem 2.1 below) imply that $K' \in L^\infty(\mathbb{R})$, hence the kernel K' is mildly singular in the sense stated in [26, Thm 2.5]. In this case, results from [26] can be summarised as follows: for every $u_0 \in L^1(\mathbb{R})$ such that $u_0 \geq 0$, there exists the unique global-in-time solution u of problem (2.1)–(2.2) satisfying

$$u \in C\left([0,+\infty),\ L^1(\mathbb{R})\right) \cap C\left((0,+\infty),\ W^{1,1}(\mathbb{R})\right) \cap C^1\left((0,+\infty),\ L^1(\mathbb{R})\right).$$

In addition, the condition $u_0(x) \geq 0$ implies $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. Moreover, we obtain the conservation of the L^1-norm of nonnegative solutions:

$$\|u(t)\|_{L^1} = \int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = \|u_0\|_{L^1}. \tag{2.3}$$

In Theorem 2.5 below, we pass to the limit $\varepsilon \to 0$, to obtain nonnegative weak solutions of problem (2.1)–(2.2) with $\varepsilon = 0$, for which the conservation of mass (2.3) holds true as well.

The goal of this work is to study the large time behaviour of solution to (2.1)–(2.2). First, we state conditions under which these solutions decay as $t \to \infty$.

Theorem 2.1 (Decay of L^p norm)**.** *Assume that $u = u(x,t)$ is a nonnegative solution to problem (2.1)–(2.2) with $\varepsilon > 0$, where the interaction kernel has the form $K'(x) = -\frac{A}{2}H(x) + V(x)$, where H is the sign function, $A > 0$ is a constant, and the function V satisfies*

$$V \in W^{1,1}(\mathbb{R}) \text{ with } \|V_x\|_{L^1} < A. \tag{2.4}$$

Suppose also that $u_0 \in L^1(\mathbb{R})$ is nonnegative. Then for every $p \in [1, \infty]$ the following inequality holds true

$$||u(t)||_p \leq (A - ||V_x||_1)^{\frac{1-p}{p}} \, ||u_0||_1^{1/p} \, t^{\frac{1-p}{p}} \tag{2.5}$$

for all $t > 0$.

Remark 2.2. *Notice that, under assumption (2.4), we have $V(x) = \int_{-\infty}^{x} V_y(y)\,\mathrm{d}y$. Hence, we get immediately that $V \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $\lim_{|x|\to\infty} V(x) = 0$, and the following estimate, $\|V\|_\infty \leq \|V_x\|_1 < A$ holds true. Consequently, our assumption on the interaction kernel K' imply that $K' + \frac{A}{2}H \in C_0(\mathbb{R})$ (continuous and decaying at infinity functions). This means that the kernel K' has a jump at zero exactly as the rescaled sign function $-\frac{A}{2}H$ and converges to the constants $\pm\frac{A}{2}$ as $x \to \mp\infty$, respectively. In some sense, this means that the potential $K(x)$ looks like $-\frac{A}{2}|x|$ at $x = 0$ and at $|x| = \infty$.*

Remark 2.3. *Our assumptions on the kernel $K'(x)$ imply that interactions between particles are similar to those in the chemorepulsion motion, namely, when regions of high chemical concentrations have a repulsive effect on particles. Such a model was studied for example in [13].*

In the next step of this work, we derive an asymptotic profile as $t \to \infty$ solutions of (2.1)–(2.2). First, notice that if the large time behaviour of a solution to problem (2.1)–(2.2) is described by either the heat kernel or the nonlinear diffusion wave (as *e.g.* in [25]) then we expect the following decay rate $||u(t)||_p \le C\ t^{\frac{1-p}{2p}}$ for all $t > 0$. Observe, that the function u from Theorem 2.1 decays faster, hence, its asymptotic behaviour as $t \to \infty$ should be different.

From now on, without loss of generality, we assume that $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 1$. Indeed, due to the conservation of mass (2.3), it suffices to replace u in equation (2.1) by $\frac{u}{\int_{\mathbb{R}} u_0\,\mathrm{d}x}$ and K' by $K'(\int_{\mathbb{R}} u_0\,\mathrm{d}x)$.

Next, let us put

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y - \frac{1}{2}, \tag{2.6}$$

where $u(x,t)$ is the solution of (2.1)–(2.2). Since $u = U_x$, using the explicit form of the kernel K' (cf. Lemma 2.7 below), we obtain that the primitive $U = U(x,t)$ satisfies the following equation

$$U_t = \varepsilon U_{xx} - AUU_x + U_x\ V * U_x, \tag{2.7}$$

which can also be considered as a nonlinear and nonlocal perturbation of the viscous Burgers equation.

Our main result says that the large time behaviour of U is described by a self-similar profile, given by a rarefaction wave, namely, the unique entropy solution of the Riemann problem for the scalar conservation law

$$W^R_t + AW^R W^R_x = 0 \tag{2.8}$$

$$W^R(x,0) = \frac{1}{2}H(x). \tag{2.9}$$

It is well-known (see *e.g.* [16]) that this rarefaction wave is given by the explicit formula

$$W^R(x,t) := \begin{cases} -\dfrac{1}{2} & \text{for} \quad x \le -\dfrac{At}{2}, \\ \dfrac{x}{At} & \text{for} \quad -\dfrac{At}{2} < x < \dfrac{At}{2}, \\ \dfrac{1}{2} & \text{for} \quad x \ge \dfrac{At}{2}. \end{cases} \tag{2.10}$$

Theorem 2.4 (Convergence towards rarefaction waves)**.** *Let the assumptions of Theorem 2.1 hold true. Assume, moreover, that a nonnegative initial datum $u_0(x)$ satisfies*

$$\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 1, \quad \textit{and} \quad \int_{\mathbb{R}} u_0(x)|x|\,\mathrm{d}x < \infty. \tag{2.11}$$

Then, there exists a constant $C > 0$ independent of ε such that for every $t > 0$ and each $p \in (1, \infty]$ the following estimate hold true

$$\|U(\cdot,t) - W^R(\cdot,t)\|_p \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left(\log(2+t)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)}, \tag{2.12}$$

where $U = U(x,t)$ is the primitive of the solution of problem (2.1)–(2.2) *given by* (2.6) *and $W^R = W^R(x,t)$ is the rarefaction wave given by* (2.10)*.*

Next, we show that the asymptotic formula (2.12) holds also true for weak solutions of problem (2.1)–(2.2) with $\varepsilon = 0$.

Theorem 2.5. *Assume that the kernel K' has the properties stated in Theorem 2.1 and the nonnegative initial condition $u_0 \in L^1(\mathbb{R})$ satisfies* (2.11)*. Then the initial value problem*

$$U_t = -AUU_x + U_x V * U_x \tag{2.13}$$

$$U(x,0) = U_0(x) = \int_{-\infty}^{x} u_0(y)\,\mathrm{d}y - \frac{1}{2} \tag{2.14}$$

has a weak solution $U \in C\big(\mathbb{R} \times (0,\infty)\big)$ such that $U_x \in L^\infty_{loc}\big((0,\infty), L^\infty(\mathbb{R})\big)$. This satisfies problem (2.13)–(2.14) *in the following integral sense*

$$\begin{aligned}-\int_0^\infty \int_{\mathbb{R}} U\varphi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x =& \frac{A}{2}\int_0^\infty \int_{\mathbb{R}} U^2\varphi_x\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}} U_x\left(V_x * U\right)\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times [0,+\infty))$. Moreover, this solution satisfies for each $p \in (1,\infty]$

$$\|U(\cdot,t) - W^R(\cdot,t)\|_p \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left(\log(2+t)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)}, \tag{2.15}$$

for a constant $C > 0$ and for all $t > 0$.

Next, we use the result from Theorems 2.4 and 2.5 to describe the large time asymptotics of solutions to problem (2.1)–(2.2).

Corollary 2.6. *Let the assumptions either of Theorem 2.1 or Theorem 2.5 hold true. For the solution* $u = u(x,t)$ *of problem* (2.1)–(2.2) *with* $\varepsilon \geq 0$ *we define its rescaled version* $u^\lambda(x,t) = \lambda u(\lambda x, \lambda t)$ *for* $\lambda > 0$, $x \in \mathbb{R}$ *and* $t > 0$. *Then, for every test function* $\varphi \in C_c^\infty(\mathbb{R})$ *and each* $t_0 > 0$

$$\int_{\mathbb{R}} u^\lambda(x,t_0)\varphi(x)\,\mathrm{d}x \to -\int_{\mathbb{R}} W^R(x,t_0)\varphi_x(x)\,\mathrm{d}x \quad \textit{as} \quad \lambda \to +\infty.$$

In other words, for each $t_0 > 0$, the family of rescaled solutions $u^\lambda(x,t_0) = \lambda u(\lambda x, \lambda t_0)$ of problem (2.1)–(2.2) with $\varepsilon \geq 0$ converges weakly as $\lambda \to \infty$ to the compactly supported self-similar profile defined as

$$\left(W^R\right)_x(x,t_0) := \begin{cases} (At)^{-1} & \text{for} \quad |x| < \dfrac{At}{2}, \\ 0 & \text{for} \quad |x| \geq \dfrac{At}{2}. \end{cases} \tag{2.16}$$

2.2 Large time asymptotics

In this section, we prove all results stated in Section 2.1. We begin by an elementary result.

Lemma 2.7. *Let* H *be the sign function. For all* $\varphi \in W^{1,1}(\mathbb{R})$ *the following equality hold true:* $H * \varphi_x = 2\varphi$.

Proof. First, we assume that $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$H * \varphi_x = \int_{\mathbb{R}} H(x-y)\varphi_y(y)\,\mathrm{d}y = \int_{-\infty}^{x} \varphi_y(y)\,\mathrm{d}y - \int_{x}^{\infty} \varphi_y(y)\,\mathrm{d}y = 2\varphi(x).$$

The proof for general $\varphi \in W^{1,1}(\mathbb{R})$ is completed by a standard approximation argument. □

Now, we are in a position to prove Theorem 2.1 concerning the decay of the solutions in L^p-spaces.

Proof of Theorem 2.1. Note that by (2.3), we have $\|u(t)\|_1 = \|u_0\|_1$ which implies (2.5) for $p = 1$. Hence, we can assume that $p > 1$.

We multiply equation (2.1) by pu^{p-1} (recall that u is nonnegative), integrate with respect to x over $\mathbb{R}$, and integrate by parts to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^p\,\mathrm{d}x = -\frac{4(p-1)\varepsilon}{p}\int_{\mathbb{R}}\left[\left(u^{p/2}\right)_x\right]^2\,\mathrm{d}x + (p-1)\int_{\mathbb{R}} u^p K' * u_x\,\mathrm{d}x.$$

The first term on the right-hand side (containing $\varepsilon > 0$) is obviously nonpositive, hence, we skip it in our estimates. Using the explicit form of the kernel $K' = -\frac{A}{2}H + V$ and Lemma 2.7, we rewrite the second term as follows

$$\int_{\mathbb{R}} u^p K' * u_x\,\mathrm{d}x = \left(-A\int_{\mathbb{R}} u^{p+1}\,\mathrm{d}x + \int_{\mathbb{R}} u^p\; V_x * u\,\mathrm{d}x\right). \tag{2.17}$$

Notice that a simple computation involving the Hölder and the Young inequalities leads to the estimates

$$\left|\int_{\mathbb{R}} u^p\; V_x * u\,\mathrm{d}x\right| \le ||V_x * u||_{p+1}||u^p||_{\frac{p+1}{p}} \le ||V_x||_1||u||_{p+1}^{p+1}. \tag{2.18}$$

Hence, using (2.17) and (2.18) we get

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)^p\,\mathrm{d}x \le (p-1)\left(-A + ||V_x||_1\right)||u(t)||_{p+1}^{p+1}. \tag{2.19}$$

Moreover, it follows from the Hölder inequality (with the exponents p and $\frac{p}{p-1}$) that

$$\int_{\mathbb{R}} u^p\,\mathrm{d}x = \int_{\mathbb{R}} u^{\frac{1}{p}}\; u^{\frac{p^2-1}{p}}\,\mathrm{d}x \le \Big(\int_{\mathbb{R}} u\,\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}} u^{p+1}\,\mathrm{d}x\Big)^{\frac{p-1}{p}},$$

which means

$$\int_{\mathbb{R}} u^{p+1}\,\mathrm{d}x \ge ||u_0||_1^{\frac{-1}{p-1}}\Big(\int_{\mathbb{R}} u^p\,\mathrm{d}x\Big)^{\frac{p}{p-1}}, \tag{2.20}$$

because $\|u(t)\|_1 = \|u_0\|_1$. Applying estimate (2.20) to (2.19), we obtain the following differential inequality for $\int_{\mathbb{R}} u^p\,\mathrm{d}x$:

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)^p\,\mathrm{d}x \le (p-1)\left(-A + ||V_x||_1\right)||u_0||_1^{-\frac{1}{p-1}}\left(\int_{\mathbb{R}} u(x,t)^p\,\mathrm{d}x\right)^{\frac{p}{p-1}}. \tag{2.21}$$

It is easy to prove that any nonnegative solution of the differential inequality

$$\frac{d}{dt}f(t) \le -Df(t)^{\frac{p}{p-1}},$$

with a constant $D > 0$, satisfies

$$f(t) \leq \left(\frac{D}{p-1}\right)^{1-p} t^{1-p}.$$

Hence, it follows from (2.21) and from the assumption $\|V_x\|_1 < A$ that

$$||u(t)||_p \leq (A - ||V_x||_1)^{\frac{1-p}{p}} \, ||u_0||_1^{1/p} \, t^{\frac{1-p}{p}} \tag{2.22}$$

for all $t > 0$. Finally, passing to the limit $p \to \infty$ in (2.22), we obtain

$$||u(t)||_\infty \leq (A - ||V_x||_1)^{-1} \ t^{-1}$$

for all $t > 0$. This completes the proof of Theorem 2.1. □

Let us now recall some results on smooth approximations of rarefaction waves, more precisely, the solution of the following Cauchy problem for the Burgers equation

$$\begin{aligned} &Z_t - \varepsilon Z_{xx} + AZZ_x = 0, \\ &Z(x,0) = Z_0(x) = \tfrac{1}{2} H(x). \end{aligned} \tag{2.23}$$

where $A > 0$.

Lemma 2.8 (Hattori-Nishihara [18]). *Problem* (2.23) *has a unique, smooth, global-in-time solution $Z(x,t)$ satisfying*

i) $-1/2 < Z(x,t) < 1/2$ *and* $Z_x(x,t) > 0$ *for all* $(x,t) \in \mathbb{R} \times (0,\infty)$*;*

ii) for every $p \in [1,\infty]$, there exists a constant $C = C(p) > 0$ independent of $\varepsilon > 0$ such that

$$\|Z_x(t)\|_p \leq Ct^{-1+1/p}$$

and

$$\|Z(t) - W^R(t)\|_p \leq Ct^{-(1-1/p)/2}$$

for all $t > 0$, where $W^R(x,t)$ is the rarefaction wave given by formula (2.10).

Sketch of the proof. All results stated in Lemma 2.8 can be found in [18] with some additional improvements contained in [27, Sec. 3], and they are deduced from an explicit formula for smooth approximation of rarefaction waves. Here, however, we should emphasise that the authors of [18] consider equation (2.23) with $\varepsilon = 1$ but, by a simple scaling argument, we can extend those results for all $\varepsilon > 0$. Indeed, we check that the function $f(x,t) = Z(\varepsilon x, \varepsilon t)$ satisfies $f_t - f_{xx} + Aff_x = 0$. Hence, by the result from [18] we have

$$\|f_x(t)\|_p \le Ct^{\frac{1-p}{p}} \qquad \text{and} \qquad \|f(t) - W^R(t)\|_p \le Ct^{-(1-1/p)/2}.$$

Now, coming back to original variables, we have

$$\varepsilon^{\frac{p-1}{p}} \|Z_x(\cdot, \varepsilon t)\|_p \le C\ (\varepsilon t)^{\frac{1-p}{p}} \varepsilon^{\frac{p-1}{p}}$$

and so, defining the new variable $\tilde{t} = \varepsilon t$, we obtain $\|Z_x(\tilde{t})\|_p \le C\ \tilde{t}^{\frac{1-p}{p}}$ with a constant C independent of ε. The second inequality in Lemma 2.8.ii. can be obtained in a similar way.

□

Next, we study the large time asymptotics of $U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y - \frac{1}{2}$, which satisfies equation (2.6). Recall that $u = U_x$. In the proof of Theorem 2.4, we need the following auxiliary result.

Lemma 2.9. *Let u_0 satisfy conditions (2.11). Assume that $U = U(x,t)$, defined by (2.6), is the solution of equation (2.7) supplemented with the initial condition $U_0(x) = \int_{-\infty}^{x} u_0(y)\,\mathrm{d}y - 1/2$ and $Z = Z(x,t)$ is the smooth approximation of the rarefaction wave, namely, the solution of problem (2.23). Then, for every $t_0 > 0$ we have*

$$\sup_{t>t_0} \frac{1}{\log(2+t)} \|U(t) - Z(t)\|_1 < \infty.$$

Proof. At the beginning, let us notice that assumption (2.11) on u_0 implies that $U_0(x) \in L^1(-\infty, 0)$ and $U_0(x) - 1 \in L^1(0, \infty)$. Hence, we have $U_0 - Z_0 \in L^1(\mathbb{R})$.

Denoting $R = U - Z$ and using equations (2.7) and (2.23), we see that this new function satisfies

$$R_t = \varepsilon R_{xx} - \frac{A}{2}(U^2 - Z^2)_x + U_x\ V * U_x.$$

We multiply this equation by $\operatorname{sgn} R$ (in fact, by a smooth approximation of $\operatorname{sgn} R$) and we integrate with respect to x to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} |R|\,\mathrm{d}x = \varepsilon \int_{\mathbb{R}} R_{xx} \operatorname{sgn} R\,\mathrm{d}x - \frac{A}{2}\int_{\mathbb{R}} (U^2 - Z^2)_x \operatorname{sgn} R\,\mathrm{d}x + \int_{\mathbb{R}} U_x\ V * U_x \operatorname{sgn} R\,\mathrm{d}x.$$

The first term on the right-hand side of the above equation is nonpositive because this is the well-known Kato inequality. The second term is equal to 0 because of the following calculations:

$$\begin{aligned}\int_{\mathbb{R}}(U^2-Z^2)_x \operatorname{sgn} R\,\mathrm{d}x &= \int_{\mathbb{R}} \left(R^2+2RZ\right)_x \operatorname{sgn} R\,\mathrm{d}x \\ &= \int_{\mathbb{R}} 2R_x|R|\,\mathrm{d}x + \int_{\mathbb{R}} 2ZR_x \operatorname{sgn} R\,\mathrm{d}x + \int_{\mathbb{R}} 2Z_x|R|\,\mathrm{d}x \\ &= -2\int_{\mathbb{R}} Z_x|R|\,\mathrm{d}x + 2\int_{\mathbb{R}} Z_x|R|\,\mathrm{d}x = 0\end{aligned}$$

since $\int_{\mathbb{R}} R_x|R|\,\mathrm{d}x = 0$. Moreover, using the Young inequality, we have

$$\left|\int_{\mathbb{R}} U_x\ V*U_x\ \operatorname{sgn} R\,\mathrm{d}x\right| \le \|U_x\ V*U_x\|_1 \le \|U_x\|_\infty\|V\|_1\|U_x\|_1.$$

Hence, by the fact that $U_x(t) = u(t)$ and using the decay estimates from Theorem 2.1 for $p=1$ and $p=\infty$, we get the following differential inequality

$$\frac{d}{dt}\|R(t)\|_1 \le Ct^{-1}$$

which completes the proof of Lemma 2.9. □

Now, we are in a position to prove our main result about the convergence of the primitive of u towards a rarefaction wave.

Proof of Theorem 2.4. Let $Z = Z(x,t)$ be a smooth approximation of the rarefaction wave from Lemma 2.8. Denote $R = Z-U$. Hence, by Lemma 2.8 and Theorem 2.1, we have

$$\|R_x(t)\|_\infty = \|U_x(t)-Z_x(t)\|_\infty \le \|u(t)\|_\infty + \|Z_x(t)\|_\infty \le C\ t^{-1}$$

for a constant $C>0$. Moreover, using the Sobolev-Gagliardo-Nirenberg inequality

$$\|R\|_p \le C\|R_x\|_\infty^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|R\|_1^{\frac{1}{2}(1+\frac{1}{p})},$$

valid for every $p\in(1,\infty]$ and Lemma 2.9 we have

$$\|U(t)-Z(t)\|_p \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+t)\right)^{\frac{1}{2}(1+\frac{1}{p})}$$

for all $t>0$.

Finally, to complete the proof, we use Lemma 2.8 to replace the smooth approximation $Z(x,t)$ by the rarefaction wave $W^R(x,t)$. □

The proof of Theorem 2.5 relies on a form of the Aubin-Simon compactness result that we recall below.

Theorem 2.10 ([46, Theorem 5]). *Let X, B and Y be Banach spaces satisfying $X \subset B \subset Y$ with compact embedding $X \subset B$. Assume, for $1 \le p \le +\infty$ and $T > 0$, that*

- *F is bounded in $L^p(0,T;X)$,*
- *$\{\partial_t f \,:\, f \in F\}$ is bounded in $L^p(0,T;Y)$.*

Then F is relatively compact in $L^p(0,T;B)$ (and even in $C(0,T;B)$ if $p = +\infty$).

Proof of Theorem 2.5. We denote by U^ε a solution of equation (2.7) with $\varepsilon > 0$ supplemented with the initial condition (2.14). The proof follows from three steps. First, we show that the family

$$\mathcal{F} \equiv \{U^\varepsilon : \varepsilon \in (0,1]\},$$

is relatively compact in $C([t_1,t_2], C[-R,R])$ for every $0 < t_1 < t_2 < \infty$ and every $R > 0$. Next, we show that there exists a function $\bar{U} = \lim_{\varepsilon\to 0} U^\varepsilon$ which is a weak solution of problem (2.13)–(2.14). Finally, we prove that $\bar{U}$ satisfies estimate (2.15).

Step 1. Compactness. We apply Theorem 2.10 with $p = \infty$, $F = \mathcal{F}$, and

$$X = C^1([-R,R]), \qquad B = C([-R,R]), \qquad Y = W^{-1,1}([-R,R]),$$

where $R > 0$ is fixed and arbitrary, and Y is the dual space of $W_0^{1,1}([-R,R])$. Obviously, the embedding $X \subset B$ is compact by the Arzela-Ascoli theorem.

First, we show that the sets $\mathcal{F}$ and $\{\partial_x U^\varepsilon \,:\, \varepsilon \in (0,1]\}$ are bounded subsets of $L^\infty([t_1,t_2], C([-R,R]))$. Indeed, it follows from the definition of the function U^ε, see (2.6), that

$$|U^\varepsilon(x,t)| \le \|(U^\varepsilon)_x(\cdot,t)\|_1 + \frac{1}{2} = \|u_0\|_1 + \frac{1}{2}. \tag{2.24}$$

Moreover, using Theorem 2.1 we have

$$\|(U^\varepsilon)_x(\cdot,t)\|_\infty \le (A - \|V_x\|_1)^{-1}\, t^{-1}. \tag{2.25}$$

To check the second condition of the Aubin-Simon compactness criterion, it suffices to show that there is a positive constant C independent of $\varepsilon \in (0,1]$

such that $\sup_{t\in[t_1,t_2]} \|\partial_t U^\varepsilon\|_Y \leq C$. Let us show this estimate by a duality argument. For every $\varphi \in C_c^\infty((-R,R))$ and $t \in [t_1,t_2]$, by (2.24), (2.25) and Theorem 2.1, we have

$$\left|\int_{\mathbb{R}} \partial_t U^\varepsilon(t)\varphi \,\mathrm{d}x\right| \leq \left|\int_{\mathbb{R}} \varepsilon U_x^\varepsilon(t)\varphi_x \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} AU^\varepsilon(t)U_x^\varepsilon(t)\varphi \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} U_x^\varepsilon(t)V * U_x^\varepsilon(t)\varphi \,\mathrm{d}x\right|$$
$$\leq \|\varphi_x\|_\infty \int_{\mathbb{R}} |U_x^\varepsilon(t)| \,\mathrm{d}x + A\|U^\varepsilon(t)\|_\infty \|\varphi\|_\infty \int_{\mathbb{R}} |U_x^\varepsilon(t)| \,\mathrm{d}x + \|U_x^\varepsilon(t)\|_\infty^2 \|V\|_1 \|\varphi\|_1$$
$$\leq \|\varphi_x\|_\infty \|u_0\|_1 + A\|u_0\|_1(\|u_0\|_1 + 1/2)\|\varphi\|_\infty + (A - \|V_x\|_1)^{-2} t_1^{-2} \|V\|_1 \|\varphi\|_1.$$

Hence, the proof of Step 1 is completed.

Step 2. Limit function. By Step 1, for every $0 < t_1 < t_2 < +\infty$, the family $\{U^\varepsilon : \varepsilon \in (0,1]\}$ is relatively compact in $C([t_1,t_2], C(-R,R))$. Consequently, by a diagonal argument, there exists a sequence of $\{U^{\varepsilon_n} : \varepsilon_n \in (0,1]\}$ and a function $\bar{U} \in C((0,+\infty), C(\mathbb{R}))$ such that

$$U^{\varepsilon_n} \to \bar{U} \quad \text{as} \quad \varepsilon_n \to 0 \quad \text{in} \quad L^\infty_{loc}\big(\mathbb{R} \times (0,+\infty)\big). \tag{2.26}$$

Moreover, it follows from the estimate (2.25) that, by the Banach-Alaoglu Theorem,

$$U_x^{\varepsilon_n} \to \bar{U}_x \quad \text{as} \quad \varepsilon_n \to 0$$

weak-$*$ in $L^\infty_{loc}\big((0,\infty), L^\infty(\mathbb{R})\big)$.

Now, multiplying equation (2.7) by a test function $\varphi \in C_c^\infty(\mathbb{R} \times [0,+\infty))$ and integrating the resulting equation over $\mathbb{R} \times [0,\infty)$, we obtain the identity

$$\begin{aligned} -\int_0^\infty \int_{\mathbb{R}} U^{\varepsilon_n} \varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} U_0(x)\varphi(x,0)\,\mathrm{d}x &= \varepsilon_n \int_0^\infty \int_{\mathbb{R}} U^{\varepsilon_n} \varphi_{xx} \,\mathrm{d}x\,\mathrm{d}t \\ +\frac{A}{2} \int_0^\infty \int_{\mathbb{R}} (U^{\varepsilon_n})^2 \varphi_x \,\mathrm{d}x\,\mathrm{d}t &+ \int_0^\infty \int_{\mathbb{R}} U_x^{\varepsilon_n} \left(V_x * U^{\varepsilon_n}\right) \varphi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.27}$$

It is easy to pass to the limit as $\varepsilon_n \to 0$ on the left-hand side of (2.27), using the Lebesgue dominated convergence theorem. To deal with the term on the right-hand side we make the following decomposition:

$$\begin{aligned} \int_{\mathbb{R}} U_x^{\varepsilon_n} (V_x * U^{\varepsilon_n}) \varphi \,\mathrm{d}x &= \int_{\mathbb{R}} U_x^{\varepsilon_n} \big(V_x * (U^{\varepsilon_n} - \bar{U})\big) \varphi \,\mathrm{d}x \\ &+ \int_{\mathbb{R}} U_x^{\varepsilon_n} \big(V_x * \bar{U}\big) \varphi \,\mathrm{d}x. \end{aligned} \tag{2.28}$$

We can estimate the first term on the right-hand side of (2.28) as follows:

$$\left|\int_{\mathbb{R}} U_x^{\varepsilon_n}\left(V_x * (U^{\varepsilon_n} - \bar{U})\right)\varphi \, \mathrm{d}x\right| \leq \|U_x^{\varepsilon_n}(t)\|_\infty \int_{\mathbb{R}} \left|V_x * (U^{\varepsilon_n} - \bar{U})\varphi\right| \, \mathrm{d}x. \tag{2.29}$$

Let us notice that $V_x * (U^{\varepsilon_n} - \bar{U})$ tends to zero as $\varepsilon_n \to 0$ by the Lebesgue dominated convergence theorem and it is bounded independently of ε_n. Hence, using the Lebesgue dominated convergence theorem and Theorem 2.1, we deduce that the right-hand side of (2.29) converges to zero. The second term on the right-hand side of (2.28) obviously converges to $\int_{\mathbb{R}} \bar{U}_x \left(V_x * \bar{U}\right)\varphi \, \mathrm{d}x$ by the weak-$*$ convergence of $U_x^{\varepsilon_n}$ in $L^\infty(\mathbb{R})$ since $(V_x * \bar{U})\varphi \in L^1(\mathbb{R})$. This completes the proof of Step 2.

Step 3. Convergence towards a rarefaction wave. To prove (2.15), we use the Fatou Lemma and (2.26); we obtain

$$\|\bar{U}(t) - W^R(t)\|_p \leq \liminf_{\varepsilon_n \to 0} \|U^{\varepsilon_n}(t) - W^R(t)\|_p$$

for all $t > 0$.

Now, it is enough to use Theorem 2.4 to estimate the quantity on right-hand side, since constant C in (2.12) is independent of ε. Hence the proof of Theorem 2.5 is finished. □

At last, we prove Corollary 2.6.

Proof of Corollary 2.6. First, we express the result stated in Theorems 2.4 and 2.5 in another way. We consider the rescaled family of functions $U^\lambda(x,t) = U(\lambda x, \lambda t)$ for all $\lambda > 0$. Let us also notice that $W^R(x,t)$ is self-similar in the sense that $\left(W^R\right)^\lambda(x,t) = W^R(x,t)$ for all $x \in \mathbb{R}$, $t > 0$, $\lambda > 0$. Hence, changing the variables and using Theorem 2.4 and Theorem 2.5 in the case $\varepsilon = 0$, we obtain

$$\begin{aligned}\|U^\lambda(\cdot, t_0) - \left(W^R\right)^\lambda(\cdot, t_0)\|_p = \lambda^{-1/p}\|U(\cdot, \lambda t_0) - W^R(\cdot, \lambda t_0)\|_p \leq \\ C\lambda^{-1/p}(\lambda t_0)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\left(\log(2+\lambda t_0)\right)^{\frac{1}{2}\left(1+\frac{1}{p}\right)} \to 0\end{aligned}$$

as $\lambda \to \infty$. It means that the family of functions U^λ converge in $L^p(\mathbb{R})$ as $\lambda \to \infty$ towards $W^R(x,t)$ for every $t_0 > 0$ and $p \in (1, \infty]$.

This scaling argument allows us to express the convergence of solutions of the original problem (2.1)–(2.2) towards a self-similar profile. Indeed, let us note that since $u = U_x$, it follows immediately that $u^\lambda(x,t) = \lambda u(\lambda x, \lambda t) = \partial_x U^\lambda(x,t)$. Hence, the weak convergence of u^λ towards the distributional derivative of the rarefaction wave $\partial_x W^R$ is an immediate consequence of the Lebesgue dominated convergence theorem and of Theorem 2.4 for $p = \infty$ since $|U^\lambda(x,t_0)| \le \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x + \frac{1}{2}$. □

Chapter 3

Stability of solutions to aggregation equation in bounded domains

3.1 Assumptions, main results and comments

3.1.1 Stability and instability of homogeneous solutions

In this Chapter we study the following initial-boundary value problem

$$u_t = \nabla \cdot (\nabla u - u \nabla \mathcal{K}(u)) \quad \text{for } x \in \Omega \subset \mathbb{R}^d,\ t > 0, \tag{3.1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{3.2}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \tag{3.3}$$

where, $\mathcal{K}(u) = \mathcal{K}(u)(x,t)$ is a linear operator defined via the following integral formula

$$\mathcal{K}(u)(x,t) = \int_\Omega K(x,y) u(y,t)\, \mathrm{d}y$$

for certain *aggregation kernel* $K = K(x,y)$.

Moreover, in this Chapter, we assume the following conditions on the aggregation kernel

$$\frac{\partial K}{\partial n}(\cdot, y) = 0 \quad \text{on } \partial\Omega \quad \text{for all} \quad y \in \Omega, \tag{3.4}$$

$$\nabla_x \left(\int_\Omega K(x,y)\, \mathrm{d}y \right) = 0, \quad \text{for all} \quad x \in \Omega \tag{3.5}$$

$$\|\nabla_x K\|_{\infty,q'} \equiv \operatorname*{ess\,sup}_{x\in\Omega} \|\nabla_x K(x,\cdot)\|_{q'} + \operatorname*{ess\,sup}_{y\in\Omega} \|\nabla_x K(\cdot,y)\|_{q'} < \infty \quad \text{for some } q' \in [1,\infty]. \tag{3.6}$$

Remark 3.1. *Notice that under the assumptions* (3.4)*, a solution of problem* (3.1)–(3.3) *conserves the integral ("mass") i.e.*

$$\|u(t)\|_1 = \int_\Omega u(x,t)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x = \|u_0\|_1 \quad \textit{for all} \quad t \geq 0. \tag{3.7}$$

Indeed, it is sufficient to integrate the equation (3.1) *with respect to x and use identities* (3.2) *and* (3.4)*. Moreover, this solution remains nonnegative if the initial condition is so, due to the maximum principle.*

Remark 3.2. *Note, that assumption* (3.5) *implies that every constant function $u \equiv M$ satisfies equation* (3.1)*. In fact, the chemotaxis model* (1.3) *is our main motivation to state this assumption. Indeed, if (U, V) is a stationary solution to* (1.3) *then U is constant if and only if V is constant as well. It means that, if the kernel K is the Green function of the operator $-\Delta + aI$ then the term $\nabla\mathcal{K}(U)$ in equation* (3.1) *for $U = M$, has to be equal* 0 *and so, K satisfies* (3.5)*.*

The main goal of this work is to study stability of homogeneous solution to problem (3.1)–(3.3). More precisely, we look for sufficient conditions for either stability of constant solutions or their instability. Our result can be summarised in the following way

- If the homogeneous solution $u(x,t) = M \geq 0$ of problem (3.1)–(3.3) is sufficiently small, then it is asymptotically stable solution in the linear and nonlinear sense, see Proposition 3.3 and Theorem 3.4 below.
- If the homogeneous solution $u(x,t) = M \geq 0$ is sufficiently large, then there is a large class of aggregation kernels (which include the kernel coming from chemotaxis system (1.3)), such that $u(x,t) = M$ is a linearly unstable solution of (3.1)–(3.3).

Thus, we focus on a solution to problem (3.1)–(3.3) in the form

$$u(x,t) = M + \varphi(x,t),$$

where M is an arbitrary constant and φ is a perturbation. Moreover, since

$$\int_\Omega u(x,t)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x = \int_\Omega M\,\mathrm{d}x = M|\Omega| \quad \text{for all } t > 0,$$

we have that $\int_\Omega \varphi(x,t)\,\mathrm{d}x = 0$ holds for all $t \geq 0$. Hence, from equation (3.1), using assumption (3.5), we obtain the following initial boundary value problem for the perturbation φ

$$\varphi_t = \Delta\varphi - \nabla\cdot\Big(M\nabla\mathcal{K}(\varphi) + \varphi\nabla\mathcal{K}(\varphi)\Big), \tag{3.8}$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{3.9}$$

$$\varphi(x,0) = \varphi_0(x). \tag{3.10}$$

We also introduce its linearized counterpart, namely, skipping the term $\nabla\cdot(\varphi\nabla\mathcal{K}(\varphi))$ on the right hand side of (3.8) we obtain

$$\varphi_t = \Delta\varphi - \nabla\cdot\Big(M\nabla\mathcal{K}(\varphi)\Big), \tag{3.11}$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{3.12}$$

$$\varphi(x,0) = \varphi_0(x). \tag{3.13}$$

In the sequel, we use the linear operator $\mathcal{L}\varphi = -\Delta\varphi + \nabla\cdot\Big(M\nabla\mathcal{K}(\varphi)\Big)$ with the Neumann boundary conditions, defined for all $\varphi, \psi \in W^{1,2}(\Omega)$ via its associated bilinear form

$$J(\varphi,\psi) = \int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}x - M\int_\Omega \nabla\mathcal{K}(\varphi)\nabla\psi\,\mathrm{d}x. \tag{3.14}$$

Here, we recall that a constant M is called a linearly asymptotically stable stationary solution to nonlinear problem (3.1)–(3.3) if the zero solution is an asymptotically stable solution of the linearized problem (3.11)–(3.13). Moreover, a constant M is called linearly unstable stationary solution to nonlinear problem (3.1)–(3.3) if zero is an unstable solution to linearized problem (3.11)–(3.13).

Proposition 3.3 (Linear stability of constant solutions)**.** *Assume, that the aggregation function $K(x,y)$ satisfy conditions (3.4) and (3.5). If, moreover, the operator $\nabla\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ given by the form $\nabla\mathcal{K}(\varphi) = \int_\Omega \nabla_x K(x,y)\varphi(y)\,\mathrm{d}y$ is bounded and if*

$$M\|\nabla\mathcal{K}\|_{L^2\to L^2} < \sqrt{\lambda_1}, \tag{3.15}$$

where λ_1 is the first nonzero eigenvalue of $-\Delta$ on Ω under the Neumann boundary condition then M is a linearly asymptotically stable stationary solution to the problem (3.1)–(3.3).

We prove this proposition in Section 3.2. Here, we only emphasise that this proof allow us to show the nonlinear stability of constant steady states, under slightly stronger assumptions imposed on the kernel K.

Theorem 3.4 (Nonlinear stability of constant solution). *Let the assumptions of Proposition 3.3 hold true. If moreover* $\|\nabla_x K\|_{\infty,2} < \infty$*, then there exists a positive constant* $\eta = \eta(\nabla_x K, M, \Omega)$ *such that for every* $\varphi_0 \in L^2(\Omega)$ *satisfying* $\|\varphi_0\|_2 < \eta$ *and* $\int_\Omega \varphi_0(x)\,\mathrm{d}x = 0$*, the perturbed problem* (3.8)–(3.10) *has a solution* $\varphi \in C([0,\infty), L^2(\Omega))$ *such that* $\int_\Omega \varphi(x,t)\,\mathrm{d}x = 0$ *for all* $t > 0$*. Moreover, we have*

$$\|\varphi(t)\|_2 \to 0 \quad as \quad t \to \infty.$$

Next, we discuss instability of constant solutions.

Theorem 3.5 (Instability of constant solutions). *Let* $w_1 = w_1(x) \geq 0$ *be the normalized eigenfunction of* $-\Delta$ *on* Ω *under the Neumann boundary condition corresponding to the first nonzero eigenvalue* λ_1*, and such that* $\|w_1\|_2 = 1$*. Assume that* $\|\nabla \mathcal{K}\|_{L^2 \to L^2} < \infty$*. If moreover, the aggregation function* $K(x,y)$ *satisfy*

$$\int_\Omega \int_\Omega K(x,y) w_1(y) w_1(x)\,\mathrm{d}x\,\mathrm{d}y = A > 0, \tag{3.16}$$

then for $M > 1/A$ *the constant solution* M *of problem* (3.1)–(3.3) *is a linearly unstable stationary solution.*

Remark 3.6. *Let us notice that the aggregation function* K *which comes from chemotaxis model* (1.3) *satisfies the condition* (3.16)*. Indeed, in this case,* $K(x,y)$ *is a fundamental solution of the operator* $-\Delta + aI$ *in a bounded domain supplemented with the Neumann boundary conditions. Thus, the function*

$$w(x) = \int_\Omega K(x,y) w_1(y)\,\mathrm{d}y$$

satisfies the following equation

$$-\Delta w + aw = w_1. \tag{3.17}$$

After multiplying equation (3.17) *by* w_1 *and integrating over* Ω *and using the Neumann boundary condition we obtain*

$$-\int_\Omega \Delta w w_1\,\mathrm{d}x + a\int_\Omega w w_1\,\mathrm{d}x = \int_\Omega (w_1)^2\,\mathrm{d}x.$$

Obviously, by the definition of A, we have $\int_\Omega ww_1\,\mathrm{d}x = A$. Thus, after integrating by parts we obtain

$$-\int_\Omega w\Delta w_1\,\mathrm{d}x = 1 - aA. \tag{3.18}$$

Finally, we use the fact that w_1 is an eigenfunction of $-\Delta$ to get

$$\lambda_1 \int_\Omega ww_1\,\mathrm{d}x = 1 - aA,$$

which implies that $A = \frac{1}{a+\lambda_1} > 0$.

Remark 3.7. *Our stability results on constant steady states corresponds to the well-known results on the global existence versus blow up of solutions to Keller-Segel system* (1.3)*. In particular, for general kernels (see Definition 3.9 below), where solutions of problem* (3.1)–(3.3) *are global-in-time, we can still resulting in different asymptotics.*

Remark 3.8. *Let us mention, that the inviscid aggregation equation* (1.9) *in the whole space $\mathbb{R}^d$ can be formally considered as a gradient flow of the energy functional*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y)u(x)u(y)\,\mathrm{d}x\,\mathrm{d}y$$

with respect to the Euclidean Wasserstein distance as introduced in [44] and generalized to a large class of PDEs in [12] and in [11]. We have proved that, in some sense, if this energy functional on the first eigenfunction w_1 of $-\Delta$ is positive then sufficiently large constant solutions of the system (1.1)-(1.3) are unstable.

The proofs of Theorems 3.4 and 3.5 are given in Section 3.2.

3.1.2 Existence of solutions

For the completeness of exposition we also study the existence of solution to (3.1)–(3.3). First, let us introduce terminology analogous to that in [26].

Definition 3.9. *The aggregation kernel $K : \Omega \times \Omega \to \mathbb{R}$ is called*

- *mildly singular if $\|\nabla_x K\|_{\infty,q'} < \infty$ for some $q' \in (d, \infty]$;*

- *strongly singular if* $\|\nabla_x K\|_{\infty,q'} < \infty$ *for some* $q' \in [1,d]$ *and* $\|\nabla_x K\|_{\infty,q'} = \infty$ *for every* $q' > d$.

Notice that aggregation kernel taken from the one dimensional chemotaxis model (1.3) is mildly singular in the sense stated above.

We begin our study of properties of solutions to the initial value problem (3.1)–(3.3) by showing the existence of solutions depending on the quantity $\|\nabla_x K\|_{\infty,q'}$ defined in (3.6).

First, we show that for mildly singular kernels, solutions to the problem (3.1)–(3.3) are global in time.

Theorem 3.10 (Global existence for mildly singular kernels)**.** *Assume that there exists* $q' \in (d,\infty]$ *such that* $\|\nabla_x K\|_{\infty,q'} < \infty$ *where* $\|\nabla_x K\|_{\infty,q'}$ *is defined in* (3.6). *Denote* $q = \frac{q'}{q'-1} \in [1, d/(d-1))$. *Then for every initial condition* $u_0 \in L^1(\Omega)$ *such that* $u_0(x) \geq 0$ *and for every* $T > 0$ *problem* (3.1)–(3.3) *has a unique mild solution in the space*

$$\mathcal{Y}_T = C([0,T], L^1(\Omega)) \cap \{u : C\big([0,T], L^q(\Omega)\big),\ \sup_{0\leq t\leq T} t^{\frac{d}{2}(1-\frac{1}{q})}\|u\|_q < \infty\}$$

equipped with the norm $\|u\|_{\mathcal{Y}_T} \equiv \sup_{0\leq t\leq T}\|u\|_1 + \sup_{0\leq t\leq T} t^{\frac{d}{2}(1-\frac{1}{q})}\|u\|_q$.

In the one dimensional case, for certain mildly singular kernels, we show that solutions to (3.1)–(3.3) are bounded in $W^{1,2}(0,1)$ for all $t > 0$.

Proposition 3.11 ($W^{1,2}$-estimates for $d = 1$)**.** *Assume that* $K \in L^\infty(\Omega\times\Omega)$ *and*

$$\|D^2\mathcal{K}(u)\|_4 \leq \tilde{C}\|u\|_4 \tag{3.19}$$

for some positive constant $\tilde{C}$. *Let* $u \in C([0,T], W^{1,2}(0,1)) \cap C([0,T], L^1(0,1))$ *for some* $T > 0$ *be a nonnegative local-in-time solution of problem* (3.1)–(3.3) *for some* $T > 0$, *with initial datum* $u_0 \in L^1(0,1)$. *Then, there exists* $C = C(\|u_0\|_1, \|\nabla_x K\|_\infty)$ *independent of* T *such that*

$$\sup_{0\leq t\leq T}\|u(t)\|_{W^{1,2}} \leq C.$$

Next, we show the local-in-time existence of solutions to (3.1)–(3.3) in the case of strongly singular kernels.

Theorem 3.12 (Local existence for strongly singular kernels)**.** *Assume that there exists* $q' \in [1, d]$ *such that* $\|\nabla_x K\|_{\infty,q'} < \infty$. *Let* $q \in [d/(d-1), \infty]$ *satisfy* $1/q + 1/q' = 1$. *Then for every positive* $u_0 \in L^1(\Omega) \cap L^q(\Omega)$ *there exists* $T = T(\|u_0\|_1, \|u_0\|_q, \|\nabla_x K\|_{\infty,q'}) > 0$ *and the unique mild solution of problem* (3.1)–(3.3) *in the space*

$$\mathcal{X}_T = C([0,T], L^1(\Omega)) \cap C([0,T], L^q(\Omega))$$

equipped with the norm $\|u\|_{\mathcal{X}_T} \equiv \sup_{0\le t\le T} \|u\|_1 + \sup_{0\le t\le T} \|u\|_q$.

Remark 3.13. *Let us mention, that our previous, stability results imply the global-in time existence of solutions for strongly singular kernels provided initial data are sufficiently small. More results on the global-in-time solutions to* (3.1)–(3.3) *in the whole space* $\Omega = \mathbb{R}^d$ *can be found,e.g., in [26].*

Remark 3.14. *Karch and Suzuki in their work [26] studied the viscous aggregation equation, namely the equation* (3.1) *considered in the whole space* $\mathbb{R}^d$. *They show that there are strongly singular kernels (in the sense similar to Definition 3.9), such that some solutions blow up in finite time. Moreover, there is a large number of works studying the blow up of solution to chemotaxis model* (1.3)*, see e.g. [7, 36, 37, 38, 39] and reference therein, as well as the review paper by Horstmann [21] for additional references.*

3.2 Stability and instability proofs

In our reasoning, we use the following Poincaré inequality

$$\lambda_1 \int_\Omega \psi^2 \,\mathrm{d}x \le \int_\Omega |\nabla \psi|^2 \,\mathrm{d}x, \tag{3.20}$$

which is valid for all $\psi \in W^{1,2}(\Omega)$ satisfying $\int_\Omega \psi \,\mathrm{d}x = 0$, where λ_1 is the first non-zero eigenvalue of $-\Delta$ on Ω under the Neumann boundary condition.

Now, we are in the position to prove the Theorem 3.3.

Proof of Proposition 3.3. After multiplying equation (3.11) by φ and integrating over Ω we get

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\varphi(\cdot,t)\|_2^2 = -\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + M\int_\Omega \nabla\mathcal{K}(\varphi)\nabla\varphi \,\mathrm{d}x.$$

Now, using the Cauchy inequality we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\varphi(\cdot,t)\|_2^2 &\le -\frac{1}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{M^2}{2}\int_\Omega (\nabla\mathcal{K}(\varphi))^2\,\mathrm{d}x \\ &\le -\frac{1}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{M^2}{2}\|\nabla\mathcal{K}\|^2_{L^2\to L^2}\int_\Omega \varphi^2\,\mathrm{d}x. \end{aligned} \tag{3.21}$$

Finally, we apply the Poincaré inequality (3.20) to get the following differential inequality

$$\frac{d}{\mathrm{d}t}\|\varphi(\cdot,t)\|_2^2 \le \Big(-\lambda_1 + M^2\|\nabla\mathcal{K}\|^2_{L^2\to L^2}\Big)\|\varphi\|_2^2$$

which, under assumption (3.15), leads us directly to the exponential decay of $\|\varphi(t)\|_2$ as $t\to\infty$. □

Proof of Theorem 3.4. After multiplying equation (3.8) by φ and integrating over Ω we get

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\varphi\|_2^2 = -J(\varphi,\varphi) + \int_\Omega \varphi\nabla\mathcal{K}(\varphi)\nabla\varphi\,\mathrm{d}x \tag{3.22}$$

where J is the bilinear form defined in (3.14). In (3.21), we have already got the inequality

$$-J(\varphi,\varphi) \le -\frac{1}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{M^2}{2}\|\nabla\mathcal{K}\|^2_{L^2\to L^2}\int_\Omega \varphi^2\,\mathrm{d}x. \tag{3.23}$$

To estimate the second (nonlinear) term on the right-hand side of (3.22), we use the ε-Cauchy inequality, as follows

$$\begin{aligned} \int_\Omega \varphi\nabla\mathcal{K}(\varphi)\nabla\varphi\,\mathrm{d}x &\le \varepsilon\int_\Omega (\nabla\varphi)^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega \varphi^2(\nabla\mathcal{K}(\varphi))^2\,\mathrm{d}x \\ &\le \varepsilon\int_\Omega (\nabla\varphi)^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\|\nabla\mathcal{K}(\varphi)\|^2_\infty\int_\Omega \varphi^2\,\mathrm{d}x \\ &\le \varepsilon\int_\Omega (\nabla\varphi)^2\,\mathrm{d}x + \frac{\|\nabla_x K\|^2_{\infty,2}}{4\varepsilon}\Big(\int_\Omega \varphi^2\,\mathrm{d}x\Big)^2, \end{aligned} \tag{3.24}$$

since

$$\|\int_\Omega \nabla_x K(\cdot,y)\varphi(y)\,\mathrm{d}y\|_\infty \le \operatorname*{ess\,sup}_{x\in\Omega}\|\nabla_x K(x,\cdot)\|_2\|\varphi\|_2 = \|\nabla_x K\|_{\infty,2}\|\varphi\|_2.$$

Applying inequalities (3.23) and (3.24) in (3.22) we obtain

$$\begin{aligned}\frac{d}{\mathrm{d}t}\int_\Omega \varphi^2\,\mathrm{d}x \leq & \Big(-1+2\varepsilon\Big)\int_\Omega (\nabla\varphi)^2\,\mathrm{d}x \\ & + (M^2\|\nabla\mathcal{K}\|^2_{L^2\to L^2})\int_\Omega \varphi^2\,\mathrm{d}x + \frac{\|\nabla_x K\|^2_{\infty,2}}{2\varepsilon}\Big(\int_\Omega \varphi^2\,\mathrm{d}x\Big)^2,\end{aligned}$$

and finally using the Poincaré inequality (3.20) we get the following differential inequality

$$\frac{d}{\mathrm{d}t}\|\varphi\|_2^2 \leq \Big(\lambda_1(2\varepsilon-1) + M^2\|\nabla\mathcal{K}\|^2_{L^2\to L^2}\Big)\|\varphi\|_2^2 + \frac{\|\nabla_x K\|^2_{\infty,2}}{2\varepsilon}\|\varphi\|_2^4.$$

Notice, that under assumption (3.15), we can find $\varepsilon > 0$ small enough such that the term $\Big(\lambda_1(2\varepsilon-1) + M^2\|\nabla\mathcal{K}\|^2_{L^2\to L^2}\Big)$ is negative. Thus, the proof is complete because every nonnegative solution of the differential inequality $f' \leq -C_1 f + C_2 f^2$ with $f(t) = \|\varphi(t)\|_2^2$ and with positive constants C_1, C_2 and $f(0)$ sufficiently small, decays exponentially to zero. $\square$

To study the instability of constant solutions, first, we consider eigenvalues of the operator $\mathcal{L}$ defined via its bilinear form (3.14).

Lemma 3.15. *Let the operator*

$$\mathcal{L}\varphi = -\Delta\varphi + \nabla\cdot\Big(M\nabla\mathcal{K}(\varphi)\Big) \tag{3.25}$$

supplemented with the Neumann boundary condition be defined on $W^{1,2}(\Omega)$ by the associated bilinear form $J(\varphi,\psi)$ given in (3.14). *Assume that $\nabla_x K \in L^2(\Omega\times\Omega)$ satisfies* (3.4). *Then, the quantity*

$$\lambda = \inf_{\substack{\varphi\in W^{1,2}(\Omega)\\ \varphi\neq 0,\ \int_\Omega \varphi\,\mathrm{d}x = 0}} \frac{J(\varphi,\varphi)}{\|\varphi\|_2^2}, \tag{3.26}$$

is finite and there exists $\tilde\varphi \in W^{1,2}(\Omega)$ such that

$$\lambda = \frac{J(\tilde\varphi,\tilde\varphi)}{\|\tilde\varphi\|_2^2}.$$

Moreover, $\mathcal{L}\tilde\varphi = \lambda\tilde\varphi$ in the weak sense.

Proof. As usual, in (3.26) we may restrict ourselves to the case $\|\varphi\|_2 = 1$. Now, let

$$\mathcal{A} = \{\varphi \in W^{1,2}(\Omega) : \|\varphi\|_2 = 1, \int_\Omega \varphi \, \mathrm{d}x = 0\}.$$

Step 1. First we show that $J(\varphi,\varphi)$ is bounded from below on $\mathcal{A}$. Repeating the estimates from the proof of Proposition 3.3 we obtain

$$\Big| M \int_\Omega \nabla\mathcal{K}(\varphi)\nabla\varphi \, \mathrm{d}x \Big| \le \frac{1}{2}\|\nabla\varphi\|_2^2 + \frac{M^2}{2}\|\nabla\mathcal{K}\|_{L^2\to L^2}^2 \|\varphi\|_2^2.$$

Hence, for every $\varphi \in \mathcal{A}$ we have

$$J(\varphi,\varphi) \ge \frac{1}{2}\|\nabla\varphi\|_2^2 - \frac{M^2}{2}\|\nabla\mathcal{K}\|_{L^2\to L^2}^2 \|\varphi\|_2^2 \ge -\frac{M^2}{2}\|\nabla\mathcal{K}\|_{L^2\to L^2}^2.$$

Step 2. Let $\{\varphi\})_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence that is

$$\lambda = \lim_{n\to\infty} J(\varphi_n,\varphi_n).$$

We show that φ_n is bounded in $W^{1,2}(\Omega)$. Since φ_n is the minimizing sequence, there exists a constant C such that

$$C \ge J(\varphi_n,\varphi_n) \ge \frac{1}{2}\|\nabla\varphi_n\|_2^2 - \frac{M^2}{2}\|\nabla\mathcal{K}\|_{L^2\to L^2}^2,$$

so we obtain

$$\|\nabla\varphi_n\|_2^2 \le 2C + M^2\|\nabla\mathcal{K}\|_{L^2\to L^2}^2.$$

Thus, using the Rellich compactness theorem we have a subsequence, again denoted by φ_n, converging to $\tilde\varphi$ strongly in $L^2(\Omega)$. Moreover, by the Banach-Alaoglu theorem, we obtain, again up to the passage to a subsequence, also weak convergence of φ_n towards to $\tilde\varphi$ in $W^{1,2}(\Omega)$.

Notice, that $\tilde\varphi \in \mathcal{A}$. Indeed, by the weak convergence in $W^{1,2}(\Omega)$ we have that $\tilde\varphi \in W^{1,2}(\Omega)$ and by the strong convergence in $L^2(\Omega)$ the limit function satisfy $\|\tilde\varphi\|_2 = 1$ and $\int_\Omega \tilde\varphi \, \mathrm{d}x = 0$.

Step 3. Now, we show that $\lim_{n\to\infty} J(\varphi_n,\varphi_n) = J(\tilde\varphi,\tilde\varphi)$.

First, notice that by the weak convergence of $\nabla\varphi_n$ in $W^{1,2}(\Omega)$ we have

$$\liminf_{n\to\infty} \|\nabla\varphi_n\|_2 \ge \|\nabla\tilde\varphi\|_2. \tag{3.27}$$

Next, by the strong convergence of φ_n in $L^2(\Omega)$ and the fact that $\nabla\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ is linear and bounded, it is easy to verify that

$$\nabla\mathcal{K}(\varphi_n) \to \nabla\mathcal{K}(\tilde{\varphi}) \quad \text{as} \quad n \to \infty \quad \text{strongly in } L^2(\Omega).$$

This property, and the weak convergence of $\tilde{\varphi}_n$, again imply that

$$\int_\Omega \nabla\mathcal{K}(\varphi_n)\nabla\varphi_n \,\mathrm{d}x \to \int_\Omega \nabla\mathcal{K}(\tilde{\varphi})\nabla\tilde{\varphi}\,\mathrm{d}x \quad \text{as} \quad n \to \infty$$

which by estimate (3.27), together with previous step, completes the proof of Step 3.

Step 4. Finally, we show that the limit function $\tilde{\varphi}$ satisfies the following eigenvalue problem $\mathcal{L}\tilde{\varphi} = \lambda\tilde{\varphi}$ in the weak sense, namely

$$J(\tilde{\varphi}, v) = \lambda \int_\Omega \tilde{\varphi} v \,\mathrm{d}x \qquad \text{for all } v \in W^{1,2}(\Omega).$$

Let us denote

$$f(t) = \frac{J(\tilde{\varphi} + \varepsilon v, \tilde{\varphi} + \varepsilon v)}{\int_\Omega (\tilde{\varphi} + \varepsilon v)^2 \,\mathrm{d}x}$$

for any $v \in W^{1,2}$ and $\varepsilon \in \mathbb{R}$. This function is differentiable with respect to ε near $\varepsilon = 0$ and has the minimum at 0. Hence, its derivative vanishes at $\varepsilon = 0$, and we get

$$0 = f'(0) = \frac{J(\tilde{\varphi}, v)}{\int_\Omega (\tilde{\varphi})^2 \,\mathrm{d}x} - \frac{J(\tilde{\varphi}, \tilde{\varphi})}{\int_\Omega (\tilde{\varphi})^2 \,\mathrm{d}x} \frac{\int_\Omega \tilde{\varphi} v \,\mathrm{d}x}{\int_\Omega (\tilde{\varphi})^2 \,\mathrm{d}x} = J(\tilde{\varphi}, v) - \lambda \int_\Omega \tilde{\varphi} v \,\mathrm{d}x.$$

Hence, the proof of Lemma 3.15 is finished. □

Now, we are in the position to prove the Theorem 3.5.

Proof of Theorem 3.5. As a standard practice, we show that under our assumptions, the linear operator $\mathcal{L}$ defined by the form (3.25) has a negative eigenvalue λ. Then, the function $\varphi(x,t) = \mathrm{e}^{-\lambda t}\tilde{\varphi}(x)$ with the eigenfunction $\tilde{\varphi}$ of $\mathcal{L}$ corresponding to the eigenvalue λ, is a solution of the linearized problem (3.11)–(3.13) such that

$$\|\varphi(\cdot, t)\|_2 = \mathrm{e}^{-\lambda t}\|\tilde{\varphi}\|_2 \to \infty \quad \text{as} \quad t \to \infty.$$

To do so, we use the definition of an eigenvalue of operator $\mathcal{L}$ from Lemma 3.15. In view of (3.26), to prove that $\lambda < 0$, it suffices to show that there exists $\varphi \in W^{1,2}(\Omega)$ that

$$J(\varphi, \varphi) < 0.$$

Here, we choose $\varphi(x) = w_1(x)$, where w_1 is the normalized eigenfunction of $-\Delta$ on Ω under the Neumann boundary condition and corresponding to the first non-zero eigenvalue λ_1. Then, we obtain the following relation

$$\begin{aligned} J(w_1, w_1) &= \int_\Omega (\nabla w_1(x))^2 \,\mathrm{d}x - M \int_\Omega \int_\Omega \nabla_x K(x,y) w_1(y) \nabla w_1(x) \,\mathrm{d}y \,\mathrm{d}x \\ &= \lambda_1 \int_\Omega (w_1(x))^2 \,\mathrm{d}x - M\lambda_1 \int_\Omega \int_\Omega K(x,y) w_1(y) w_1(x) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Now, since $\lambda_1 > 0$ and $\int_\Omega (w_1)^2 \,\mathrm{d}x = 1$, using assumption (3.16) and choosing $M > 1/A$, we conclude the proof. □

3.3 Existence proofs

We construct local-in-time *mild* solutions of (3.1)–(3.3) which are solutions of the following integral equation

$$u(t) = \mathrm{e}^{t\Delta} u_0 - \int_0^t \nabla \mathrm{e}^{(t-s)\Delta} \big(u \nabla v\big)(s) \;\mathrm{d}s \tag{3.28}$$

where $\mathrm{e}^{t\Delta}$ is the Neumann linear heat semigroup on $L^p(\Omega)$. Moreover, we subsequently use estimates of $\{\mathrm{e}^{t\Delta}\}_{t\geq 0}$ stated in Lemma A.1.

First, we construct global-in-time solutions in the case of mildly singular kernel.

Proof of Theorem 3.10. We split the proof into two parts. First we construct the local-in-time solution to problem (3.1)–(3.3) and later on we show how to extend this solution to the whole time interval $[0, T]$, $T > 0$.

Step 1. Local-in-time solution. Here, we follow the reasoning from [26, Theorem 2.2]. We construct the local-in-time solution to the equation (3.28), written as $u(t) = \mathrm{e}^{t\Delta} u_0 + B(u, u)(t)$ with the bilinear form

$$B(u, v)(t) = - \int_0^t \nabla \mathrm{e}^{(t-s)\Delta} \cdot \big(u \nabla \mathcal{K}(v)\big)(s) \;\mathrm{d}s, \tag{3.29}$$

in the space $\mathcal{Y}_T$. Notice that $\mathrm{e}^{t\Delta}u_0 \in \mathcal{Y}_T$ by (A.2). To apply ideas from [26, Theorem 2.2], one should prove the following estimates of the bilinear form (3.29).

First, let us notice that by the Minkowski inequality we have that

$$\begin{aligned} \|\nabla\mathcal{K}(v)\|_{q'} \le \| \int_\Omega |\nabla_x K(\cdot,y)| v(y)\,\mathrm{d}y\|_{q'} &\le \int_\Omega \|\nabla_x K(\cdot,y)\|_{q'}|v(y)|\,\mathrm{d}y \\ &\le \|\nabla_x K\|_{\infty,q'}\|v\|_1. \end{aligned} \tag{3.30}$$

Now, for every $u, v \in \mathcal{Y}_T$, using (A.3) combined with relation (3.30) we obtain

$$\begin{aligned} \|B(u,v)(t)\|_1 &\le C\int_0^t (t-s)^{-1/2}\|u\nabla\mathcal{K}(v)(s)\|_1\,\mathrm{d}s \\ &\le C\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|\nabla\mathcal{K}(v)(s)\|_{q'}\,\mathrm{d}s \\ &\le C\|\nabla_x K\|_{\infty,q'}\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|v(s)\|_1\,\mathrm{d}s. \end{aligned}$$

Therefore, by the argument used in [26] we obtain

$$\|B(u,v)(t)\|_1 \le CT^{\frac{1}{2}(1-d(1-\frac{1}{q}))}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{Y}_T}\|v\|_{\mathcal{Y}_T} \tag{3.31}$$

where $\frac{1}{2}(1-d(1-\frac{1}{q})) > 0$.

In a similar way, we prove the following L^q-estimate

$$\begin{aligned} t^{\frac{d}{2}(1-\frac{1}{q})}\|B(u,v)(t)\|_q &\le Ct^{\frac{d}{2}(1-\frac{1}{q})}\int_0^t (t-s)^{-1/2}\|u\nabla\mathcal{K}(v)(s)\|_q\,\mathrm{d}s \\ &\le Ct^{\frac{d}{2}(1-\frac{1}{q})}\int_0^t (t-s)^{-1/2}\|u\|_q\|\nabla\mathcal{K}(v)(s)\|_\infty\,\mathrm{d}s \\ &\le C\|\nabla_x K\|_{\infty,q'}t^{\frac{d}{2}(1-\frac{1}{q})}\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|v(s)\|_q\,\mathrm{d}s \end{aligned}$$

since

$$\| \int_\Omega \nabla_x K(\cdot,y)v(y)\,\mathrm{d}y\|_\infty \le \operatorname*{ess\,sup}_{x\in\Omega}\|\nabla_x K(x,\cdot)\|_{q'}\|v\|_q.$$

Again, by the argument used in [26] we obtain

$$t^{\frac{d}{2}(1-\frac{1}{q})}\|B(u,v)(t)\|_q \le CT^{\frac{1}{2}(1-d(1-\frac{1}{q}))}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{Y}_T}\|v\|_{\mathcal{Y}_T}. \tag{3.32}$$

By inequalities (3.31) and (3.32) we obtain the following estimate of the bilinear form

$$\|B(u,v)\|_{\mathcal{Y}_T} \leq CT^{\frac{1}{2}(1-d(1-\frac{1}{q}))}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{Y}_T}\|v\|_{\mathcal{Y}_T}.$$

Hence, choosing $T > 0$ such that $4CT^{\frac{1}{2}(1-d(1-\frac{1}{q}))}\|\nabla_x K\|_{\infty,q'}\|u_0\|_1 < 1$, we obtain a solution in $\mathcal{Y}_T$ by [26, Lemma 3.1].

Step 2. Global solution. Now, it suffices to follow a standard procedure which consists in applying repeatedly the previous step to equation (3.1) supplemented with the initial datum $u(x, kT)$ to obtain a unique solution on the interval $[kT, (k+1)T]$ for every $k \in \mathbb{N}$. Notice, that we can perform this procedure since the local existence time T depends only on $\|u_0\|_1$ and $\|\nabla_x K\|_{\infty,q'}$ which implies that it does not change for all nonnegative $u_0 \in L^1(\Omega)$ with the same controlled L^1-norm, see Remark 3.1.

□

Next, we show that in one dimensional case, all solutions $u = u(t)$ are bounded in L^p for every $p \in [1,\infty]$.

Proof of Proposition 3.11. First we estimate $\|u(t)\|_2$. In order to do that we multiply equation (3.1) by u and integrate over $[0,1]$ to obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^1 u^2\,\mathrm{d}x = -\int_0^1 (u_x)^2\,\mathrm{d}x + \int_0^1 u(\partial_x\mathcal{K}(u))u_x\,\mathrm{d}x. \tag{3.33}$$

Applying the Cauchy inequality to the second term in the right hand-side of equation (3.33) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^1 u^2\,\mathrm{d}x &\leq -\frac{1}{2}\int_0^1 (u_x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u^2(\partial_x\mathcal{K}(u))^2\,\mathrm{d}x\\ &\leq -\frac{1}{2}\int_0^1 (u_x)^2\,\mathrm{d}x + \frac{\|u_0\|_1^2\|\nabla_x K\|_\infty^2}{2}\int_0^1 u^2\,\mathrm{d}x.\end{aligned} \tag{3.34}$$

Now, adding the term $\|u(t)\|_2^2$ to the both sides of (3.34) we obtain

$$\frac{d}{\mathrm{d}t}\|u(t)\|_2^2 + \|u(t)\|_{W^{1,2}}^2 \leq (1 + \|u_0\|_1^2\|\nabla_x K\|_\infty^2)\|u(t)\|_2^2.$$

Now, we use the following Gagliardo-Nirenberg-Sobolev inequality $\|u\|_2^6 \leq C\|u\|_{W^{1,2}}^2\|u\|_1^4$, and the conservation of the integral (3.7) to get

$$\frac{d}{\mathrm{d}t}\|u(t)\|_2^2 + C_1\Big(\|u(t)\|_2^2\Big)^3 \leq C_2\|u(t)\|_2^2,$$

with positive constants $C_1 = C_1(\|u_0\|_1)$ and $C_2 = C_2(\|u_0\|_1, \|\nabla_x K\|_\infty)$.

We leave for the reader the proof that any nonnegative solution of the differential inequality $f' \leq -Cf^3 + Cf$ is bounded, which gives us the boundedness of $\|u(t)\|_2$ for all $t > 0$.

In order to get the boundedness of $\|u_x(t)\|_2$ is also bounded we multiply equation (3.1) by u_{xx} and integrate over $[0,1]$ to obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^1 (u_x)^2\,\mathrm{d}x = -\int_0^1 (u_{xx})^2\,\mathrm{d}x + \int_0^1 (u\partial_x\mathcal{K}(u))_x u_{xx}\,\mathrm{d}x.$$

Now, we use the Cauchy inequality as it was in (3.33) and obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^1 (u_x)^2\,\mathrm{d}x \leq -\frac{1}{2}\int_0^1 (u_{xx})^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 ((u\partial_x\mathcal{K}(u))_x)^2\,\mathrm{d}x$$

and thus we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}\int_0^1 (u_x)^2\,\mathrm{d}x \leq &-\int_0^1 (u_{xx})^2\,\mathrm{d}x + \int_0^1 (u_x\partial_x\mathcal{K}(u))^2\,\mathrm{d}x\\ &+ 2\int_0^1 uu_x\partial_x\mathcal{K}(u)\partial_{xx}\mathcal{K}(u)\,\mathrm{d}x + \int_0^1 (u\partial_{xx}\mathcal{K}(u))^2\,\mathrm{d}x.\end{aligned} \tag{3.35}$$

To deal with the second integral in the right-hand side of (3.35) we use conservation of mass and boundedness of $\partial_x\mathcal{K}(u)$ to get

$$\int_0^1 (u_x\partial_x\mathcal{K}(u))^2\,\mathrm{d}x \leq \|\nabla_x K\|_\infty^2 \|u_0\|_1^2 \int_0^1 (u_x)^2\,\mathrm{d}x. \tag{3.36}$$

Moreover, using the Cauchy inequality we have

$$\begin{aligned}\int_0^1 uu_x\partial_x\mathcal{K}(u)\partial_{xx}\mathcal{K}(u)\,\mathrm{d}x &\leq \|\nabla_x K\|_\infty \|u_0\|_1 \int_0^1 uu_x\partial_{xx}\mathcal{K}(u)\,\mathrm{d}x\\ &\leq \frac{1}{2}\|\nabla_x K\|_\infty^2 \|u_0\|_1^2 \int_0^1 (u_x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 u^2(\partial_{xx}\mathcal{K}(u))^2.\end{aligned} \tag{3.37}$$

To deal with the last term in the right-hand side of (3.35) we use the Schwarz inequality and assumption (3.19) to get

$$\int_0^1 (u\partial_{xx}\mathcal{K}(u))^2\,\mathrm{d}x \leq \|u\|_4^2 \|\partial_{xx}\mathcal{K}(u)\|_4^2 \leq \tilde{C}\|u\|_4^4. \tag{3.38}$$

Applying inequalities (3.36)–(3.38) to (3.35) we obtain

$$\frac{d}{\mathrm{d}t}\int_0^1 (u_x)^2\,\mathrm{d}x \le -\int_0^1 (u_{xx})^2\,\mathrm{d}x + 2\|\nabla_x K\|_\infty^2 \|u_0\|_1^2 \int_0^1 (u_x)^2\,\mathrm{d}x + 2\tilde{C}\int_0^1 u^4\,\mathrm{d}x. \tag{3.39}$$

To deal with the first term on the right-hand side of (3.39) we use the following Gagliardo-Nirenberg-Sobolev inequality $\|u_x\|_2^4 \le C\|u_x\|_{W^{1,2}}^2\|u\|_2^2$ and the boundedness of $\|u\|_2$ just proved, to get relation

$$-\|u_{xx}\|_2^2 \le -C\|u_x\|_2^4 + \|u_x\|_2^2. \tag{3.40}$$

Now, we use the following Gagliardo-Nirenberg-Sobolev inequality $\|u\|_4^2 \le C\|u\|_{W^{1,2}}\|u\|_1$ and the conservation of the integral (3.7) to get

$$\|u\|_4^4 \le C\|u\|_{W^{1,2}}^2\|u_0\|_1^2 \le C_1\|u_x\|_2^2 + C_2 \tag{3.41}$$

since $\|u\|_2$ is bounded.

Using relations (3.40) and (3.41) in (3.39) we obtain

$$\frac{d}{\mathrm{d}t}\int_0^1 (u_x)^2\,\mathrm{d}x \le -C_1\Big(\int_0^1 (u_x)^2\,\mathrm{d}x\Big)^2 + C_2\int_0^1 (u_x)^2\,\mathrm{d}x + C_3$$

for constants C_1, $C_2 = C_2(\tilde{C}, \|u_0\|_1, \|\nabla_x K\|_\infty)$ and $C_3 = C_3(\tilde{C}, \|u_0\|_1)$. The proof is completed by a similar argument which was used to show the boundedness of $\|u(t)\|_2$ for all $t > 0$. □

Finally, we prove the local-in-time existence of solutions in the case when $\mathcal{K}$ is strongly singular.

Proof of Theorem 3.12. We assume now, that $q' \in [1, d]$. Again notice that $\mathrm{e}^{t\Delta}u_0 \in \mathcal{X}_T$ since by (A.2) we have

$$\|\mathrm{e}^{t\Delta}u_0\|_{\mathcal{X}_T} \le C(\|u_0\|_1 + \|u_0\|_q).$$

Next, for every $u, v \in \mathcal{Y}_T$, we get

$$\begin{aligned}
\|B(u,v)(t)\|_1 &\le C\int_0^t (t-s)^{-1/2}\|u\nabla\mathcal{K}(v)(s)\|_1\,\mathrm{d}s\\
&\le C\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|\nabla\mathcal{K}(v)(s)\|_{q'}\,\mathrm{d}s\\
&\le C\|\nabla_x K\|_{\infty,q'}\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|v(s)\|_1\,\mathrm{d}s\\
&\le CT^{1/2}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}
\end{aligned}$$

where C is a positive constant.

To deal with the L^q-norm of $B(u,v)$ we proceed similarly

$$\begin{aligned}\|B(u,v)(t)\|_q &\leq C\int_0^t (t-s)^{-1/2}\|u\nabla\mathcal{K}(v)(s)\|_q\,\mathrm{d}s\\ &\leq C\|\nabla_x K\|_{\infty,q'}\int_0^t (t-s)^{-1/2}\|u(s)\|_q\|v(s)\|_q\,\mathrm{d}s\\ &\leq CT^{1/2}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}.\end{aligned}$$

Summing up these inequalities, we obtain the following estimate of the bilinear form

$$\|B(u,v)\|_{\mathcal{X}_T} \leq CT^{1/2}\|\nabla_x K\|_{\infty,q'}\|u\|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}.$$

Hence, choosing $T>0$ such that $4CT^{1/2}\|\nabla_x K\|_{\infty,q'}(\|u_0\|_1+\|u_0\|_q)<1$, we obtain the solution in $\mathcal{X}_T$ by [26, Lemma 3.1]. □

Chapter 4

Mathematical properties of solutions to the model of formation chemotactic E. coli colonies

4.1 Results and comments

In this Chapter we focus on the following initial-boundary value problem

$$u_t = \Delta u - \nabla \cdot (u \nabla \chi(c)) + g(u)nu - b(n)u \tag{4.1}$$

$$c_t = d_c \Delta c + \alpha u - \beta c \tag{4.2}$$

$$n_t = d_n \Delta n - \gamma g(u)nu \tag{4.3}$$

$$w_t = b(n)u. \tag{4.4}$$

System (4.1)–(4.4) is considered in a bounded domain $\Omega \subset \mathbb{R}^d$ with a sufficiently smooth boundary $\partial\Omega$, and is supplemented with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \qquad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0, \tag{4.5}$$

as well as with nonnegative initial data

$$\begin{matrix} u(x,0) = u_0(x), & c(x,0) = c_0(x), \\ n(x,0) = n_0(x), & w(x,0) = w_0(x) \end{matrix} \qquad \text{for} \quad x \in \Omega. \tag{4.6}$$

We begin by formulating assumptions on all coefficients and functions in (4.1)–(4.6).

Standing Assumptions: The diffusion coefficients $d_c > 0$ and $d_n > 0$ in (4.2)–(4.3) are constant. The coefficients $\alpha > 0$, $\beta > 0$, $\gamma > 0$ in equations (4.2)–(4.3) also denote given constants. Moreover, we impose the following assumptions on the functions $g, b, \chi \in C^1([0,\infty))$:

(i) $g(0) = 0$ and $g = g(s)$ is increasing for $s > 0$ and bounded with $G_0 \equiv \sup_{s\geq 0} g(s)$;

(ii) $b(0) = B_0 > 0$ and $b = b(s)$ is decreasing for $s > 0$ and nonnegative;

(iii) $\chi' \in L^\infty([0,\infty))$.

First we show that, for all sufficiently regular initial conditions, problem (4.1)–(4.6) has a unique local-in-time solution. Moreover, this solution is nonnegative if initial conditions (4.6) are nonnegative. This is more-or-less standard reasoning which we postpone to Section 4.2 (see Theorems 4.6 and 4.8). In this work, we focus mainly on the behaviour of nonnegative solutions to problem (4.1)–(4.6) for large values of time.

First, we notice that if initial conditions (4.6) are independent of x, namely, if

$$u_0(x) = \bar{u}_0, \quad c_0(x) = \bar{c}_0, \quad \bar{n}_0(x) = \bar{n}_0, \quad \overline{w}_0(x) = \overline{w}_0, \tag{4.7}$$

for certain constants $\bar{u}_0, \bar{c}_0, \bar{n}_0, \overline{w}_0 \in [0,\infty)$, then the corresponding solution is independent of x, as well. This property is an immediate consequence of the uniqueness of solution to problem (4.1)–(4.6). The following theorem describes the large time behaviour of such nonnegative, space homogeneous solutions.

Theorem 4.1. *Let Standing Assumptions are satisfied. For every nonnegative, constant initial condition* (4.7), *the corresponding solution*

$$\big(\bar{u}(t), \bar{c}(t), \bar{n}(t), \bar{w}(t)\big)$$

to problem (4.1)–(4.6) *is global-in-time and converges exponentially towards the constant vector* $(0, 0, \bar{n}_\infty, \bar{w}_\infty)$ *for some* $\bar{n}_\infty \geq 0$ *and* $\bar{w}_\infty \geq 0$ *depending on initial conditions.*

This theorem is proved in Section 4.3 by analysing the phase portrait of the corresponding system of ordinary differential equations, see (4.18)–(4.21) below. In Section 4.3, we also study a large time behaviour of “mass" of space inhomogeneous solutions, and we show in Theorem 4.10 below, that the vector

$$\left(\int_\Omega u(x,t)\,\mathrm{d}x, \int_\Omega c(x,t)\,\mathrm{d}x, \int_\Omega n(x,t)\,\mathrm{d}x, \int_\Omega w(x,t)\,\mathrm{d}x\right)$$

behaves for large values of time like a space homogeneous solution.

Next, we consider problem (4.1)–(4.6) in the one dimensional case and we show that all solutions corresponding to sufficiently regular, nonnegative initial conditions are global-in-time and converge uniformly towards certain steady states. This result has been already proved in [22, 23] for problem (4.1)–(4.6) with particular functions g, b and χ. Here, however, we propose a different approach which allows us to consider more general nonlinearities.

Theorem 4.2. *Assume that $d = 1$ and $\Omega \subset \mathbb{R}$ is an open and bounded interval. Let the constants α, β, γ and the functions g, b and χ satisfy Standing Assumptions. For every initial condition $u_0, n_0, w_0 \in L^\infty(\Omega)$ and $c_0 \in W^{1,\infty}(\Omega)$, the corresponding solution (u, c, n, w) to problem (4.1)–(4.6) exists for all $t > 0$. Moreover, there exists a constant $n_\infty \geq 0$ and a nonnegative function $w_\infty \in L^\infty(\Omega)$ such that*

$$\big(u(x,t), c(x,t), n(x,t), w(x,t)\big) \xrightarrow{t\to\infty} (0, 0, n_\infty, w_\infty(x))$$

exponentially in $L^\infty(\Omega)$.

An analogous result holds true in higher dimensions under a smallness assumption on initial conditions.

Theorem 4.3. *Let $d \in \{2, 3\}$ and (u, c, n, w) be a nonnegative local-in-time solution to problem (4.1)–(4.6) with the parameters satisfying Standing Assumptions. Fix, $p_0 \in (\frac{d}{2}, \frac{d}{d-2})$. There exist $\varepsilon > 0$ such that if*

$$\max(\|u_0\|_{p_0}, \|n_0\|_1, \|\nabla c_0\|_{2p_0}) < \varepsilon,$$

then the solution (u, c, n, w) exist for all $t > 0$ and satisfies

$$\sup_{t>0} \|u(t)\|_\infty < \infty. \tag{4.8}$$

Moreover, there exist a constant $n_\infty \geq 0$ and a nonnegative function $w_\infty \in L^\infty(\Omega)$ such that

$$\big(u(x,t), c(x,t), n(x,t)\big) \xrightarrow{t\to\infty} (0, 0, n_\infty, w_\infty(x)) \tag{4.9}$$

exponentially in $L^\infty(\Omega)$.

Remark 4.4. *Because of methods used in the proof of Theorem 4.3, we have to limit ourselves to the dimension $d \in \{2, 3\}$ (note that the interval $(\frac{d}{2}, \frac{d}{d-2})$ is nonempty only in this case). Obviously, this is not a constraint from the point of view of application. In Remark 4.15 below, we explain how to show an analogous result for $d > 3$.*

There is a natural question if the smallness assumptions in Theorem 4.3 are really necessary to show the global-in-time existence of nonnegative solutions to problem (4.1)–(4.6) and their exponential convergence toward steady states as in (4.9). We conjecture that our Standing Assumptions are not sufficient to show such a claim. Here, we use an idea, which is well-known from the study of the Keller-Segel model of chemotaxis. We show that solutions to (4.1)–(4.6) in two dimensions and where the parabolic equation for $c = c(x, t)$ is replaced by its elliptic counterpart, may blow up in finite time.

Thus, we focus on the following system

$$u_t = \Delta u - \chi_0 \nabla \cdot (u \nabla c) + g(u)nu - b(n)u \tag{4.10}$$

$$0 = \Delta c + u - c \tag{4.11}$$

$$n_t = \Delta n - \gamma g(u)nu \tag{4.12}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$, supplemented with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial n}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0, \tag{4.13}$$

and with nonnegative initial data

$$u(x, 0) = u_0(x), \quad c(x, 0) = c_0(x), \quad n(x, 0) = n_0(x). \tag{4.14}$$

Compared problem (4.1)–(4.6) we now consider a linear function $\chi(c) = \chi_0 c$ with a constant $\chi_0 > 0$. The functions $g = g(u)$, $b = b(n)$ and the constant γ satisfy Standing Assumptions. Moreover, we set $\alpha = \beta = d_c = d_n = 1$ and we skip equation (4.4) on the function w because these constants and this function do not play any role in our reasoning.

By a standard reasoning, completely analogous to that one in Section 4.2, one can show that problem (4.10)–(4.14) has a local-in-time nonnegative solution. In the following theorem, we show that some of those solutions cannot be extended for all $t > 0$.

Theorem 4.5. *Let d=2 and let* (u, c, n) *be a local-in-time nonnegative and classical solution of problem* (4.10)–(4.14). *Assume that*

$$M_0 = \int_\Omega u_0(x)\,\mathrm{d}x > \frac{8\pi}{\chi_0}.$$

For every $q \in \Omega$ *there exists* $\varepsilon(q) > 0$ *such that if* $\int_\Omega u_0(x)|x-q|^2\,\mathrm{d}x < \varepsilon(q)$ *then the solution* (u, c, n) *cannot be extended to a global one.*

4.2 Local existence of nonnegative solutions

Let us first show that the initial-boundary value problem (4.1)–(4.6) has a local-in-time, unique, nonnegative and regular solution. This is more-or-less standard reasoning and we sketch it only.

As a standard practice, local-in-time solutions may be obtained via the Banach fixed point argument applied to the Duhamel formulation of problem (4.1)–(4.6) as the following integral equations

$$\begin{aligned} u(t) = \mathrm{e}^{\Delta t}u_0 &+ \int_0^t \partial_x \mathrm{e}^{\Delta(t-s)}u(s)\nabla\chi(c(s))\,\mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{\Delta(t-s)}u(s)(g(u)n - b(n))(s)\,\mathrm{d}s, \end{aligned} \tag{4.15}$$

$$c(t) = \mathrm{e}^{(\Delta-\beta)t}c_0 + \alpha\int_0^t \mathrm{e}^{(\Delta-\beta)(t-s)}u(s)\,\mathrm{d}s, \tag{4.16}$$

$$n(t) = \mathrm{e}^{\Delta t}n_0 - \gamma\int_0^t \mathrm{e}^{\Delta(t-s)}g(u(s))n(s)u(s)\,\mathrm{d}s. \tag{4.17}$$

Such an approach has been applied in several works on systems of reaction–diffusion equations, see the classical monographs by Henry [19], Rothe [45] and, more recently, by Yagi [49]. To recall such a result from [49], we introduce the following spaces

$$\begin{aligned} H^2_N(\Omega) &= \{u \in W^{1,2}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\}, \\ \mathcal{H}^4_{N^2}(\Omega) &= \{u \in H^2_N(\Omega) : \Delta u \in H^2_N(\Omega)\}. \end{aligned}$$

Theorem 4.6 (Local-in-time solutions)**.** *Let Standing Assumptions hold true. For every initial datum* $(u_0, c_0, n_0, w_0) \in L^2(\Omega) \times H^2_N(\Omega) \times L^2(\Omega) \times L^\infty(\Omega)$, *there exists $T > 0$ such that problem* (4.1)–(4.6) *possesses a unique local-in-time solution satisfying*

$$\begin{aligned} &u \in C((0,T]; H^2_N(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)),\\ &c \in C((0,T]; \mathcal{H}^4_{N^2}(\Omega)) \cap C([0,T]; H^2_N(\Omega)) \cap C^1((0,T]; H^2_N(\Omega)),\\ &n \in C((0,T]; H^2_N(\Omega)) \cap C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)),\\ &w \in C^1([0,T]; L^\infty(\Omega)). \end{aligned}$$

For the proof of this theorem, it suffices to apply a construction of local-in-time solutions to an abstract semilinear evolution equation in the monograph [49, Ch. 4 and Ch. 12, Sec. 1.2].

Remark 4.7. *The local-in-time solution from Theorem 4.6 may be continued to global-in-time provided we find suitable a priori estimates on its norms, see [49, Ch. 4, Sec. 1.4].*

Theorem 4.8 (Nonnegativity)**.** *If $u_0 \geq 0$, $c_0 \geq 0$, $n_0 \geq 0$, $w_0 \geq 0$ almost everywhere, then, the local-in-time solution obtained in Theorem 4.6 is nonnegative for all $0 < t < T$.*

Proof. We employ a standard truncation method. Let $h = h(u)$ be a cutoff function such that

$$h(u) := \begin{cases} \frac{1}{2}u^2 & \text{for} \quad -\infty < u < 0,\\ 0 & \text{for} \quad 0 \leq u < \infty. \end{cases}$$

Then, the function $\psi(t) = \int_\Omega h(u(t))\,\mathrm{d}x$ is continuously differentiable and $\psi(0) = 0$ for nonnegative u_0. Following the reasoning in [49, Ch. 12, Sec. 1.3] one may show that $\psi'(t) \leq C\psi(t)$ for all $t \in [0,T]$. Then, $\psi(0) = 0$ implies $\psi(t) \equiv 0$ and consequently $u(t) \geq 0$ for all $t \in [0,T]$. An analogous reasoning should be applied to the functions $c(t)$ and $n(t)$. Moreover, now, it is clear that by equation (4.4) $w(t) \geq 0$ for all $t \in [0,T]$ if u is nonnegative. □

Remark 4.9. *Notice, that applying the maximum principle to equation* (4.3) *with $\gamma g(u)u \geq 0$ and $n_0 \geq 0$, we may show that*

$$0 \leq n(x,t) \leq n_0(x) \quad \textit{for all} \quad x \in \Omega,\ t \in [0,T].$$

In particular, for each $p \in [1,\infty]$, we have $\|n(t)\|_p \leq \|n_0\|_p$ for all $t \in [0,T]$.

4.3 Space homogeneous solutions and large time behaviour of mass

Proof of Theorem 4.1. Obviously, the chemotactic term $-\nabla \cdot (u\nabla\chi(c))$ as well as the terms in equations (4.1)–(4.4) containing Laplacian disappear in the case of x-independent solutions. Hence, we focus on the following system of ordinary differential equations

$$\frac{d}{\mathrm{d}t}\bar{u} = g(\bar{u})\bar{n}\bar{u} - b(\bar{n})\bar{u} \tag{4.18}$$
$$\frac{d}{\mathrm{d}t}\bar{c} = \alpha\bar{u} - \beta\bar{c} \tag{4.19}$$
$$\frac{d}{\mathrm{d}t}\bar{n} = -\gamma g(\bar{u})\bar{n}\bar{u} \tag{4.20}$$
$$\frac{d}{\mathrm{d}t}\bar{w} = b(\bar{n})\bar{u}. \tag{4.21}$$

It is clear that we should begin with the analysis of the first and the third equation

$$\frac{d}{\mathrm{d}t}\bar{u} = g(\bar{u})\bar{n}\bar{u} - b(\bar{n})\bar{u} \tag{4.22}$$
$$\frac{d}{\mathrm{d}t}\bar{n} = -\gamma g(\bar{u})\bar{n}\bar{u}, \tag{4.23}$$

where the vector $(0, \bar{n}_\infty)$ is a steady state for each nonnegative constant $\bar{n}_\infty$. Here, we notice that, by a standard reasoning, every solution $(\bar{u}(t), \bar{n}(t))$ of (4.22)–(4.23) which starts in the first quadrant ($\bar{u} > 0, \bar{n} > 0$) at $t = 0$ has to remain in this quadrant of the $(\bar{u}, \bar{n})$-plane for all future times (in fact, this is also proved in Theorem 4.8, below). Observe also that, by equation (4.23) and by Standing Assumptions, the derivative $\frac{d}{\mathrm{d}t}\bar{n}$ is always nonpositive in the first quadrant.

Now, we split the first quadrant into two regions, where $\frac{d}{\mathrm{d}t}\bar{u}$ has a fixed sign, in the following way. Since g is a bounded and increasing function, it has a limit $G_0 = \lim_{s\to\infty} g(s)$. Thus, the bijection $g^{-1} : [0, G_0) \to [0, \infty)$ is an increasing function and the equation $g(\bar{u})\bar{n}\bar{u} - b(\bar{n})\bar{u} = 0$ for $\bar{u} > 0$, can be written as $\bar{u} = f(\bar{n}) \equiv g^{-1}(b(\bar{n})/\bar{n})$. By properties of the functions g and b, we immediately obtain that f is a decreasing function such that

$$f(\bar{n}) \to 0 \quad \text{as} \quad n \to \infty \qquad \text{and} \qquad f(\bar{n}) \to +\infty \quad \text{as} \quad n \to N_0,$$

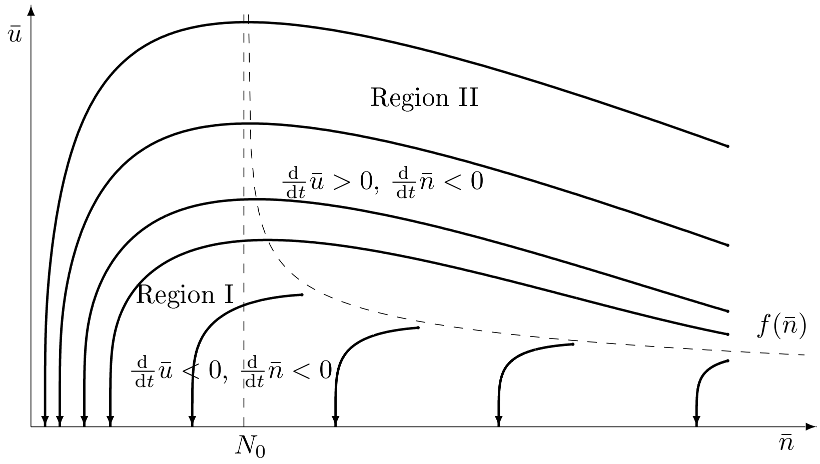

Figure 4.1: Phase portrait of system (4.22)–(4.23) with $g(u) = \frac{u^4}{7^4+u^4}$ and $b(n) = \frac{10}{1+\frac{n}{7}}$.

where the constant $N_0 > 0$ is defined by the equation $\frac{b(N_0)}{N_0} = G_0$, see Figure 4.1. The curve $\bar{u} = f(\bar{n})$ divides the first quadrant of the $(\bar{u}, \bar{n})$-plane into two regions, where $\frac{d}{\mathrm{d}t}\bar{u} < 0$ in region I and $\frac{d}{\mathrm{d}t}\bar{u} > 0$ in region II.

For every $(\bar{u}(0), \bar{n}(0))$ in region I, the functions $\bar{n}(t)$ and $\bar{u}(t))$ are non increasing, hence, $\lim_{t\to\infty}(\bar{u}(t), \bar{n}(t)) = (0, \bar{n}_\infty)$ for some $\bar{n}_\infty \geq 0$. In fact, this is an exponential convergence because, for a solution $(\bar{u}(t), \bar{n}(t))$ in region I we have $\frac{d}{\mathrm{d}t}\bar{u} \leq \big(g(\bar{u}(0))\bar{n}(0) - b(\bar{n}(0))\big)\bar{u}$, where $g(\bar{u}(0))\bar{n}(0) - b(\bar{n}(0)) < 0$ by properties of the functions g and b. Moreover, by equation (4.23), we have that $\bar{n}_\infty = \bar{n}_0 - \gamma \int_0^\infty g(\bar{u}(s))\bar{n}(s)\bar{u}(s)\,\mathrm{d}s$. Hence

$$|\bar{n}(t) - \bar{n}_\infty| = \gamma \int_t^\infty g(\bar{u}(s))\bar{n}(s)\bar{u}(s)\,\mathrm{d}s \leq C \int_t^\infty \bar{u}(s)\,\mathrm{d}s \to 0$$

exponentially as $t \to \infty$, due to exponential decay of $\bar{u}(t)$.

Now, let us prove that if a trajectory starts in region II, then this has to enter region I. Indeed, if we suppose that it is not the case, and we have $\bar{u}(t) \geq \bar{u}(0)$ for all $t > 0$. Then, from equation (4.23) we have that $\bar{n}_t \leq -\gamma g(\bar{u}(0))\bar{u}(0)\bar{n}$ and so $\bar{n}(t) \leq \bar{n}(0)\mathrm{e}^{-t(\gamma g(\bar{u}(0))\bar{u}(0))} \to 0$ as $t \to \infty$ which leads to a contradiction. Thus, we have proved that $(\bar{u}(t), \bar{n}(t)) \to (0, \bar{n}_\infty)$ exponentially as $t \to \infty$.

Let us now describe the large time behaviour of $\bar{c}(t)$ and $\bar{w}(t)$. Solving equation (4.19) with respect to $\bar{c} = \bar{c}(t)$, we may easily show that

$\lim_{t\to\infty} \bar{c}(t) = 0$ exponentially, using exponential decay of $\bar{u}(t)$. Finally, we have

$$\lim_{t\to\infty} \bar{w}(t) = \bar{w}_0 + \int_0^\infty b(\bar{n}(s))\bar{u}(s)\,\mathrm{d}s \equiv \bar{w}_\infty, \tag{4.24}$$

where the quantity on the right-hand side is finite and positive, because $b(\bar{n}(t))$ is bounded for $t > 0$, and $\bar{u}(t)$ decays exponentially. □

Now, we consider solutions of problem (4.1)–(4.6) with nonconstant initial conditions and we prove a similar result to the one in Theorem 4.1 on the large time behaviour of the integrals $\int_\Omega u(x,t)\,\mathrm{d}x$, $\int_\Omega c(x,t)\,\mathrm{d}x$, $\int_\Omega n(x,t)\,\mathrm{d}x$ and $\int_\Omega w(x,t)\,\mathrm{d}x$, which correspond to masses with the densities u, c, n and w, respectively.

Theorem 4.10. *Assume that a nonnegative solution (u, c, n, w) of problem (4.1)–(4.6) exists for all $t > 0$. Let the Standing Assumptions hold true. Then*

$$\int_\Omega u(t)\,\mathrm{d}x \to 0 \quad \textit{and} \quad \int_\Omega c(t)\,\mathrm{d}x \to 0 \qquad \textit{as} \ \ t \to \infty,$$

and there are constants $n_\infty > 0$ and $w_\infty > 0$ such that

$$\int_\Omega n(t)\,\mathrm{d}x \to n_\infty \quad \textit{and} \quad \int_\Omega w(t)\,\mathrm{d}x \to w_\infty \qquad \textit{as} \ \ t \to \infty.$$

Proof. First, integrating equations (4.1)–(4.4) with respect to x, we obtain

$$\frac{d}{dt}\int_\Omega u\,\mathrm{d}x = \int_\Omega g(u)nu\,\mathrm{d}x - \int_\Omega b(n)u\,\mathrm{d}x \tag{4.25}$$

$$\frac{d}{dt}\int_\Omega c\,\mathrm{d}x = \alpha\int_\Omega u\,\mathrm{d}x - \beta\int_\Omega c\,\mathrm{d}x \tag{4.26}$$

$$\frac{d}{dt}\int_\Omega n\,\mathrm{d}x = -\gamma\int_\Omega g(u)nu\,\mathrm{d}x \tag{4.27}$$

$$\frac{d}{dt}\int_\Omega w\,\mathrm{d}x = \int_\Omega b(n)u\,\mathrm{d}x. \tag{4.28}$$

Since, $\frac{d}{dt}\left(\int_\Omega u(t)\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega n(t)\,\mathrm{d}x + \int_\Omega w(t)\,\mathrm{d}x\right) = 0$, we get the conservation of mass in the following sense

$$\int_\Omega u(t)\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega n(t)\,\mathrm{d}x + \int_\Omega w(t)\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega n_0\,\mathrm{d}x + \int_\Omega w_0\,\mathrm{d}x, \tag{4.29}$$

for all $t > 0$. In particular, since all functions are nonnegative, we have

$$\int_\Omega u(x,t)\,\mathrm{d}x \le \int_\Omega u_0\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega n_0\,\mathrm{d}x + \int_\Omega w_0\,\mathrm{d}x \quad \text{for all} \quad t > 0. \tag{4.30}$$

Now, we improve this estimate by adding equation (4.25) to equation (4.27) multiplied by γ^{-1} and integrating resulting equation over $[0,t]$ to obtain the relation

$$\int_\Omega u(t) + \gamma^{-1} n(t)\,\mathrm{d}x = \int_\Omega u_0 + \gamma^{-1} n_0\,\mathrm{d}x - \int_0^t \int_\Omega b(n(s))u(s)\,\mathrm{d}x\,\mathrm{d}s, \tag{4.31}$$

which by positivity of b and u implies that

$$\int_\Omega u(t)\,\mathrm{d}x \le \int_\Omega u_0\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega n_0\,\mathrm{d}x. \tag{4.32}$$

Next, we observe that, since $g(u)nu \ge 0$, it follows from equation (4.27) that $\int_\Omega n(t)\,\mathrm{d}x$ is nonincreasing and since it is also bounded from below, the following finite limit exists

$$\lim_{t\to\infty} \int_\Omega n(t)\,\mathrm{d}x = n_\infty > 0. \tag{4.33}$$

Now, since $b(n)u \ge 0$, equation (4.31) implies that the mapping $t \mapsto \int_\Omega u(t) + \gamma^{-1}n(t)\,\mathrm{d}x$ is also nonincreasing, hence, it has a limit as $t \to \infty$. Consequently, using (4.33), we conclude that there exists a constant $u_\infty \ge 0$ such that

$$\lim_{t\to\infty} \int_\Omega u(t)\,\mathrm{d}x = u_\infty.$$

Moreover, since $\int_\Omega u(t) + \gamma^{-1}n(t)\,\mathrm{d}x$ is bounded for $t \ge 0$, identity (4.31) implies that $b(n)u \in L^1\big((0,\infty); L^1(\Omega)\big)$. However, since $b(n) \ge b(\|n_0\|_\infty) > 0$, it follows that

$$u \in L^1((0,\infty); L^1(\Omega)). \tag{4.34}$$

Consequently, we have $u_\infty = 0$.

Since $b(n)u \in L^1\big((0,\infty), L^1(\Omega)\big)$ we obtain from equation (4.28)

$$\lim_{t\to\infty} \int_\Omega w(t)\,\mathrm{d}x \equiv \int_\Omega w_0\,\mathrm{d}x + \int_0^\infty \int_\Omega b(n)u\,\mathrm{d}x\,\mathrm{d}s.$$

Finally, $\lim_{t\to\infty} \int_\Omega c(t)\,\mathrm{d}x = 0$ due to equation (4.26) because $\lim_{t\to\infty} \|u(t)\|_1 = 0$. This completes the proof of Theorem 4.10. □

Remark 4.11. *Under the following assumption*

$$G_0\|n_0\|_\infty - b(\|n_0\|_\infty) < 0,$$

where $b(\|n_0\|_\infty) = \inf_n b(n)$ *and* $G_0 = \sup_u g(u)$, *we obtain the exponential decay of* $\int_\Omega u(x,t)\,\mathrm{d}x$. *This is an immediate consequence of equation* (4.25) *and the estimate*

$$g(u)nu - b(n)u \le (G_0\|n_0\|_\infty - b(\|n_0\|_\infty))u < 0,$$

since $n(x,t) \le \|n_0\|_\infty$ *by equation* (4.3) *(cf. Remark 4.9).*

Remark 4.12. *Let us point out that the method from the proof of Theorem 4.10 can be also used to show Theorem 4.1.*

4.4 Problem in one space dimension

The proof of Theorem 4.2 requires the following two auxiliary lemmas. First, we find an estimate of $c_x(t)$ which is uniform in time.

Lemma 4.13. *Let the assumptions of Theorem 4.2 hold true and denote by (u,c,n,w) the corresponding nonnegative local-in-time solution to problem* (4.1)–(4.6) *on* $[0,T]$ *constructed in Theorem 4.6. For each* $p \in [1,\infty)$ *there exists a constant* $C = C(p) > 0$ *independent of* T *such that* $\|c_x(t)\|_p \le C$ *for all* $t \in (0,T]$. *Moreover, if the solution is global-in-time, then* $\lim_{t\to\infty}\|c_x(t)\|_p = 0$ *for each* $p \in [1,\infty)$.

Proof. Using the Duhamel principle (4.16) and the estimate of the heat semigroup (A.3) we obtain

$$\begin{aligned} \|c_x(t)\|_p &\le \|\partial_x \mathrm{e}^{t\Delta-\beta t}c_0\|_p + \alpha\int_0^t \|\partial_x \mathrm{e}^{(\Delta-\beta)(t-s)}u(s)\|_p\,\mathrm{d}s \\ &\le C\mathrm{e}^{-\beta t}\|c_{0,x}\|_p + C\int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\mathrm{e}^{-(\beta+\lambda_1)(t-s)}\|u(s)\|_1\,\mathrm{d}s \end{aligned} \tag{4.35}$$

for all $t \in (0,T]$ and a constant $C > 0$ independent of $t > 0$. The right-hand side of this inequality is bounded uniformly in $t > 0$ and independent of $T > 0$ because of estimate (4.32). Moreover, if solution is global-in-time, this converges to zero (*cf.* Lemma A.2 below) since $\lim_{t\to\infty}\|u(t)\|_1 = 0$ by Theorem 4.10. □

Next, we show the boundedness of the L^2-norm of u using usual energy estimates. This result was already obtained in [22, 23] and we recall it for the completeness of the exposition.

Lemma 4.14. *Let the assumptions of Theorem 4.2 hold true. Moreover, let* (u, c, n, w) *be the nonnegative local-in-time solution to problem* (4.1)–(4.6) *constructed in Theorem 4.6. Then, there exists* C *independent of* t *such that* $\|u(t)\|_2 \le C$ *for all* $t \in [0, T]$.

Proof. After multiplying equation (4.1) by u and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}x + \int_\Omega (u_x)^2\,\mathrm{d}x + \int_\Omega b(n)u^2\,\mathrm{d}x = \int_\Omega g(u)nu^2\,\mathrm{d}x + \int_\Omega uc_x\chi'(c)u_x\,\mathrm{d}x.$$

Thus, by the Cauchy inequality and Standing Assumptions on the functions g, b and χ, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega (u_x)^2\,\mathrm{d}x + b(\|n_0\|_\infty)\int_\Omega u^2\,\mathrm{d}x \\ \le G_0\|n_0\|_\infty \int_\Omega u^2\,\mathrm{d}x + \frac{\|\chi'\|^2_{L^\infty(\mathbb{R})}}{2}\int_\Omega u^2(c_x)^2\,\mathrm{d}x,\end{aligned} \tag{4.36}$$

where constants $b(\|n_0\|_\infty) = \inf_n b(n) > 0$ and $G_0 = \sup_u g(u) > 0$ are finite by Standing Assumptions. To deal with the last term on the right-hand side of (4.36) we use estimate (4.32) and Lemma 4.13 combined with the Hölder, Sobolev and the ε-Cauchy inequalities in the following way

$$\int_\Omega u^2(c_x)^2\,\mathrm{d}x \le \|u\|_4^2\|c_x\|_4^2 \le C\|u\|_{W^{1,2}}\|u\|_1\|c_x\|_4^2 \le \varepsilon\|u\|^2_{W^{1,2}} + C(\varepsilon),$$

where $C(\varepsilon) = C(\varepsilon, \|u\|_1, \|c_x\|_4)$ is uniformly bounded in t. Moreover, by the Sobolev inequality and the Young inequality, we have

$$\int_\Omega u^2\,\mathrm{d}x \le C\|u\|^{2/3}_{W^{1,2}}\|u\|_1^{4/3} \le \varepsilon\|u\|^2_{W^{1,2}} + C_\varepsilon\|u\|_1^2, \tag{4.37}$$

Therefore, for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega (u_x)^2\,\mathrm{d}x + b(\|n_0\|_\infty)\int_\Omega u^2\,\mathrm{d}x \le \varepsilon\|u\|^2_{W^{1,2}} + C(\varepsilon). \tag{4.38}$$

The term on the right-hand side of (4.38) containing small $\varepsilon > 0$ can be absorbed by the corresponding two terms on the left-hand side. Thus, we obtain the following differential inequality

$$\frac{d}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}x + C\|u\|^2_{W^{1,2}} \le C,$$

with a constant $C > 0$, which, in particular, implies that $\|u(t)\|_2$ has to be bounded uniformly in t. □

The reminder of this section is devoted to the proof of Theorem 4.2 on the large time behaviour of solutions to problem (4.1)–(4.6) in a one dimensional domain.

Proof of Theorem 4.2. The local-in-time solutions constructed in Theorem 4.6 can be extended to all $t > 0$ due to *a priori* estimates, which we will obtain below in the study of the large time behaviour of solution. We skip this standard reasoning and we proceed directly to estimates of solutions for large values of $t > 0$.

Step 1: $\lim_{t\to\infty}\|u(t)\|_\infty = 0$. We apply the Duhamel principle (4.15) in the following way

$$\begin{aligned}\Big\|u(t) - \mathrm{e}^{\Delta t}u_0 - \int_0^t \mathrm{e}^{\Delta(t-s)}u(s)(g(u)n - b(n))(s)\,\mathrm{d}s\Big\|_\infty \\ = \Big\|\int_0^t \partial_x \mathrm{e}^{\Delta(t-s)}u(s)c_x(s)\chi'(c)\,\mathrm{d}s\Big\|_\infty.\end{aligned} \tag{4.39}$$

Using the property of the heat semigroup (A.3), the Hölder inequality, and Standing Assumptions on the function χ we estimate the right-hand side of (4.39) as follows

$$\begin{aligned}\Big\|\int_0^t \partial_x \mathrm{e}^{\Delta(t-s)}u(s)c_x(s)\chi'(c)\,\mathrm{d}s\Big\|_\infty &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{5}{6}}\mathrm{e}^{-\lambda_1(t-s)}\|u(s)c_x(s)\|_{3/2}\,\mathrm{d}s \\ &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{5}{6}}\mathrm{e}^{-\lambda_1(t-s)}\|u(s)\|_2\|c_x(s)\|_6\,\mathrm{d}s.\end{aligned} \tag{4.40}$$

Thus, by Lemma A.2 below, the integral on the right-hand side of (4.40) tends to zero because $\|u(t)\|_2$ is bounded by Lemma 4.14 and because $\|c_x(t)\|_6$

tends to zero, which is proved in Lemma 4.13. Hence, coming back to identity (4.39), we see that

$$\|u(t) - v(t)\|_\infty \to 0 \quad \text{as} \quad t \to \infty, \tag{4.41}$$

where $v(x,t)$ is a solution to the problem

$$v_t = v_{xx} + g(u)nu - b(n)u, \tag{4.42}$$

$$v(x,0) = u_0(x), \tag{4.43}$$

supplemented with the Neumann boundary conditions. We denote the nonlinear term on the right-hand side of (4.42) by $f \equiv g(u)nu - b(n)u$ and since g, b and n are bounded, there exist a constant $C > 0$ such that

$$\|f(\cdot,t)\|_1 \le C\|u(t)\|_1 \to 0 \quad \text{as} \quad t \to \infty,$$

by Theorem 4.10. Hence, by Lemma A.3 we obtain

$$\left\| v(t) - \frac{1}{|\Omega|}\int_\Omega v(t)\,\mathrm{d}x \right\|_\infty \to 0 \quad \text{as} \quad t \to \infty. \tag{4.44}$$

However, integrating equation (4.42) with respect to x and comparing it with (4.25), it is easy to see that $\int_\Omega u(t)\,\mathrm{d}x = \int_\Omega v(t)\,\mathrm{d}x$ for all $t > 0$. Therefore, using (4.41) and (4.44) we obtain the convergence

$$\left\| u(t) - \frac{1}{|\Omega|}\int_\Omega u(t)\,\mathrm{d}x \right\|_\infty \le \|u(t) - v(t)\|_\infty + \left\| v(t) - \frac{1}{|\Omega|}\int_\Omega u(t)\,\mathrm{d}x \right\|_\infty \to 0$$

as $t \to \infty$, which, in virtue of Theorem 4.10, completes the proof that $\lim_{t\to\infty} \|u(t)\|_\infty = 0$.

Step 2: Exponential decay of $\int_\Omega u(t)\,\mathrm{d}x$. Recall that the function $b(n(x,t))$ is bounded from below by $b(\|n_0\|_\infty) > 0$ because b is decreasing, *cf.* Standing Assumptions. Hence, since $\|u(t)\|_\infty \to 0$ as $t \to \infty$ and since $g(0) = 0$, there exist constants $T > 0$ and $\mu > 0$ such that for all $t \ge T$ and all $x \in \Omega$ we have

$$\big(g(u)n - b(n)\big)(x,t) \le -\mu.$$

Thus, using equation (4.25) we get the following differential inequality

$$\frac{d}{\mathrm{d}t}\int_\Omega u(t)\,\mathrm{d}x \le -\mu \int_\Omega u(t)\,\mathrm{d}x,$$

which implies the exponential decay

$$\|u(t)\|_1 \le \|u_0\|_1 \mathrm{e}^{-\mu t} \quad \text{for all} \quad t > 0. \tag{4.45}$$

Now, we use this estimate to improve Lemma 4.13.

Step 3: Exponential decay of $\|c_x(t)\|_p$ for each $p \in [1, \infty)$. Using the exponential decay of $\|u(t)\|_1$ from (4.45) in estimate (4.35), we obtain

$$\|c_x\|_p \le C\mathrm{e}^{-\beta t}\|c_{0,x}\|_p + C\int_0^t (t-s)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}} \mathrm{e}^{-(\beta+\lambda_1)(t-s)}\mathrm{e}^{-\mu s}\,\mathrm{d}s,$$

where the integral on the right-hand side decays exponentially by Lemma A.2.

Step 4: Exponential decay of $\|c(t)\|_\infty$. Applying the Duhamel principle (4.16), computing the L^∞-norm, and using the heat semigroup estimate (A.2) we have

$$\|c(t)\|_\infty \le C\mathrm{e}^{-\beta t}\|c_0\|_\infty + C\int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) \mathrm{e}^{-\beta(t-s)}\|u(s)\|_1\,\mathrm{d}s$$

for all $t > 0$ and a constant $C > 0$ independent of $t > 0$. Since $\|u(t)\|_1$ decays exponentially, see (4.45), we complete the proof of this step by Lemma A.2, again.

Step 5: Exponential decay of $\|u(t)\|_\infty$. Here, it suffices to repeat all the estimates from Step 1 using the exponential decay estimates of $\|c_x(t)\|_6$ established in Step 3 and the decay of $\|u(t)\|_1$ from Step 2.

Step 6: Exponential convergence $\lim_{t\to\infty}\|n(t)\|_\infty = n_\infty$. By Theorem 4.10, the limit

$$\lim_{t\to\infty}\int_\Omega n(t)\,\mathrm{d}x \equiv n_\infty = \int_\Omega n_0\,\mathrm{d}x - \int_0^\infty\int_\Omega \gamma g(u) n u\,\mathrm{d}x\,\mathrm{d}s$$

exists and is nonnegative. This is, in fact, exponential convergence, because by equation (4.27) and by Step 2 we have

$$\begin{aligned}\left|\int_\Omega n(t)\,\mathrm{d}x - n_\infty\right| &\le \gamma\int_t^\infty\int_\Omega |g(u)nu|\,\mathrm{d}x\,\mathrm{d}s \\ &\le \gamma G_0\|n_0\|_\infty \int_t^\infty \|u(s)\|_1\,\mathrm{d}s \le C\mathrm{e}^{-\mu t}.\end{aligned}$$

Now, applying Lemma A.3 with $f(x,t) = -\gamma g(u)nu$ to equation (4.3), since $\|f(\cdot,t)\|_1 \to 0$ exponentially as $t \to \infty$, we obtain

$$\left\|n(t) - \frac{1}{|\Omega|}\int_\Omega n(t)\,\mathrm{d}x\right\|_\infty \to 0 \quad \text{exponentially as} \quad t\to\infty.$$

Combining these two convergence results we complete the proof of Step 6.

Step 7: $\|w(t) - w_\infty\|_\infty \to 0$ *exponentially as* $t \to \infty$*.* Here, we define

$$w_\infty(x) = w_0(x) + \int_0^\infty b(n(x,t))u(x,t)\,\mathrm{d}t. \tag{4.46}$$

Notice, that since b is bounded and $\|u(t)\|_\infty$ decays exponentially, the right-hand side of (4.46) belongs to $L^\infty(\Omega)$. Moreover, it is easy to see that

$$\begin{aligned}\|w(t) - w_\infty(x)\|_\infty = \left\| \int_t^\infty b(n)u(x,s)\,\mathrm{d}s \right\|_\infty &\le C \int_t^\infty \|u(s)\|_\infty\,\mathrm{d}s \\ &\le C \int_t^\infty \mathrm{e}^{-\mu s}\,\mathrm{d}s \to 0\end{aligned}$$

exponentially as $t \to \infty$. This completes the proof of Step 7 and of Theorem 4.2. □

4.5 Problem in higher dimensions

Proof of Theorem 4.3. As in the one dimensional case, we consider a non-negative local-in-time solution to problem (4.1)–(4.6) which is constructed in Theorem 4.6. This solution can be continued to the global one due to estimates proved below.

Our first goal is to obtain an estimate for an L^p-norm of $u(t)$ for a certain fixed p, which is uniform in time. Here, we use the Duhamel formula (4.15) in the following way

$$\begin{aligned}\|u(t)\|_p \le &\left\| \mathrm{e}^{\Delta t}u_0 + \int_0^t \mathrm{e}^{\Delta(t-s)}u(s)(g(u)n - b(n))(s)\,\mathrm{d}s \right\|_p \\ &+ \left\| \int_0^t \nabla \mathrm{e}^{\Delta(t-s)}u(s)\nabla\chi(c(s))\,\mathrm{d}s \right\|_p.\end{aligned} \tag{4.47}$$

Step 1: Fixed $p = p_0 \in \left(\frac{d}{2}, \frac{d}{d-2}\right)$*.* We are going to apply inequality (4.47) with $p = p_0$. As in Step 1 of the proof of Theorem 4.2, the first term on the right-hand side of (4.47) will be denoted by $\|v(t)\|_{p_0}$, where $v(x,t)$ is a solution to the auxiliary problem (A.4)–(A.6) with $f = u(g(u)n - b(n))$ and $v_0 = u_0$. Recall, that

$$\|f(t)\|_1 \le \|u(t)(g(u)n - b(n))(t)\|_1 \le C\|u(t)\|_1 \quad \text{for all} \quad t > 0.$$

Hence, using Lemma A.3 (note that $p_0 < \frac{d}{d-2}$), inequality (4.32), and the elementary estimate $\|u_0\|_1 \le C(\Omega)\|u_0\|_{p_0}$, we obtain

$$\left\| e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}u(s)(g(u)n - b(n))(s)\,ds \right\|_{p_0} \le C(\|u_0\|_{p_0} + \|n_0\|_1), \quad (4.48)$$

for some constant $C > 0$ independent of $t > 0$.

Now, we deal with the second term on the right-hand side in (4.47). First, using the heat semigroup estimate (A.3), the assumption $\chi' \in L^\infty(\mathbb{R})$ and the Hölder inequality with $\frac{1}{q} = \frac{1}{p_0} + \frac{1}{2p_0}$ we obtain

$$\begin{aligned} \left\| \int_0^t \nabla e^{\Delta(t-s)}u(s)\nabla\chi(c(s))\,ds \right\|_{p_0} &\le C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p_0})-\frac{1}{2}} e^{-\lambda_1(t-s)}\|u\nabla\chi(c(s))\|_q\,ds \\ &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{d}{4p_0}-\frac{1}{2}} e^{-\lambda_1(t-s)}\|u(s)\|_{p_0}\|\nabla c(s)\|_{2p_0}\,ds. \end{aligned} \quad (4.49)$$

Notice, that, since $p_0 > d/2$, the function $(t-s)^{-\frac{d}{4p_0}-\frac{1}{2}}$ is integrable at $s = t$.

We proceed in a similar way using (4.16) and (A.2) to estimate

$$\begin{aligned} \|\nabla c(t)\|_{2p_0} &\le \left\|e^{(\Delta-\beta)t}\nabla c_0\right\|_{2p_0} + \int_0^t \left\|\nabla e^{(\Delta-\beta)(t-s)}u(s)\right\|_{2p_0}\,ds \\ &\le e^{-\beta t}\|\nabla c_0\|_{2p_0} + C\int_0^t (t-s)^{-\frac{d}{4p_0}-\frac{1}{2}} e^{-\lambda_1(t-s)}\|u(s)\|_{p_0}\,ds. \end{aligned} \quad (4.50)$$

Now, we define function

$$f(t) \equiv \sup_{0\le s\le t} \|u(s)\|_{p_0}.$$

Then, by inequality (4.50) we have that

$$\|\nabla c(t)\|_{2p_0} \le \|\nabla c_0\|_{2p_0} + Cf(t). \quad (4.51)$$

Finally, applying estimates (4.48), (4.49) and (4.51) into (4.47) we obtain

$$\|u(t)\|_{p_0} \le C\big(\|u_0\|_{p_0} + \|n_0\|_1\big) + Cf(t)\big(\|\nabla c_0\|_{2p_0} + Cf(t)\big),$$

which implies the following inequality

$$f(t) \le C_1(\|u_0\|_{p_0} + \|n_0\|_1) + C_2\|\nabla c_0\|_{2p_0}f(t) + C_3f^2(t) \quad (4.52)$$

for positive constants C_1, C_2 and C_3 independent of $t>0$ and of the solution. Now, we prove that, for a sufficiently small initial datum, inequality (4.52) implies that $f(t)$ has to be bounded function.

Indeed, denote $H(y) = C_3 y^2 + (B-1)y + A$, where $B = C_2\|\nabla c_0\|_{2p_0}$ and $A = C_1(\|u_0\|_{p_0} + \|n_0\|_1)$. It is easy to check that for $4AC_3 < (B-1)^2$, the equation $H(y)=0$ has two roots, say y_1 and y_2. Moreover, for $H'(0) = B-1 < 0$, those roots are both positive. Hence, since $f(t)$ is nonnegative and continuous if we assume that $f(0) = \|u_0\|_{p_0} \in (0, y_1)$ then $f(t) \in [0, y_1]$ for all $t>0$. Note here that $f(0) \le A$ because we can assume that $C_1 \ge 1$ without loss of generality. Moreover, by a direct calculation, we have $A < y_1$. Hence, $f(0) \in (0, y_1)$, and this completes the proof of Step 1.

Step 2: Estimate of $\sup_{t>0}\|u(t)\|_\infty$. We come back to inequality (4.47) with $p=\infty$. Note that $2 \in (\frac{d}{2}, \frac{d}{d-2})$ for $d \in \{2,3\}$. Hence, by Step 1, we have that $\sup_{t>0}\|u(t)\|_2 < \infty$. Thus, we use Lemma A.4 with $p=\infty$ and $q=2$ to obtain the following estimate of the first term on the right-hand side of (4.47)

$$\begin{aligned}\left\| \mathrm{e}^{\Delta t} u_0 + \int_0^t \mathrm{e}^{\Delta(t-s)} u(s)(g(u)n - b(n))(s)\,\mathrm{d}s \right\|_\infty \\ \le C\big(\|u_0\|_{p_0} + \|n_0\|_1 + \sup_{t>0}\|u(t)\|_2\big).\end{aligned}$$

Now, we deal with the second term on the right-hand side of (4.47). First, we consider the case $d=2$. By Step 1, for each $p \in [1,\infty)$ there is a constant $C>0$ such that $\|u(t)\|_p \le C$ for all $t>0$. Using relation (4.51) we also have that $\|\nabla c(t)\|_p \le C$ for all $t>0$ and for each $p \in [1,\infty)$. Hence, by the heat semigroup estimate (A.3) and the Hölder inequality, we obtain the inequalities

$$\begin{aligned}\left\| \int_0^t \nabla \mathrm{e}^{\Delta(t-s)} u(s)\nabla\chi(c(s))\,\mathrm{d}s \right\|_\infty &\le C\int_0^t (t-s)^{-\frac{1}{3}-\frac{1}{2}} \mathrm{e}^{-\lambda_1(t-s)} \|u\nabla\chi(c(s))\|_3\,\mathrm{d}s \\ &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{5}{6}} \mathrm{e}^{-\lambda_1(t-s)} \|u(s)\|_6 \|\nabla c(s)\|_6\,\mathrm{d}s,\end{aligned}$$

where the right-hand side is bounded uniformly in $t>0$.

Next, we consider the case $d=3$, where by Step 1, we have $\sup_{t>0}\|u(t)\|_p < \infty$ for each $p \in [1,3)$. Hence, using estimate (A.3) and the Hölder inequality, *cf.* (4.50), we get

$$\|\nabla c(t)\|_q \le \mathrm{e}^{-\beta t}\|\nabla c_0\|_q + C\int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \mathrm{e}^{-\lambda_1(t-s)} \|u(s)\|_p\,\mathrm{d}s. \quad (4.53)$$

Note, that the function $(t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}$ is integrable at $s=t$ for $q<\frac{3p}{3-p}$. Hence, for each $q\in[1,\infty)$ there exists a constant $C>0$ such that $\|\nabla c(t)\|_q \le C$ for all $t>0$.

Now, we are in a position to estimate the second term on the right-hand side of (4.47) for $d=3$ and we use the same reasoning as in the case $d=2$. First, for every $p\in[1,6)$ we obtain

$$\begin{aligned}\left\|\int_0^t \nabla \mathrm{e}^{\Delta(t-s)} u(s)\nabla\chi(c(s))\,\mathrm{d}s\right\|_p &\le C\int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\mathrm{e}^{-\lambda_1(t-s)}\|u\nabla\chi(c(s))\|_2\,\mathrm{d}s\\ &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\mathrm{e}^{-\lambda_1(t-s)}\|u(s)\|_{5/2}\|\nabla c(s)\|_{10}\,\mathrm{d}s.\end{aligned}$$

Since the function $(t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}$ is integrable at $s=t$ for each $p<6$, and since $\|u(s)\|_{5/2}$ and $\|\nabla c(s)\|_{10}$ are uniformly bounded in $s>0$, we have proved that for each $p\in[1,6)$ we have $\|u(t)\|_p\le C$ for all $t>0$.

Repeating these estimates for $p=\infty$, we obtain

$$\begin{aligned}\left\|\int_0^t \nabla \mathrm{e}^{\Delta(t-s)} u(s)\nabla\chi(c(s))\,\mathrm{d}s\right\|_\infty &\le C\int_0^t (t-s)^{-\frac{3}{2}\cdot\frac{1}{4}-\frac{1}{2}}\mathrm{e}^{-\lambda_1(t-s)}\|u\nabla\chi(c(s))\|_4\,\mathrm{d}s\\ &\le C\|\chi'\|_\infty \int_0^t (t-s)^{-\frac{7}{8}}\mathrm{e}^{-\lambda_1(t-s)}\|u(s)\|_{5}\|\nabla c(s)\|_{20}\,\mathrm{d}s,\end{aligned}$$

where the right-hand side is uniformly bounded in $t>0$. This completes the proof of Step 2.

Step 3: Exponential convergence of $(u(t),c(t),n(t),w(t))$. First, we show that $\lim_{t\to\infty}\|u(t)\|_p=0$ for every $p\in[1,\infty)$. Here, it suffices to combine the standard interpolation inequality of L^p-norms

$$\|u(t)\|_p \le C\|u(t)\|_1^{1/p}\|u(t)\|_\infty^{1-1/p}, \tag{4.54}$$

together with $\lim_{t\to\infty}\|u(t)\|_1=0$ provided by Theorem 4.10 and the estimate $\sup_{t>0}\|u(t)\|_\infty<\infty$ by Step 2.

Using this fact, we show that $\lim_{t\to\infty}\|u(t)\|_\infty=0$ following the reasoning from Step 2 again. Next, we prove the exponential decay of $\|u(t)\|_1$ in the same way as in Step 3 of the proof of Theorem 4.2. Therefore, using interpolation equation (4.54) again, we get the exponential decay of $\|u(t)\|_p$ for every $p\in[1,\infty)$ as well. By this fact, one can follow the reasoning from Step 2 once again, to obtain that $\|u(t)\|_\infty\to 0$ exponentially as $t\to\infty$. Moreover, by equation (4.16) we immediately show the exponential decay of $\|c(t)\|_\infty$.

Finally, to obtain exponential convergence of $\|n(t)\|_\infty$ and $\|w(t)\|_\infty$ it suffices to repeat arguments from Step 6 and 7 of the proof of Theorem 4.2.

□

Remark 4.15. *For $d \geq 3$ and under suitable smallness assumptions on initial data, we may show an exponential decay of the vector*

$$(\|u(t)\|_\infty, \|c(t)\|_\infty, \|n(t)\|_\infty, \|w(t)\|_\infty)$$

in the following way. First, multiplying equation (4.1) by u^{p-1} and integrating over Ω we obtain

$$\begin{aligned}\frac{d}{\mathrm{d}t}\int_\Omega u^p(t)\,\mathrm{d}x \leq -C\int_\Omega |\nabla u^{\frac{p}{2}}(t)|^2\,\mathrm{d}x + C\int_\Omega u^{\frac{p}{2}}\nabla u^{\frac{p}{2}}(t)\cdot\nabla\chi(c(t))\,\mathrm{d}x \\ +p\int_\Omega u^p(t)(g(u)n-b(n))(t)\,\mathrm{d}x,\end{aligned} \tag{4.55}$$

for some positive constant $C = C(p)$. Now, let us notice that by Standing Assumptions we have $\inf_n b(n) = b(\|n_0\|_\infty) > 0$. Hence, choosing $\|n_0\|_\infty$ so small to have

$$g(u)n - b(n) \leq G_0\|n_0\|_\infty - b(\|n_0\|_\infty) = -r < 0,$$

we obtain the estimate

$$\int_\Omega u^p(g(u)n-b(n))\,\mathrm{d}x \leq -r\int_\Omega u^p\,\mathrm{d}x = -r\|u^{\frac{p}{2}}(t)\|_2^2.$$

Thus, we get from equation (4.55) the following estimate

$$\frac{d}{\mathrm{d}t}\int_\Omega u^p(t)\,\mathrm{d}x \leq -C_1\|u^{p/2}(t)\|^2_{W^{1,2}(\Omega)} + C\|\chi'\|_\infty\int_\Omega |\nabla u^{p/2}(t)|u^{p/2}(t)|\nabla c(t)|\,\mathrm{d}x$$

with the constant $C_1 = \min\{r, 4(p-1)/p\}$. Moreover, we use the Hölder inequality to obtain

$$\frac{d}{\mathrm{d}t}\int_\Omega u^p(t)\,\mathrm{d}x \leq -C_1\|u^{\frac{p}{2}}(t)\|^2_{W^{1,2}(\Omega)} + 2(p-1)\|u^{\frac{p}{2}}(t)\|_{\frac{2d}{d-2}}\|\nabla u^{\frac{p}{2}}(t)\|_2\|\nabla c(t)\|_d. \tag{4.56}$$

Next, it suffices to estimate $\|\nabla c(t)\|_d$ using the Duhamel formula (4.16) and arguments analogous to those in the proof of Theorem 4.3 to obtain, for each

$q \in (\frac{d}{2}, d]$, the estimates

$$\begin{aligned}\|\nabla c(t)\|_d &\le C\mathrm{e}^{-(\beta+\lambda_1)t}\|\nabla c_0\|_d + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{d})-\frac{1}{2}}\mathrm{e}^{-(\beta+\lambda_1)(t-s)}\|u(s)\|_q\,\mathrm{d}s \\ &\le C\mathrm{e}^{-(\beta+\lambda_1)t}\|\nabla c_0\|_d + C\sup_{0\le s\le t}\|u(s)\|_q.\end{aligned} \tag{4.57}$$

Moreover, we recall the following Sobolev inequality

$$\|u^{\frac{p}{2}}\|_{\frac{2d}{d-2}} \le C\|u^{\frac{p}{2}}\|_{W^{1,2}(\Omega)}. \tag{4.58}$$

Hence, using inequalities (4.57) and (4.58) in (4.56) and choosing $p = q \in (\frac{d}{2}, d]$ we obtain

$$\frac{d}{\mathrm{d}t}\int_\Omega u^p(t)\,\mathrm{d}x \le \|u^{\frac{p}{2}}(t)\|^2_{W^{1,2}(\Omega)}\left(C\big(\|\nabla c_0\|_d + \sup_{0\le s\le t}\|u(s)\|_p\big) - C_1\right). \tag{4.59}$$

Observe now that, if the norms $\|\nabla c_0\|_d$ and $\|u_0\|_p$ are small enough, the right-hand side of inequality (4.59) is negative for $t \in [0, \varepsilon]$ for small $\varepsilon > 0$. Thus, $\|u(t)\|_p$ decreases on $[0, \varepsilon]$. We can repeat this argument on $[\varepsilon, 2\varepsilon]$ and, by induction, for all $t > 0$. Since $\|u(t)\|_p$ is decreasing, we have

$$C\big(\|\nabla c_0\|_d + \sup_{0\le s\le t}\|u(s)\|_p\big) - C_1 \le C\big(\|\nabla c_0\|_d + \|u_0\|_p\big) - C_1 < 0.$$

Hence, using the obvious inequality $\|u^{p/2}\|^2_{W^{1,2}(\Omega)} \ge \int_\Omega u^p\,\mathrm{d}x$, we obtain from (4.59) the exponential decay of $\|u(t)\|_p$ for each $p \in (d/2, d]$. To show the exponential decay of this norm for other $p \in [1, \infty]$, and to show the exponential convergence of (c, n, w) toward $\big(0, n_\infty, w_\infty(x)\big)$, it suffices to use a reasoning similar to that one in the proof of Theorem 4.3.

4.6 Blow up of solutions in two dimensional case

Proof of Theorem 4.5. Here, we are inspired by analogous proofs of a blow up of solutions to the parabolic-elliptic model of chemotaxis and we follow the work of Nagai [36]. For given numbers r_1 and r_2 satisfying $0 < r_1 < r_2 < \mathrm{dist}(q, \partial\Omega)$, we define the function $\phi \in C^1([0,\infty)) \cap W^{2,\infty}((0,\infty))$ by the formula

$$\phi(r) := \begin{cases} r^2 & \text{for} \quad 0 \le r \le r_1, \\ a_1 r^2 + a_2 r + a_3 & \text{for} \quad r_1 \le r \le r_2, \\ r_1 r_2 & \text{for} \quad r > r_2, \end{cases}$$

where

$$a_1 = -\frac{r_1}{r_2 - r_1}, \quad a_2 = \frac{2r_1 r_2}{r_2 - r_1}, \quad a_3 = -\frac{r_1^2 r_2}{r_2 - r_1}.$$

Hence, for fixed $q \in \Omega$, the function $\varphi(x) = \phi(|x - q|)$ satisfies $\varphi \in C^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$. Moreover, by direct computations, we obtain

$$\nabla\varphi(x) = \begin{cases} 2x & \text{for} \quad |x| \le r_1, \\ \dfrac{2r_1}{r_2 - r_1}(r_2 - |x|)\dfrac{x}{|x|} & \text{for} \quad r_1 \le |x| \le r_2, \\ 0 & \text{for} \quad |x| > r_2, \end{cases} \tag{4.60}$$

$$|\nabla\varphi(x)| \le 2(\varphi(x))^{1/2}, \tag{4.61}$$

$$\Delta\varphi(x) = 4 \quad \text{for} \quad |x| \le r_1 \qquad \text{and} \qquad \Delta\varphi(x) \le 2 \quad \text{for} \quad |x| > r_1. \tag{4.62}$$

Now, we consider a nonnegative solution (u, c, n) of problem (4.10)–(4.14) on an interval $[0, T_{\max})$ and define mass and the generalized moment by the formulas

$$M(t) = \int_\Omega u(x,t)\,\mathrm{d}x \quad \text{and} \quad I(t) = \int_\Omega u(x,t)\varphi(x)\,\mathrm{d}x.$$

By relation (4.62), it is clear that

$$\int_\Omega u(x,t)\Delta\varphi(x)\,\mathrm{d}x \le 4M(t). \tag{4.63}$$

Moreover, since the functions b, g and n are bounded and nonnegative, we obtain the following estimate

$$\int_\Omega (g(u)n - b(u))(x,t)u(x,t)\varphi(x)\,\mathrm{d}x \le C_3 I(t), \tag{4.64}$$

where $C_3 = G_0\|n_0\|_\infty$.

Next, we recall an estimate which is a straightforward adaptation of the result from [36] to system (4.10)–(4.12).

Lemma 4.16 ([36, Lemma 3.1]). *Let $q \in \Omega$, $0 < r_1 < r_2 < dist(q, \partial\Omega)$ and $\varphi(x) = \phi(|x - q|)$ be defined as above. Then, for all $t \in (0, T_{max})$, we have the following estimate*

$$\int_\Omega u(x,t)\nabla\varphi(x) \cdot \nabla c(x,t)\,\mathrm{d}x \le -\frac{1}{2\pi}M(t)^2 + C_1 M(t)I(t) + C_2 M(t)^{3/2} I(t)^{1/2}$$

for some constants C_1, C_2 depending on r_1, r_2 and $dist(q, \partial\Omega)$, only.

Now, multiplying equation (4.10) by $\varphi(x)$, integrating over Ω and using estimates (4.63)–(4.64) together with Lemma 4.16 we obtain

$$\frac{d}{\mathrm{d}t}I(t) \le 4M(t) - \frac{\chi_0}{2\pi}M^2(t) + (C_1\chi_0 M(t) + C_3)I(t) + C_2\chi_0 M(t)^{3/2} I(t)^{1/2}.$$

Note that for all $s > 0$ and $\varepsilon > 0$ we have the inequality $s^{1/2} \le \varepsilon + \frac{1}{4\varepsilon}s$. Hence, for fixed $\varepsilon > 0$, which will be chosen later, we use inequality (4.32) to obtain

$$\frac{d}{\mathrm{d}t}I(t) \le 4M(t) + \varepsilon - \frac{\chi_0}{2\pi}M^2(t) + C_4 I(t), \tag{4.65}$$

where

$$C_4 = C_3 + C_1\chi_0(\|u_0\|_1 + \frac{1}{\gamma}\|n_0\|_1) + \frac{C_2^2\chi_0^2(\|u_0\|_1 + \frac{1}{\gamma}\|n_0\|_1)^3}{4\varepsilon}. \tag{4.66}$$

Estimate (4.65) immediately implies that

$$\frac{d}{\mathrm{d}t}\Big(I(t)\mathrm{e}^{-C_4 t}\Big) \le \Big(4M(t) + \varepsilon - \frac{\chi_0}{2\pi}M^2(t)\Big)\mathrm{e}^{-C_4 t}. \tag{4.67}$$

Next, integrating equation (4.10) over Ω and using the inequalities $0 \le g(u)n \le C_3 = G_0\|n_0\|_\infty$ and $0 < b(t) \le B_0$, we deduce that

$$\frac{d}{\mathrm{d}t}M(t) \le C_3 M(t) \qquad \text{and} \qquad \frac{d}{\mathrm{d}t}M(t) \ge -B_0 M(t),$$

hence,

$$M(t) \le M_0\mathrm{e}^{C_4 t} \qquad \text{and} \qquad M(t) \ge M_0\mathrm{e}^{-B_0 t} \qquad \text{for all} \qquad t > 0. \tag{4.68}$$

Substituting estimates (4.68) in (4.67) we obtain the inequality

$$\frac{d}{\mathrm{d}t}\Big(I(t)\mathrm{e}^{-C_4 t}\Big) \le 4M_0 + \varepsilon - \frac{\chi_0 M_0^2}{2\pi}\mathrm{e}^{-(C_4+2B_0)t},$$

which implies

$$I(t)\mathrm{e}^{-C_4 t} \leq I(0) + (4M_0 + \varepsilon)t - \frac{\chi_0 M_0^2}{2\pi(C_4 + 2B_0)}(1 - \mathrm{e}^{-(C_4+2B_0)t}). \quad (4.69)$$

To complete the proof of the nonexistence of global-in-time solutions, it suffices to show that right-hand side of inequality (4.69) is negative for some $t > 0$. Hence, it suffices to study the function $f(t) = A + Bt - D(1 - \mathrm{e}^{-kt})$. First, note that f attains its minimum at a certain point if and only if $B < kD$, which is the case if the number $\frac{1}{2\pi}M_0(8\pi - \chi_0 M_0) + \varepsilon$ is negative. Here, one can chose for instance $\varepsilon = \frac{1}{4\pi}M_0(\chi_0 M_0 - 8\pi)$. Thus, for sufficiently small $f(0) = I(0) = A$ there exist $t > 0$ such that $f(t) < 0$.

Hence, under these assumptions the function $I(t)$ becomes negative in a finite time, which is impossible due to positivity of $\int_\Omega u(x,t)\varphi(x)\,\mathrm{d}x$. This means that a solution $u(t)$ with sufficiently small initial generalized moment $I(0)$ and the initial mass satisfying $M_0 > 8\pi/\chi_0$ cannot be continued for all $t > 0$. □

Appendix A

Complementary material

First, we recall some standard estimates on heat semigroup in bounded domain with the Neumann boundary condition.

Lemma A.1. *Let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist constants C_1, C_2 independent of t, f which have the following properties.*

(i) If $1 \le q \le p \le +\infty$ then

$$\|e^{t\Delta} f\|_{L^p(\Omega)} \le C t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\lambda_1 t} \|f\|_{L^q(\Omega)} \tag{A.1}$$

holds for all $f \in L^q(\Omega)$ satisfying $\int_\Omega f \, \mathrm{d}x = 0$.

(ii) If $1 \le q \le p \le +\infty$ then

$$\|e^{t\Delta} f\|_{L^p(\Omega)} \le C\left(1 + t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right) \|f\|_{L^q(\Omega)} \tag{A.2}$$

holds for all $f \in L^q(\Omega)$.

(iii) If $1 \le q \le p \le +\infty$ then

$$\|\nabla e^{t\Delta} f\|_{L^p(\Omega)} \le C t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} e^{-\lambda_1 t} \|f\|_{L^q(\Omega)} \tag{A.3}$$

is true for all $f \in L^q(\Omega)$.

Proofs of above inequalities (A.1)–(A.3) are well-known and can be found *e.g.* in [45], see also [50, Lemma 1.3].

Next, we recall a technical lemma which we use systematically in Chapter 4. We omit its elementary proof.

Lemma A.2. *Let $f \in L^\infty(0,\infty)$ satisfy $\lim_{t\to\infty} f(t) = 0$. Then, for each $k > -1$ and $M > 0$, we have $\lim_{t\to\infty} \int_0^t (t-s)^k e^{-M(t-s)} f(s)\,\mathrm{d}s = 0$. Moreover, the decay rate is exponential if the function $f(t) \to 0$ as $t \to \infty$ exponentially.*

The following result on the large time behaviour of solutions to the nonhomogeneous heat equation seems to be known. However, for the completeness of the exposition, we present its proof.

Lemma A.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let*

$$p \in [1,\infty] \quad \text{if} \quad d = 1, \quad p \in [1,\infty) \quad \text{if} \quad d = 2, \quad p \in \left[1, \frac{d}{d-2}\right) \quad \text{if} \quad d \geq 3.$$

Assume that $v_0 \in L^p(\Omega)$ and $f(x,t) \in L^\infty([0,\infty), L^1(\Omega))$ Then, the solution to the following initial value problem

$$v_t = \Delta v + f(x,t) \quad \text{for} \quad x \in \Omega,\ t > 0, \tag{A.4}$$

$$\frac{\partial v}{\partial \nu} = 0, \quad \text{for} \quad x \in \partial\Omega,\ t > 0, \tag{A.5}$$

$$v(x,0) = v_0(x) \quad \text{for} \quad x \in \Omega \tag{A.6}$$

satisfies

$$\|v(t)\|_p \leq C\big(\|v_0\|_p + \|v(t)\|_1 + \sup_{s>0} \|f(s)\|_1\big) \quad \text{for all} \quad t > 0, \tag{A.7}$$

where a constant C is independent of $t > 0$. Moreover, if $\|f(\cdot,t)\|_1 \to 0$ as $t \to \infty$ then we have

$$\left\| v(t) - \frac{1}{|\Omega|} \int_\Omega v(t)\,\mathrm{d}x \right\|_p \to 0 \quad \text{as} \quad t \to \infty. \tag{A.8}$$

In addition, if $\|f(\cdot,t)\|_1 \to 0$ exponentially as $t \to \infty$, then the convergence in (A.8) *is exponential, as well.*

Proof. The function

$$w(x,t) = v(x,t) - \frac{1}{|\Omega|} \int_\Omega v(x,t)\,\mathrm{d}x \tag{A.9}$$

is a solution to the following initial value problem

$$w_t = \Delta w + f(x,t) - \frac{1}{|\Omega|} \int_\Omega f(x,t)\,\mathrm{d}x \quad \text{for} \quad x \in \Omega,\ t > 0,$$

$$w(x,0) = w_0(x) = v_0(x) - \frac{1}{|\Omega|} \int_\Omega v_0(x)\,\mathrm{d}x,$$

supplemented with the Neumann boundary condition. We estimate the L^p-norm of w using its Duhamel representation

$$w(t) = e^{\Delta t} w_0 + \int_0^t e^{\Delta(t-s)} \left(f - \frac{1}{|\Omega|} \int_\Omega f \, \mathrm{d}x \right) \mathrm{d}s. \tag{A.10}$$

Obviously we have $\left\| f(s) - \frac{1}{|\Omega|} \int_\Omega f(x,s) \, \mathrm{d}x \right\|_1 \le 2 \|f(s)\|_1$. Thus, we may use estimate (A.1) (note that $\int_\Omega w(x,t) \, \mathrm{d}x = 0$ for all $t \ge 0$) in the following way

$$\|w(t)\|_p \le C e^{-\lambda_1 t} \|w_0\|_p + C \int_0^t (t-s)^{-\frac{d}{2}(1-\frac{1}{p})} e^{-\lambda_1 (t-s)} \|f(s)\|_1 \, \mathrm{d}s. \tag{A.11}$$

Now, the inequality $-\frac{d}{2}(1-\frac{1}{p}) > -1$ holds true due to the assumption on p. Moreover, notice that by the definition of w in (A.9), we have the following elementary inequalities

$$\|v(t)\|_p \le \|w(t)\|_p + |\Omega|^{\frac{1-p}{p}} \|v(t)\|_1 \tag{A.12}$$

$$\|w_0\|_p \le \|v_0\|_p + |\Omega|^{\frac{1-p}{p}} \|v_0\|_1 \le C \|v_0\|_p. \tag{A.13}$$

Thus, applying estimates (A.12)–(A.13) in inequality (A.11) we obtain bound (A.7) because $\sup_{t>0} \int_0^t (t-s)^{-\frac{d}{2}(1-\frac{1}{p})} e^{-\lambda_1 (t-s)} \, \mathrm{d}s < \infty$.

To show (A.8), we apply Lemma A.2 in inequality (A.11) which completes the proof of Lemma A.3. □

Next, we slightly generalise estimates from Lemma A.3.

Lemma A.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded and let $p \in [1,\infty]$. Assume that $v_0 \in L^p(\Omega)$ and $f \in L^\infty((0,\infty), L^q(\Omega))$ for some $\frac{dp}{2p+d} \le q \le p$. Then there exists a constant $C > 0$ such that the solution of problem* (A.4)–(A.6) *satisfies*

$$\|v(t)\|_p \le C \big(\|v_0\|_p + \|v(t)\|_1 + \sup_{t>0} \|f(s)\|_1 + \sup_{t>0} \|f(s)\|_q \big)$$

for all $t > 0$. Moreover, if $\|f(\cdot,t)\|_1 \to 0$ and $\|f(\cdot,t)\|_q \to 0$ as $t \to \infty$ then we have

$$\left\| v(t) - \frac{1}{|\Omega|} \int_\Omega v(t) \, \mathrm{d}x \right\|_p \to 0 \quad \text{as} \quad t \to \infty. \tag{A.14}$$

In addition, if $\|f(\cdot,t)\|_1 \to 0$ and $\|f(\cdot,t)\|_q \to 0$ exponentially as $t \to \infty$, then the convergence in (A.14) *is exponential as well.*

Proof. We proceed in the same way as in the proof of Lemma A.3. The only difference consists in rewriting inequality (A.11) in the following way

$$\begin{aligned}\|w(t)\|_p &\le Ce^{-\lambda_1 t}\|w_0\|_p \\ &\quad + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\lambda_1(t-s)} \Big(\|f(s)\|_q + |\Omega|^{\frac{1-q}{q}}\|f(s)\|_1\Big)\,\mathrm{d}s.\end{aligned}$$

This ends the proof of Lemma A.4. □

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